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Hausdorff dimension of limsup random fractals

Liang Zhang*

Abstract

In this paper we find a critical condition for nonempty intersection of a limsup random fractal and an independent fractal percolation set defined on the boundary of a spherically symmetric tree. We then use a codimension argument to derive a formula for the Hausdorff dimension of limsup random fractals.

Keywords: Hausdorff dimension; limsup random fractals; packing dimension; fractal percolation sets.
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1 Introduction

A limsup random fractal on $\mathbb{R}^N$ can be constructed as follows: (i) for each $n \geq 1$, let $\mathcal{D}_n$ denote the collection of all N-dimensional dyadic hyper-cubes of the form $[k^1 2^{-n}, (k^1+1)2^{-n}] \times \cdots \times [k^N 2^{-n}, (k^N+1)2^{-n}]$, where $k \in \mathbb{Z}_+^N$ is an N-dimensional positive integer; (ii) for each $n \geq 1$, let $\{Z_n(I) : I \in \mathcal{D}_n\}$ denote a collection of Bernoulli random variables with distribution $\mathbb{P}(Z_n = 1) = q_n$; (iii) a limsup random fractal, denoted by A, is then defined by

$$A := \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k \qquad \text{with} \qquad A_n := \bigcup_{I \in \mathcal{D}_n, Z_n(I)=1} I^\circ, \tag{1.1}$$

where I° denotes the interior of I.

Limsup random fractals arise in the study of stochastic processes. Many interesting random sets are limsup random fractals. For example, the fast points of Brownian motion considered by Orey and Taylor [16], the thick points of Brownian motion investigated by Dembo, Peres, Rosen, and Zeitouni [1], and the Dvoretzky covering set on the unit circle studied by Li, Shieh and Xiao [10], to name only a few.

Limsup random fractals have intimate connection to packing dimension. In particular, if we assume that $t = -\lim_{n\to\infty} n^{-1} \log_2 q_n$ exists and call it the index of A, it is shown by Khoshnevisan, Peres, and Xiao [9] that for all Borel set $F \subset \mathbb{R}^N$, $\dim_{\mathrm{P}}(F \cap A) = \dim_{\mathrm{P}}(F)$ almost surely, provided that $\dim_{\mathrm{P}}(F) > t$ and certain correlation bounds on the

*University of Utah, USA. E-mail: lzhang@math.utah.edu

random variables $\{Z_n(I) : I \in \mathcal{D}_n\}$ hold. Here $\dim_{\rm P}$ denotes packing dimension. However, the Hausdorff dimension of a limsup random fractal is unknown in general. It is shown in [9] that for all Borel sets $F \subset \mathbb{R}^N$ with $\dim_{\rm H}(F) > t$,

$$\dim_{\rm H}(F) - t \le \dim_{\rm H}(F \cap A) \le \dim_{\rm P}(F) - t \qquad \text{a.s.,} \tag{1.2}$$

where $\dim_{\rm H}$ denotes Hausdorff dimension. For sets with equal Hausdorff and packing dimension, (1.2) gives the Hausdorff dimension of a limsup random fractal. However, it is well known that there are sets whose Hausdorff dimension is strictly less than its packing dimension.

In this paper we strive to derive a formula for the Hausdorff dimension of limsup random fractals. We notice that the construction of a limsup random fractal on the unit hypercube $[0,1]^N$ generates a 2^N-nary tree T. More specifically, we can associate each sub-hypercube with a vertex, and connect two sub-hypercubes with an edge if one contains the other and the ratio of their side lengths is 2. The collection of all infinite rays of T is called the boundary of T, denoted by ∂T, and it can be made into a nice metric space. Under the assumption that the Bernoulli random variables Z_n's are independent, we have succeeded in obtaining a formula for the Hausdorff dimension of limsup random fractals defined on the boundary of a spherically symmetric tree, where a tree is said to be spherically symmetric if and only if all the vertices at the same generation have same number of children. The boundaries of these trees include many sets whose Hausdorff and packing dimension are different. Thus our result improves (1.2).

For each $s \ge 0$, we define a new index on the boundary ∂T of a spherically symmetric tree T via the prescription

$$\delta_s(\partial T) := \overline{\lim_{n\to\infty}} \frac{1}{-n} \log \left(\inf_{\mu \in \mathcal{P}(\partial S)} \iint_{d(x,y)\le e^{-n}} d(x,y)^{-s} \mu(dx)\mu(dy) \right), \tag{1.3}$$

where d is the tree metric on the boundary and $\mathcal{P}(\partial T)$ denotes the collection of all Borel probability measures supported on ∂T. By using this index, we are able to obtain a formula of the Hausdorff dimension of limsup random fractals.

Theorem 1.1. *Let T be a spherically symmetric tree and A be a limsup random fractal with parameters $\{q_n\}_{n\ge1}$. Assume that $t = -\lim_{n\to\infty} n^{-1} \ln q_n$ exists and $0 < t < \dim_{\rm P}(\partial T)$. Furthermore, assume the Bernoulli random variable Z_n's in the definition of the limsup random fractal are independent. Then*

$$||\dim_{\rm H}(A)||_{L^\infty(\mathbb{P})} = \sup\{s \ge 0 : \delta_s(\partial T) > t\}, \tag{1.4}$$

with the convention that $\sup \emptyset := 0$.

We use a codimension argument developed in Lyons [11] and Peres [17] to prove the above theorem. We consider a fractal percolation set defined on the same boundary of a spherically symmetric tree and independently with respect to the limsup random fractal. A fractal percolation set is defined in a similar manner as limsup random fractals, except that the Bernoulli random variables are independent and identically distributed and this random fractal consists of rays along which the Bernoulli random variables all equal to 1. The codimension argument (see Corollary 3.2) tells us that a nonrandom set will hit a fractal percolation set with positive probability if the Hausdorff dimension of this nonrandom set is large enough and the set will not hit the fractal percolation set almost surely if its Hausdorff dimension is too small. In Theorem 5.6, we strive to find a critical condition for the limsup random fractal hitting an independent fractal percolation set with positive probability. This critical condition will give us estimates of the Hausdorff dimension of the limsup random fractal.

The result of Khoshnevisan, Peres, and Xiao [9] also yields a codimension argument (see Theorem 3.5): a nonrandom set will hit a limsup random fractal almost surely if the packing dimension of this nonrandom set is large enough and it will not hit the limsup random fractal if its packing dimension is too small. As a result of the critical condition derived in Theorem 5.6, we also obtain the packing dimension of a fractal percolation set defined on a spherically symmetric tree in Corollary 5.7.

The remainder of the article is organized as follows. In Section 2, we give some background materials, including definitions for trees, Riesz capacity and fractal dimensions as well as their properties. Then we define random fractals on the boundary of a tree and review some known results in Section 3. In Section 4, we define two new indices and discuss their properties. Finally in Section 5, we estimates the hitting probability of a limsup random fractal and an independent fractal percolation set, and use codimension arguments and the new indices to compute the Hausdorff dimension of limsup random fractals and packing dimension of fractal percolation sets. We also give an example in which we explicitly calculate the dimension of the two random fractals.

2 Preliminaries

2.1 Tree Topology

Let $T = (\mathsf{V}, \mathsf{E})$ denote a tree with distinct root o, where V is the collection of all vertices and $\mathsf{E} \subset \mathsf{V} \times \mathsf{V}$ is the collection of all edges. Figure 1 gives an illustration of a typical tree. For $x, y \in \mathsf{V}$, if x is on the path from o to y, then we call x an ***ancestor*** of y, call y a ***descendant*** of x, and use $y \succ x$ to denote this relation. In particluar if $(x, y) \in \mathsf{E}$, then we call x the ***parent*** of y and y a ***child*** of x. For each $x \in \mathsf{V}$, let $\deg(x)$ denote the number of children x has, that is,

$$\deg(x) := \#\{v \in \mathsf{V} : (x, v) \in \mathsf{E}\}. \tag{2.1}$$

If $\deg(x)$ is finite for all $x \in \mathsf{V}$, then T is called ***locally finite***. We are interested in locally finite trees with infinitely many vertices. Moreover, if $\deg(x) = \deg(y)$ whenever x and y have the same distance from root o, then the tree is called ***spherically symmetric***.

For a tree T, a ray is an infinite path starting from o, that is, a sequence of vertices $\{o, v_1, v_2, \dots\} \subset \mathsf{V}$ such that $(o, v_1) \in \mathsf{E}$ and $(v_n, v_{n+1}) \in \mathsf{E}$ for all $n \geq 1$. The collection of all rays is called the ***boundary*** of T and denoted by ∂T. For a ray $\sigma = (o, v_1, \dots)$, define $\sigma(n) := v_n$ and denote $x \in \sigma$ if $x = v_n$ for some $n \geq 1$. For two rays $\sigma = (o, v_1, \dots)$ and $\gamma = (o, u_1, \dots)$, $\sigma = \gamma$ if and only if $v_n = u_n$ for all $n \geq 1$. Furthermore, let $\sigma \curlywedge \gamma$ denote the common vertex on σ and γ which is farthest from o. We can define a metric d on ∂T by

$$d(\sigma, \gamma) := e^{-|\sigma \curlywedge \gamma|} \qquad \forall\, \sigma, \gamma \in \partial T, \tag{2.2}$$

where $|x|$ denote the number of edges between x and o. It follows immediately that d is an ultrametric on ∂T in the sense that $d(\sigma, \gamma) \leq \max\{d(\sigma, \eta), d(\gamma, \eta)\}$ for all $\sigma, \gamma, \eta \in \partial T$. For $\sigma \in \partial T$ and $r > 0$, let $B(\sigma, r)$ denote the closed ball $\{\gamma \in \partial T : d(\sigma, \gamma) \leq r\}$. Moreover, define

$$B(x) := \{\sigma \in \partial T : x \in \sigma\}. \tag{2.3}$$

In words, $B(x)$ is the collection of all rays that pass through the vertex x. By definition, we have $B(x) = B(\sigma, r)$ with $r = e^{-|x|}$ and any σ such that $x \in \sigma$. Finally, let $\mathcal{T}$ denote the Borel σ-alegbra generated by all closed balls.

The following proposition is well known.

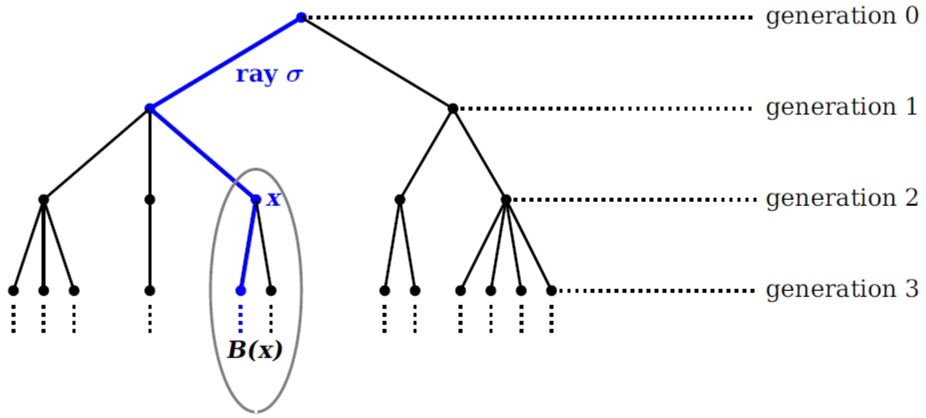

Figure 1: A General Infinite Tree

Proposition 2.1. *$(\partial T, d)$ is a complete separable metric space. Moreover, $(\partial T, d)$ is compact and totally disconnected.*

Remark 2.2. *In* (2.2)*, if we replace e by any $b > 1$, the resulting topology does not change. In particular, Proposition 2.1 remains valid.*

2.2 Riesz Energy and Capacity

Let (X, d) be a general metric space. For every Borel subset $G \subset X$, let $\mathcal{P}(G)$ denote the collection of all Borel probability measures supported on G. For $s \geq 0$ and $\mu \in \mathcal{P}(X)$, the ***s*-dimensional Riesz energy** of μ is defined as

$$\mathcal{E}_s(\mu) := \iint d(\sigma, \gamma)^{-s} \mu(d\sigma)\mu(d\gamma). \tag{2.4}$$

And the ***s*-dimensional Riesz capacity** of G is defined as

$$\mathrm{Cap}_s(G) := \left(\inf_{\mu \in \mathcal{P}(G)} \mathcal{E}_s(\mu) \right)^{-1}, \tag{2.5}$$

with the convention $\mathrm{Cap}_s(\emptyset) := 0$. When X is the boundary of a tree and d is define in (2.2), we have special forms for the Riesz energy and capacity. For each $\mu \in \mathcal{P}(\partial T)$, we write $\mu(x) := \mu(B(x))$. Furthermore, for $x, y \in \mathsf{V}$, let σ_x and σ_y denote two rays such that $x \in \sigma_x$ and $y \in \sigma_y$. Then we can define

$$x \curlywedge y := \begin{cases} \sigma_x \curlywedge \sigma_y, & \text{if } x \neq y \\ x, & \text{if } x = y \end{cases}, \text{ and } x \curlywedge \gamma := \begin{cases} \sigma_x \curlywedge \gamma, & \text{if } x \notin \gamma \\ x, & \text{if } x \in \gamma \end{cases}. \tag{2.6}$$

Note that $x \curlywedge y$ and $x \curlywedge \gamma$ do not depend on the choices of σ_x and σ_y.

