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Spectral gap for Glauber type dynamics for a special class of potentials

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Abstract

We consider an equilibrium birth and death type process for a particle system in infinite volume, the latter is described by the space of all locally finite point configurations on $\mathbb{R}^d$. These Glauber type dynamics are Markov processes constructed for pre-given reversible measures. A representation for the "carré du champ" and "second carré du champ" for the associate infinitesimal generators L are calculated in infinite volume and for a large class of functions in a generalized sense. The corresponding coercivity identity is derived and explicit sufficient conditions for the appearance and bounds for the size of the spectral gap of L are given. These techniques are applied to Glauber dynamics associated to Gibbs measures and conditions are derived extending all previous known results and, in particular, potentials with negative parts can now be treated. The high temperature regime is extended essentially and potentials with non-trivial negative part can be included. Furthermore, a special class of potentials is defined for which the size of the spectral gap is as least as large as for the free system and, surprisingly, the spectral gap is independent of the activity. This type of potentials should not show any phase transition for a given temperature at any activity.

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1 Introduction

The process studied in this paper is an analogue for continuous systems of the well-known Glauber dynamics for lattice systems. The main focus of the paper is on the spectral properties of the associated infinitesimal generator L. Such kind of dynamics were introduced for the first time by C. Preston in [21, 10] for systems in finite volume, such that for each finite time interval at most a finite number of particles appear in

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the system. By construction, equilibrium states of classical statistical mechanics, Gibbs measures, are formally reversible measures for such processes. Gibbs measures are perturbations of Poisson point processes, though they are in general inequivalent to all Poisson point processes, highly correlated and do not have necessarily nice decay of correlation properties. Gibbs measures are constructed using a pair potential ϕ and an activity z. In [16], Yu. Kondratiev and E. Lytvynov constructed the Glauber dynamics in infinite volume using Dirichlet-form techniques. In any finite time interval, an infinite number of birth and death events happen, therefore this process cannot be considered as a birth and death process in the classical sense. In infinite volume, the processes exist only in an L^2-sense with respect to a chosen invariant measure μ. For more specific constructions in the non-reversible case, see [7, 18, 14, 15].

The infinitesimal generator L associated to these dynamics have a spectral gap for small positive potentials and small activity (high temperature regime). In [4], L. Bertini, N. Cancrini and F. Cesi derived a Poincaré inequality in finite volume and a bound on the spectral gap uniform in the volume. They pointed out that typically a log-Sobolev type inequality will not hold, cf. [19] for Poisson processes. In [16] the technique of coercivity identity was used to improve the result and to give a clear estimate for the spectral gap. In [5], A.-S. Boudou, P. Caputo, P. dai Pra and G. Posta derived a general framework for this technique for general jump-type processes and rederived the result for the Glauber dynamics in finite volume. In [17], Yu. Kondratiev, R. Minlos and E. Zhizhina show that one can split the L^2-space and the spectrum accordingly into three parts: one part associated to the eigenvalue zero describing the ground state; a second part, restricted to which the generator is unitary equivalent to a multiplication operator by a simple functions describing a quasi-one-particle system. The spectrum for this part is concentrated near -1. It has also been shown that the upper bound of the remaining part of the spectrum is almost -2.

In [2], D. Bakry and M. Emery calculated the “second carré du champ” generalizing the Bochner-Lichérowicz-Weitzenböck formula and in this way related the spectral gap of the Laplacian on a manifold with the underlying curvature. Therefore, it seems quite natural to apply these techniques also in the case of Glauber dynamics in the continuum.

In Section 3, we consider, slightly more general, all measure which have an integration by parts formula with respect to the considered difference operator, in other words measures which have a Papangelou kernel. We calculate the “second carré du champ” in infinite volume under very mild assumptions on the Papangelou kernel. In Appendix A.1, we introduce the “second carré du champ” in a generalized weak sense which is sufficient to derive the explicit expression of the “second carré du champ” for any function from any domain of any self-adjoint extension of L exploiting fundamentally the pointwise nature of the “second carré du champ”.

