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A CLT for winding angles of the arms for critical planar percolation

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Abstract

Consider critical percolation in two dimensions. Under the condition that there are k disjoint alternating black and white arms crossing the annulus $A(\ell, n)$, we prove a central limit theorem and variance estimates for the winding angles of the arms (as $n \to \infty, \ell$ fixed). This result confirms a prediction of Beffara and Nolin (Ann. Probab. 39: 1286–1304, 2011). Using this theorem, we also get a CLT for the multiple-armed incipient infinite cluster (IIC) measures.

Keywords: critical percolation ; incipient infinite cluster ; winding angle ; central limit theorem ; martingale ; arm events.
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1 Introduction

Percolation is a central model of statistical physics. Recall that performing a site percolation with parameter p on a lattice means that each site is chosen independently to be black (open) with probability p and white (closed) with probability $1-p$. It is well-known that site percolation on the regular triangular lattice exhibits a phase transition at a critical point $p_c = 1/2$: when $p \le p_c$ there is almost surely no infinite black connected component, whereas when $p > p_c$ there is almost surely a unique infinite black connected component.

Consider percolation on a planar lattice. In the literature, given an annulus in the lattice, the arm events are referred to the existence of some number of disjoint paths (arms, see below for a formal definition) crossing the annulus, the color of each path (black or white) being prescribed. These events are very useful for studying critical and near-critical percolation, the so-called arm exponents can be used to describe some fractal properties of critical percolation (see [1, 21, 27]).

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In this paper, we investigate the winding angles of the arms. The motivation mainly came from [2]. In that paper, Beffara and Nolin proved the existence of the monochromatic exponents, and the monochromatic j-arm exponent is strictly between the polychromatic j-arm and $(j+1)$-arm exponents. Their proof relied on analyzing the winding angles of the monochromatic and polychromatic arms. They believed that a central limit theorem should hold on the winding angles but did not give the proof.

The winding angles are also interesting in their own rights. In fact, the winding angles for several different planar models have been studied in the literature (e.g., random walk [3], Brownian motion [20, 25], loop-erased random walk (LERW) [14], self-avoiding walk (SAW) [8], radial Schramm-Loewner evolution (SLE) [7, 23, 26], see also Remark 1.4). In these models, from a macroscopic view, the conformal invariance properties were extensively used to derive the winding angle variance and CLT.

We focus on site percolation on the triangular lattice at criticality. We will realize the triangular lattice with site (or vertex) set $\mathbb{Z}^2$. For a given $(x,y)\in\mathbb{Z}^2$, its neighbors are defined as $(x\pm1,y),(x,y\pm1),(x+1,y-1)$ and $(x-1,y+1)$. Edges (bonds) between neighboring or adjacent sites therefore correspond to vertical or horizontal displacements of one unit, or diagonal displacements between two nearest sites along a line making an angle of $135°$ with the positive x-axis. Let each site of $\mathbb{Z}^2$ be black or white with probability $1/2$ independently of each other, and denote $P=P_{1/2}$ the corresponding product probability measure on the set of configurations. We also represent the measure as a (black or white) random coloring of the faces of the dual hexagonal lattice. Let us mention that the results in this paper also hold for critical bond percolation on $\mathbb{Z}^2$.

A path is a sequence of distinct sites connected by nearest neighbor bonds. The event that two sets of sites $X_1,X_2\subset\mathbb{Z}^2$ are connected by a black path is denoted by $X_1\leftrightarrow X_2$, and X_1,X_2 are connected by a black or white path is denoted by $X_1\leftrightarrow_1 X_2$. Given a set X of sites, let ∂X denote the boundary of X which contains sites in X that are adjacent to some site not in X. A circuit is a path which starts and ends at the same site and does not visit the same site twice, except for the starting site. For a circuit $\mathcal{C}$, define

$$\overline{\mathcal{C}}:=\mathcal{C}\cup\text{ interior sites of }\mathcal{C}.$$

Let $\sigma=(\sigma_i)$ be a sequence of colors. Given two circuits $\mathcal{C}_1,\mathcal{C}_2$ such that $\overline{\mathcal{C}_1}\subset\overline{\mathcal{C}_2}$, we say that $\mathcal{C}_1$ is σ-connected to $\mathcal{C}_2$, if there exist $|\sigma|$ disjoint paths (**arms**) connecting $\mathcal{C}_1$ and $\mathcal{C}_2$, ordered counterclockwise in a cyclic way, and the color of the i-th path is σ_i. Denote this event by $\mathcal{C}_1\leftrightarrow_\sigma\mathcal{C}_2$.

Define $||\mathbf{x}||_\infty:=\max\{|x|,|y|\}$ for $\mathbf{x}=(x,y)\in\mathbb{Z}^2$. For any $r\geq0$, define the square box of sites $B(r):=\{\mathbf{x}\in\mathbb{Z}^2:||\mathbf{x}||_\infty\leq r\}$. For $0<n<m$, define the annulus

$$A(n,m):=B(m)\backslash B(n).$$

For a crossing arm γ in an annulus $A(n,m)$, we often consider γ as a continuous curve by connecting the neighbor sites with line segments and assume the direction of γ is from $\partial B(n)$ to $\partial B(m)$. The **winding angle** of γ is the overall (algebraic) variation of the argument along it and is denoted by $\theta(\gamma)$.

For a polychromatic configuration in the annulus $A(n,m)$ (i.e., with at least one arm of each color), it is easy to see that the winding angles of the arms differ by at most 2π. In the following, we fix a deterministic way to choose a unique arm $\gamma_{n,m}$ and focus on $\theta(\gamma_{n,m})$, since there is essentially a unique winding angle from a macroscopic point of view.

For two positive functions f and g, the notation $f\asymp g$ means that f and g remain of the same order of magnitude, in other words that there exist two positive and finite constants c_1 and c_2 such that $c_1g\leq f\leq c_2g$.

Now we give our main result in Theorem 1.1, from which we see that a crossing arm of a polychromatic configuration in a long annulus looks like a random logarithmic spiral.

Theorem 1.1. *Assume that σ is alternating and $|\sigma|$ is even. Let l be the minimal number such that $|\partial B(l)| \geq |\sigma|$. We condition on the event $\partial B(l) \leftrightarrow_\sigma \partial B(n), n > l$. Let $\theta_n := \theta(\gamma_{l,n})$, and $a_n := \sqrt{Var[\theta_n]}$. Then we have*

$$a_n \asymp \sqrt{\log n}, \quad n > l,$$

and under the conditional measure $P(\cdot|\partial B(l) \leftrightarrow_\sigma \partial B(n))$

$$\frac{\theta_n}{a_n} \to_d N(0,1).$$

Remark 1.2. Following the conjecture made by Wieland and Wilson [26] (see (1.1) below), it is expected that

$$a_n^2 = \left(\frac{6}{|\sigma|^2} + o(1)\right) \log n \ \text{ as } n \to \infty,$$

which might be proved by conformal invariance and SLE approach. Heuristically, one can decompose a typical arm into a short path near the origin and for which the winding angle contribution is of a smaller order than $\sqrt{\log n}$ and a long path far from the origin and for which the winding angle contribution can be approximated by the winding angle of multiple (mutually-avoiding) SLE paths (for multiple SLE paths, see Remark 1.4). However, it is still not clear how to prove it rigorously. We will actually use another sequence $h_n \sim a_n$ instead of a_n, for the expressions for h_n, see (3.27).

Remark 1.3. Our proof mainly relies on the Strong Separation Lemma and the coupling argument in [10]. These two ingredients may be extended to the following more general case without too much work: σ is polychromatic and σ either does not contain neighboring white colors or does not contain neighboring black colors (here we take the first and last elements of σ to be neighbors). See Remark 7 in [5] and subsection 5.4 in [10]. Thus Theorem 1.1 can also be extended to this case.

Remark 1.4. There exist some analogous results on the winding angles of various random paths. For the classic results on random walk and Brownian motion, the interested reader is referred to a short survey [4]. We address some results concerning SLE as follows. For radial SLE_κ, Schramm [23] showed that the variance of the winding angle of the radial SLE_κ path truncated at distance ε from the origin grows like $(\kappa+o(1))\log(1/\varepsilon)$ (see also [24]), a CLT was proved simultaneously. However, as the authors said in [26], conditioned there are k disjoint random paths in a long annulus, there are few results about the windings compared with the one path case. Conditioned on the event that there are k mutually-avoiding SLE_κ paths crossing the annulus $A(1,R)$ of $\mathbb{R}^2$, Wieland and Wilson [26] made the conjecture that the winding angle variance of the paths is

$$\left(\frac{\kappa}{k^2} + o(1)\right) \log R \ \text{ as } R \to \infty. \tag{1.1}$$

In [14], conditioned on the annulus $A(1,R)$ of $\delta\mathbb{Z}^2$ has 2 (resp. 3) disjoint LERWs, Kenyon showed that the winding angle of the paths has variance tending to $(\frac{1}{2}+o(1))\log R$ (resp. $(\frac{2}{9}+o(1))\log R$) as $R \to \infty$ while $\delta \to 0$ (see "Remarks on LERW" in [26] about Kenyon's incorrect values). This confirms the formula (1.1) in the cases of $\kappa = 2$ and $k = 2, 3$, since LERW converges to SLE_2 [18]. We also note that in section 8 and subsection 10.6 in [6], using the method from quantum gravity, Duplantier showed the formula 1.1. See also [7] for the proof from Coulomb gas method.

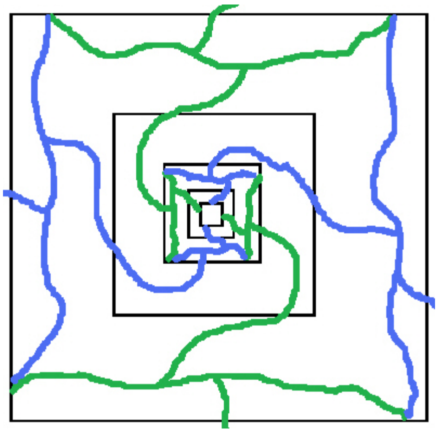

Figure 1: We construct a sequence of Markovian good faces to get a martingale structure of the winding angle.