Proposition 2.3. *For $s \geq 0$, let $p = e^{-s}$. Then for all $\mu \in \mathcal{P}(\partial T)$ and $n \geq 1$,*

$$\mathcal{E}_s(\mu) = \sum_{\substack{|x|=|y|=n \\ x \neq y}}\sum p^{-|x \curlywedge y|} \mu(x)\mu(y) + \sum_{\substack{|x|=n \\ \mu(x)>0}} \mathcal{E}_s(\mu_x)\mu(x)^2, \tag{2.7}$$

where $\mu_x \in \mathcal{P}(B(x))$ satisfies $\mu_x(G) = \mu(G \cap B(x))/\mu(x)$ for all $G \in \mathcal{T}$, provided that $\mu(x) > 0$.

Proof. First note that $d(\sigma,\gamma)^{-s} = (e^{-|\sigma\curlywedge\gamma|})^{-s} = p^{-|\sigma\curlywedge\gamma|}$. Consider $x \neq y$ with $|x| = |y| = n$. For all $\sigma \in B(x)$ and $\gamma \in B(y)$, we have

$$\sigma \curlywedge \gamma = \sigma \curlywedge y = x \curlywedge \gamma = x \curlywedge y. \tag{2.8}$$

Thus

$$\iint\limits_{B(x)\times B(y)} d(\sigma,\gamma)^{-s}\mu(d\gamma)\mu(d\sigma) = \int_{B(x)} p^{-|\sigma\curlywedge y|}\mu(y)\mu(d\sigma) = p^{-|x\curlywedge y|}\mu(y)\mu(x). \tag{2.9}$$

Moreover, if $\mu(x) > 0$, then

$$\begin{aligned}\iint\limits_{B(x)\times B(x)} d(\sigma,\gamma)^{-s}\mu(d\gamma)\mu(d\sigma) &= \mu(x)^2 \cdot \iint d(\sigma,\gamma)^{-s}\mu_x(d\gamma)\mu_x(d\sigma) \\ &= \mu(x)^2\mathcal{E}_s(\mu_x).\end{aligned} \tag{2.10}$$

If $\mu(x) = 0$, then we simply have $\iint_{B(x)\times B(x)} d(\sigma,\gamma)^{-s}\mu(d\gamma)\mu(d\sigma) = 0$. Therefore

$$\begin{aligned}\mathcal{E}_s(\mu) = &\sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} \iint\limits_{B(x)\times B(y)} d(\sigma,\gamma)^{-s}\mu(d\gamma)\mu(d\sigma) \\ &+ \sum_{|x|=n} \iint\limits_{B(x)\times B(x)} d(\sigma,\gamma)^{-s}\mu(d\gamma)\mu(d\sigma) \\ = &\sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} p^{-|x\curlywedge y|}\mu(x)\mu(y) + \sum_{\substack{|x|=n \\ \mu(x)>0}} \mathcal{E}_s(\mu_x)\mu(x)^2.\end{aligned} \tag{2.11}$$

This completes the proof. ☐

Corollary 2.4. *For all $G \in \mathcal{T}$, $s \geq 0$, and $n \geq 1$,*

$$\mathit{Cap}_s(G) = \left(\inf_{\mu\in\mathcal{P}(G)} \left\{ \sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} p^{-|x\curlywedge y|}\mu(x)\mu(y) + \sum_{\substack{|x|=n \\ \mu(x)>0}} \frac{\mu(x)^2}{\mathit{Cap}_s(B(x)\cap G)} \right\}\right)^{-1}. \tag{2.12}$$

Proof. From Proposition 2.3, we have

$$\mathrm{Cap}_s(G) = \left(\inf_{\mu\in\mathcal{P}(G)} \left\{ \sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} p^{-|x\curlywedge y|}\mu(x)\mu(y) + \sum_{\substack{|x|=n \\ \mu(x)>0}} \mathcal{E}_s(\mu_x)\mu(x)^2 \right\}\right)^{-1}, \tag{2.13}$$

where $\mu_x \in \mathcal{P}(B(x)\cap G)$ satisfies $\mu_x(H) = \mu(H\cap B(x))/\mu(x)$ for all $H \in \mathcal{T}$, provided that $\mu(x) > 0$. Let C_1 denote the reciprocal of the right hand side of (2.13) and C_2 denote the reciprocal of the right hand side of (2.12).

On one hand, if $\mu(x) > 0$, then $\mathcal{E}_s(\mu_x) \geq \mathrm{Cap}_s(B(x)\cap G)^{-1}$. Thus we have $C_1 \geq C_2$. On the other hand, for every $\varepsilon > 0$, we can find some $\nu \in \mathcal{P}(G)$ such that

$$\sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} p^{-|x\curlywedge y|}\nu(x)\nu(y) + \sum_{\substack{|x|=n \\ \nu(x)>0}} \frac{\nu(x)^2}{\mathrm{Cap}_s(B(x)\cap G)} \leq C_2 + \varepsilon. \tag{2.14}$$

For each x with $|x| = n$ and $\nu(x) > 0$, by the definition of Riesz capacity, we can find some $\lambda_x \in \mathcal{P}(B(x) \cap G)$ such that $\mathcal{E}_s(\lambda_x) \leq \text{Cap}_s(B(x) \cap G)^{-1} + \varepsilon$. Define a Borel probability measure $\mu^* \in \mathcal{P}(G)$ by $\mu^*(y) = \nu(y)$ if $|y| \leq n$ and $\mu^*(y) = \nu(x) \cdot \lambda_x(y)$ if $|y| > n$, where $|x| = n$ and $y \succ x$. Then we have

$$\begin{aligned}
&\sum\sum_{\substack{|x|=|y|=n \\ x \neq y}} p^{-|x \curlywedge y|} \mu^*(x)\mu^*(y) + \sum_{\substack{|x|=n \\ \mu^*(x)>0}} \mathcal{E}_s(\mu_x^*)\mu^*(x)^2 \\
&\leq \sum\sum_{\substack{|x|=|y|=n \\ x \neq y}} p^{-|x \curlywedge y|} \nu(x)\nu(y) + \sum_{\substack{|x|=n \\ \nu(x)>0}} \big((\text{Cap}_s(B(x) \cap G))^{-1} + \varepsilon\big)\nu(x)^2 \qquad (2.15) \\
&\leq C_2 + 2\varepsilon.
\end{aligned}$$

Let $\varepsilon \downarrow 0$ to see that $C_2 \geq C_1$. □

When the tree T is spherically symmetric, Riesz capacities can be obtained by computing energies of the uniform probability measure.

Lemma 2.5. *Let T be a spherically symmetric tree and ν be the uniform probability measure on ∂T, that is, ν satisfies*

$$\nu(x) = \nu(y), \qquad \textit{for all } |x| = |y| = m \textit{ and all } m \geq 1. \tag{2.16}$$

Then

$$\mathcal{E}_s(\nu) = (\textit{Cap}_s(\partial T))^{-1} \quad \forall\, 0 \leq s < \dim_{\mathrm{H}}(\partial T). \tag{2.17}$$

Proof. Since T is spherically symmetric, the degrees of the vertices at the same generation are equal. Thus we can let K_n denote the degree of a vertex at generation n. If for each vertex we fix an order for its children, then each ray $\sigma \in \partial T$ can be identified by a sequence of integers $(l_n)_{n\geq 1}$ such that $\sigma(n)$ is the l_nth child of $\sigma(n-1)$. We will simply use $\sigma = (l_n)_{n\geq 1}$ to denote this identification. Define a binary operation "+" on ∂T by

$$\sigma + \gamma = ((l_n + m_n) \mod K_n)_{n\geq 1}\,, \tag{2.18}$$

where $\sigma = (l_n)_{n\geq 1}$ and $\gamma = (m_n)_{n\geq 1}$. It follows that $(\partial T, +)$ is a group. Furthermore, $(\partial T, +)$ is a topological group under the metric d. The proof of Theorem 3.1 of Khoshnevisan [8] implies that the equilibrium measure that minimizes the Riesz energies on a topological group is the Haar measure. This completes the proof. □

For a Borel set $G \subset \partial T$ and $s \geq 0$, the collection of Borel probability measures supported on G with finite s-dimensional Riesz energy is of special interest. Let $\mathcal{P}_s(G)$ denote this collection, that is,

$$\mathcal{P}_s(G) := \{\mu \in \mathcal{P}(G) : \mathcal{E}_s(\mu) < \infty\}. \tag{2.19}$$

We have a necessary condition for $\mu \in \mathcal{P}(G)$ to have finite s-dimensional Riesz energy.

Proposition 2.6. *If $\textit{Cap}_s(B(z) \cap G) = 0$, then $\mu(z) = 0$ for all $\mu \in \mathcal{P}_s(G)$.*

Proof. Since $\text{Cap}_s(B(z) \cap G) = 0$, $\mathcal{E}_s(\nu) = \infty$ for all $\nu \in \mathcal{P}(B(z) \cap G)$ by the definition of Riesz capacity. From Proposition 2.3 we have,

$$\mathcal{E}_s(\mu) = \sum\sum_{\substack{|x|=|y|=|z| \\ x \neq y}} p^{-|x \curlywedge y|} \mu(x)\mu(y) + \sum_{\substack{|x|=|z| \\ \mu(x)>0}} \mathcal{E}_s(\mu_x)\mu(x)^2, \tag{2.20}$$

where $\mu_x \in \mathcal{P}(B(x) \cap G)$ satisfies $\mu_x(H) = \mu(H \cap B(x))/\mu(x)$ for all $H \in \mathcal{T}$, provided that $\mu(x) > 0$. If $\mu(z) > 0$, then we have

$$\mathcal{E}_s(\mu) \geq \mathcal{E}_s(\mu_z)\mu(z)^2 = \infty. \tag{2.21}$$

Therefore, we must have $\mu(z) = 0$. □

Lemma 2.7. *For fixed $s > 0$ and $\mu \in \mathcal{P}_s(\partial T)$, we have*

$$\mathcal{E}_s(\mu) = \lim_{n\to\infty} \sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} p^{-|x\curlywedge y|}\mu(x)\mu(y). \tag{2.22}$$

In particular,

$$\lim_{n\to\infty} \sum_{|x|=n,\mu(x)>0} \mathcal{E}_s(\mu_x)\mu(x)^2 = 0, \tag{2.23}$$

where $\mu_x \in \mathcal{P}(B(x))$ satisfies $\mu_x(G) = \mu(G \cap B(x))/\mu(x)$ for all $G \in \mathcal{T}$, provided that $\mu(x) > 0$.

Proof. Since $\mu \in \mathcal{P}_s(\partial T)$, we have $\mu \times \mu(\{\sigma = \gamma\}) = 0$. Thus

$$\iint_{\{\sigma=\gamma\}} d(\sigma,\gamma)^{-s}\mu(d\sigma)\mu(d\gamma) = 0. \tag{2.24}$$

The fact $\lim_{n\to\infty} \mathbb{1}\{d(\sigma,\gamma) \geq e^{-n}\}d(\sigma,\gamma)^{-s} = \mathbb{1}\{\sigma \neq \gamma\}d(\sigma,\gamma)^{-s}$ and an application of the monotone convergence theorem complete the proof.

□

2.3 Fractal Dimensions

In this part, we recall the definitions and properties of fractal dimensions. We refer to Mattila [14] for more details. Let (X, ρ) be a locally compact metric space and $\mathcal{B}$ be its Borel σ-algebra induced by the metric. For every subset F, let $|F|$ denote the diameter of F, that is, $|F| := \sup\{\rho(x, y) : x, y \in F\}$. For fixed $s \geq 0$, define the ***s-dimensional Hausdorff measure*** $\mathcal{H}^s$ by

$$\mathcal{H}^s(F) := \lim_{\delta\downarrow 0} \inf \left\{ \sum_{n\geq 1} |F_n|^s : F \subset \bigcup_{n\geq 1} F_n, |F_n| < \delta \text{ for all } n \geq 1 \right\}. \tag{2.25}$$

Then for every $F \subset X$ the ***Hausdorff dimension*** of F is defined by

$$\dim_{\mathrm{H}}(F) := \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}. \tag{2.26}$$

We can consider packings instead of coverings to derive a packing measure. This was first done by Tricott in 1982 [18]. For every $F \subset X$ and $\delta > 0$, a δ-packing of F is a countable collection of disjoint closed balls $\{B(x_n, r_n)\}_{n\geq 1}$ such that $x_n \in F$ and $r_n < \delta$ for all $n \geq 1$. For each fixed $s \geq 0$, define

$$P^s(F) = \lim_{\delta\downarrow 0} \sup \left\{ \sum_{n\geq 1} (2r_n)^s : \{B(x_n, r_n)\}_{n\geq 1} \text{ is a } \delta\text{-packing of } F \right\}. \tag{2.27}$$

The set function P^s is a premeasure and we regularize it to obtain the ***s-dimensional packing measure*** $\mathcal{P}^s$:

$$\mathcal{P}^s(F) := \inf \left\{ \sum_{n=1}^{\infty} P^s(F_n) : F \subset \bigcup_{n=1}^{\infty} F_n \right\}. \tag{2.28}$$

Then for every $F \subset X$ we define the ***packing dimension*** of F by

$$\dim_{\mathrm{P}}(F) := \inf\{s \geq 0 : \mathcal{P}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{P}^s(F) = \infty\}. \tag{2.29}$$

Packing dimension can also be defined in terms upper Minkowski dimension. For every bounded set $F \subset X$ and for each $\varepsilon > 0$, let $N(F, \varepsilon)$ denote the smallest number of closed balls with radius ε that are needed to cover F:

$$N(F, \varepsilon) := \min\left\{N : F \subset \bigcup_{n=1}^{N} B(x_n, \varepsilon)\right\}. \tag{2.30}$$

Then the ***upper Minkowski dimension*** and ***lower Minkowski dimension*** of F are defined by

$$\overline{\dim}_{\mathrm{M}}(F) := \varlimsup_{\varepsilon \downarrow 0} \frac{\ln N(F, \varepsilon)}{-\ln \varepsilon} \quad \text{and} \quad \underline{\dim}_{\mathrm{M}}(F) := \varliminf_{\varepsilon \downarrow 0} \frac{\ln N(F, \varepsilon)}{-\ln \varepsilon}, \tag{2.31}$$

respectively. Theorem 5.11 of Matilla [14] states that

$$\dim_{\mathrm{P}}(F) = \inf\left\{\sup_{n \geq 1} \overline{\dim}_{\mathrm{M}}(F_n) : F \subset \bigcup_{n=1}^{\infty} F_n, F_n \text{ is bounded}\right\}, \tag{2.32}$$

for all $F \subset X$.