Integrating this expression of the “second carré du champ” with respect to a measure which is invariant for L gives a coercivity identity, which in such a generality cannot be derived directly. We recover in an equivalent form the coercivity identity given in [16] and exactly the one given in [5], however in infinite volume. Proceeding, as in this paper, via a generalization of the Bochner-Lichérowicz-Weitzenböck formula, has the additional advantage to provide a mechanism to select particular one among the different forms of the coercivity identity to use. Although a geometrical justification could not be given, the results presented in this paper may motivate further studies to introduce an adequate geometrical structure on configuration spaces. Sufficient criteria for the presence of a spectral gap are derived from the coercivity identity. Readers interested in the spectral gap result for Gibbs measure may skip the first two subsection and start with Corollary 3.2.

In Section 4, we study the case of operators L associated to Gibbs measures in more details. Sufficient conditions for the presence of a spectral gap are derived and bounds on the size of the gap in terms of the potential and the activity are given. We introduce a class of non-trivial potentials for which the spectral gap has at least the same size as in the free case and, even more surprisingly, the derived bound on the size of the spectral gap is independent of the activity. The definition of this class is based upon Fourier transform and hence the continuous space structure of the system is essential. Even more surprisingly, there are potentials with non-trivial negative part in this class. Furthermore, do we show that an increase in the temperature will not alter these estimates as well. This is the first result of a spectral gap in infinite volume which is not restricted to a kind of high temperature regime.

Finally, we derive a bound for potentials which are the sum of a potential from the aforementioned special class and a usual regular and stable potential in an extended high temperature regime. This result gives an improvment even if one just considers generic stable and regular potentials alone. Till now only non-negative potentials could be treated and it seems to be impossible to cover potentials even with the smallest non-negative part with the techniques used previously. Even just for general positive potentials the previous results are improved, see e.g. [16].

Precisely speaking we do not derive a spectral gap but a coercivity inequality on cylinder functions. If L is essentially self-adjoint on this domain, as proven for positive potentials in [16], then the coercivity identity is equivalent to spectral gap. In the Appendix A.1, we derive the expression for the “second carré du champ” in such a general sense that all self-adjoint extensions are covered. However, this is not sufficient to establish the coercitivity identity for general self-adjoint extensions of L. Essential self-adjointness for non-positive potentials is a non-trivial problem and will be subject of future investigations, see [8] and [6] for the analogous problem in the case of gradient diffusion.

Assuming essential self-adjointness, we found a class of potentials with a very interesting thermodynamical property. These potentials have a non-trivial attractive part, nevertheless there will be no phase transition of any kind for all values of the activity z.

2 States and dynamics

2.1 Configuration space

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$ is defined as the set of all Radon measures with values in $\mathbb{N} \cup \{0, \infty\}$, i.e. for any $\gamma \in \Gamma$ there exists a sequence $(x_i)_{i\in I}$ of vectors from $\mathbb{R}^d$ and an index set $I \subset \mathbb{N}$ such that $\gamma = \sum_{i\in I} \delta_{x_i}$, where δ_x denotes the Dirac measure concentrated at x. Conversely, any sequence without accumulation points can be associated to a configuration by the above formula. Modulo renumeration there is only one sequence representing γ. The space Γ is Polish in the relative topology as a subset of the space off all Radon measures $\mathcal{M}(\mathbb{R}^d)$ endowed with the vague topology, i.e. the topology generated by the mappings

$$\gamma \mapsto \langle f, \gamma\rangle := \int_{\mathbb{R}^d} f(x)\gamma(dx) \qquad \mathcal{C}_0(\mathbb{R}^d),$$

where $\mathcal{C}_0(\mathbb{R}^d)$ denotes the set of all continuous functions on $\mathbb{R}^d$ with compact support. The corresponding Borel σ-algebra on Γ is denoted by $\mathcal{B}(\Gamma)$. A probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ is called a point process (random field). A measurable function $r : \mathbb{R}^d \times \Gamma \longrightarrow [0, \infty]$ is the Papangelou intensity of a point process μ if