Idea of the proof. The strategy of the proof is similar to [17]. In that paper, Kesten and Zhang constructed a sequence of black circuits surrounding the origin in a Markovian way (the circuits could be thought of as stopping times). Using these circuits, they got a martingale structure on the maximal number of disjoint black circuits in a large box, and then they applied McLeish's CLT [19] for the martingales. However, for our setting clearly we can not use Markovian black circuits to get a martingale structure of the winding angle, thanks to [10], we can use faces instead. As introduced in Section 3 of [10], the faces are some type of circuits which are composed of alternating color paths. With some conditions added to the faces, we construct a sequence of good faces to get a martingale structure of the winding angle. See Fig. 1. Since we are considering conditional measure, it is hard to estimate some events and check the conditions in McLeish's CLT. Thanks to the coupling argument in [10], we can get some weak dependence of the faces and carry out the method from [17]. In the proof of Theorem 1.6 in [5], the authors used black circuits with defects, we note that these circuits may not adapt to the proof of our setting.

Let us give a direct corollary of Theorem 1.1 in the following. First we introduce some definitions. In the celebrated paper [15], Kesten gave the mathematical rigorous definition of the incipient infinite cluster, which describes large critical percolation clusters from the microscopic (lattice scale) perspective [13] and the configuration at a "typical exceptional time" of dynamical critical percolation [11]. More precisely, let $\mathbf{0}$ denote the origin, it is shown in [15] that the limit

$$\nu(E) := \lim_{n\to\infty} P(E|\mathbf{0} \leftrightarrow \partial B(n))$$

exists for any event E that depends on the state of finitely many sites in $\mathbb{Z}^2$. The unique extension of ν to a probability measure on configurations of $\mathbb{Z}^2$ exists and we call ν the **incipient infinite cluster (IIC) measure** or one-armed IIC measure. Following Kesten's spirit, Damron and Sapozhnikov introduced multiple-armed IIC measures in [5]. Suppose that σ is alternating and let ℓ be the minimal number such that $|\partial B(l)| \geq |\sigma|$. For every cylinder event E, it is shown in Theorem 1.6 in [5] the limit

$$\nu_\sigma(E) := \lim_{n\to\infty} P(E|\partial B(l) \leftrightarrow_\sigma \partial B(n)) \tag{1.2}$$

exists. The unique extension of ν_σ to a probability measure on the configurations of $\mathbb{Z}^2$ exists. We call ν_σ the **σ-IIC measure**.

Corollary 1.5. *Suppose that σ is alternating and let l be the minimal number such that $|\partial B(l)| \geq |\sigma|$. Suppose $a_n, n > l$ is the sequence defined in Theorem 1.1. Under $P(\cdot|\partial B(l) \leftrightarrow \partial B(m)), m > n$ and ν_σ, we define θ_n similarly as in Theorem 1.1. Under the above measures, we have*

$$\frac{\theta_n}{a_n} \to_d N(0,1).$$

Remark 1.6. Under $P(\cdot|\partial B(l) \leftrightarrow \partial B(m))$ and ν_σ, using Lemma 2.1 and the coupling result in Lemma 2.3, it is not hard to check that $\sqrt{Var(\theta_n)} = (1 + o(1))a_n$. We shall not give the proof here, though.

Remark 1.7. Following the spirit of [13, 26], choosing a typical site from the boundary (or external perimeter) of a large cluster or the pivotal sites of a crossing event in a large box uniformly, we can consider how the arms wind around the chosen site. Since it is expected that the local measure viewed from the typical site converges to the corresponding σ-IIC measure, one would expect a CLT from this corollary. (For the existence of the limiting measures, see Remark of Theorem 1 in [13]. In the 4-arm case, see also Remark 1.7 in [9] for the analog of the tightness result in Theorem 8 in [13].)

For a monochromatic σ, there are many ways to select the arms and the winding angles of these arms may differ a lot. Let $j = |\sigma|$. We denote by $I_{j,n}$ the set of all the winding angles of the arms in the annulus $A(\ell, n)$, where $|\partial B(l)| \geq j$. Let $\theta_{max,n} = \theta_{max,j,n} := \max\{\alpha, \alpha \in I_{j,n}\}$ and $\theta_{min,n} := \min\{\alpha, \alpha \in I_{j,n}\}$. It is easy to show (see [2]) that $\theta_{max,n}$ and $\theta_{min,n}$ are of order $\pm \log n$. Furthermore, by Proposition 7 in [2], if one sorts the elements of $I_{j,n}$ in increasing order: $\alpha_1 < \alpha_2 < \cdots < \alpha_{|I_{j,n}|}$, then for every $1 \leq i \leq |I_{j,n}| - 1$, $\alpha_{i+1} - \alpha_i < 2\pi$.

In the 1-arm case, one can also get central limit theorems for $\theta_{max,n}$ and $\theta_{max,n} - \theta_{min,n}$ by similar methods for the proof of Theorem 1.1. Using good black circuits and the coupling argument for the 1-arm case in [10], the proof is similar and simpler, we leave it to the reader and just give the following statements for $\theta_{max,n}$.

Under the conditional measure $P(\cdot|\mathbf{0} \leftrightarrow_\sigma \partial B(n))$ and the IIC measure ν we both have

$$E[\theta_{max,n}] \asymp \log n, \quad Var[\theta_{max,n}] \asymp \log n,$$

and

$$\frac{\theta_{max,n} - E[\theta_{max,n}]}{\sqrt{Var[\theta_{max,n}]}} \to_d N(0,1).$$

In this paper, we only prove the alternating four arm case, since the proof for this case applies to all cases that σ is alternating, with no essential changes. In general, we assume $\sigma = (black, white, black, white)$ in the following.

Throughout this paper, $c, c_1, c_2, \ldots$ denote positive finite constants that may change from line to line or page to page according to the context.

2 Preliminary results

As remarked above, we focus on the alternating four arm case. Firstly, following the terminology of [10], let us introduce some definitions. Suppose Γ is the set of percolation interfaces which cross the annulus $A = A(m, n)$. If there are $p \geq 2$ interfaces crossing A and if $x_1, \ldots, x_p$ denote the endpoints of these interfaces on $\partial B(n)$, define the **quality**

$$Q(\Gamma) := \frac{1}{n} \inf_{k \neq l} |x_k - x_l|,$$

where $|\cdot|$ denotes Euclidean distance. If $\Gamma = \emptyset$, we define $Q(\Gamma) = 0$.

Let $x_1,\dots,x_4$ be four midpoints of four distinct bonds in $\partial B(n)$. We will adopt here cyclic notation, i.e., for any $i,j\in\mathbb{Z}$, we have $x_j=x_i$ if $j\equiv i \mod 4$. For any $i\in\mathbb{Z}$, let γ_i be a simple path of hexagons joining x_i to x_{i+1} and $\gamma_i\subset B(n)$ (here we see γ as a sequence of sites). Assume γ_i is black if i is odd and white otherwise. Then we call the circuit Θ which is composed of these four paths a configuration of **(interior) faces**, and say Θ are faces of $\partial B(n)$. Define the quality of a configuration of faces $Q(\Theta)$ to be the least distance between the endpoints (i.e., $x_1,\dots,x_4$) , normalized by n. Similar to the definition of faces, we call a circuit around $\partial B(n)$ **exterior faces**, if the circuit is composed of four alternating color paths contained in $(\mathbb{Z}^2\backslash B(n))\cup\partial B(n)$ with endpoints on $\partial B(n)$. Note that our definition of exterior faces is exactly the same as the definition of faces in [10]. Similar to the quality of faces, we can also define quality of exterior faces.

The following properties of arm events are well-known, see [16, 21]. We assume that the reader is familiar with the FKG-inequality, the Russo-Seymour-Welsh (RSW) technology. See [12, 27]. Using FKG, RSW and Theorem 11 in [21], the statements related to faces can be easily obtained from the classic results, the proof is omitted here. Note that for general alternating color sequence σ with even $|\sigma|$, the corresponding notion of faces can be defined, and analogous results hold in this more general case.

1. A priori bound for arm events: For any color sequence σ, there exist constants $c(|\sigma|),\beta(|\sigma|)>0$ such that for all $n_1<n_2$,
$$P(\partial B(n_1)\leftrightarrow_\sigma \partial B(n_2))\geq c\left(\frac{n_1}{n_2}\right)^\beta. \tag{2.1}$$
Furthermore, given $\sigma=(black,white,black,white)$, for any faces Θ of $\partial B(n_1)$ with $Q(\Theta)>\frac{1}{4}$,
$$P(\Theta\leftrightarrow_\sigma \partial B(n_2))\geq c\left(\frac{n_1}{n_2}\right)^\beta. \tag{2.2}$$
2. **Quasi-multiplicativity**: For any color sequence σ, there is a constant $c(|\sigma|)>0$, such that for all $n_1<n_2<n_3$,
$$\begin{aligned} cP(\partial B(n_1)\leftrightarrow_\sigma \partial B(n_2))P(\partial B(n_2)\leftrightarrow_\sigma \partial B(n_3)) &\leq P(\partial B(n_1)\leftrightarrow_\sigma \partial B(n_3))\\ &\leq P(\partial B(n_1)\leftrightarrow_\sigma \partial B(n_2))P(\partial B(n_2)\leftrightarrow_\sigma \partial B(n_3)).\end{aligned}$$
Furthermore, given $\sigma=(black,white,black,white)$, for any faces Θ of $\partial B(n_1)$ with $Q(\Theta)>\frac{1}{4}$,
$$P(\Theta\leftrightarrow_\sigma \partial B(n_2))P(\partial B(n_2)\leftrightarrow_\sigma \partial B(n_3))\asymp P(\Theta\leftrightarrow_\sigma \partial B(n_3)).$$

Define $R(m,n):=\{z\in\mathbb{Z}^2: |\arg(z)|<\frac{\pi}{10}\}\cap A(m,n)$. We say a path $\gamma\subset R(m,n)$ is a crossing path in $R(m,n)$ if the endpoints of γ lie adjacent (Euclidean distance smaller than $\sqrt{2}$) to the rays of argument $\pm\frac{\pi}{10}$ respectively. By step 3 of the proof of Theorem 5 in [2], we obtain the following lemma.