Now let T be a tree and consider $(X, \rho) = (\partial T, d)$. Since $(\partial T, d)$ is an ultrametric space, Theorem 3 of Haase [4] shows that for each $s > 0$, $\mathcal{H}^s(F) \leq \mathcal{P}^s(F)$ for all $F \subset \partial T$. Therefore

$$\dim_{\mathrm{H}}(F) \leq \dim_{\mathrm{P}}(F) \qquad \forall\, F \subset \partial T. \tag{2.33}$$

In fact, there exists $F \subset \partial T$ such that $\dim_{\mathrm{H}}(F) < \dim_{\mathrm{P}}(F)$.

Lemma 2.8. *Let T be a spherically symmetric tree. Then $\dim_{\mathrm{H}}(\partial T) = \underline{\dim}_{\mathrm{M}}(\partial T)$ and $\dim_{\mathrm{P}}(\partial T) = \overline{\dim}_{\mathrm{M}}(\partial T)$.*

Proof. The fact $\dim_{\mathrm{H}}(\partial T) = \underline{\dim}_{\mathrm{M}}(\partial T)$ when T is spherically symmetric is a standard result in Chapter 1 of Lyons and Peres [13]. In order to show the other equality, we use a Baire category argument. Let N_n denote the total number of vertices at generation n. Then a monotone argument shows that

$$\overline{\dim}_{\mathrm{M}}(\partial T) = \varlimsup_{n \to \infty} \frac{\ln N_n}{n}. \tag{2.34}$$

For each $x \in \mathsf{V}$, the closed ball $B(x)$ can be regarded as the boundary of a subtree $T^{B(x)} = (\mathsf{V}^{B(x)}, \mathsf{E}^{B(x)})$, where $\mathsf{V}^{B(x)}$ includes x's ancestors, x, and x's descendants and $\mathsf{E}^{B(x)} = (\mathsf{V}^{B(x)} \times \mathsf{V}^{B(x)}) \cap \mathsf{E}$. For each $m \geq 1$, let $N_m^{B(x)}$ denote the total number of vertices at generation m of the subtree $T^{B(x)}$. Since T is spherically symmetric, we have

$$N_m^{B(x)} \cdot N_n = N_m \qquad \forall m \geq n, \tag{2.35}$$

where $n = |x|$. Thus

$$\begin{aligned} \overline{\dim}_{\mathrm{M}}(B(x)) = \overline{\dim}_{\mathrm{M}}(\partial T^{B(x)}) &= \varlimsup_{m \to \infty} \frac{\ln N_m^{B(x)}}{m} = \varlimsup_{m \to \infty} \frac{\ln(N_m \cdot N_{|x|}^{-1})}{m} \\ &= \overline{\dim}_{\mathrm{M}}(\partial T). \end{aligned} \tag{2.36}$$

Since $B(x)$ is also an open set, (2.36) implies that $\overline{\dim}_{\mathrm{M}}(U) = \overline{\dim}_{\mathrm{M}}(\partial T)$ for all open set $U \subset \partial T$. Then Proposition 3.6 of Falconer [3] implies that $\overline{\dim}_{\mathrm{M}}(\partial T) = \dim_{\mathrm{P}}(\partial T)$. $\square$

Example 2.9. *We can use Lemma 2.8 to construct a tree T such that $\dim_{\mathrm{H}}(\partial T) < \dim_{\mathrm{P}}(\partial T)$. Let $\{n_m\}_{m\geq 1}$ be a sequence of integers so that*

$$\lim_{m\to\infty} n_m = \infty \quad \textit{and} \quad \lim_{m\to\infty} \frac{n_m}{n_{m+1}-n_m} = 0. \tag{2.37}$$

For example, let $k_1 = 1, k_2 = 2$ and $k_m = (\sum_{i=1}^{m-1} k_i)^2$ for all $m \geq 3$. Then the sequence $\{n_m\}_{m\geq 1}$ with $n_m = \sum_{i=1}^{m} k_i$ satisfies this requirement. Now construct a tree T by the following scheme: if $m = 2k-1$, then all vertices between generation $n_m + 1$ and n_{m+1} have exactly 3 children; if $m = 2k$, then all vertices between generation $n_m + 1$ and n_{m+1} have exactly 2 children. It follows that

$$\overline{\dim}_{\mathrm{M}}(\partial T) \geq \lim_{k\to\infty} \frac{\ln N_{n_{2k-1}}}{n_{2k-1}} = \ln 3 \;\; \textit{and} \;\; \underline{\dim}_{\mathrm{M}}(\partial T) \leq \lim_{k\to\infty} \frac{\ln N_{n_{2k}}}{n_{2k}} = \ln 2. \tag{2.38}$$

Since this tree is spherically symmetric, we can apply Lemma 2.8 to see that $\dim_{\mathrm{H}}(\partial T) < \dim_{\mathrm{P}}(\partial T)$.

3 Random Fractals On Trees

3.1 Definition of Random Fractals

In this section we define the main object we are studying, namely the ***limsup random fractal***. For each $x \in \mathsf{V}$, define a random variable Z_x with distribution

$$\mathbb{P}(Z_x = 1) = q_n = 1 - \mathbb{P}(Z_x = 0)\,, \tag{3.1}$$

where $n = |x|$ and $0 \leq q_n \leq 1$. Note that q_n may vary as n changes. Define the limsup random fractal A with parameters $\{q_n\}_{n\geq 1}$ by

$$A := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad \text{with} \quad A_k := \bigcup_{|x|=k, Z_x=1} B(x) \quad \forall\, k \geq 1. \tag{3.2}$$

Thus if $\sigma \in A$, then $Z_{\sigma(n)} = 1$ for infinitely many n. Throughout this paper we will assume that

$$t = -\lim_{n\to\infty} \frac{1}{n} \ln q_n \;\; \text{exists,} \tag{3.3}$$

and call t the ***index*** of the limsup random fractal A. Moreover, we assume that A satisfies the ***independence assumption***:

> Let $\{W_i\}_{i\in I}$ be a collection of subsets of V so that x_i is the youngest (the root o is the oldest) common ancestor of all $x \in W_i$. We assume that the collections of random variables $\{\{Z_x\}_{x\in W_i}\}_{i\in I}$ are mutually independent if $x_i \not\leq x_j$ for all $i, j \in I$.

There is a dual object of limsup random fractal, namely the ***fractal percolation set***. Define i.i.d. random variables $\{Y_x\}_{x\in\mathsf{V}}$ with

$$\mathbb{P}(Y_x = 1) = p = 1 - \mathbb{P}(Y_x = 0)\,, \tag{3.4}$$

where $0 \leq p \leq 1$. Define the fractal percolation set E with parameter p by

$$E := \bigcap_{n=1}^{\infty} E_n \quad \text{with} \quad E_n := \{\sigma \in \partial T : Y_{\sigma(i)} = 1 \text{ for } 1 \leq i \leq n\} \quad \forall\, n \geq 1. \tag{3.5}$$

Thus if $\sigma \in E$, then $Y_{\sigma(n)} = 1$ for all $n \geq 1$. We will call $s := -\ln p$ the ***index*** of the fractal percolation set E.

Figure 2 illustrates part of a limsup random fractal and/or a fractal percolation set.

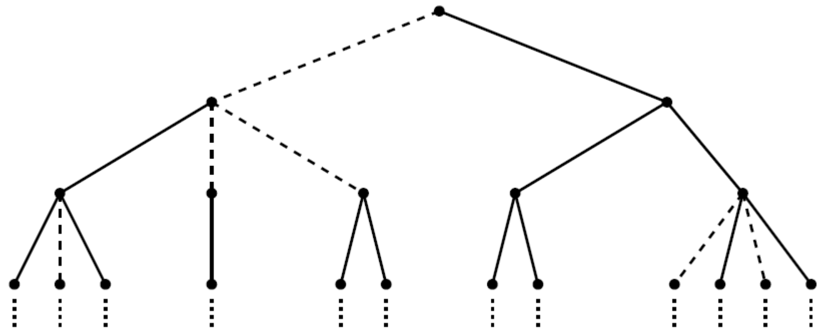

Figure 2: A Graphical Interpretation: Solid Line for 1 and Dashed Line for 0

3.2 Fractal Percolation Sets and Hausdorff Dimension

There is a close connection between fractal percolation sets defined in (3.5) and Hausdorff dimension. The following theorem is due to Lyons [11].

Theorem 3.1. *(Lyons [11]) Let T be a tree and E be a fractal percolation set defined on ∂T with index s. Then:*

(i) If $\dim_{\mathrm{H}}(\partial T) > s$, then $\mathbb{P}(E \neq \emptyset) > 0$; and

(ii) If $\dim_{\mathrm{H}}(\partial T) < s$, then $\mathbb{P}(E \neq \emptyset) = 0$.

We can generalize this result to all Borel subsets of ∂T.

Corollary 3.2. *Let T be a tree and E be a fractal percolation set defined on ∂T with index s. Then for all Borel set $F \subset \partial T$:*

(i) If $\dim_{\mathrm{H}}(F) > s$, then $\mathbb{P}(E \cap F \neq \emptyset) > 0$; and

(ii) If $\dim_{\mathrm{H}}(F) < s$, then $\mathbb{P}(E \cap F \neq \emptyset) = 0$.

Proof. (i) First, if F is a closed subset, then we can regard F as the boundary of a subtree T^F. In fact let $T^F := (\mathsf{V}^F, \mathsf{E}^F)$, where

$$\mathsf{V}^F := \{x \in \mathsf{V} : x \in \sigma \text{ for some } \sigma \in F\} \text{ and } \mathsf{E}^F := (\mathsf{V}^F \times \mathsf{V}^F) \cap \mathsf{E}. \tag{3.6}$$

It follows immediately that $F \subset \partial T^F$. Conversely, for every $\sigma = (v_0, v_1, \dots) \in \partial T^F$ with $v_0 = o$, (3.6) implies that $v_n \in V_F$ and there exists a $\sigma_n \in F$ such that $v_n \in \sigma_n$ for each $n \geq 0$. Thus $d(\sigma_n, \sigma) \leq e^{-n}$. Since F is closed, we must have $\sigma \in F$. Therefore $\partial T^F \subset F$. Now we can apply Theorem 3.1 to the subtree T^F to obtain $\mathbb{P}\{E \cap F \neq \emptyset\} = \mathbb{P}\{E \cap \partial T^F \neq \emptyset\} > 0$, provided that $\dim_{\mathrm{H}}(F) > s$.

In general, for every Borel subset F with $\dim_{\mathrm{H}}(F) > s$, we can find some t such that $s < t < \dim_{\mathrm{H}}(F)$. Then Corollary 7 of Howroyd [5] implies the existence of a closed subset $F_0 \subset F$ such that $0 < \mathcal{H}^t(F_0) < \infty$. Thus $\dim_{\mathrm{H}}(F_0) = t > s$. Since $\mathbb{P}\{E \cap F_0 \neq \emptyset\} > 0$ and $F_0 \subset F$, we have $\mathbb{P}\{E \cap F \neq \emptyset\} > 0$.

(ii) For every fixed $s > \dim_{\mathrm{H}}(F)$, the definition of Hausdorff dimension implies $\mathcal{H}^s(F) = 0$. Then for every $\varepsilon > 0$, we can find a ball covering $\{B_n\}_{n\geq 1}$ of F such that $\sum_{n\geq 1} |B_n|^s < \varepsilon$, where $B_n = B(x_n)$ for some $x_n \in \mathsf{V}$. Since $(\partial T, d)$ is an ultrametric space, (2.3) implies that $|B_n| = e^{-|x_n|}$. Then (3.5) implies that

$$\mathbb{P}\{E \cap B_n \neq \emptyset\} \leq \mathbb{P}\{E_{|x_n|} \cap B(x_n) \neq \emptyset\} = p^{|x_n|} = e^{-s|x_n|}. \tag{3.7}$$

Therefore

$$\mathbb{P}\{E \cap F \neq \emptyset\} \leq \sum_{n\geq 1} \mathbb{P}\{E \cap B_n \neq \emptyset\} \leq \sum_{n\geq 1} e^{-s|x_n|} = \sum_{n\geq 1} |B_n|^s < \varepsilon. \tag{3.8}$$

Let $\varepsilon \downarrow 0$ to see that $\mathbb{P}\{E \cap F \neq \emptyset\} = 0$. □

When $(\partial T, d)$ is replaced by the Euclidean space $\mathbb{R}^N$ equipped with the classic metric, the above corollary is same as Lemma 5.1 of Peres [17]. Moreover we can apply the above corollary to an independent fractal percolation set and estimate the Hausdorff dimension of E.