$$\int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) F(x, \gamma) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} dx \cdot r(x, \gamma) F(x, \gamma + \delta_x) \tag{2.1}$$

for any measurable function $F : \mathbb{R}^d \times \Gamma \to [0, +\infty[$. Let us fix a point process μ which has Papangelou intensity r and for which the first correlation function exists. The first n correlation functions exist exactly iff μ has all local moments up to degree n, that is, for all bounded measurable subsets $\Lambda \subset \mathbb{R}^d$ the following integral $\int_\Gamma \gamma(\Lambda)^n \mu(d\gamma)$ is finite. Gibbs measures are a particular class of point processes for which an explicit formula for the Papangelou intensity exist, cf. Subsection 4.1.

2.2 Glauber dynamics

In this subsection we introduce the Glauber dynamics, a birth and death type dynamics in the continuum via Dirichlet form techniques, for details cf. [16]. For this purpose we first introduce the set $\mathcal{FC}_\mathrm{b}(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ of all functions of the form

$$\Gamma \ni \gamma \mapsto F(\gamma) = g_F(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle),$$

where $N \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_N \in \mathcal{C}_0(\mathbb{R}^d)$ and $g_F \in \mathcal{C}_\mathrm{b}(\mathbb{R}^N)$. Here $\mathcal{C}_\mathrm{b}(\mathbb{R}^N)$ denotes the set of all continuous bounded functions on $\mathbb{R}^N$. The dynamics is constructed using two types of difference operators which are in some sense adjoint to each other: for $F : \Gamma \to \mathbb{R}$, $\gamma \in \Gamma$, and $x, y \in \mathbb{R}^d$

$$(D_x^- F)(\gamma) := F(\gamma - \delta_x) - F(\gamma), \quad (D_x^+ F)(\gamma) := F(\gamma) - F(\gamma + \delta_x). \tag{2.2}$$

As we want to consider the dynamics only in an L^2-framework, we use the following bilinear form, cf. [16]

$$\mathcal{E}(F, G) := \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) (D_x^- F)(\gamma)(D_x^- G)(\gamma), \qquad F, G \in \mathcal{FC}_\mathrm{b}(\mathcal{C}_0(\mathbb{R}^d), \Gamma). \tag{2.3}$$

The following properties of the $\mathcal{E}$, which are useful for our considerations, where proved in [16]. Using the associated integration by parts formula for a measure μ with a Papangelou intensity r and first local moments, in [16], it was proven that the bilinear form $(\mathcal{E}, \mathcal{FC}_\mathrm{b}(\mathcal{C}_0(\mathbb{R}^d), \Gamma))$ is closable on $L^2(\Gamma, \mu)$ and its closure is a Dirichlet form also denoted by $(\mathcal{E}, D(\mathcal{E}))$. The generator $(L, D(L))$ associated to$(\mathcal{E}, D(\mathcal{E}))$, i.e. $\mathcal{E}(F, G) = (-LF, G)_{L^2(\Gamma,\mu)}$ is for functions $F \in \mathcal{FC}_\mathrm{b}(\mathcal{C}_0(\mathbb{R}^d), \Gamma) \subset D(L)$ given by

$$(LF)(\gamma) = \int_{\mathbb{R}^d} \gamma(dx)\, (D_x^- F)(\gamma) - \int_{\mathbb{R}^d} r(x, \gamma)(D_x^+ F)(\gamma) dx \qquad \mu\text{-a.e.}. \tag{2.4}$$

Following the usual techniques for Dirichlet forms, in [16], for the case, that μ is a Gibbs measure, for definition cf. Subsection 4.1, the associated conservative Hunt process was constructed, that is,