Lemma 2.1. *Define event*

$$\mathcal{B}:=\{\text{there exist at least } K\log(n/m) \text{ disjoint black crossing paths in } R(m,n)\}.$$

There exist constants $c_1,c_2>0$, such that for all $K>0$ and $n>m>0$,

$$P(\mathcal{B})\leq c_1\exp[(-c_2K+c_1\log K)\log(n/m)].$$

In particular, there exist constants $c_3,K_0>0$, such that for all $K>K_0$ and $n>m>0$,

$$P(\mathcal{B})\leq c_1\exp[-c_3K\log(n/m)].$$

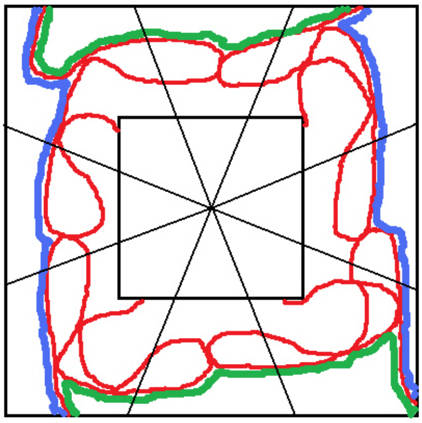

Figure 2: Four interfaces crossing the annulus induce a natural configuration of good faces.

Define $\mathcal{C}_i := \{z \in \mathbb{C} : -\frac{3\pi}{8} + \frac{i\pi}{4} < arg(z) < -\frac{\pi}{8} + \frac{i\pi}{4}\}, 1 \le i \le 8$. Lemma 2.2 is the straightforward analog of Lemma 3.4 in [10] with a little modification. See Fig. 2. The proof is exactly the same as the second proof of Lemma 3.4 in [10], see Figure 3.2 in [10] for the strategy. We will adopt cyclic notation in Lemma 2.2, i.e., for any $i, j \in \mathbb{Z}$, we have $\mathcal{C}_j = \mathcal{C}_i$ if $j \equiv i \mod 8$.

Lemma 2.2. *In the annulus $A = A(n, 2n)$, let $\mathcal{R}$ be the event that there are exactly 4 disjoint alternating arms crossing A, and the resulting 4 interfaces are contained respectively in $\mathcal{C}_{i-1} \cup \mathcal{C}_i \cup \mathcal{C}_{i+1}, i = 2, 4, 6, 8$, with the endpoints of the interfaces on the two boundaries of A belonging to $\mathcal{C}_i$. Then $P(\mathcal{R}) > c$ for an absolute constant $c > 0$.*

Define

$$A(p) := A(2^p, 2^{p+1}), \quad p \ge 0.$$

For each $A(p)$, if the event $\mathcal{R}$ (see the definition in Lemma 2.2) happens, then the four interfaces induce a natural configuration of faces $\Theta \subset A(p)$ of $\partial B(2^{p+1})$. We call Θ **good faces**, and say there are good faces in $A(p)$. See Fig. 2. Θ are composed of four paths $\{\Theta(j)\}_{1 \le j \le 4}$, where the path $\Theta(j) \subset \{z \in \mathbb{C} : (j-2)\frac{\pi}{2} - \frac{\pi}{8} < arg(z) < j\frac{\pi}{2} + \frac{\pi}{8}\}$.

Lemma 2.3. *There is a constant $c_1 > 0$, such that for all $r = 2^{p_0}, R > 10r$, $0 \le t \le \log_2(R/r)$ and any faces Θ of $\partial B(r)$,*

$$P\left(\bigcap_{0 \le i \le t} \{\textit{there are no good faces in } A(p_0 + i)\} | \Theta \leftrightarrow_\sigma \partial B(R)\right) \le \exp(-c_1 t). \tag{2.3}$$

*Furthermore, there exists a constant $c_2 > 0$, such that for any $r_1 = 2^{p_1}, r_2 = 2^{p_2}$ with $r_1 \le r_2 < R/10$, given any faces Θ_1 of $\partial B(r_1)$ and faces Θ_2 of $\partial B(r_2)$, for any $0 \le t \le \log_2(R/r_2)$, there is a **coupling** of the measures $P(\cdot|\Theta_1 \leftrightarrow_\sigma \partial B(R))$(or $P(\cdot|\partial B(1) \leftrightarrow_\sigma \partial B(R))$) and $P(\cdot|\Theta_2 \leftrightarrow_\sigma \partial B(R))$ so that with probability at least $1 - \exp(-c_2 t)$, there exists $0 \le i \le t$ such that $A(p_2 + i)$ has identical good faces Θ_0 for both measures, and the configurations in the domain $B(R)\backslash\overline{\Theta}_0$ are also identical.*

Proof. Since the proof is basically the same as for Proposition 3.1 in [10] with little modifications for our setting, we only sketch the proof and omit some details. First let us prove inequality (2.3).

Without loss of generality, let $N = \lfloor t/8 \rfloor \ge 1$. For $0 \le i \le N$, let $r_i = 8^i r$. Now we sample under $P(\cdot|\Theta \leftrightarrow \partial B(R))$ the set of interfaces $\Gamma(r_i)$ which start from the four endpoints of Θ until they reach radius r_i. We proceed by induction on the scale $r_i, i \geq 0$. By the Strong Separation Lemma (Lemma 3.3 in [10], Lemma 6.2 in [5] is another version), we have

$$P(Q(\Gamma(2r_i) > 1/4)|\Theta \leftrightarrow_\sigma \partial B(R), \Gamma(r_i)) > c_3. \tag{2.4}$$

It can be checked that Θ plus $\Gamma(2r_i)$ induce configurations of faces of $\partial B(2r_i)$, which is denoted by Θ_i. For each $A(4r_i, r_{i+1})$, define

$$\mathcal{R}_i := \{\text{there are good faces in } A(4r_i, r_{i+1})\}.$$

Let S_i be the union of all sites whose color is determined by the crossing interfaces according to the measure $P(\cdot|\Theta_i, \Theta_i \leftrightarrow_\sigma \partial B(R))$. Let S_i be a possible value for S_i such that $\mathcal{R}_i$ holds. If $Q(\Gamma(2r_i)) > 1/4$, by Lemma 2.2, using a gluing technique, it can be showed (the same to the proof of (3.1) in [10]) there is a universal constant $c > 0$ such that

$$2^{-|S|}/c \leq P(S_i = S|\Theta_i, \Theta_i \leftrightarrow_\sigma \partial B(R)) \leq c2^{-|S|}. \tag{2.5}$$

Then we get

$$P(\mathcal{R}_i|\Theta_i, \Theta_i \leftrightarrow_\sigma \partial B(R)) \geq c_4 P(\mathcal{R}_i) > c_5. \tag{2.6}$$

Combining (2.4) and (2.6), we have

$$P(\mathcal{R}_i|\Theta \leftrightarrow_\sigma \partial B(R), \Gamma(r_i)) > c_3 c_5. \tag{2.7}$$

By (2.7), choosing c_2 appropriately, we get

$$\begin{aligned}
&P\left(\bigcap_{0\leq i\leq t}\{\text{there are no good faces in } A(p_0+i)\}|\Theta \leftrightarrow_\sigma \partial B(R)\right)\\
&\leq P(\mathcal{R}_0^c|\Theta \leftrightarrow_\sigma \partial B(R))P(\mathcal{R}_1^c|\Theta \leftrightarrow_\sigma \partial B(R), \mathcal{R}_0^c)\ldots\\
&\quad P(\mathcal{R}_{N-1}^c|\Theta \leftrightarrow_\sigma \partial B(R), \mathcal{R}_0^c\mathcal{R}_1^c\ldots\mathcal{R}_{N-2}^c)\\
&\leq P(\mathcal{R}_0^c|\Theta \leftrightarrow_\sigma \partial B(R))\max_{\Gamma(r_1)}\{P(\mathcal{R}_1^c|\Theta \leftrightarrow_\sigma \partial B(R), \Gamma(r_1))\}\ldots\\
&\quad \max_{\Gamma(r_{N-1})}\{P(\mathcal{R}_{N-1}^c|\Theta \leftrightarrow_\sigma \partial B(R), \Gamma(r_{N-1}))\}\\
&\leq \exp(-c_2 t).
\end{aligned}$$

Now let us prove the coupling result. Sampling the interfaces for the measures $P(\cdot|\Theta_1 \leftrightarrow_\sigma \partial B(R))$ and $P(\cdot|\Theta_2 \leftrightarrow_\sigma \partial B(R))$ by induction similarly to the above argument, using the Strong Separation Lemma and (2.5), one can show the coupling result with the strategy very similar to the proof of Proposition 3.1 in [10]. We omitted the details here and refer the reader to "Proof of Proposition 3.1, continued." in [10]. To couple $P(\cdot|\partial B(1) \leftrightarrow_\sigma \partial B(R))$ and $P(\cdot|\Theta_2 \leftrightarrow_\sigma \partial B(R))$, one can use an argument analogous to the proof of Proposition 3.6 in [10], the details are also omitted here. □

Remark 2.4. The coupling argument was introduced in [10], which is a very useful tool to gain weak independence of events. The coupling argument is based on the Strong Separation Lemma, which was first proposed in [5] (see a broad overview for the strong separation phenomenon in many planar statistical physics models in Appendix A of [10]), and is an extension of Kesten's arm separation lemma [16, 21].