Theorem 3.3. *(Falconer [2], Mauldin and Williams [15]) Let T be a tree and E be a fractal percolation set defined on ∂T with index s. Then*

$$||\dim_{\mathrm{H}}(E)||_{L^\infty(\mathbb{P})} = \dim_{\mathrm{H}}(\partial T) - s. \tag{3.9}$$

The fractal percolation set E is also closely related to the Riesz capacity of the boundary ∂T.

Theorem 3.4. *(Lyons [12]) Let T be a tree and E be a fractal percolation set defined on ∂T with index s. Then*

$$\frac{1}{2} Cap_s(\partial T) \leq \mathbb{P}\{E \neq \emptyset\} \leq 2Cap_s(\partial T). \tag{3.10}$$

3.3 Limsup Random Fractals and Packing Dimension

Limsup random fractals and packing dimension are closely related. The following theorem is due to Khoshnevisan, Peres, and Xiao [9].

Theorem 3.5. *(Khoshnevisan, Peres, and Xiao [9]) Let T be a tree and A a limsup random fractal defined on ∂T with index t. Then for all Borel set $F \subset \partial T$:*

(i) If $\dim_{\mathrm{P}}(F) > t$, then $\mathbb{P}(E \cap F \neq \emptyset) = 1$; and

(ii) If $\dim_{\mathrm{P}}(F) < t$, then $\mathbb{P}(E \cap F \neq \emptyset) = 0$.

Remark 3.6. *The limsup random fractals studied in [9] are defined on Euclidean spaces. The proofs there rely on the existence of closed sets with positive finite packing measure in Euclidean spaces (see Joyce and Preiss [6]). The existence of such closed sets in an ultrametric space is proved by Haase [4]. Thus Theorem 3.1 of [9] still holds on the boundary of a tree.*

We can compare this theorem to Lyon's Theorem (Corollary 3.2). Similar to Theorem 3.3, we can apply Theorem 3.5 to an independent limsup random fractal and estimate the packing dimension of A.

Theorem 3.7. *(Khoshnevisan, Peres, and Xiao [9]) Let T be a tree and A a limsup random fractal defined on ∂T with index t. Assume that $0 < t < \dim_{\mathrm{P}}(\partial T)$. Then with probability one,*

$$\dim_{\mathrm{P}}(A) = \dim_{\mathrm{P}}(\partial T). \tag{3.11}$$

We may ask the Hausdorff dimension of a limsup random fractal and the packing dimension of a fractal percolation set. This is answered partially in [9].

Theorem 3.8. *(Khoshnevisan, Peres, and Xiao [9]) Let T be a tree. Define on the boundary ∂T a limsup random fractal A with index t and a fractal percolation set E with index s. Then:*

(i) $\dim_{\mathrm{H}}(\partial T) - t \leq \dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{P}}(\partial T) - t$ almost surely, and

(ii) $\dim_{\mathrm{P}}(E) \leq \dim_{\mathrm{P}}(\partial T) - s$ almost surely.

4 New Indices

In order to compute the Hausdorff dimension of limsup random fractals, we introduce two families of indices on the boundary of a tree.

Definition 4.1. *Let ∂T be the boundary of a tree T. For each $s \geq 0$ and $n \geq 1$, define the* **optimized energy form** $J_s(n; \partial T)$ *by*

$$J_s(n; \partial T) := \inf_{\mu \in \mathcal{P}(\partial T)} \iint\limits_{d(\sigma,\gamma) \leq e^{-n}} d(\sigma, \gamma)^{-s} \mu(d\sigma)\mu(d\gamma). \tag{4.1}$$

Define a family of indices $\{\delta_s(\partial T)\}_{s \geq 0}$ by

$$\delta_s(\partial T) := \overline{\lim_{n\to\infty}} \frac{\ln J_s(n; \partial T)}{-n}. \tag{4.2}$$

Remark 4.2. *The optimized energy form $J_s(n; \partial T)$ satisfies*

$$J_s(n; \partial T) = \inf_{\mu \in \mathcal{P}(\partial T)} \sum_{|x|=n, \mu(x)>0} \mathcal{E}_s(\mu_x)\mu(x)^2 \tag{4.3}$$

where $\mathcal{E}_s(\mu)$ is the s-dimensional Riesz energy of a probability measure μ defined in (2.4)*, x denotes any vertex at generation n and $\mu_x \in \mathcal{P}(B(x))$ satisfies $\mu_x(G) = \mu(G \cap B(x))/\mu(x)$ for all $G \in \mathcal{T}$, provided that $\mu(x) > 0$.*

Lemma 4.3. *For a fixed tree T, if $0 \leq s_1 < s_2$, then*

$$\delta_{s_2}(\partial T) + s_2 \leq \delta_{s_1}(\partial T) + s_1. \tag{4.4}$$

In particular, $\delta_s(\partial T)$ is non-increasing in s.

Proof. For all σ, γ with $d(\sigma, \gamma) \leq e^{-n}$, the definition of the metric d (see eq. (2.2)) implies that

$$\begin{aligned} d(\sigma,\gamma)^{-s_2} &= e^{s_2|\sigma\curlywedge\gamma|} = e^{(s_2-s_1)|\sigma\curlywedge\gamma|}e^{s_1|\sigma\curlywedge\gamma|} \\ &\geq e^{(s_2-s_1)n}e^{s_1|\sigma\curlywedge\gamma|} = e^{(s_2-s_1)n}d(\sigma,\gamma)^{-s_1}. \end{aligned} \tag{4.5}$$

Thus by combining (4.1) and (4.5), we obtain $J_{s_2}(n; \partial T) \geq e^{(s_2-s_1)n} J_{s_1}(n; \partial T)$ for all $n \geq 1$. Then (4.4) follows by taking limits. □

Lemma 4.4. *For a fixed tree T,*

(i) $0 \leq \delta_s(\partial T) \leq \overline{\dim}_{\mathrm{M}}(\partial T)$ *for all* $0 \leq s < \dim_{\mathrm{H}}(\partial T)$*, and* $\delta_0(\partial T) = \overline{\dim}_{\mathrm{M}}(\partial T)$*;*

(ii) $\delta_s(\partial T) < 0$ *if* $s > \dim_{\mathrm{H}}(\partial T)$*; and*

(iii) $\delta_{s_2}(\partial T) < \delta_{s_1}(\partial T)$*, for all* $0 \leq s_1 < s_2 < \dim_{\mathrm{H}}(\partial T)$*, that is,* $\delta_s(\partial T)$ *is strictly decreasing in s for* $0 \leq s < \dim_{\mathrm{H}}(\partial T)$*.*

Proof. (i) On one hand (4.3) implies that

$$J_0(n; \partial T) = \inf_{\mu \in \mathcal{P}(\partial T)} \sum_{|x|=n} \mu(x)^2 = N_n^{-1}, \tag{4.6}$$

where N_n is the number of vertices at generation n. It follows from this and (2.34) that $\delta_0(\partial T) = \overline{\dim}_{\mathrm{M}}(\partial T)$. On the other hand for every $s < \dim_{\mathrm{H}}(\partial T)$, Frostman's lemma ([13], Corollary 15.6) guarantees the existence of a Borel probability measure μ whose s-dimensional Riesz energy is finite. Then (2.23) implies that $\lim_{n\to\infty} J_s(n; \partial T) = 0$.

Therefore $\delta_s(\partial T) \geq 0$ for $s < \dim_{\mathrm{H}}(\partial T)$. The monotonicity of $\delta_s(\partial T)$ completes the proof of the first part.

(ii) For every $s > \dim_{\mathrm{H}}(\partial T)$, we have $s > \dim_{\mathrm{H}}(B(x))$ for all vertices $x \in \mathsf{V}$. Then Frostman's lemma implies that $\mathcal{E}_s(\mu) = \infty$ for all $\mu \in \mathcal{P}(B(x))$ and $x \in \mathsf{V}$. Therefore $J_s(n; \partial T) = \infty$ for all $n \geq 1$. This completes the proof of the second part.

(iii) This follows from the finiteness of $\delta_s(\partial T)$ for $0 \leq s < \dim_{\mathrm{H}}(\partial T)$ and Lemma 4.3. □

For each $x \in \mathsf{V}$, we can treat the ball $B(x)$ as the boundary of a subtree just as in the proof of Corollary 3.2. Then we can extend the index δ_s to closed balls.

Lemma 4.5. *If T is a spherically symmetric tree, then $\delta_s(\partial T) = \delta_s(B(x))$ for all $0 \leq s < \dim_{\mathrm{H}}(\partial T)$ and all $x \in \mathsf{V}$.*

Proof. Since $J_s(n; \partial T)$ is an energy form obtained by optimizing over all Borel probability measures supported on ∂T, we can use a similar argument as in the proof of Corollary 2.4 to obtain

$$J_s(n; \partial T) = \inf_{\mu \in \mathcal{P}(\partial T)} \sum_{|x|=n, \mu(x)>0} \frac{\mu(x)^2}{\mathrm{Cap}_s(B(x))}. \tag{4.7}$$

Since T is spherically symmetric, we have $\mathrm{Cap}_s(B(x)) = \mathrm{Cap}_s(B(y))$ for all x, y such that $|x| = |y|$. Thus

$$J_s(n; \partial T) = (N_n \mathrm{Cap}_s(B(x)))^{-1}, \tag{4.8}$$

where N_n is the total number of vertices at generation n and x is an arbitrary vertex at generation n. If we fix $n \geq 1$ and treat each closed ball as the boundary of a subtree, then for all $|x| = n$ and $m \geq n$,

$$J_s(m; \partial T) = N_n^{-1} J_s(m; B(x)). \tag{4.9}$$

Since N_n is finite, the above equation implies that $\delta_s(\partial T) = \delta_s(B(x))$. □

Definition 4.6. *Let ∂T be the boundary of a tree T. A family of indices $\{\mathfrak{D}_t(\partial T)\}_{t \geq 0}$ is defined by*

$$\mathfrak{D}_t(\partial T) := \sup\{s \geq 0 : \delta_s(\partial T) > t\}, \tag{4.10}$$

with the convention $\sup \emptyset := 0$.

Remark 4.7. *The index $\mathfrak{D}_t(\partial T)$ is well-defined according to Lemma 4.3.*

Lemma 4.8. *For a fixed tree T:*

(i) If $0 \leq t \leq \overline{\dim}_{\mathrm{M}}(\partial T)$, then $\mathfrak{D}_t(\partial T) \leq \overline{\dim}_{\mathrm{M}}(\partial T) - t$;

(ii) If $0 \leq t \leq \dim_{\mathrm{H}}(\partial T)$, then $\mathfrak{D}_t(\partial T) \geq \dim_{\mathrm{H}}(\partial T) - t$;

(iii) $\mathfrak{D}_t(\partial T)$ is non-decreasing in t for $0 \leq t \leq \overline{\dim}_{\mathrm{M}}(\partial T)$; and

(iv) $\mathfrak{D}_t(\partial T) \leq \dim_{\mathrm{H}}(\partial T)$ for $0 \leq t \leq \overline{\dim}_{\mathrm{M}}(\partial T)$.

Proof. (i) Let $s_0 := \overline{\dim}_{\mathrm{M}}(\partial T) - t$. Then $s_0 \geq 0$ and Lemma 4.3 implies that $\delta_{s_0}(\partial T) + s_0 \leq \delta_0(\partial T) + 0$. According to Lemma 4.4 (i), we have $\delta_0(\partial T) = \overline{\dim}_{\mathrm{M}}(\partial T)$. Therefore $\delta_{s_0}(\partial T) \leq t$. This implies $\mathfrak{D}_t(\partial T) \leq s_0 = \overline{\dim}_{\mathrm{M}}(\partial T) - t$.

(ii) Without loss of generality, we assume that $\dim_{\mathrm{H}}(\partial T) - t > 0$, otherwise there is nothing to prove. For s_1, s_2 such that $0 < s_1 < s_2 < \dim_{\mathrm{H}}(\partial T) - t$, we apply Lemma 4.3 and Lemma 4.4 (i) to get

$$\delta_{s_1}(\partial T) + s_1 \geq \delta_{s_1+t}(\partial T) + (s_1 + t) \geq \delta_{s_2+t}(\partial T) + (s_2 + t) \geq s_2 + t. \tag{4.11}$$

Therefore $\delta_{s_1}(\partial T) \geq (s_2 - s_1) + t > t$. This implies $\mathfrak{D}_t(\partial T) \geq s_1$. Let $s_1 \uparrow \dim_{\mathrm{H}}(\partial T) - t$ to complete the proof.

(iii) For $0 \leq t_1 < t_2 \leq \overline{\dim}_{\mathrm{M}}(\partial T)$, let $s_1 = \mathfrak{D}_{t_1}(\partial T)$ and $s_2 = \mathfrak{D}_{t_2}(\partial T)$. Then for any $s < s_2$, we have $\delta_s(\partial T) > t_2 > t_1$. Therefore $s_1 \geq s$. This shows that $s_1 \geq s_2$.