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t \geq 0}, (\mathbf{\Theta}_t)_{t \geq 0}, (\mathbf{X}(t))_{t \geq 0}, (\mathbf{P}_\gamma)_{\gamma \in \Gamma})$$

on Γ (see e.g. [20, p. 92]) which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$, i.e., for all ($\mu$-versions of) $F \in L^2(\Gamma, \mu)$ and all $t > 0$ the function

$$\Gamma \ni \gamma \mapsto p_t F(\gamma) := \int_{\mathbf{\Omega}} F(\mathbf{X}(t))\, d\mathbf{P}_\gamma$$

is an $\mathcal{E}$-quasi-continuous version of $\exp(tL)F$. $\mathbf{\Omega}$ is the set of all *cadlag* functions $[0, \infty[\to \Gamma$. The processes $\mathbf{M}$ is up to μ-equivalence unique (cf. [20, Chap. IV, Sect. 6]). In particular, $\mathbf{M}$ is μ-symmetric (i.e., $\int G\, p_t F\, d\mu = \int F\, p_t G\, d\mu$ for all $F, G : \Gamma \to \mathbb{R}_+$, $\mathcal{B}(\Gamma)$-measurable) and thus has μ as an invariant measure.

3 Coercivity identity for Glauber dynamics

In the Subsection 3.1 and 3.2, we compute two quadratic forms associated to L, the generator of Glauber dynamics given by (2.4), the so-called “carré du champ”, the “second carré du champ” and furthermore an analogue of the Bochner-Lichnérowicz-Weitzenböck formula in this context, cf. e.g. [1]. In Subsection 3.2, we derive the associated coercivity identity. As this is essentially an algebraic calculation, details are omitted. Readers interested in the spectral gap result for Gibbs measure may jump directly to Corollary 3.2. The aim of the two subsections is to motivate why the particular form of the coercivity identity given in Corollary 3.2 is natural among different possible variants. In Appendix A.1, we introduce the “second carré du champ” in a generalized sense which covers, in particular, the results in this section. We give there just the main steps of the computation and we describe the way how to order the terms appropriately, which should allow the interested reader to easily reconstruct the missing details.

3.1 Carré du champ

In this subsection we essentially need only the following assumption on $r : \mathbb{R}^d \times \Gamma \to [0, \infty]$: There exists a subset $\Gamma_{\mathrm{temp}} \subset \Gamma$ such that

1. $r(x, \gamma) < \infty$ for all $(x, \gamma) \in \mathbb{R}^d \times \Gamma_{\mathrm{temp}}$
2. for all $\gamma \in \Gamma_{\mathrm{temp}}$, the function $x \mapsto r(x, \gamma)$ is locally integrable
3. for all $\gamma \in \Gamma_{\mathrm{temp}}$ and all $x \in \gamma$ and $y \in \mathbb{R}^d$ also $\gamma - \delta_x$ and $\gamma + \delta_y$ are in Γ_{temp}.

For $F, G \in \mathcal{FC}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ we define the “carré du champ” corresponding to L as

$$\square(F, G) := \frac{1}{2}(L(FG) - FLG - GLF). \tag{3.1}$$

Due to linearity, one can split $\square$ into a birth and a death “part”

$$\square^-(F, G) := \frac{1}{2}\int_{\mathbb{R}^d} \gamma(dx) D_x^- F(\gamma) D_x^- G(\gamma), \quad \square^+(F, G) := \frac{1}{2}\int_{\mathbb{R}^d} r(x, \gamma) D_x^+ F(\gamma) D_x^+ G(\gamma) dx,$$

where then $\square(F, G) = \square^-(F, G) + \square^+(F, G)$.