Consider measure $P(\cdot|\partial B(1) \leftrightarrow_\sigma \partial B(n))$, $n \geq 2^{q+q^{\frac{1}{3}}+2}$ (the exponent $1/3$ can be replaced with any fixed positive constant which is smaller than $1/2$). Define for $1 \leq p \leq q$

$$m(p) = m(p,\omega) := \min\{t \in \{p, p+1, \cdots\} : \text{there exist good faces in } A(t)\}.$$

Define

$$\mathcal{A}_q := \{m(q) \le q + q^{\frac{1}{3}}\}.$$

The following lemma implies that when we go out of a box (or faces) to search good faces, we can quickly find them with high probability.

Lemma 2.5. *There exists constant* $c_1 > 0$ *such that for* $1 \le p \le q$, $0 \le t \le q + q^{\frac{1}{3}} - p$ *and* $n \ge 2^{q+q^{\frac{1}{3}}+2}$,

$$P(m(p) - p \ge t | \partial B(1) \leftrightarrow_\sigma \partial B(n)) \le \exp(-c_1 t). \tag{2.8}$$

Furthermore, there exists constant $c_2 > 0$*, such that for* $1 \le p_1 < p \le q$*, given faces* Θ_{p_1} *of* $\partial B(2^{p_1})$,

$$P(m(p) - p \ge t | \Theta_{p_1} \leftrightarrow_\sigma \partial B(n)) \le \exp(-c_2 t). \tag{2.9}$$

Let $\Theta_0 := \partial B(1)$*. In particular, there exists constant* $c_3 > 0$*, such that for* $0 \le p \le q-1$*, given faces* Θ_p *of* $\partial B(2^p)$,

$$P(\mathcal{A}_q^c | \Theta_p \leftrightarrow_\sigma \partial B(n)) \le \exp(-c_3 q^{\frac{1}{3}}). \tag{2.10}$$

Proof. If there exist 4 alternating arms from $\partial B(1)$ to $\partial B(2^{p+\lfloor t/2 \rfloor})$ and there is no extra disjoint arm, the crossing interfaces between $\partial B(1)$ to $\partial B(2^{p+\lfloor t/2 \rfloor})$ induce faces Θ of $\partial B(2^{p+\lfloor t/2 \rfloor})$. So we look at whether there is an extra arm or not, and if not, we can condition on faces Θ. This means

$$\begin{aligned}
&P(m(p) - p \ge t | \partial B(1) \leftrightarrow_\sigma \partial B(n)) \\
&\le \frac{P(\partial B(1) \leftrightarrow_1 \partial B(2^{p+\lfloor t/2 \rfloor}) \Box \partial B(1) \leftrightarrow_\sigma \partial B(n))}{P(\partial B(1) \leftrightarrow_\sigma \partial B(n))} \\
&\quad + \sum_\Theta P(\Theta | \partial B(1) \leftrightarrow_\sigma \partial B(n)) P(m(p + \lfloor t/2 \rfloor) - p - \lfloor t/2 \rfloor \ge t/2 | \Theta, \Theta \leftrightarrow_\sigma \partial B(n)).
\end{aligned}$$

Using Reimer's inequality [22], RSW and Lemma 2.3, choosing $c_1, c_4 > 0$ appropriately, we get

$$\begin{aligned}
&P(m(p) - p \ge t | \partial B(1) \leftrightarrow_\sigma \partial B(n)) \\
&\le P(\partial B(1) \leftrightarrow_1 \partial B(2^{p+\lfloor t/2 \rfloor}) + \sum_\Theta P(\Theta | \partial B(1) \leftrightarrow_\sigma \partial B(n)) \exp(-c_4 t) \\
&\le \exp(-c_1 t).
\end{aligned}$$

The proof of (2.9) is similar and simpler, we leave it to the reader. Applying (2.8) and (2.9), we obtain (2.10) immediately. □

3 Proof of theorem

As remarked in the Introduction, we focus on the alternating 4-arm case throughout this section. Let Θ_p denote the good faces in $A(m(p))$, $1 \le p \le q$. Recall $\Theta_0 = \partial B(1)$. Define

$$\mathcal{F}_p := \sigma\text{-field generated by } \{0,1\}^{\overline{\Theta}_p \setminus \mathbf{0}}, \quad 1 \le p \le q.$$

Let $\mathcal{F}_0$ be the trivial σ-field.

Define

$$P_q(\cdot) = P_{q,n}(\cdot) := P(\cdot | \mathcal{A}_q, \partial B(1) \leftrightarrow_\sigma \partial B(n)),$$

$$E_q(\cdot) = E_{q,n}(\cdot) := E(\cdot | \mathcal{A}_q, \partial B(1) \leftrightarrow_\sigma \partial B(n)).$$

Let us concentrate on the measure P_q in the following.

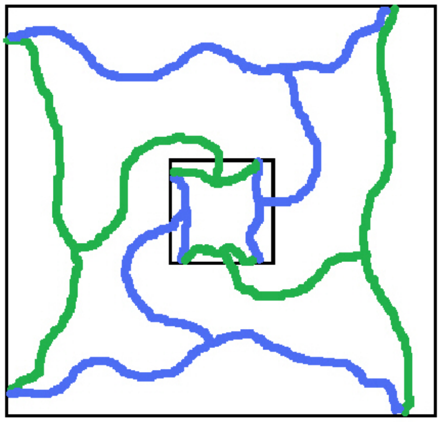

Figure 3: Denote the good faces in this figure by Θ_1 and Θ_2. It is clear that $\theta(\Theta_1, \Theta_2) = \frac{\pi}{2}$.

For $1 \le p_1 < p_2 \le q$, we define the winding angle between good faces Θ_{p_1} and Θ_{p_2} as follows. If $\Theta_{p_1} = \Theta_{p_2}$ (note that this happens when $m(p_1) = m(p_2)$, which is possible), let $\theta(\Theta_{p_1}, \Theta_{p_2}) = 0$. Now suppose $\Theta_{p_1} \ne \Theta_{p_2}$ and $\gamma([0,1]) \subset (\overline{\Theta}_{p_2} \backslash \overline{\Theta}_{p_1}) \cup \Theta_{p_1}$ is a black or white path connecting Θ_{p_1} and Θ_{p_2}. Recall the definition of $\Theta(j)$ which is defined after the definition of good faces. Assume the starting point $\gamma(0) \in \Theta_{p_1}(k_1)$ and the endpoint $\gamma(1) \in \Theta_{p_2}(k_2)$, $1 \le k_1, k_2 \le 4$. We connect $\gamma'(0) := 2^{m(p_1)+1} \exp(i(k_1 - 1)\frac{\pi}{2})$ and $\gamma(0)$ by the segment $\overline{\gamma'(0)\gamma(0)}$ and connect $\gamma(1)$ and $\gamma'(1) := 2^{m(p_2)+1} \exp(i(k_2 - 1)\frac{\pi}{2})$ by $\overline{\gamma(1)\gamma'(1)}$. Then we construct a new path $\gamma' := \overline{\gamma'(0)\gamma(0)}\gamma\overline{\gamma(1)\gamma'(1)}$. Define

$$\theta(\Theta_{p_1}, \Theta_{p_2}) := \theta(\gamma'), \quad 1 \le p_1 < p_2 \le q.$$

See Fig. 3. By the definition of good faces, it is easy to see that $\theta(\cdot,\cdot)$ is well defined and independent of the choice of γ.

Recall $\Theta_0 = \partial B(1)$. For $1 \le p \le q$, now we define $\theta(\Theta_0, \Theta_p)$ similarly. Among the black or white paths connecting Θ_0 and Θ_p in $\overline{\Theta}_p \backslash \mathbf{0}$, we choose the unique one in some definite way, and denote it by γ. Assume the endpoint $\gamma(1) \in \Theta_p(k), 1 \le k \le 4$. Then we connect $\gamma(1)$ and $\gamma'(1) := 2^{m(p)+1} \exp(i(k-1)\frac{\pi}{2})$ by $\overline{\gamma(1)\gamma'(1)}$. Let $\gamma'(0) = \gamma(0)$. Then we construct a new path $\gamma' := \gamma\overline{\gamma(1)\gamma'(1)}$. Define

$$\theta(\Theta_0, \Theta_p) := \theta(\gamma'), \quad 1 \le p \le q.$$

By a simple topological argument, we get

$$\theta(\Theta_0, \Theta_q) = \sum_{i=0}^{q-1} \theta(\Theta_i, \Theta_{i+1}). \tag{3.1}$$

Define for $1 \le p \le q$

$$\Delta_p = \Delta_{p,q,n} := E_q(\theta(\Theta_0, \Theta_q)|\mathcal{F}_p) - E_q(\theta(\Theta_0, \Theta_q)|\mathcal{F}_{p-1}). \tag{3.2}$$

Thus

$$\theta(\Theta_0, \Theta_q) - E_q(\theta(\Theta_0, \Theta_q)) = \sum_{p=1}^{q} \Delta_p.$$

See Fig. 1 for an illustration.