(iv) For every $s > \dim_{\mathrm{H}}(\partial T)$, Lemma 4.4 (ii) shows that $\delta_s(\partial T) < 0$. Thus $\mathfrak{D}_t(\partial T) \leq s$ for all $t \geq 0$. Let $s \downarrow \dim_{\mathrm{H}}(\partial T)$ to complete the proof. □

Example 4.9. *Consider the spherically symmetric tree T constructed in Example 2.9. We show that*

$$\delta_s(\partial T) = \dim_{\mathrm{P}}(\partial T) - s = \ln 3 - s, \tag{4.12}$$

and

$$\mathfrak{D}_t(\partial T) = (\dim_{\mathrm{P}}(\partial T) - t) \wedge \dim_{\mathrm{H}}(\partial T) = (\ln 3 - t) \wedge \ln 2. \tag{4.13}$$

For each $x \in \mathsf{V}$, let $T^x := (\mathsf{V}^x, \mathsf{E}^x)$ be the subtree rooted at x such that

$$\mathsf{V}^x = \{x \text{ and all the descendents of } x\} \quad \text{and} \quad \mathsf{E}^x = (\mathsf{V}^x \times \mathsf{V}^x) \cap \mathsf{E}. \tag{4.14}$$

Note that there is a natural one-to-one correspondence between ∂T^x and $B(x)$. Let σ^x denote the ray in ∂T^x which corresponds to $\sigma \in B(x)$. Then

$$e^{-|x|} d(\sigma^x, \gamma^x) = d(\sigma, \gamma) \qquad \forall\, \sigma, \gamma \in B(x). \tag{4.15}$$

For each Borel probability measure μ^x on ∂T^x, it naturally induces a Borel probability measure μ_x on $B(x)$, and vice versa. Then (4.15) *implies that*

$$e^{s|x|} \mathcal{E}_s(\mu^x) = \mathcal{E}_s(\mu_x) \qquad \forall\, \mu_x \in B(x). \tag{4.16}$$

According to Lemma 2.5, we have

$$\mathcal{E}_s(\nu^x) = \mathit{Cap}_s(\partial T^x) \quad \text{and} \quad \mathcal{E}_s(\nu_x) = \mathit{Cap}_s(B(x)), \tag{4.17}$$

where ν^x and ν_x are the uniform probability measures on ∂T^x and $B(x)$, respectively. Thanks to (4.8) *and* (4.17)*, we obtain*

$$J_s(n, \partial T) = N_n^{-1} \mathcal{E}_s(\nu_x) = e^{s|x|} N_n^{-1} \mathcal{E}_s(\nu^x), \tag{4.18}$$

for any $x \in \mathsf{V}$ with $|x| = n$, where the second equality follows from (4.16)*. Since $\mathcal{E}_s(\nu^x)$ only depends on $|x|$, we denote it by $I_{|x|}$. Thus*

$$J_s(n, \partial T) = e^{sn} N_n^{-1} I_n. \tag{4.19}$$

For a k-nary tree (all vertices have degree k), it is well known that its boundary has Hausdorff dimension $\ln k$. The structure of T implies that T^x contains a binary tree and T^x is contained in a ternary tree for all $x \in \mathsf{V}$. Since $s < \ln 2 = \dim_{\mathrm{H}}(\partial T)$, there exist constants c and C such that

$$c \leq I_n \leq C \qquad \forall\, n \geq 1. \tag{4.20}$$

This and (4.19) *imply that*

$$\delta_s(\partial T) = \varlimsup_{n \to \infty} \frac{\ln J_s(n; \partial T)}{-n} = \varlimsup_{n \to \infty} \frac{-\ln N_n}{-n} - s = \overline{\dim}_{\mathrm{M}}(\partial T) - s. \tag{4.21}$$

Thanks to Lemma 2.8, we have $\delta_s(\partial T) = \dim_{\mathrm{P}}(\partial T) - s$.

Remark 4.10. *It is not clear whether $\delta_s(\partial T) = \dim_{\mathrm{P}}(\partial T) - s$ holds for a general spherically symmetric tree with bounded degrees.*

5 Hitting Probabilities and Proof for the Main Results

Let T be a tree and A be a limsup random fractal defined on the boundary ∂T. According to Corollary 3.2, if E is an independent fractal percolation set defined on ∂T, then we can find estimates of the Hausdorff dimension of A by computing the probability of the event $\{A \cap E \neq \emptyset\}$.

Recall the definition of limsup random fractals and fractal percolation sets from (3.2) and (3.5), respectively. We adopt the following notations for certain events:

$$\begin{aligned} \{o \to x\} &:= \{\exists\, \sigma \in B(x) \text{ so that } Y_{\sigma(i)} = 1 \text{ for all } 1 \leq i \leq |x|\}, \text{ and} \\ \{x \to \infty\} &:= \{\exists\, \sigma \in B(x) \text{ so that } Y_{\sigma(i)} = 1 \text{ for all } i \geq |x|\}, \end{aligned} \tag{5.1}$$

where $\sigma(i)$ denotes the ith vertex on the ray σ. Then we have

$$\{o \to x\} \cap \{x \to \infty\} = \{B(x) \cap E \neq \emptyset\}, \tag{5.2}$$

and

$$\{A_n \cap E \neq \emptyset\} = \bigcup_{|x|=n} \left(\{o \to x\} \cap \{x \to \infty\} \cap \{Z_x = 1\}\right). \tag{5.3}$$

In order to compute $\mathbb{P}\{A \cap E \neq \emptyset\}$, we estimate the probabilities of events $\{A_n \cap E \neq \emptyset\}$ for all $n \geq 1$. According to Theorem 3.4, we have $\mathbb{P}\{B(x) \cap E \neq \emptyset\} > 0$ if and only if $\text{Cap}_s(B(x)) > 0$, where s is the index of the fractal percolation set E. We define for each $\mu \in \mathcal{P}(\partial T)$

$$I_\mu^n(\partial T) \equiv I_\mu^n := \sum_{|x|=n} \mathbb{1}\{o \to x\} \mathbb{1}\{x \to \infty\} \mathbb{1}\{Z_x = 1\} q_n^{-1} a_\mu(x), \tag{5.4}$$

where $\{q_n\}_{n \geq 1}$ are the parameters associated with the limsup random fractal A and

$$a_\mu(x) = \begin{cases} 0, & \text{if } \text{Cap}_s(B(x)) = 0, \\ \mu(x)(\mathbb{P}\{B(x) \cap E \neq \emptyset\})^{-1}, & \text{if } \text{Cap}_s(B(x)) > 0. \end{cases} \tag{5.5}$$

Lemma 5.1. *For fixed $s \geq 0$,*

$$\mathbb{P}\{A_n \cap E \neq \emptyset\} \geq \sup_{\mu \in \mathcal{P}_s(\partial T)} \mathbb{P}\{I_\mu^n > 0\}, \quad \forall\, n \geq 1, \tag{5.6}$$

where $\mathcal{P}_s(\partial T)$ denote the collection of Borel probability measures with finite s-dimensional Riesz energy.

Proof. This follows from Proposition 2.6 and (5.3). □

Before we proceed to estimate the probability of the event $\{A_n \cap E \neq \emptyset\}$, we define a new capacity and give some technical lemmas for this capacity. For fixed constants $s \geq 0$, $K > 0$ and $n \geq 1$, define a kernel function on the boundary ∂T by

$$f(\sigma, \gamma; s, K, n) = \begin{cases} d(\sigma, \gamma)^{-s}, & \text{if } d(\sigma, \gamma) > e^{-n}, \\ K d(\sigma, \gamma)^{-s}, & \text{if } d(\sigma, \gamma) \leq e^{-n}. \end{cases} \tag{5.7}$$

Then for each $\mu \in \mathcal{P}(\partial T)$ define an energy form

$$\mathcal{E}(\mu; s, K, n) := \iint f(\sigma, \gamma; s, K, n) \mu(d\sigma) \mu(d\gamma). \tag{5.8}$$

Similar to Proposition 2.3, $\mathcal{E}(\mu; s, K, n)$ satisfies

$$\mathcal{E}(\mu; s, K, n) = \sum_{\substack{|x|=|y|=n \\ x \neq y}}\sum p^{-|x \wedge y|} \mu(x) \mu(y) + K \sum_{\substack{|x|=n \\ \mu(x)>0}} \mathcal{E}_s(\mu_x) \mu(x)^2, \tag{5.9}$$

where $\mu_x \in \mathcal{P}(B(x))$ satisfies $\mu_x(G) = \mu(G \cap B(x))/\mu(x)$ for all $G \in \mathcal{T}$, provided that $\mu(x) > 0$. For each Borel set $G \subset \partial T$, we define a capacity

$$\mathrm{Cap}(G; s, K, n) := \left[\inf_{\mu \in \mathcal{P}(G)} \mathcal{E}(\mu; s, K, n)\right]^{-1}. \tag{5.10}$$

Note that when $K = 1$, the above energy form and capacity are the same as the s-dimensional Riesz energy and capacity, respectively. We will be interested in the capacity $\mathrm{Cap}(G; s, K, n)$ with large K and n.

Lemma 5.2. *For fixed constant $s \geq 0$, $K > 0$ and $n \geq 1$, and for all Borel set $G \subset \partial T$,*

$$\mathit{Cap}(G; s, K, n) = \left[\inf_{\mu \in \mathcal{P}_s(G)} \mathcal{E}(\mu; s, K, n)\right]^{-1}. \tag{5.11}$$

Moreover

$$\mathit{Cap}(G; s, K, n) = \left(\inf_{\mu \in \mathcal{P}(G)} \left\{ \sum\sum_{\substack{|x|=|y|=n \\ x \neq y}} p^{-|x \curlywedge y|} \mu(x)\mu(y) + K \sum_{\substack{|x|=n \\ \mu(x)>0}} \frac{\mu(x)^2}{\mathit{Cap}_s(B(x) \cap G)} \right\}\right)^{-1}. \tag{5.12}$$

Proof. First, by comparing (5.9) to (2.7), we see that $\mathcal{E}(\mu; s, K, n) = \infty$ whenever $\mu \in \mathcal{P}(G) \backslash \mathcal{P}_s(G)$. Therefore $\inf_{\mu \in \mathcal{P}_s(G)} \mathcal{E}(\mu; s, K, n) = \inf_{\mu \in \mathcal{P}(G)} \mathcal{E}(\mu; s, K, n)$ and (5.11) follows immediately. Second, the proof for (2.12) also works for (5.12). □

Lemma 5.3. *Let T be a tree. For fixed $s > 0$ and $\{q_n\}_{n \geq 1}$ such that $t = -\lim_{n\to\infty} n^{-1} \ln q_n$ exists and $t > 0$:*

(i) $\sum_{n \geq 1} \mathit{Cap}(\partial T; s, q_n^{-1}, n) < \infty$, provided that $\delta_s(\partial T) < t$; and

(ii) $\overline{\lim}_{n\to\infty} \mathit{Cap}(\partial T; s, q_n^{-1}, n) \geq \mathit{Cap}_s(\partial T)$, provided that T is spherically symmetric and $\delta_s(\partial T) > t$.

Proof. (i) Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$ and $q_n = e^{-n(t+a_n)}$. We can choose r and r_0 such that $\delta_s(\partial T) < r_0 < r < t$ and $t - r < t - r_0 + a_n$ for all sufficiently large n. Then there exists $N > 1$ such that $J_s(n; \partial T) > e^{-nr_0}$ and $q_n^{-1} J_s(n; \partial T) > e^{n(t-r)}$ for all $n > N$. This implies that

$$\begin{aligned} &\inf_{\mu \in \mathcal{P}(\partial T)} \left\{ \sum\sum_{\substack{|x|=|y|=n \\ x \neq y}} p^{-|x \curlywedge y|} \mu(x)\mu(y) + q_n^{-1} \sum_{|x|=n} \mathcal{E}_s(\mu_x)\mu(x)^2 \right\} \\ &\qquad \geq \inf_{\mu \in \mathcal{P}(\partial T)} \left\{ q_n^{-1} \sum_{|x|=n} \mathcal{E}_s(\mu_x)\mu(x)^2 \right\} = q_n^{-1} J_s(n; \partial T) > e^{n(t-r)}, \end{aligned} \tag{5.13}$$

for all $n > N$. Thanks to (5.9), we have $\mathrm{Cap}(\partial T; s, q_n^{-1}, n) < e^{-n(t-r)}$ for all $n > N$. Since $t - r > 0$, we obtain $\sum_{n \geq 1} \mathrm{Cap}(\partial T; s, q_n^{-1}, n) < \infty$.