Iterating in some sense the definition of “carré du champ” one may introduce the so-called “second carré du champ” $\square_2$, cf. [1], as follows

$$2\square_2(F, F) := L\square(F, F) - 2\square(F, LF). \tag{3.2}$$

Using the explicit formula for L we obtain the following Bochner-Lichnérowicz-Weitzenböck formula

Theorem 3.1. *For all $F, G \in \mathcal{FC}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ it holds that*

$$\begin{aligned}
&\square_2(F, F)(\gamma) \qquad (3.3)\\
&= \frac{1}{4} \sum_{\substack{x \in \gamma_{\Lambda_2} \\ y \in \gamma_{\Lambda_1} : x \neq y}} \left(D_x^- D_y^- F\right)^2(\gamma) + \frac{1}{2}\square^+(F, F)(\gamma) - \frac{1}{2}\square^-(F, F)(\gamma) + \square(F, F)(\gamma)(\gamma)\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(x, \gamma) r(y, \gamma + \delta_x)(D_x^+ D_y^+ F)^2(\gamma) dx dy\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} r(y, \gamma) D_y^+ r(x, \cdot)(\gamma) \left[-(D_x^+ F)^2(\gamma) + 2D_x^+ F(\gamma) D_y^+ F(\gamma)\right] dx dy\\
&\quad + \frac{1}{2}\sum_{x \in \gamma}\int_{\mathbb{R}^d} r(y, \gamma)(D_x^- D_y^+ F)^2(\gamma) dy\\
&\quad + \frac{1}{4}\sum_{x \in \gamma}\int_{\mathbb{R}^d} D_x^- r(y, \cdot)(\gamma) \left[(D_y^+ F)^2(\gamma - \delta_x) + 2D_y^+ F(\gamma - \delta_x) D_x^- F(\gamma)\right] dy
\end{aligned}$$

This representation is not canonical. For Gaussian type measures there is a Bochner-Lichnérowicz-Weitzenböck kind formula and an associated Bakry-Emery criterium for $\square_2$ in terms of geometrical quantities like the underlying curvature and the Hessian. Unfortunately, in our case we lack this understanding of the associated geometrical structure. However, we observe that we have four terms of fourth order in the differential operator. Note that the first summand in the last line can actually be rewritten as the sum of a fourth order term and second and third order terms. One may expect that, in a natural representation, all fourth order terms would have the same integral w.r.t. the reversible measure μ. For the three first of them this is the case in our representation. The exception is the fourth order term in the last line which will be used for further cancelations in the next subsection, cf. 3.4. We have no geometrical explanation for this choice.

3.2 Coercivity identity

In order to study spectral properties of L we consider integrals of $\square$ and $\square_2$ with respect to an associated probability μ, that is a probability measure with a Papangelou intensities r, cf. (2.1). The representation given in Theorem 3.1 is a particularly useful for this purpose.

In this subsection we need to assume that μ has local moments up to second order. In particular, then for all compact $\Lambda \subset \mathbb{R}^d$ holds that $\gamma \mapsto \int_\Lambda \int_\Lambda r(y,\gamma) r(y, \gamma+\delta_x)$ is integrable w.r.t μ. In order that $\mathcal{F}\mathcal{C}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma) \subset D(L^2)$, we have additionally to assume that $\gamma \mapsto \int_\Lambda r(x,\gamma)dx$ is in $L^2(\Gamma,\mu)$. Then one can choose a pointwise version of r which fulfills all assumptions required in Subsection 3.1 for a set Γ_{temp} of full measure. (The generalized sense in which the formula is derived in Appendix A.1 allows to extend the identity to a much wider class of function than $\mathcal{F}\mathcal{C}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$, but the used sense is too weak to guarantee the identity on a domain of self-adjointness directly without further consideration)

Recall that L is symmetric with respect to μ and L applied to constant functions is zero. Using that we get the following relations for $\square$ and $\square_2$:

$$\begin{aligned} \mathcal{E}(F,F) = -\int_\Gamma F(\gamma) LF(\gamma)\mu(d\gamma) &= \int_\Gamma \square(F,F)(\gamma)\mu(d\gamma). \\ \int_\Gamma (LF)^2(\gamma)\mu(d\gamma) &= \int_\Gamma \square_2(F,F)(\gamma)\mu(d\gamma), \end{aligned}$$

for all $F \in \mathcal{F}\mathcal{C}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$.