Similarly to [17], we have to become more specific about the probability space. Set $\Omega = \{black, white\}^{\mathbb{Z}^2}$. The σ-field $\mathscr{B}$ is generated by the cylinder sets of Ω. Recall the notation $P(\cdot)$ we introduced earlier, the underlying probability is just $(\Omega, \mathscr{B}, P)$. Let $(\Omega', \mathscr{B}', P')$ be a copy of $(\Omega, \mathscr{B}, P)$. We need some cumbersome but unavoidable expressions in the following. For example, to determine

$$\theta(\Theta_p(\omega), \Theta_q(\omega'))(\omega') \tag{3.3}$$

in Lemma 3.1 below one first determines $\Theta_p(\omega)$ in the configuration $\omega \in \Omega$ under $(\Omega, \mathscr{B}, P_q)$, and then $\Theta_q(\omega')$ in the configuration $\omega' \in \Omega'$ under $(\Omega', \mathscr{B}', P'(\cdot|\mathcal{A}_q, \Theta_p(\omega) \leftrightarrow_\sigma \partial B(n))$ (note that these two conditional laws are different). With faces $\Theta_p(\omega)$ and $\Theta_q(\omega')$ fixed, one can determine (3.3) in the configuration ω'. For convenience, we define

$$P'_{q,\Theta_p(\omega)}(\cdot) = P'_{q,n,\Theta_p(\omega)}(\cdot) := P'(\cdot|\mathcal{A}_q, \Theta_p(\omega) \leftrightarrow_\sigma \partial B(n)),$$

$$E'_{q,\Theta_p(\omega)}(\cdot) = E'_{q,n,\Theta_p(\omega)}(\cdot) := E'(\cdot|\mathcal{A}_q, \Theta_p(\omega) \leftrightarrow_\sigma \partial B(n)).$$

It will also be necessary to introduce a copy $(\Omega'', \mathscr{B}'', P''_{q,\Theta_p(\omega)})$ of $(\Omega', \mathscr{B}', P'_{q,\Theta_p(\omega)})$. The element of Ω'' is denoted by ω'' and expectation with respect to $P''_{q,\Theta_p(\omega)}$ is denoted by $E''_{q,\Theta_p(\omega)}$. The following lemma is the analog of (2.11) in [17]. However, we can not get the analogous results corresponding to (2.12) and Lemma 2 in [17]: in [17], exact independence can be revealed, which is not possible here due to the "global" conditioning.

Lemma 3.1. *We have*

$$\Delta_p(\omega) = \theta(\Theta_{p-1}(\omega), \Theta_p(\omega))(\omega) + E''_{q,\Theta_p(\omega)}\theta(\Theta_p(\omega), \Theta_q(\omega''))(\omega'') \\ - E'_{q,\Theta_{p-1}(\omega)}\theta(\Theta_{p-1}(\omega), \Theta_q(\omega'))(\omega'), \quad 1 \le p \le q.$$

Proof. The proof is similar to the proof of (2.11) in [17]. Fix a configuration ω. By (3.1) and the definition of Δ_p in (3.2), we have

$$\Delta_p = \theta(\Theta_{p-1}, \Theta_p) + E_q(\theta(\Theta_p, \Theta_q)|\mathcal{F}_p) - E_q(\theta(\Theta_{p-1}, \Theta_q)|\mathcal{F}_{p-1}), \quad 1 \le p \le q. \tag{3.4}$$

Combining (3.4) and

$$E_q(\theta(\Theta_p, \Theta_q)|\mathcal{F}_p)(\omega) = E''_{q,\Theta_p(\omega)}\theta(\Theta_p(\omega), \Theta_q(\omega''))(\omega''),$$

$$E_q(\theta(\Theta_{p-1}, \Theta_q)|\mathcal{F}_{p-1})(\omega) = E'_{q,\Theta_{p-1}(\omega)}\theta(\Theta_{p-1}(\omega), \Theta_q(\omega'))(\omega'),$$

the conclusion follows immediately. □

Lemma 3.2. *There exist constants $c_i > 0, 1 \le i \le 9$ such that for $q \ge 1$ and $n \ge 2^{q+q^{\frac{1}{3}}+2}$*

$$P_q(m(p) - p \ge x) \le c_1 \exp(-c_2 x), \quad x \ge 0, 0 \le p \le q, \tag{3.5}$$

$$P_q(|\Delta_p| \ge x) \le c_3 \exp(-c_4 x), \quad x \ge 0, 1 \le p \le q, \tag{3.6}$$

$$P_q\left(\max_{1\le p\le q} |\Delta_p| \ge \varepsilon q^{\frac{1}{2}}\right) \le c_5 q \exp(-c_6 \varepsilon q^{\frac{1}{2}}), \quad \varepsilon > 0, \tag{3.7}$$

$$E_q\left(\max_{1\le p\le q} \Delta_p^2\right) \le c_7 q, \tag{3.8}$$

$$c_8 q \le \sum_{p=1}^{q} E_q \Delta_p^2 \le c_9 q. \tag{3.9}$$

Proof. By the definitions of $\mathcal{A}_q$ and $P_q(\cdot)$, it's obvious that for $x > q+q^{\frac{1}{3}}-p$, $P_q(m(p)-p \geq x) = 0$. By (2.10) and (2.8), there exist constants $c_1, c_2 > 0$ such that for $0 \leq x \leq q+q^{\frac{1}{3}}-p$,

$$P_q(m(p)-p \geq x) \leq c_1 P(m(p)-p \geq x | \partial B(1) \leftrightarrow_\sigma \partial B(n)) \leq c_1 \exp(-c_2 x).$$

Then we conclude (3.5).

Using the symmetric property of the polychromatic setting of the windings and coupling argument, one may give a short proof of (3.6). However, the following method we used to prove (3.6) can be modified easily to prove the analogous result of the 1-arm case.

By Lemma 3.1, for $1 \leq p \leq q$,

$$\begin{aligned}|\Delta_p(\omega)| \leq &|E'_{q,\Theta_{p-1}(\omega)}\theta(\Theta_{p-1}(\omega), \Theta_q(\omega'))(\omega') - E''_{q,\Theta_p(\omega)}\theta(\Theta_p(\omega), \Theta_q(\omega''))(\omega'')| \\ &+ |\theta(\Theta_{p-1}(\omega), \Theta_p(\omega))(\omega)|. \end{aligned} \tag{3.10}$$

For fixed ω, now we will prove there exists a constant $c_{10} > 0$ such that

$$|E'_{q,\Theta_{p-1}(\omega)}\theta(\Theta_{p-1}(\omega), \Theta_q(\omega')) - E''_{q,\Theta_p(\omega)}\theta(\Theta_p(\omega), \Theta_q(\omega''))| \leq c_{10}(m(p,\omega)-p+1). \tag{3.11}$$

Remark 3.3. When $p = 1$, we just let $P'_{q,\Theta_{p-1}(\omega)} = P'_q, E'_{q,\Theta_{p-1}(\omega)} = E'_q$, the arguments in the following also adapt to this case.

First we show there exist $c_{11}, c_{12}, c_{13} > 0$ such that for $0 \leq p_1 < p_2 \leq q$ and $x \geq c_{13}(p_2 - p_1)$,

$$P'_{q,\Theta_{p_1}(\omega)}(|\theta(\Theta_{p_1}(\omega), \Theta_{p_2}(\omega'))| \geq x) \leq c_{11}\exp(-c_{12}x). \tag{3.12}$$

Let us note first that,

$$\begin{aligned}&P'_{q,\Theta_{p_1}(\omega)}(|\theta(\Theta_{p_1}(\omega), \Theta_{p_2}(\omega'))| \geq x) \\ &\leq P'_{q,\Theta_{p_1}(\omega)}(m(p_2,\omega') - p_2 \geq c_{14}x) \\ &\quad + P'_{q,\Theta_{p_1}(\omega)}(m(p_2,\omega') - p_2 \leq c_{14}x, |\theta(\Theta_{p_1}(\omega), \Theta_{p_2}(\omega'))| \geq x),\end{aligned}$$

where c_{14} will be fixed later. Let us now estimate each term separately. For the first term, by Lemma 2.5, similar to the proof of (3.5), we get that there exist $c_{15}, c_{16} > 0$ such that for $x \geq 0$,

$$P'_{q,\Theta_{p_1}(\omega)}(m(p_2,\omega') - p_2 \geq x) \leq c_{15}\exp(-c_{16}x).$$

For short, let $p'_2 := \min\{p_2 + \lfloor c_{14}x \rfloor + 1, q + \lfloor q^{\frac{1}{3}} \rfloor\}$. Recall $R(m,n) := \{z \in \mathbb{Z}^2 : |\arg(z)| < \frac{\pi}{10}\} \cap A(m,n)$ and we call a path in $R(m,n)$ crosses $R(m,n)$ if it connects the two rays of argument $\pm\frac{\pi}{10}$. For the second term,

$$\begin{aligned}&P'_{q,\Theta_{p_1}(\omega)}(m(p_2,\omega') - p_2 \leq c_{14}x, |\theta(\Theta_{p_1}(\omega), \Theta_{p_2}(\omega'))| \geq x) \\ &\leq \frac{c_{17}P'(m(p_2,\omega') - p_2 \leq c_{14}x, |\theta(\Theta_{p_1}(\omega), \Theta_{p_2}(\omega'))| \geq x, \Theta_{p_1}(\omega) \leftrightarrow_\sigma \partial B(n))}{P'(\Theta_{p_1}(\omega) \leftrightarrow_\sigma \partial B(n))} \quad \text{by (2.10)} \\ &\leq \frac{c_{18}P'(m(p_2,\omega') - p_2 \leq c_{14}x, |\theta(\Theta_{p_1}(\omega), \Theta_{p_2}(\omega'))| \geq x)}{P'(\Theta_{p_1}(\omega) \leftrightarrow_\sigma \partial B(2^{p'_2}))} \quad \text{by quasi-multiplicativity} \\ &\leq \frac{c_{50}P'(\text{there exist at least } \frac{x}{2\pi} - 4 \text{ disjoint black crossing paths in } R(2^{p_1-1}, 2^{p'_2}))}{P'(\Theta_{p_1}(\omega) \leftrightarrow_\sigma \partial B(2^{p'_2}))}.\end{aligned}$$

Since $x \geq c_{13}(p_2 - p_1)$, by Lemma 2.1 and (2.2), then we can choose c_{13}, c_{14} appropriately, such that

$$P'_{q,\Theta_{p_1}(\omega)}(m(p_2,\omega') - p_2 \leq c_{14}x, |\theta(\Theta_{p_1}(\omega), \Theta_{p_2}(\omega'))| \geq x) \leq c_{19}\exp(-c_{20}x).$$

Now we have bounded the two terms and completed the proof of (3.12).