(ii) Choose r and r_0 such that $\delta_s(\partial T) > r_0 > r > t$ and $r - t < r_0 - t - a_n$ for all sufficiently large n. Then we can find a subsequence $\{n_k\}_{k \geq 1}$ such that $J_s(n_k; \partial T) < e^{-n_k r_0}$ and $q_{n_k}^{-1} J_s(n_k; \partial T) < e^{-n_k(r-t)}$ for all large k. This implies that

$$\varliminf_{n\to\infty} q_n^{-1} \inf_{\mu \in \mathcal{P}(\partial T)} \left\{ \sum_{|x|=n} \mathcal{E}_s(\mu_x)\mu(x)^2 \right\} = 0. \tag{5.14}$$

Consider the uniform probability measure ν. Since T is spherically symmetric, Lemma 2.5 implies that

$$\mathcal{E}_s(\nu) = \inf_{\mu \in \mathcal{P}(\partial T)} \mathcal{E}_s(\mu) = (\mathrm{Cap}_s(\partial T))^{-1} < \infty. \tag{5.15}$$

Moreover,

$$\sum_{|x|=n} \mathcal{E}_s(\nu_x)\nu(x)^2 = \inf_{\mu \in \mathcal{P}(\partial T)} \left\{ \sum_{|x|=n} \mathcal{E}_s(\mu_x)\mu(x)^2 \right\}. \tag{5.16}$$

Therefore,

$$\begin{aligned}
&\varliminf_{n\to\infty} \inf_{\mu \in \mathcal{P}(\partial T)} \left\{ \sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} p^{-|x\curlywedge y|}\mu(x)\mu(y) + q_n^{-1} \sum_{|x|=n} \mathcal{E}_s(\mu_x)\mu(x)^2 \right\} \\
&\le \varliminf_{n\to\infty} \left\{ \sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} p^{-|x\curlywedge y|}\nu(x)\nu(y) + q_n^{-1} \sum_{|x|=n} \mathcal{E}_s(\nu_x)\nu(x)^2 \right\} \qquad (5.17) \\
&= \lim_{n\to\infty} \sum\sum_{\substack{|x|=|y|=n \\ x\neq y}} p^{-|x\curlywedge y|}\nu(x)\nu(y) + \varliminf_{n\to\infty} q_n^{-1} \sum_{|x|=n} \mathcal{E}_s(\nu_x)\nu(x)^2 \\
&= (\mathrm{Cap}_s(\partial T))^{-1},
\end{aligned}$$

where the first equality follows from Lemma 2.7 and the last one follows from (5.14), (5.15), and (5.16). ☐

With the newly defined capacity, we can estimate the probability of the event $\{A_n \cap E \neq \emptyset\}$. The following two lemmas generalize the idea of the proof of Theorem 2.1 of Khoshnevisan [7].

Lemma 5.4. *Let A be a limsup random fractal with index t and E be an independent fractal percolation set with index s. Assume that $0 < t < \dim_{\mathrm{P}}(\partial T)$ and $0 < s < \dim_{\mathrm{H}}(\partial T)$. Then for all $n \geq 1$,*

$$\mathbb{P}\{A_n \cap E \neq \emptyset\} \geq \mathit{Cap}(\partial T; s, 2q_n^{-1}, n). \tag{5.18}$$

Proof. We first estimate $\mathbb{P}\{I_\mu^n > 0\}$ for every fixed $\mu \in \mathcal{P}_s(\partial T)$ and then use Lemma 5.1 to derive the desired lower bound. According to the Paley-Zygmund inequality ([7], Lemma 1.2), if X is a nonnegative random variable with finite second moment, then for all $\varepsilon \in (0,1)$,

$$\mathbb{P}\{X > \varepsilon \mathbb{E}[X]\} \geq (1-\varepsilon)^2 \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}. \tag{5.19}$$

Thus in order to estimate $\mathbb{P}\{I_\mu^n > 0\}$, we will calculate the first two moments of I_μ^n. On one hand, from the independence of the random variables $\{Y_x\}_{x\in \mathsf{V}}$ and $\{Z_x\}_{x\in\mathsf{V}}$, we have

$$\begin{aligned}
\mathbb{E}[I_\mu^n] &= \sum_{|x|=n} \mathbb{P}\{o \to x, x \to \infty\}\mathbb{P}\{Z_x = 1\} q_n^{-1} a_\mu(x) \\
&= \sum_{|x|=n} \mathbb{P}\{B(x) \cap E \neq \emptyset\} a_\mu(x) \qquad (5.20) \\
&= 1,
\end{aligned}$$

where the second equality follows from (5.2) and the last equality follows from the definition of a_μ (see (5.5)).

On the other hand, we apply the independence of the random variables $\{Y_x\}_{x\in\mathsf{V}}$ and $\{Z_x\}_{x\in\mathsf{V}}$ again to obtain

$$\begin{aligned}
&\mathbb{E}[(I_\mu^n)^2] \\
&= q_n^{-1}\sum_{|x|=n}\mathbb{P}\{o\to x, x\to\infty\}(a_\mu(x))^2 \\
&\quad + \sum\sum_{\substack{|x|=|y|=n\\ x\neq y}} p^{-|x\curlywedge y|}\mathbb{P}\{o\to x, x\to\infty\}\mathbb{P}\{o\to y, y\to\infty\}a_\mu(x)a_\mu(y) \\
&= q_n^{-1}\sum_{|x|=n}\frac{\mu(x)^2}{\mathbb{P}\{B(x)\cap E\neq\emptyset\}} + \sum\sum_{\substack{|x|=|y|=n\\ x\neq y}} p^{-|x\curlywedge y|}\mu(x)\mu(y).,
\end{aligned} \tag{5.21}$$

where we used the identity $\mathbb{P}\{o\to x, o\to y\} = p^{-|x\curlywedge y|}\mathbb{P}\{o\to x\}\mathbb{P}\{o\to y\}$ to derive the first equality. Thanks to Lyons' Theorem (Theorem 3.4), we have

$$\mathbb{E}[(I_\mu^n)^2] \le 2q_n^{-1}\sum_{|x|=n}\frac{\mu(x)^2}{\mathrm{Cap}_s(B(x))} + \sum\sum_{\substack{|x|=|y|=n\\ x\neq y}} p^{-|x\curlywedge y|}\mu(x)\mu(y). \tag{5.22}$$

Now combine (5.19), (5.20) and (5.22), and let $\varepsilon\downarrow 0$ to get

$$\mathbb{P}\{I_\mu^n>0\} \ge \left[2q_n^{-1}\sum_{|x|=n}\frac{\mu(x)^2}{\mathrm{Cap}_s(B(x))} + \sum\sum_{\substack{|x|=|y|=n\\ x\neq y}} p^{-|x\curlywedge y|}\mu(x)\mu(y)\right]^{-1}. \tag{5.23}$$

Finally, we take supremum over all $\mu\in\mathcal{P}_s(B(x))$ and apply Lemma 5.1 and Lemma 5.2 to obtain the desired result. $\square$

In order to find an upper bound for $\mathbb{P}\{A_n\cap E\neq\emptyset\}$ we need to introduce some new notations to describe the structure of the tree. For each $n\geq 1$, we label all the vertices at generation n from left to right by $x_1^n, x_2^n, \ldots, x_{N_n}^n$, where N_n is the total number of vertices at generation n. (We use supscript to denote the generation and subscript to denote the position from left to right.) See Figure 3 for an illustration.

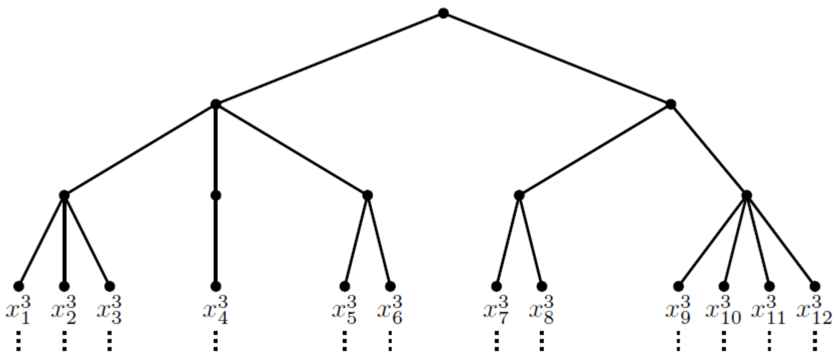

Figure 3: Labeling Vertices from Left to Right

Recall that $\{Y_x\}_{x\in\mathsf{V}}$ and $\{Z_x\}_{x\in\mathsf{V}}$ are the random variables used to define the fractal percolation set E and the limsup random fractal A, respectively. For each fixed $n\geq 1$, define $X_i^n := (Y_{x_{i_1}^1}, \ldots, Y_{x_{i_n}^n})$ for $1\le i\le N_n$, where $i_n = i$ and $x_{i_k}^k$ is the parent of $x_{i_{k+1}}^{k+1}$

for $1 \le k \le n-1$. In words, X_i^n denotes the vector consisting of the random variables $\{Y_v\}$ on the path from root o to the vertex x_i^n. In particular, $\{X_i^n = \mathbb{1}_n := (1,\ldots,1) \in \mathbb{R}^n\} = \{o \to x_i^n\}$. It follows that $\{X_i^n\}_{1\le i\le N_n}$ is a Markov process. For $1 \le k \le N_n$, define filtrations

$$\begin{aligned} &\mathcal{A}_k^n := \sigma\big(\{Z_{x_i^n}\}_{1\le i\le k}\big),\\ &\mathcal{B}_k^n := \sigma\big(\{X_i^n\}_{1\le i\le k}\big),\\ &\mathcal{C}_k^n := \sigma\big(\{Y_y : y \succ x_i^n\}_{1\le i\le k}\big), \text{ and}\\ &\mathcal{F}_k^n := \mathcal{A}_k^n \vee \mathcal{B}_k^n \vee \mathcal{C}_k^n, \end{aligned} \tag{5.24}$$

where $y \succ x$ means y is a descendant of x. In words, $\mathcal{A}_k^n$ is the information generated by the Z_x random variables at x_i^n for $1 \le i \le k$; $\mathcal{B}_k^n$ is the information generated by the Y_x random variables at x_k^n, x_k^n's ancestors, x_k^n's left siblings, and x_k^n's left siblings' ancestors; $\mathcal{C}_k^n$ is the information generated by the Y_x random variables at x_k^n's descendants and x_k^n's left siblings' descendants; and $\mathcal{F}_k^n$ is the information of all three. By the independence of the Y_x and Z_x random variables, we know that $\mathcal{A}_k^n$, $\mathcal{B}_k^n$ and $\mathcal{C}_k^n$ are independent for fixed $n \ge 1$ and $1 \le k \le N_n$.

Lemma 5.5. *Let A be a limsup random fractal with index t and E be an independent fractal percolation set with index s. Assume that $0 < t < \dim_{\mathrm{P}}(\partial T)$ and $0 < s < \dim_{\mathrm{H}}(\partial T)$. Then for all $n \ge 1$,*

$$\mathbb{P}\{A_n \cap E \ne \emptyset\} \le 2\mathit{Cap}(\partial T; s, q_n^{-1}, n). \tag{5.25}$$

Proof. Define $T_n := \inf\{k \ge 1 : Z_{x_k^n} = 1, o \to x_k^n, \text{ and } x_k^n \to \infty\}$ with the convention $\inf\emptyset := \infty$. In words, T_n is the first time (from left to right) such that $A_n \cap E \ne \emptyset$ at generation n. Then T_n is a stopping time with respect to the filtration $\{\mathcal{F}_n^k\}_{1\le k\le N_n}$ and

$$\{T_n < \infty\} = \{A_n \cap E \ne \emptyset\}. \tag{5.26}$$

For every fixed $\mu \in \mathcal{P}_s(\partial T)$ and $n \ge 1$, (5.4) implies that I_μ^n is bounded almost surely. Thus we can define a bounded martingale $\{M_\mu^n(k)\}_{1\le k\le N_n}$ by

$$M_\mu^n(k) := \mathbb{E}[I_\mu^n | \mathcal{F}_k^n], \qquad \text{for } 1 \le k \le N_n. \tag{5.27}$$

By iterated conditioning and independence, we have

$$\begin{aligned} M_\mu^n(k) = &\sum_{1\le i\le k} \mathbb{1}\{o \to x_i^n\}\mathbb{1}\{x_i^n \to \infty\}\mathbb{1}\{Z_{x_i^n} = 1\}q_n^{-1}a_\mu(x_i^n) + \\ &+ \sum_{k+1\le i\le N_n} \mathbb{E}[\mathbb{1}\{o \to x_i^n\}|\mathcal{F}_k^n]\mathbb{P}(x_i^n \to \infty)a_\mu(x_i^n). \end{aligned} \tag{5.28}$$

Since $\mathcal{A}_k^n$, $\mathcal{B}_k^n$ and $\mathcal{C}_k^n$ are independent and $\{X_k^n\}_{1\le k\le N_n}$ is a Markov process, we have

$$\begin{aligned} \mathbb{E}[\mathbb{1}\{o \to x_i^n\}|\mathcal{F}_k^n] &= \mathbb{E}[\mathbb{1}\{o \to x_i^n\}|\mathcal{B}_k^n] = \mathbb{E}[\mathbb{1}\{o \to x_i^n\}|X_k^n] \\ &\ge p^{n-|x_i^n \curlywedge x_k^n|}\mathbb{1}\{X_k^n = \mathbb{1}_n\}, \end{aligned} \tag{5.29}$$

for $k+1 \le i \le N$. We apply this inequality in (5.28), multiply (5.28) by $\mathbb{1}\{T_n = k\}$ on both sides, and notice the facts $\{T_n = k\} \subset \{o \to x_k^n, x_k^n \to \infty, Z_{x_k^n} = 1\}$ and $\{T_n =$

$k\} \cap \{o \to x_i^n, x_i^n \to \infty, Z_{x_i^n} = 1\} = \emptyset$ for $1 \le i \le k-1$, in order to obtain

$$\begin{aligned} &M_\mu^n(k)\mathbb{1}\{T_n = k\} \\ &\ge \mathbb{1}\{o \to x_k^n\}\mathbb{1}\{x_k^n \to \infty\}\mathbb{1}\{Z_{x_k^n} = 1\}q_n^{-1}a_\mu(x_k^n)\mathbb{1}\{T_n = k\} \\ &\quad + \sum_{k+1 \le i \le N_n} p^{n-|x_i^n \curlywedge x_k^n|}\mathbb{1}\{X_k^n = \mathbb{1}_n\}\mathbb{P}\{x_i^n \to \infty\}a_\mu(x_i^n)\mathbb{1}\{T_n = k\} \\ &= \mathbb{1}\{T_n = k\}q_n^{-1}a_\mu(x_k^n) \\ &\quad + \mathbb{1}\{T_n = k\}\sum_{T_n+1 \le i \le N_n} p^{n-|x_i^n \curlywedge x_{T_n}^n|}\mathbb{P}\{x_i^n \to \infty\}a_\mu(x_i^n). \end{aligned} \tag{5.30}$$

Since $\dim_{\mathrm{H}}(\partial T) > s$ and $A_n := \bigcup_{|x|=n, Z_x=1} B(x)$, the σ-stability of Hausdorff dimension implies that $\mathbb{P}\{\dim_{\mathrm{H}}(A_n) > s\} > 0$. Thus by first conditioning on A_n and then applying Corollary 3.2 and (5.26), we get $\mathbb{P}(T_n < \infty) > 0$. Then we can choose a special Borel probability measure $\mu_n^* \in \mathcal{P}(\partial T)$ that satisfies

$$\mu_n^*(x_k^n) = \mathbb{P}\{T_n = k | T_n < \infty\}, \text{ for } 1 \le k \le N_n. \tag{5.31}$$

By the definition of T_n and Theorem 3.4, we have

$$\mu_n^*(x_k^n) \begin{cases} = 0, & \text{if } \mathrm{Cap}_s(B(x_k^n)) = 0, \\ > 0, & \text{if } \mathrm{Cap}_s(B(x_k^n)) > 0. \end{cases} \tag{5.32}$$

Thus for all x_k^n with $\mu_n^*(x_k^n) > 0$, from the definition of Riesz capacity, we can choose some $\mu_{n,x_k^n}^* \in \mathcal{P}(B(x_k^n))$ so that its s-dimensional Riesz energy is finite. Moreover, Proposition 2.3 shows that if $\mu_{n,x_k^n}^*$ has finite s-dimensional Riesz energy for all x_k^n with $\mu_n^*(x_k^n) > 0$, then μ_n^* has finite s-dimensional Riesz energy too. Therefore $\mu_n^* \in \mathcal{P}_s(\partial T)$.