The following identities are derived using repeatedly the identity $D_x^+ F(\gamma - \delta_x) = D_x^- F(\gamma)$ and the definition of the Papangelou intensities, cf. (2.1). For all $F \in \mathcal{F}\mathcal{C}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$ holds

$$\int_\Gamma \square(F,F)(\gamma)\mu(d\gamma) = 2\int_\Gamma \square^{\pm}(F,F)(\gamma)\mu(d\gamma) = 2\int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma)(D_x^+ F)^2(\gamma)dx\mu(d\gamma),$$

where the last equality corresponds to the case $\square^+$. The case $\square^-$ is to the representation (2.3) of the Dirichlet form $\mathcal{E}$. The main estimate in the derivation of the sufficient condition for spectral gap is to bound below the first three fourth order terms in Theorem 3.1 by zero. One can find a cancelation between the μ-integral of the fourth line in the expression in Theorem 3.1 and the integral of the sixth line. The integral in the last line in Theorem 3.1, using (2.1) and $D_x^- F(\gamma+\delta_x) = D_x^+ F(\gamma)$, can be rewritten as

$$\begin{aligned} &\int_\Gamma \int_{\mathbb{R}^d} \sum_{x\in\gamma} D_x^- r(y,\cdot)(\gamma) \left[(D_y^+ F)^2(\gamma-\delta_x) + 2D_y^+ F(\gamma-\delta_x) D_x^- F(\gamma)\right] dy\mu(d\gamma) \qquad (3.4) \\ &= \int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma) \int_{\mathbb{R}^d} D_x^- r(y,\cdot)(\gamma+\delta_x) \left[(D_y^+ F)^2(\gamma) + 2D_y^+ F(\gamma) D_x^- F(\gamma+\delta_x)\right] dydx\mu(d\gamma) \\ &= \int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma) \int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma) \left[(D_y^+ F)^2(\gamma) + 2D_y^+ F(\gamma) D_x^+ F(\gamma)\right] dydx\mu(d\gamma). \end{aligned}$$

Note that the first summand in the last term has the opposite sign as the first summand in the third line of the representation given in Theorem 3.1.

Finally, let us give an elegant expression for the coercivity identity. This representation will not be used in the following, but it will allow a comparison with other versions of the identity. Note that the first three of the fourth order terms were rearanged in Theorem 3.1 in such a way that their expectations coincide, cf. Subsection A.2 for more details. Here we choose to represent these fourth order terms by the double sum to rewrite the coercivity identity as follows:

Corollary 3.2. *For all* $F \in \mathcal{FC}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$ *holds that*

$$\begin{aligned} \int_\Gamma (LF)^2(\gamma)\mu(d\gamma) &= \int_\Gamma \Box_2(F,F)(\gamma)\mu(d\gamma) \\ &= \int_\Gamma \Box(F,F)(\gamma)\mu(d\gamma) + \int_\Gamma \sum_{x\in\gamma} \sum_{y\in\gamma-\delta_x} \left(D_x^- D_y^- F\right)^2(\gamma)\mu(d\gamma) \\ &+ \int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma) \int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma) D_y^+ F(\gamma) D_x^+ F(\gamma) dydx\mu(d\gamma). \end{aligned}$$

3.3 Sufficient condition for spectral gap

Instead of proving spectral gap directly using the Poincaré inequality, we consider the following approach, see [12] and [3, Chapter. 6, Section 4].

Let L be a nonnegative self-adjoint operator which maps the constant functions to zero. Let $D(L)$ be a core of L and $c>0$. Then L has a spectral gap of at least c if and only if the following so-called coercivity inequality holds

$$\int_\Gamma (LF)^2(\gamma)\mu(d\gamma) \geq c\mathcal{E}(F,F), \quad \forall F\in D(L). \qquad (3.5)$$

The latter inequality can be expressed in terms of the "carré du champ" $\Box$ and $\Box_2$

$$\int_\Gamma \Box_2(F,F)(\gamma)\mu(d\gamma) \geq c \int_\Gamma \Box(F,F)(\gamma)\mu(d\gamma). \qquad (3.6)$$

For diffusions D. Bakry and M. Emery could derive directly an inequality for $\Box$ and $\Box_2$, cf. [2], which we are not able to do.