To obtain inequality (3.11), let us consider the two cases in the following.

Case 1 ($m(p,\omega) \geq q-1$). By (3.12), for all $x \geq c_{13}(q-p+1)$,

$$P'_{q,\Theta_{p-1}(\omega)}(|\theta(\Theta_{p-1}(\omega), \Theta_q(\omega'))| \geq x) \leq c_{11}\exp(-c_{12}x).$$

Thus there exists $c_{21} > 0$ such that

$$E'_{q,\Theta_{p-1}(\omega)}|\theta(\Theta_{p-1}(\omega), \Theta_q(\omega'))| \leq c_{21}(q-p+1).$$

Similarly we have

$$E''_{q,\Theta_p(\omega)}|\theta(\Theta_p(\omega), \Theta_q(\omega''))| \leq c_{22}(q-p+1).$$

Combining the above two inequalities, choosing c_{10} appropriately, we obtain (3.11) since $m(p,\omega) \geq q-1$.

Case 2 ($m(p,\omega) \leq q-2$). By the coupling result of Lemma 2.3 and (2.10), there exists a constant $c_{23} > 0$ such that for $1 \leq p \leq q-1$, we can couple $P'_{q,\Theta_{p-1}(\omega)}$ and $P''_{q,\Theta_p(\omega)}$ so that with probability at least $1-\exp(-c_{23}x)$, there exists $l(p,\omega,\omega') = l(p,\omega,\omega'')$, such that $m(p,\omega)+1 \leq l(p,\omega,\omega') \leq m(p,\omega)+1+x \leq q$, the good faces $\Theta_{l(p,\omega,\omega')}(\omega')$ and $\Theta_{l(p,\omega,\omega'')}(\omega'')$ are identical, and the configurations in $B(n)\backslash\overline{\Theta}_{l(p,\omega,\omega')}(\omega')$ and $B(n)\backslash\overline{\Theta}_{l(p,\omega,\omega'')}(\omega'')$ are also identical. Denote by $\mathcal{S}$ the event that the above coupling succeeds for $m(p,\omega)+1 \leq l(p,\omega,\omega') \leq q$.

Let us note first that,

$$\begin{aligned}
&|E'_{q,\Theta_{p-1}(\omega)}\theta(\Theta_{p-1}(\omega), \Theta_q(\omega')) - E''_{q,\Theta_p(\omega)}\theta(\Theta_p(\omega), \Theta_q(\omega''))| \\
&\leq E'_{q,\Theta_{p-1}(\omega)}|I_{\mathcal{S}}\theta(\Theta_{p-1}(\omega), \Theta_{l(p,\omega,\omega')}(\omega'))| + E''_{q,\Theta_p(\omega)}|I_{\mathcal{S}}\theta(\Theta_p(\omega), \Theta_{l(p,\omega,\omega')}(\omega''))| \\
&\quad + E'_{q,\Theta_{p-1}(\omega)}|I_{\mathcal{S}^c}\theta(\Theta_{p-1}(\omega), \Theta_q(\omega'))| + E''_{q,\Theta_p(\omega)}|I_{\mathcal{S}^c}\theta(\Theta_p(\omega), \Theta_q(\omega''))|.
\end{aligned}$$

We now estimate the four terms separately.

For short, define

$$M(p,\omega) := min\{m(p,\omega)+2+\lfloor c_{24}x\rfloor, q+1\},$$

$$\mathcal{B} := \{\text{there exist at least } \tfrac{x}{2\pi}-4 \text{ disjoint black crossing paths in } R(2^{p-1}, 2^{M(p,\omega)})\},$$

where c_{24} will be fixed later. For $x \geq c_{25}(m(p,\omega)-p+1)$, we have the following inequality, where c_{25} will be fixed later.

$$\begin{aligned}
&P'_{q,\Theta_{p-1}(\omega)}(\mathcal{S}, |\theta(\Theta_{p-1}(\omega), \Theta_{l(p,\omega,\omega')}(\omega'))| \geq x) \\
&\leq P'_{q,\Theta_{p-1}(\omega)}(\mathcal{S}, l(p,\omega,\omega') \geq m(p,\omega)+1+c_{24}x) + P'_{q,\Theta_{p-1}(\omega)}(\mathcal{B}) \\
&\leq c_{40}\exp(-c_{26}x) + \frac{c_{27}P'(\mathcal{B}, \Theta_p(\omega) \leftrightarrow_\sigma \partial B(n))}{P'(\Theta_p(\omega) \leftrightarrow_\sigma \partial B(n))} \quad \text{by (2.10) and coupling result} \\
&\leq c_{40}\exp(-c_{26}x) + \frac{c_{28}P'(\mathcal{B})}{P'(\Theta_p(\omega) \leftrightarrow_\sigma \partial B(2^{M(p,\omega)}))} \quad \text{by quasi-multiplicativity.}
\end{aligned}$$

By Lemma 2.1 and (2.2), then we can choose appropriate c_{24}, c_{25} such that

$$P'_{q,\Theta_{p-1}(\omega)}(\mathcal{S}, |\theta(\Theta_{p-1}(\omega), \Theta_{l(p,\omega,\omega')}(\omega'))| \geq x) \leq c_{29}\exp(-c_{30}x).$$

Choosing c_{10} large enough, then we obtain

$$E'_{q,\Theta_{p-1}(\omega)}|I_{\mathcal{S}}\theta(\Theta_{p-1}(\omega), \Theta_{l(p,\omega,\omega')}(\omega'))| \leq c_{10}(m(p,\omega)-p+1)/4.$$

Similarly

$$E''_{q,\Theta_p(\omega)}|I_{\mathcal{S}}\theta(\Theta_p(\omega), \Theta_{l(p,\omega,\omega')}(\omega''))| \leq c_{10}(m(p,\omega)-p+1)/4.$$

For the third term, let c_{10} be a large enough constant, then

$$\begin{aligned}
&E'_{q,\Theta_{p-1}(\omega)}|I_{\mathcal{S}^c}\theta(\Theta_{p-1}(\omega),\Theta_q(\omega'))| \\
&\le [P'_{q,\Theta_{p-1}(\omega)}(\mathcal{S}^c)]^{\frac{1}{2}}[E'_{q,\Theta_{p-1}(\omega)}|\theta(\Theta_{p-1}(\omega),\Theta_q(\omega'))|^2]^{\frac{1}{2}} \text{ by Cauchy-Schwarz inequality} \\
&\le c_{31}\exp(-c_{32}(q-1-m(p,\omega)))(q-p+1) \text{ by the coupling result and (3.12)} \\
&\le c_{10}(m(p,\omega)-p+1)/4 \text{ by } m(p,\omega)\le q-2.
\end{aligned}$$

Similarly we have

$$E''_{q,\Theta_p(\omega)}|I_{\mathcal{S}^c}\theta(\Theta_p(\omega),\Theta_q(\omega''))| \le c_{10}(m(p,\omega)-p+1)/4.$$

Now the four terms have been bounded, which ends the proof of (3.11).

By (3.10) and (3.11), we can choose an appropriate constant c_{33} such that

$$P_q(|\Delta_p|\ge x)\le P_q(m(p,\omega)-p\ge c_{33}x)+P_q(|\theta(\Theta_{p-1}(\omega),\Theta_p(\omega))|\ge x/2).$$

Now we bound the two terms in the r.h.s of above inequality. For the first term, by (3.5) we get

$$P_q(m(p,\omega)-p\ge c_{33}x)\le c_1\exp(-c_2c_{33}x).$$

For the second term, if $2\le p\le q$, by (3.12), there exist $c_{34},c_{35}>0$ such that

$$\begin{aligned}
&P_q(|\theta(\Theta_{p-1}(\omega),\Theta_p(\omega))(\omega)|\ge x/2) \\
&=\sum_{\Theta_{p-1}(\omega)} P_q(\Theta_{p-1}(\omega))P'_{q,\Theta_{p-1}(\omega)}(\theta(\Theta_{p-1}(\omega),\Theta_p(\omega'))>x/2)\le c_{34}\exp(-c_{35}x);
\end{aligned}$$

if $p=1$, we can bound the second term directly by (3.12) and Note 3.3. Thus (3.6) is concluded. Using (3.6), we conclude (3.7),(3.8) and the second inequality in (3.9) immediately. Now let us prove the first inequality in (3.9). By Lemma 3.1 and (3.11), we have

$$\Delta_p\ge\theta(\Theta_{p-1},\Theta_p)-c_{10}(m(p)-p+1).$$

Applying (2.10) and inequality in the above, gives

$$\begin{aligned}
&E_q\Delta_p^2\ge P_q(\Delta_p\ge 1) \\
&\ge P_q(m(p-1)=p-1,m(p)=p+1,\theta(\Theta_{p-1},\Theta_p)\ge 2c_{10}+1) \\
&\ge P(m(p-1)=p-1,m(p)=p+1,\theta(\Theta_{p-1},\Theta_p)\ge 2c_{10}+1)|\partial B(1)\leftrightarrow_\sigma\partial B(n)) \\
&\quad -c_{41}\exp(-c_{36}q^{\frac{1}{3}}) \qquad (3.13)
\end{aligned}$$

Define event

$$\begin{aligned}
\mathcal{A}:=\{&A(p-1)\text{ and }A(p+1)\text{ have good faces, }\Theta_{p-1}\leftrightarrow_\sigma\Theta_{p+1}, \\
&\theta(\Theta_{p-1},\Theta_p)\ge 2c_{10}+1\text{, there exist five disjoint paths crossing }A(p)\}.
\end{aligned}$$

