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Erratum: A central limit theorem for random ordered factorizations of integers

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Abstract

A gap in the proof of our estimates for odd moments in [3] is fixed.

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Dr. Ian Morris (University of Surrey) kindly pointed out that our application of Delange's Tauberian theorem contains a gap, which arises from the fact that $D_3(s)$ (and thus $D_1(s)$) has a branch-type singularity at ρ; see (0.1) below. Thus the function G (pp. 350–351 in our paper [3]) in the statement of Delange's Tauberian theorem fails to be analytic at ρ. This gap can be readily filled by the following arguments.

We first show that D_1 has a branch singularity at $s = \rho$. Let $k = 2\ell - 1$, $\ell \geqslant 1$. Consider (same notations as in [3])

$$\begin{aligned} D_1(s) &:= \sum_{n\geqslant 1} n^{-s} \sum_{m\geqslant 0} a_m(n) \left((m-\mu\log n)^k + (\log n)^{k/2}\right)^2, \\ D_2(s) &:= \sum_{n\geqslant 1} n^{-s} \sum_{m\geqslant 0} a_m(n) \left((m-\mu\log n)^{2k} + (\log n)^{k}\right) \\ &= \mathcal{M}_{2k}(s) + (-1)^k \mathcal{A}^{(k)}(s), \end{aligned}$$

and

$$D_3(s) := \tfrac{1}{2}(D_1(s) - D_2(s)) = (-1)^\ell \pi^{-1/2} \int_0^\infty \mathcal{M}_k^{(\ell)}(s+t) t^{-1/2}\,\mathrm{d}t.$$

By induction using the recurrence (Eq. (2.11) in [3])

$$\mathcal{M}_k(s) = \frac{1}{1-\mathcal{P}(s)} \sum_{0\leqslant j<k} \binom{k}{j} \mathcal{M}_j(s)\mathcal{B}_{k-j}(s) \qquad (k \geqslant 1)$$

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with $\mathcal{M}_0(s) = 1/(1-\mathcal{P}(s))$, where $\mathcal{B}_k(s) := \sum_{0\leqslant \ell\leqslant k}\binom{k}{\ell}\mu^\ell\mathcal{P}^{(\ell)}(s)$, we deduce the local expansion

$$\mathcal{M}_k(s) = \sum_{1\leqslant j\leqslant k+1} c_j(s-\rho)^{-j} + H_\rho(s),$$

for some coefficients c_j, where the generic symbol $H_c(s)$ represents an analytic function for $\Re(s)\geqslant c$, not necessarily the same at each occurrence. This in turn yields

$$D_3(s) = (-1)^\ell \sum_{1\leqslant j\leqslant k+1} \frac{c_j\Gamma(j-1/2)}{(j-1)!}(s-\rho)^{-j+1/2} + H_\rho(s). \tag{0.1}$$

Now

$$\begin{aligned} D_1(s) &= D_2(s) + 2D_3(s) \\ &= \sum_{1\leqslant j\leqslant k+1} \bar{c}_j(s-\rho)^{-j} + 2(-1)^\ell \sum_{1\leqslant j\leqslant k+1} \frac{c_j\Gamma(j-1/2)}{(j-1)!}(s-\rho)^{-j+1/2} + H_\rho(s), \end{aligned} \tag{0.2}$$

for some coefficients $\bar{c}_j$.

Thus, due to the presence of the branch singularity at $s=\rho$, we cannot apply the Tauberian theorem as that stated in [3]. However, as pointed out to us by Dr. Morris, we can apply the more general version of Delange's Tauberian theorem (also due to Delange; see [1, Theorem III] or [2, Theorem A]).