Let $J_1(k,\mu)$ denote the first summand in the last line of (5.30) and $J_2(k,\mu)$ the second summand. If we use the above μ_n^* to replace μ, sum over k and take expectations, then we have

$$\begin{aligned} \mathbb{E}\left[\sum_{k=1}^{N_n} J_1(k,\mu_n^*)\right] &= \sum_{k=1}^{N_n} \mathbb{P}\{T_n = k\}q_n^{-1}a_{\mu_n^*}(x_k^n) \\ &= \sum_{k=1}^{N_n} \mu^*(x_k^n) \cdot \mathbb{P}\{T_n < \infty\}q_n^{-1}a_{\mu_n^*}(x_k^n) \\ &\ge \frac{1}{2}q_n^{-1}\mathbb{P}\{T_n < \infty\}\sum_{k=1}^{N_n} \frac{\mu^*(x_k^n)^2}{\mathrm{Cap}_s(B(x_k^n))}, \end{aligned} \tag{5.33}$$

where we used Theorem 3.4 to derive the last inequality, and

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^{N_n} J_2(k,\mu^*)\right] \\
&= \mathbb{E}\left[\mathbb{1}\{T_n<\infty\}\left(\sum_{2\le i\le N_n} p^{n-|x_i^n \curlywedge x_{T_n}^n|}\mathbb{P}\{x_i^n\to\infty\}a_{\mu_n^*}(x_i^n)\mathbb{1}\{T_n<i\}\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{2\le i\le N_n}\sum_{l=1}^{i-1} p^{n-|x_i^n \curlywedge x_l^n|}\mathbb{P}\{x_i^n\to\infty\}a_{\mu_n^*}(x_i^n)\mathbb{1}\{T_n=l\}\right)\right] \\
&= \mathbb{P}\{T_n<\infty\}\left(\sum_{2\le i\le N_n}\sum_{l=1}^{i-1} p^{n-|x_i^n \curlywedge x_l^n|}\mathbb{P}\{x_i^n\to\infty\}a_{\mu_n^*}(x_i^n)\mu_n^*(x_l^n)\right) \\
&= \mathbb{P}\{T_n<\infty\}\left(\sum_{2\le i\le N_n}\sum_{l=1}^{i-1} p^{-|x_i^n \curlywedge x_l^n|}\mu_n^*(x_i^n)\mu_n^*(x_l^n)\right) \\
&= \frac{1}{2}\mathbb{P}\{T_n<\infty\}\left(\sum_{\substack{|x|=|y|=n\\ x\ne y}} p^{-|x\curlywedge y|}\mu_n^*(x)\mu_n^*(y)\right),
\end{aligned}
\tag{5.34}
$$

where the last equality follows from the symmetry of the summation indices. We also change our notation from x_i^n back to x. Now combining (5.30), (5.33) and (5.34), we get

$$
\begin{aligned}
&\mathbb{E}[M_{\mu_n^*}^n(T_n)\mathbb{1}\{T_n<\infty\}] \\
&\ge \frac{1}{2}\mathbb{P}\{T_n<\infty\}\left(q_n^{-1}\sum_{|x|=n}\frac{\mu_n^*(x)^2}{\mathrm{Cap}_s(B(x))}+\sum_{\substack{|x|=|y|=n\\ x\ne y}} p^{-|x\curlywedge y|}\mu_n^*(x)\mu_n^*(y)\right).
\end{aligned}
\tag{5.35}
$$

Since $\{M_{\mu_n^*}^n(k)\}_{1\le k\le N}$ is a nonnegative bounded martingale and T_n is a stopping time, by Bounded Convergence Theorem and Optional Stopping Theorem, we have

$$
\begin{aligned}
&\mathbb{E}[M_{\mu_n^*}^n(T_n)\mathbb{1}\{T_n<\infty\}] \\
&\quad\le \mathbb{E}\left[\lim_{K\to\infty} M_{\mu_n^*}^n(T_n\wedge K)\right] = \lim_{K\to\infty}\mathbb{E}[M_{\mu_n^*}^n(T_n\wedge K)] \\
&\quad= \lim_{K\to\infty}\mathbb{E}[M_{\mu_n^*}^n(1)] = \mathbb{E}[I_{\mu_n^*}^n] = 1,
\end{aligned}
\tag{5.36}
$$

where the last equality follows from (5.20). Combining this inequality with (5.35) and (5.26), we get

$$
\mathbb{P}\{A_n\cap E\ne\emptyset\}\le 2\left[q_n^{-1}\sum_{|x|=n}\frac{\mu_n^*(x)^2}{\mathrm{Cap}_s(B(x))}+\sum_{\substack{x\ne y\\ |x|=|y|=n}}\sum p^{-|x\curlywedge y|}\mu_n^*(x)\mu_n^*(y)\right]^{-1}, \tag{5.37}
$$

where $0/0:=0$. Finally take $\sup$ over $\mu\in\mathcal{P}(\partial T)$ and apply Lemma 5.2 to obtain the desired result. □

Now we can use Lemma 5.4 and Lemma 5.5 to estimate the probability of the event $\{A\cap E\ne\emptyset\}$.

Theorem 5.6. *Let T be a tree. On the boundary ∂T, define a limsup random fractal A with index t and an independent fractal percolation set E with index s. Assume that $0 < t < \dim_{\mathrm{P}}(\partial T)$ and $0 < s < \dim_{\mathrm{H}}(\partial T)$.*

(i) If $\delta_s(\partial T) < t$, then $\mathbb{P}\{A \cap E \neq \emptyset\} = 0$; and

(ii) if T is spherically symmetric and $\delta_s(\partial T) > t$, then $\mathbb{P}\{A \cap E \neq \emptyset\} > 0$.

Proof. (i) From Lemma 5.5, we have $\mathbb{P}\{A_n \cap E \neq \emptyset\} \le 2\mathrm{Cap}(\partial T; s, q_n^{-1}, n)$. Since $\delta_s(\partial T) < t$, Lemma 5.3 (i) shows that $\sum_{n\ge1} \mathbb{P}\{A_n \cap E \neq \emptyset\} < \infty$. The definition $A = \cap_{n\ge1} \cup_{k\ge n} A_k$ and an application of Borel-Cantelli lemma show that $\mathbb{P}\{A \cap E \neq \emptyset\} = 0$.

(ii) We construct an independent limsup random fractal A' with parameters $\{q'_n\}_{n\ge1}$ such that $t' = -\lim_{n\to\infty} n^{-1} \ln q'_n$ exists and $t < t' < \delta_s(\partial T)$. We will show that either

$$\mathbb{P}\{A' \cap E \neq \emptyset\} > 0, \tag{5.38}$$

or

$$\mathbb{P}\{\dim_{\mathrm{P}}(E) \ge t'\} = \mathbb{P}\{E \neq \emptyset\}. \tag{5.39}$$

In the case $\mathbb{P}\{A' \cap E \neq \emptyset\} > 0$, by first conditioning on E and then applying Theorem 3.5, we see that $\mathbb{P}\{\dim_{\mathrm{P}}(E) \ge t'\} > 0$, therefore $\mathbb{P}\{\dim_{\mathrm{P}}(E) > t\} > 0$. Now we can condition on E and apply Theorem 3.5 again to obtain that $\mathbb{P}\{A \cap E \neq \emptyset\} > 0$. On the other hand, if $\mathbb{P}\{\dim_{\mathrm{P}}(E) \ge t'\} = \mathbb{P}\{E \neq \emptyset\}$, then a similar argument shows that $\mathbb{P}\{A \cap E \neq \emptyset\} > 0$.

In order to show $\mathbb{P}\{A' \cap E \neq \emptyset\} > 0$, we employ a Baire category argument. Define $C(n) := \cup_{k\ge n} A'_k$ for all $n \ge 1$. Then $A' = \cap_{n\ge1} C(n)$. Since $(\partial T, d)$ is an ultrametric space, the ball $B(x)$ is both open and closed for each $x \in \mathsf{V}$. Therefore both A'_n and $C(n)$ are open sets for all $n \ge 1$. If we can show that with positive probability $C(n) \cap E$ is dense in E for all $n \ge 1$, then the Baire category theorem guarantees that with positive probability $A' \cap E$ is dense in E. In particular, $\mathbb{P}\{A' \cap E \neq \emptyset\} > 0$. Therefore we strive to show

$$\mathbb{P}\{C(n) \cap E \text{ is dense in } E \text{ for all } n \ge 1\} > 0. \tag{5.40}$$

Lemma 5.4 shows that

$$\mathbb{P}\{A'_n \cap E \neq \emptyset\} \ge \mathrm{Cap}(\partial T; s, 2q_n'^{-1}, n) \qquad \forall\, n \ge 1. \tag{5.41}$$

Then the fact $\delta_s(\partial T) > t'$, Lemma 5.3 (ii) and Theorem 3.4 imply that

$$\varlimsup_{n\to\infty} \mathbb{P}\{A'_n \cap E \neq \emptyset\} \ge \mathrm{Cap}_s(\partial T) \ge \frac{1}{2}\mathbb{P}\{E \neq \emptyset\}. \tag{5.42}$$

Since $\{A'_n \cap E \neq \emptyset \text{ for infinitely many } n\} = \cap_{n\ge1} \cup_{k\ge n} \{A'_k \cap E \neq \emptyset\}$, we have

$$\begin{aligned}\mathbb{P}\{C(n) \cap E \neq \emptyset \text{ for all } n \ge 1\} &= \mathbb{P}\{A'_n \cap E \neq \emptyset \text{ for infinitely many } n\} \\ &\ge \varlimsup_{n\to\infty} \mathbb{P}\{A'_n \cap E \neq \emptyset\} \ge \frac{1}{2}\mathbb{P}\{E \neq \emptyset\}.\end{aligned} \tag{5.43}$$

We adopt the notation used in the proof of Lemma 5.5 again. We use supscript to denote the generation and subscript to denote the position from left to right. Thus x_i^m denotes the ith vertex at generation m. For each $m \ge 1$, define the event

$$\Omega_m := \{C(n) \cap E \cap B(x_i^m) \neq \emptyset \text{ for all } n \ge 1 \text{ whenever } B(x_i^m) \cap E \neq \emptyset\}. \tag{5.44}$$

It follows that $\Omega_{m+1} \subset \Omega_m$ for $m \ge 1$. Moreover

$$\bigcap_{m\ge1} \Omega_m = \{C(n) \cap E \text{ is dense in } E \text{ for all } n \ge 1\} \cup \{E = \emptyset\}. \tag{5.45}$$

Thus in order to show (5.40), it is equivalent to show

$$\lim_{m\to\infty} \mathbb{P}\{\Omega_m\} - \mathbb{P}\{E=\emptyset\} > 0 \tag{5.46}$$

Recall the subtree defined in Example 4.9. For each $x \in \mathsf{V}$, let $T^x := (\mathsf{V}^x, \mathsf{E}^x)$ be the subtree rooted at x such that

$$\mathsf{V}^x = \{x \text{ and all the descendents of } x\} \quad \text{and} \quad \mathsf{E}^x = (\mathsf{V}^x \times \mathsf{V}^x) \cap \mathsf{E}. \tag{5.47}$$

Then an application of Lemma 4.5 shows that $\delta_s(\partial T^x) = \delta_s(\partial T) > t'$ for all $x \in \mathsf{V}$. For each $m \geq 1$ and $1 \leq i \leq N_m$ (N_m is the number of vertices at generation m), we define the event

$$\Omega_{m,i} = \{E(m) \cap C(n) \cap \partial T^{x_i^m} \neq \emptyset \text{ for all } n \geq m\}, \tag{5.48}$$

where $E(m)$ denotes the fractal percolation set on $\partial T^{x_i^m}$ constructed from the same set of random variables $\{Y_x\}_{x\in\mathsf{V}}$. Then we can apply (5.43) to the subtree $T^{x_i^m}$ to obtain

$$\mathbb{P}\{\Omega_{m,i}\} \geq \frac{1}{2}\mathbb{P}\{E(m) \cap \partial T^{x_i^m} \neq \emptyset\}. \tag{5.49}$$