Assume $A(p-1)$ and $A(p+1)$ have good faces. By Lemma 2.2, the four interfaces crossing $A(p+1)$ also induce "exterior good faces" Θ'_{p+1} around $\partial B(2^{p+1})$ in $A(p+1)$, see Fig. 2. By RSW, FKG and Lemma 2.2, we can connect Θ'_{p+1} and Θ_{p-1} by four alternating color arms with large enough winding angles and obtain

$$P(\mathcal{A})\ge c_{37}. \qquad (3.14)$$

Then by FKG, RSW and Separation Lemma (see Theorem 11 in [21]), using standard gluing argument, we have

$$\begin{aligned}
&P(\mathcal{A},\partial B(1)\leftrightarrow_\sigma\partial B(n)) \\
&\quad\ge c_{38}P(\mathcal{A})P(\partial B(1)\leftrightarrow_\sigma\partial B(2^{p-2}))P(\partial B(2^{p+3})\leftrightarrow_\sigma\partial B(n)). \qquad (3.15)
\end{aligned}$$

Combining (3.13), (3.14), (3.15), we know that there exists a constant $c_{39} > 0$, such that for all large q, we have

$$E_q\Delta_p^2 \geq c_{39}, \quad 4 \leq p \leq q-4.$$

Thus the first inequality in (3.9) follows. □

Lemma 3.4. *Denote by $E_n(\cdot)$ the expectation with respect to $P(\cdot|\partial B(1) \leftrightarrow_\sigma \partial B(n))$. For all large q and all $n \geq 2^{q+q^{\frac{1}{3}}+2}$,*

$$|E_{q,n}\theta(\partial B(1), \Theta_{q,n})| \leq 2\pi \text{ and } |E_n\theta_n| \leq 2\pi.$$

Proof. Define

$$\theta_{max} := \max\{\theta(\gamma) : \gamma \text{ is an arm connecting } \partial B(1) \text{ and } \Theta_{q,n}\},$$

$$\theta_{min} := \min\{\theta(\gamma) : \gamma \text{ is an arm connecting } \partial B(1) \text{ and } \Theta_{q,n}\}.$$

By (3.12), it is easy to see that $E_{q,n}\theta(\partial B(1), \Theta_{q,n})$ exists. Since the winding angles of the arms between $\partial B(1)$ and $\Theta_{q,n}$ differ at most 2π, hence $E_{q,n}\theta_{max}, E_{q,n}\theta_{min}$ also exist. By the symmetry of the lattice and the definition of $\Theta_{q,n}$, it is obvious that $E_{q,n}\theta_{max} = -E_{q,n}\theta_{min}$. Hence $E_{q,n}[\theta_{max} + \theta_{min}] = 0$. Then we conclude $|E_{q,n}\theta(\partial B(1), \Theta_{q,n})| \leq 2\pi$ by

$$|\theta(\partial B(1), \Theta_{q,n}) - (\theta_{max} + \theta_{min})/2| \leq 2\pi.$$

$|E_n\theta_n| \leq 2\pi$ can be proved similarly. □

Lemma 3.5. *Assume $n \geq 2^{q+q^{\frac{1}{3}}+2}$. Under $P_{q,n}$, as $q \to \infty$,*

$$\frac{\theta(\partial B(1), \Theta_{q,n})}{\left(\sum_{p=1}^q E_{q,n}\Delta_{p,q,n}^2\right)^{1/2}} \to_d N(0,1).$$

Proof. By Lemma 3.4 and (3.9), Lemma 3.5 is equivalent to

$$\frac{\theta(\partial B(1), \Theta_{q,n}) - E_{q,n}\theta(\partial B(1), \Theta_{q,n})}{\left(\sum_{p=1}^q E_{q,n}\Delta_{p,q,n}^2\right)^{1/2}} \to_d N(0,1). \tag{3.16}$$

Let us now check the three conditions of Theorem 2.3 in [19]. First we set

$$X_{p,q,n} := \frac{\Delta_{p,q,n}}{\left(\sum_{p=1}^q E_{q,n}\Delta_{p,q,n}^2\right)^{1/2}},$$

then we can write

$$\frac{\theta(\partial B(1), \Theta_{q,n}) - E_{q,n}\theta(\partial B(1), \Theta_{q,n})}{\left(\sum_{p=1}^q E_{q,n}\Delta_{p,q,n}^2\right)^{1/2}} = \sum_{p=1}^q X_{p,q,n}.$$

By (3.9),

$$|X_{p,q,n}| \leq |\Delta_{p,q,n}|/(c_8 q)^{1/2}.$$

The conditions (2.3a) and (2.3b) of McLeish [19] are implied by (3.7) and (3.8). By (3.9), the condition (2.3c) is equivalent to

$$\frac{1}{q}\sum_{p=1}^q (\Delta_{p,q,n}^2 - E_{q,n}\Delta_{p,q,n}^2) \to 0 \text{ in probability.} \tag{3.17}$$

We can not use the method from the proof of (2.60) in [17] since there is a lack of independence of our setting. Thanks to the coupling result, we can gain some weak independence of our model. Recall that we let $E_q = E_{q,n}$ and $\Delta_p = \Delta_{p,q,n}$ for short. Note that if we prove

$$E_q\left[\sum_{p=1}^{q}(\Delta_p^2 - E_q\Delta_p^2)\right]^2 = o(q^2), \tag{3.18}$$

then (3.17) follows. For a fixed constant $c > 0$, let us split the above sum into two terms:

$$\begin{aligned} E_q\left[\sum_{p=1}^{q}(\Delta_p^2 - E_q\Delta_p^2)\right]^2 &= \sum_{|p-r|\le c\log q} E_q[(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)] \\ &\quad + \sum_{|p-r|>c\log q} E_q[(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)]. \end{aligned}$$

For the first term, using (3.6), we have

$$\sum_{|p-r|\le c\log q} E_q[(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)] \le c_1 q\log q.$$

Now we estimate the second term. Assume $r - p > c\log q$ in the following. Define events

$$\mathcal{A} := \{m(p,\omega) \le p + \frac{c}{2}\log q\},$$

$$\begin{aligned} \mathcal{B} := \{&\text{there exists } m(p,\omega)+1 \le l = l(p,\omega,\omega') \le r-2, \text{ we can couple } P'_{q,\Theta_p(\omega)} \\ &\text{and } P''_q \text{ such that } \Theta_l(\omega') \text{ and } \Theta_l(\omega'') \text{ are identical, and the configurations} \\ &\text{in } B(n)\backslash\overline{\Theta}_l(\omega') \text{ and } B(n)\backslash\overline{\Theta}_l(\omega'') \text{ are also identical}\}. \end{aligned}$$

First, we write

$$\begin{aligned} &E_q[(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)] \\ &= E_q[I_{\mathcal{A}^c}(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)] + E_q[I_{\mathcal{A}}(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)]. \end{aligned} \tag{3.19}$$

Let us estimate the two terms in the r.h.s. of above inequality separately. For the first term, with Hölder's inequality, (3.5) and (3.6), we get

$$\begin{aligned} E_q[I_{\mathcal{A}^c}(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)] &\le P(\mathcal{A}^c)^{\frac{1}{3}}(E_q|\Delta_p^2 - E_q\Delta_p^2|^3)^{\frac{1}{3}}(E_q|\Delta_r^2 - E_q\Delta_r^2|^3)^{\frac{1}{3}} \\ &\le c_2\exp(-c_3\log q) = o(1). \end{aligned} \tag{3.20}$$

For the second term, write

$$\begin{aligned} &E_q[I_{\mathcal{A}}(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)] \\ &= E_q[I_{\mathcal{A}}(\Delta_p^2(\omega) - E_q\Delta_p^2(\omega))E'_{q,\Theta_p(\omega)}[I_{\mathcal{B}}(\Delta_r^2(\omega') - E'_q\Delta_r^2(\omega'))]] \\ &\quad + E_q[I_{\mathcal{A}}(\Delta_p^2(\omega) - E_q\Delta_p^2(\omega))E'_{q,\Theta_p(\omega)}[I_{\mathcal{B}^c}(\Delta_r^2(\omega') - E'_q\Delta_r^2(\omega'))]]. \end{aligned} \tag{3.21}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &E'_{q,\Theta_p(\omega)}|I_{\mathcal{B}^c}(\Delta_r^2(\omega') - E'_q\Delta_r^2(\omega'))| \\ &\qquad \le [P'_{q,\Theta_p(\omega)}(\mathcal{B}^c)]^{\frac{1}{2}}[E'_{q,\Theta_p(\omega)}[\Delta_r^2(\omega') - E'_q\Delta_r^2(\omega')]^2]^{\frac{1}{2}}. \end{aligned} \tag{3.22}$$

Using a very similar argument as the proof of (3.6), we know that there exist universal constants $c_4, c_5 > 0$ such that for all $x \ge 0$,

$$P'_{q,\Theta_p(\omega)}(|\Delta_r(\omega')| \ge x) \le c_4\exp(-c_5 x).$$

Combining (3.22), the coupling result, (2.10) and above inequality, we get

$$E'_{q,\Theta_p(\omega)}[I_{\mathcal{B}^c}(\Delta_r^2(\omega') - E'_q\Delta_r^2(\omega'))] = o(1). \tag{3.23}$$

Similarly we have

$$E''_q[I_{\mathcal{B}^c}(\Delta_r^2(\omega'') - E''_q\Delta_r^2(\omega''))] = o(1).$$

Then $E''_q[\Delta_r^2(\omega'') - E''_q\Delta_r^2(\omega'')] = 0$ implies

$$E''_q[I_{\mathcal{B}}(\Delta_r^2(\omega'') - E''_q\Delta_r^2(\omega''))] = o(1).$$

By the definition of $\mathcal{B}$, we immediately obtain

$$E'_{q,\Theta_p(\omega)}[I_{\mathcal{B}}(\Delta_r^2(\omega') - E'_q\Delta_r^2(\omega'))] = E''_q[I_{\mathcal{B}}(\Delta_r^2(\omega'') - E''_q\Delta_r^2(\omega''))] = o(1). \tag{3.24}$$