> *Let $F(s) := \sum_{n\geqslant 1}\alpha(n)n^{-s}$ be a Dirichlet series with nonnegative coefficients and convergent for $\Re(s) > \varrho > 0$. Assume (i) $F(s)$ is analytic for all points on $\Re(s)=\varrho$ except at $s=\varrho$; (ii) for $s\sim\varrho$, $\Re(s)>\varrho$,*
>
> $$F(s) = \frac{G(s)}{(s-\varrho)^\beta} + \sum_{1\leqslant j\leqslant m}(s-\varrho)^{-\beta_j}G_j(s) + H(s) \qquad (\beta>0),$$
>
> *where $m\geqslant 0$, $\Re(\beta_j)<\beta$ and G, H and the G_j's are analytic at $s=\varrho$ with $G(\varrho)\neq 0$. Then*
>
> $$\sum_{n\leqslant N}\alpha(n) \sim \frac{G(\varrho)}{\varrho\Gamma(\beta)}N^\varrho(\log N)^{\beta-1},$$
>
> *as $N\to\infty$.*

An alternative approach to fill the gap, still relying on the Tauberian theorem stated in [3], is to subtract from D_1 suitable functions having the same local expansion near ρ. More precisely, define

$$Z_\alpha(s) := \sum_{n\geqslant 2} n^{-s}(\log n)^\alpha \qquad (\alpha>0).$$

Then ($m := \lfloor\alpha\rfloor$ and $\theta := \{\alpha\}$)

$$Z_{m+\theta}(s) = \frac{(-1)^{m+1}}{\Gamma(1-\theta)}\int_0^\infty \zeta^{(m+1)}(s+t)t^{-\theta}\,\mathrm{d}t,$$

where ζ denotes Riemann's zeta function. Note that

$$\zeta(s) = \frac{1}{s-1} + \text{entire function},$$

so that

$$\begin{aligned} Z_\alpha(s) &= \frac{(m+1)!}{\Gamma(1-\theta)}(s-1)^{-1-m-\theta}\int_0^\infty x^{-\theta}(1+x)^{-m-2}\,\mathrm{d}x + H_1(s) \\ &= \Gamma(1+\alpha)\,(s-1)^{-1-\alpha} + H_1(s). \end{aligned}$$

Now let

$$D_4(s) := -2(-1)^\ell \sum_{1\leqslant j\leqslant k+1} \frac{c_j}{(j-1)!}\, Z_{j-3/2}(s+1-\rho) + CZ_k(s+1-\rho),$$

where C is chosen so large that D_4 has only nonnegative coefficients. This and (0.2) yield

$$D_1(s) + D_4(s) = \sum_{1\leqslant j\leqslant k+1} \bar{c}_j(s-\rho)^{-j} + C\Gamma(k+1)(s-\rho)^{-k-1} + H_\rho(s).$$

Thus we can apply Delange's Tauberian theorem (in the form stated in [3]) to $D_1(s) + D_4(s)$ and obtain an asymptotic approximation to the partial sum of the coefficients. More precisely, let $[n^{-s}]f(s)$ denote the coefficient of n^{-s} in the Dirichlet series $f(s) = \sum_{n\geqslant 1} f_n n^{-s}$. Then

$$\sum_{n\leqslant N} [n^{-s}]\,(D_1(s) + D_4(s)) \sim \rho^{-1}\left(\frac{\bar{c}_{k+1}}{k!} + C\right) N^\rho(\log N)^k.$$

But we also have, by definition,

$$\begin{aligned} \sum_{n\leqslant N} [n^{-s}]D_4(s) &= -2(-1)^\ell \sum_{1\leqslant j\leqslant k+1} \frac{c_j}{(j-1)!} \sum_{n\leqslant N} n^{\rho-1}(\log n)^{j-3/2} + C\sum_{n\leqslant N} n^{\rho-1}(\log n)^k \\ &\sim C\rho^{-1}N^\rho(\log N)^k. \end{aligned}$$

Thus we conclude that

$$\sum_{n\leqslant N} [n^{-s}]D_1(s) \sim \frac{\bar{c}_{k+1}}{\rho k!}\, N^\rho(\log n)^k,$$

as required.

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