The spherical symmetry of T guarantees that $\mathbb{P}\{\Omega_{m,i}\} = \mathbb{P}\{\Omega_{m,j}\}$ and $\mathbb{P}\{E(m)\cap\partial T^{x_i^m} \neq \emptyset\} = \mathbb{P}\{E(m)\cap\partial T^{x_j^m} \neq \emptyset\}$ for all $1 \leq i, j \leq N_m$. For each $m \geq 1$, define a random variable

$$U_m := \sum_{1\leq i\leq N_m} \mathbb{1}\{o \to x_i^m\}. \tag{5.50}$$

From the independence of the random variables $\{Y_x\}_{x\in\mathsf{V}}$ and $\{Z_x\}_{x\in\mathsf{V}}$, we have following observations:

$$\{\Omega_{m,i}\}_{1\leq i\leq N_m} \text{ are independent events;} \tag{5.51}$$

and

$$\{\Omega_{m,i}\}_{1\leq i\leq N_m} \text{ are independent of } \sigma(U_m). \tag{5.52}$$

Moreover, if $I, J \subset \{1, \dots, N_m\}$ satisfies $I \cap J = \emptyset$, then

$$\{\Omega_{m,i}\}_{i\in I} \text{ are independent of } \{E(m) \cap \partial T^{x_j^m} \neq \emptyset\}_{j\in J}. \tag{5.53}$$

These independences imply that

$$\mathbb{P}\{\Omega_m | U_m = l\} = \left(\sum_{k=0}^{l} \binom{l}{k} r_m^k (1-p_m)^{l-k}\right) \geq \left(1 - \frac{1}{2}p_m\right)^l, \tag{5.54}$$

where $p_m = \mathbb{P}\{E(m) \cap \partial T^{x_i^m} \neq \emptyset\}$, $r_m = \mathbb{P}\{\Omega_{m,i}\}$, and $r_m \geq p_m/2$ by (5.49). Taking expectation for U_m gives

$$\mathbb{P}\{\Omega_m\} \geq \mathbb{E}\left[\left(1 - \frac{1}{2}p_m\right)^{U_m}\right]. \tag{5.55}$$

Thus if $\overline{\lim}_{m\to\infty} \mathbb{E}[(1-p_m/2)^{U_m}] - \mathbb{P}\{E=\emptyset\} > 0$, then (5.46) is proved, which in turn proves (5.40).

Next we show that if $\overline{\lim}_{m\to\infty} \mathbb{E}[(1-p_m/2)^{U_m}] - \mathbb{P}\{E=\emptyset\} \leq 0$, then

$$\mathbb{P}\{\overline{\dim}_{\mathrm{M}}(E) \geq t'\} = \mathbb{P}\{E \neq \emptyset\}. \tag{5.56}$$

According to the proof of Theorem 3.1 of [9], for all nonrandom Borel set $F \subset \partial T$, if $\overline{\dim}_{\mathrm{M}}(F) < t'$, then $\mathbb{P}\{F \cap C(n) \neq \emptyset \text{ for all } n \geq 1\} = 0$. Thus by conditioning on $E(m)$, (5.49) implies that $\mathbb{P}\{\overline{\dim}_{\mathrm{M}}(E(m) \cap \partial T(x_i^m)) \geq t'\} \geq 2^{-1}\mathbb{P}\{E(m) \cap \partial T(x_i^m) \neq \emptyset\}$, for all $m \geq 1$ and $1 \leq i \leq N_m$. Notice that for each fixed $m \geq 1$

$$\overline{\dim}_{\mathrm{M}}(E(m) \cap \partial T(x_i^m)) = \overline{\dim}_{\mathrm{M}}(E \cap B(x_i^m)) \qquad \text{a.s. on } \{o \to x_i^m\}, \tag{5.57}$$

and

$$\overline{\dim}_{\mathrm{M}}(E) = \max_{1 \leq i \leq N_m} \overline{\dim}_{\mathrm{M}}(E \cap B(x_i^m)) \qquad \text{a.s.} \tag{5.58}$$

Since T is spherically symmetric, $\{\overline{\dim}_{\mathrm{M}}(E(m) \cap \partial T(x_i^m))\}_{1 \leq i \leq N_m}$ are i.i.d. random variables. Therefore

$$\begin{aligned} \mathbb{P}\{\overline{\dim}_{\mathrm{M}}(E) < t'\} &= \mathbb{E}\left[\mathbb{P}\{\overline{\dim}_{\mathrm{M}}(E(m) \cap \partial T^{x_i^m}) < t'\}^{U_m}\right] \\ &\leq \mathbb{E}\left[(1 - \frac{1}{2}p_m)^{U_m}\right], \end{aligned} \tag{5.59}$$

where U_m is defined in (5.50) and $p_m = \mathbb{P}\{E(m) \cap \partial T(x_i^m) \neq \emptyset\}$. Then (5.59) implies that $\mathbb{P}\{\overline{\dim}_{\mathrm{M}}(E) \geq t'\} = \mathbb{P}\{E \neq \emptyset\}$, since $\overline{\lim}_{m\to\infty} \mathbb{E}[(1 - p_m/2)^{U_m}] - \mathbb{P}\{E = \emptyset\} \leq 0$.

Finally, since each ball $B(x)$ can be regarded as the boundary of a subtree of T, we can apply Lemma 4.5 and the above arguments to deduce that either

$$\mathbb{P}\{C(n) \cap E \cap B(x) \text{ is dense in } E \cap B(x) \text{ for all } n \geq 1\} > 0, \tag{5.60}$$

or

$$\mathbb{P}\{\overline{\dim}_{\mathrm{M}}(E \cap B(x)) \geq t'\} = \mathbb{P}\{E \cap B(x) \neq \emptyset\}. \tag{5.61}$$

If (5.60) holds for some ball $B(x_0)$, then the Baire category argument carried at the beginning of (ii) shows that $\mathbb{P}\{A' \cap E \cap B(x_0) \neq \emptyset\} > 0$. In particular,

$$\mathbb{P}\{A' \cap E \neq \emptyset\} > 0. \tag{5.62}$$

Otherwise (5.61) holds for all $x \in \mathsf{V}$. Since V is countable, we can first remove a null event and then apply Proposition 3.6 of Falconer [3] to deduce that

$$\mathbb{P}\{\dim_{\mathrm{P}}(E) \geq t'\} = \mathbb{P}\{E \neq \emptyset\}. \tag{5.63}$$

This completes the proof of the theorem. □

Now we can use Theorem 5.6 to prove Theorem 1.1. In fact we prove a little more.

Corollary 5.7. *Let T be a spherically symmetric tree. Then:*

(i) If A is a limsup random fractal defined on ∂T with index t and $0 < t < \dim_{\mathrm{P}}(\partial T)$, then $||\dim_{\mathrm{H}}(A)||_{L^\infty(\mathbb{P})} = \mathfrak{D}_t(\partial T)$.

(ii) If E is a fractal percolation set defined on ∂T with index s and $0 < s < \dim_{\mathrm{H}}(\partial T)$, then $||\dim_{\mathrm{P}}(E)||_{L^\infty(\mathbb{P})} = \delta_s(\partial T)$.

Proof. (i) For each $s > 0$, let $E(s)$ be a fractal percolation set with parameter $p = e^{-s}$ so that $E(s)$ is independent of A. Since T is spherically symmetric, Lemma 2.8 implies that $\dim_{\mathrm{P}}(\partial T) = \overline{\dim}_{\mathrm{M}}(\partial T)$.

First, consider $\mathfrak{D}_t(\partial T) = \dim_{\mathrm{H}}(\partial T)$. On one hand by the monotonicity of Haudorff dimension, we have $\dim_{\mathrm{H}}(A) \leq \mathfrak{D}_t(\partial T)$ almost surely. On the other hand, for every $0 < s < \mathfrak{D}_t(\partial T)$, we have $\delta_s(\partial T) > t$. Then Theorem 5.6 (ii) shows that $\mathbb{P}\{A \cap E(s) \neq \emptyset\} > 0$. Now condition on A and apply Corollary 3.2 to obtain $\mathbb{P}\{\dim_{\mathrm{H}}(A) > s\} > 0$. This shows that $\|\dim_{\mathrm{H}}(A)\|_{L^\infty(\mathbb{P})} = \mathfrak{D}_t(\partial T)$ when $\mathfrak{D}_t(\partial T) = \dim_{\mathrm{H}}(\partial T)$.

Second, consider $0 < \mathfrak{D}_t(\partial T) < \dim_{\mathrm{H}}(\partial T)$. On one hand for any s and s_0 with $\mathfrak{D}_t(\partial T) < s_0 < s < \dim_{\mathrm{H}}(\partial T)$, we have $\delta_s(\partial T) < \delta_{s_0}(\partial T) \le t$, thanks to Lemma 4.4 (iii). Now we can apply Theorem 5.6 (i) to derive $\mathbb{P}\{A \cap E(s) \ne \emptyset\} = 0$. Then Corollary 3.2 shows that with probability one $\dim_{\mathrm{H}}(A) \le s$. Let $s \downarrow \mathfrak{D}_t(\partial T)$ to see that $\dim_{\mathrm{H}}(A) \le \mathfrak{D}_t(\partial T)$ almost surely. On the other hand we can apply a similar argument as for the case $\mathfrak{D}_t(\partial T) = \dim_{\mathrm{H}}(\partial T)$ to obtain $\|\dim_{\mathrm{H}}(A)\|_{L^\infty(\mathbb{P})} = \mathfrak{D}_t(\partial T)$.

Finally, consider $\mathfrak{D}_t(\partial T) = 0$. Then a similar argument as for the case $0 < \mathfrak{D}_t(\partial T) < \dim_{\mathrm{H}}(\partial T)$ shows that $\dim_{\mathrm{H}}(A) \le \mathfrak{D}_t(\partial T)$ almost surely. Therefore $\|\dim_{\mathrm{H}}(A)\|_{L^\infty(\mathbb{P})} = 0 = \mathfrak{D}_t(\partial T)$. This completes the proof of the first part.

(ii) For each $t > 0$, let $A(t)$ be a limsup random fractal with parameters $\{q_n\}_{n\ge1}$ such that $t = -\lim_{n\to\infty} n^{-1} \ln q_n$ exists. Furthermore, we assume that $A(t)$ is independent of E. Since $\dim_{\mathrm{P}}(\partial T) = \overline{\dim}_{\mathrm{M}}(\partial T)$ and $s > 0$, Lemma 4.4 (i) and (iii) guarantee that $\delta_s(\partial T) < \dim_{\mathrm{P}}(\partial T)$.

If $\delta_s(\partial T) = 0$, then for every $t > 0$, Theorem 5.6 (i) implies that $\mathbb{P}\{E \cap A(t) \ne \emptyset\} = 0$. Then we can first condition on E and then apply Theorem 3.5 to obtain $\dim_{\mathrm{P}}(E) \le t$ almost surely. Let $t \downarrow 0$ to see that $\|\dim_{\mathrm{P}}(E)\|_{L^\infty(\mathbb{P})} = \delta_s(\partial T)$.

If $0 < \delta_s(\partial T) < \dim_{\mathrm{P}}(\partial T)$, then Theorem 5.6 (ii) implies that $\mathbb{P}\{E \cap A(t) \ne \emptyset\} > 0$ for every $0 < t < \delta_s(\partial T)$. An application of Theorem 3.5 conditioned on E gives $\mathbb{P}\{\dim_{\mathrm{P}}(E) \ge t\} > 0$. This means that $\|\dim_{\mathrm{P}}(E)\|_{L^\infty(\mathbb{P})} \ge \delta_s(\partial T)$. On the other hand, for all $\delta_s(\partial T) < t < \dim_{\mathrm{P}}(\partial T)$, we can apply a similar argument as for the case $\delta_s(\partial T) = 0$ to obtain $\dim_{\mathrm{P}}(E) \le \delta_s(\partial T)$ almost surely. This completes the proof of the second part. □

Remark 5.8. *If T is spherically symmetric, then $\overline{\dim}_{\mathrm{M}}(\partial T) = \dim_{\mathrm{P}}(\partial T)$, thanks to Lemma 2.8. Thus Lemma 4.8 implies $\dim_{\mathrm{H}}(\partial T) - t \le \mathfrak{D}_t(\partial T) \le \dim_{\mathrm{P}}(\partial T) - t$ and Lemma 4.4 implies $\delta_s(\partial T) \le \dim_{\mathrm{P}}(\partial T) - s$. Therefore Corollary 5.7 extends Theorem 3.8.*

Example 5.9. *On the boundary of the spherically symmetric tree T considered in Example 2.9 and 4.9, we have*

$$||\dim_{\mathrm{P}}(E)||_{L^\infty(\mathbb{P})} = \dim_{\mathrm{P}}(\partial T) - s = \ln 3 - s, \tag{5.64}$$

and

$$||\dim_{\mathrm{H}}(A)||_{L^\infty(\mathbb{P})} = (\dim_{\mathrm{P}}(\partial T) - t) \wedge \dim_{\mathrm{H}}(\partial T) = (\ln 3 - t) \wedge \ln 2. \tag{5.65}$$

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