Combining (3.21), (3.23), (3.24) and (3.6), we get

$$E_q[I_{\mathcal{A}}(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)] = E_q[I_{\mathcal{A}}(\Delta_p^2 - E_q\Delta_p^2)]o(1) = o(1). \tag{3.25}$$

Consequently, for the second term, by (3.19), (3.20) and (3.25), we get

$$\sum_{|p-r|>c\log q} E_q[(\Delta_p^2 - E_q\Delta_p^2)(\Delta_r^2 - E_q\Delta_r^2)] = o(q^2).$$

□

Proof of Theorem 1.1. Let $2^{q+q^{\frac{1}{3}}+2} \le n < 2^{(q+1)+(q+1)^{\frac{1}{3}}+2}$ and $0 < \varepsilon < \frac{1}{6}$. Define event

$$\mathcal{B} := \{\text{there exist at least } q^{\frac{1}{2}-\varepsilon}/(2\pi) - 4 \text{ disjoint black crossing paths in } R(2^q, n)\}.$$

Recall we let $P_q = P_{q,n}$ and $\Delta_p = \Delta_{p,q,n}$ for short. By the definition of θ_n and $\theta(\cdot,\cdot)$, we obtain

$$\begin{aligned}
&P_q(|\theta_n - \theta(\partial B(1), \Theta_q)| \ge q^{\frac{1}{2}-\varepsilon})\\
&\le P_q(\mathcal{B})\\
&\le \frac{c_1 P(\mathcal{B}, \partial B(1) \leftrightarrow_\sigma \partial B(n))}{P(\partial B(1) \leftrightarrow_\sigma \partial B(n))} \quad \text{by (2.10)}\\
&\le \frac{c_2 P(\mathcal{B}, \partial B(2^q) \leftrightarrow_\sigma \partial B(n))}{P(\partial B(2^q) \leftrightarrow_\sigma \partial B(n))} \quad \text{by quasi-multiplicativity}\\
&\le c_3 \exp(-c_4 q^{\frac{1}{2}-\varepsilon}) \quad \text{by Lemma 2.1 and (2.1).}
\end{aligned} \tag{3.26}$$

For $2^{q+q^{\frac{1}{3}}+2} \le n < 2^{q+(q+1)^{\frac{1}{3}}+3}$, define

$$h_n := \left(\sum_{p=1}^{q} E_q\Delta_p^2\right)^{1/2}. \tag{3.27}$$

Then by (3.26), Lemma 3.5, (2.10) and (3.9), under $P(\cdot|\partial B(1) \leftrightarrow_\sigma \partial B(n))$ we have

$$\frac{\theta_n}{h_n} \to_d N(0,1).$$

Hence Theorem 1.1 is concluded if $a_n = h_n + o(\sqrt{\log n})$. Let us prove this now. For short, let $\theta_q := \theta(\partial B(1), \Theta_q)$. By Lemma 3.4 , (2.10) and (3.9),

$$h_n^2 = E_q[\theta_q^2] - [E_q\theta_q]^2 = E_q[\theta_q^2] + O(1) = (1 + o(1))E[\theta_q^2 I_{\mathcal{A}_q}] \asymp \log n, \tag{3.28}$$

$$a_n^2 = E[\theta_n^2] - [E\theta_n]^2 = E[\theta_n^2] + O(1) = E[\theta_n^2 I_{\mathcal{A}_q}] + E[\theta_n^2 I_{\mathcal{A}_q^c}] + O(1). \quad (3.29)$$

By Lemma 2.1 and (2.1), it is easy to show that there exist $c_5, c_6, c_7 > 0$, such that for $x > c_5 \log n$,

$$P(|\theta_n| \geq x | \partial B(1) \leftrightarrow_\sigma \partial B(n)) \leq c_6 \exp(-c_7 x). \quad (3.30)$$

By the Cauchy-Schwarz inequality, (3.30) and (2.10),

$$E[\theta_n^2 I_{\mathcal{A}_q^c}] \leq (E[\theta_n^4])^{1/2} (P(\mathcal{A}_q^c))^{1/2} \leq c_8 (\log n)^2 \exp(-c_9 (\log n)^{1/3}) = o(1). \quad (3.31)$$

By (3.28), (3.29) and (3.31), we have

$$|a_n^2 - h_n^2| = |E[\theta_n^2 I_{\mathcal{A}_q}] - E[\theta_q^2 I_{\mathcal{A}_q}]| + o(\log n).$$

In the following we prove $|E[\theta_n^2 I_{\mathcal{A}_q}] - E[\theta_q^2 I_{\mathcal{A}_q}]| = o(\log n)$, which implies $a_n = h_n + o(\sqrt{\log n})$. Analogous to the proof of (3.26), it can be shown that there exist $c_{10}, c_{11}, c_{12} > 0$, such that for all $x \geq c_{10} (\log n)^{1/3}$,

$$P(|\theta_n I_{\mathcal{A}_q} - \theta_q I_{\mathcal{A}_q}| \geq x | \partial B(1) \leftrightarrow_\sigma \partial B(n)) \leq c_{11} \exp(-c_{12} x). \quad (3.32)$$

Then by the Cauchy-Schwarz inequality, (3.28), (3.32) and (3.9), we have

$$\begin{aligned} &|E[\theta_n^2 I_{\mathcal{A}_q}] - E[\theta_q^2 I_{\mathcal{A}_q}]| \\ &\quad = |2E[\theta_q I_{\mathcal{A}_q} (\theta_n I_{\mathcal{A}_q} - \theta_q I_{\mathcal{A}_q})] + E[\theta_n I_{\mathcal{A}_q} - \theta_q I_{\mathcal{A}_q}]^2| \\ &\quad \leq 2(E[\theta_q I_{\mathcal{A}_q}]^2)^{1/2} (E[\theta_n I_{\mathcal{A}_q} - \theta_q I_{\mathcal{A}_q}]^2)^{1/2} + E[\theta_n I_{\mathcal{A}_q} - \theta_q I_{\mathcal{A}_q}]^2 \\ &\quad \leq c_{13} (\log n)^{5/6} + c_{14} (\log n)^{2/3}. \end{aligned}$$

□

Proof of Corollary 1.5. Let $10 \leq n \leq m$ and $2^{q+q^{\frac{1}{3}}+1} \leq n \leq 2^{q+q^{\frac{1}{3}}+2}$. By a slight modification of Proposition 3.6 in [10] and its proof (analog to the coupling result in Lemma 2.3), there exists a universal constant $c_1 > 0$, we can couple $P(\cdot | \partial B(1) \leftrightarrow \partial B(n))$ and $P(\cdot | \partial B(1) \leftrightarrow \partial B(m))$ such that with probability at least $1 - \exp(-c_1 q^{\frac{1}{3}})$, there exist identical exterior faces Θ with quality $Q(\Theta) \geq \frac{1}{4}$ (well separated) around $\partial B(2^{q+p})$ for some $0 \leq p \leq q^{\frac{1}{3}}$ and identical configurations on $\overline{\Theta}$ for $P(\cdot | \partial B(1) \leftrightarrow \partial B(n))$ and $P(\cdot | \partial B(1) \leftrightarrow \partial B(m))$. Let $\mathcal{A}$ denote the event that the above coupling succeeds and $P_{n,m}(\cdot)$ denote the coupling measure. For a configuration of $P(\cdot | \partial B(1) \leftrightarrow \partial B(m))$, we denote by $\theta_{n,m}$ the winding angle of the arm (chosen uniquely by some definite way) connecting $\partial B(1)$ and $\partial B(n)$. Let $0 < \varepsilon < \frac{1}{6}$. Define event

$$\mathcal{B} := \{\text{there exist at least } q^{\frac{1}{2}-\varepsilon}/(4\pi) - 4 \text{ disjoint black crossing paths in } R(2^q, n)\}.$$

Then by the coupling argument we discuss above,

$$\begin{aligned} &P_{n,m}(|\theta_{n,m} - \theta_{n,n}| > 2q^{\frac{1}{2}-\varepsilon}) \\ &\quad \leq P_{n,m}(\mathcal{A}^c) + P_{n,m}(\mathcal{A}, |\theta_{n,m} - \theta_{n,n}| > 2q^{\frac{1}{2}-\varepsilon}) \\ &\quad \leq \exp(-c_1 q^{\frac{1}{3}}) + P(\mathcal{B} | \partial B(1) \leftrightarrow_\sigma \partial B(n)) + P(\mathcal{B} | \partial B(1) \leftrightarrow_\sigma \partial B(m)). \end{aligned}$$

Since

$$\begin{aligned} P(\mathcal{B} | \partial B(1) \leftrightarrow_\sigma \partial B(m)) &= \frac{P(\mathcal{B}, \partial B(1) \leftrightarrow_\sigma \partial B(m))}{P(\partial B(1) \leftrightarrow_\sigma \partial B(m))} \\ &\leq \frac{c_2 P(\mathcal{B})}{P(\partial B(2^q) \leftrightarrow_\sigma \partial B(n))} \quad \text{by quasi-multiplicativity} \\ &\leq c_3 \exp(-c_4 q^{\frac{1}{2}-\varepsilon}) \quad \text{by Lemma 2.1 and (2.1),} \end{aligned}$$

then we get

$$P_{n,m}(|\theta_{n,m} - \theta_{n,n}| > q^{\frac{1}{2}-\varepsilon}) = o(1) \text{ as } q \to \infty.$$

Recall $a_n \asymp \sqrt{\log n}$, $q \asymp \log n$. Using Theorem 1.1, under $P(\cdot|\partial B(1) \leftrightarrow \partial B(m))$, let $m \geq n \to \infty$, we have

$$\frac{\theta_{n,m}}{a_n} \to_d N(0,1).$$

By the definition of ν_σ (see (1.2)), the conclusion follows. □

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