By inserting in (3.6) the representations of the previous sections and using that the first three terms are non-negative, in particular using Corollary 3.2 with

$$\sum_{x\in\gamma} \sum_{y\in\gamma-\delta_x} \left(D_x^- D_y^- F\right)^2(\gamma) \geq 0,$$

one obtains the following sufficient condition for the coercivity inequality with constant c

$$\begin{aligned} &(1-c)\int_\Gamma \int_{\mathbb{R}^d} r(x,\gamma)(D_x^+ F)^2(\gamma) dx\mu(d\gamma) \qquad (3.7) \\ &+ \int_\Gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} r(x,\gamma) D_x^+ r(y,\cdot)(\gamma) D_y^+ F(\gamma) D_x^+ F(\gamma) dydx\mu(d\gamma) \geq 0. \end{aligned}$$

Considering the integrand (3.7) for fixed γ and denoting by

$$K_\gamma(x,y) = r(x,\gamma)(r(y,\gamma) - r(y,\gamma+\delta_x)), \ \ \psi_\gamma(x) = D_x^+ F(\gamma).$$

we can give a sufficient condition for the inequality (3.7) to hold for all $F \in \mathcal{F}\mathcal{C}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$, namely for all $\psi \in \mathcal{C}_0(\mathbb{R}^d)$ holds

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(K_\gamma(x,y) + (1-c)\sqrt{r(x,\gamma)}\sqrt{r(y,\gamma)}\delta(x-y))\psi(y)\psi(x)dxdy \geq 0. \tag{3.8}$$

This can be formulate more elegantly using the following definition

Definition 3.3. *A Radon measure K on $\mathbb{R}^d \times \mathbb{R}^d$ is called a positive definite kernel if for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ holds*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x)\psi(y)K(dx,dy) \geq 0. \tag{3.9}$$

Theorem 3.4. *If there is a $c > 0$ such that for μ-a.a. γ the kernel*

$$r(x,\gamma)(r(y,\gamma) - r(y,\gamma+\delta_x)) + (1-c)\sqrt{r(x,\gamma)}\sqrt{r(y,\gamma)}\delta(x-y) \tag{3.10}$$

is positive definite then the coercivity inequality (3.5) *for L with constant c holds for all $F \in \mathcal{F}\mathcal{C}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$.*

4 Coercivity identity for Gibbs measures

In this section we demonstrate that the sufficient condition for the coercivity inequality developed in Theorem 3.4 gives surprising results for the Glauber dynamics associated to Gibbs measures.

4.1 Gibbs measures

Gibbs measures are just the measures with Papangelou intensities of the form $r(x,\gamma) = z\exp[-E(x,\gamma)]$, where $z > 0$ and

$$E(x,\gamma) := \begin{cases} \sum\limits_{y\in\gamma}\phi(x-y), & \text{if } \sum\limits_{y\in\gamma}|\phi(x-y)| < \infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

for a measurable symmetric function $\phi : \mathbb{R}^d \to (-\infty,\infty]$. One calls such a measure a Gibbs measure to the activity z and pair potential ϕ. Sometimes it is useful to introduce an extra parameter, the inverse temperature β, and consider Gibbs measures for $\beta\phi$.

To guarantee existence of a measure with such Papangelou intensities, we need to require further conditions on the pair potential ϕ. For every $r \in \mathbb{Z}^d$, define a cube $\Delta_r = \left\{x \in \mathbb{R}^d : r_i - \frac{1}{2} \leq x_i < r_i + \frac{1}{2}\right\}$. These cubes form a partition of $\mathbb{R}^d$. Denote by $N_r(\gamma) = \gamma(\Delta_r)$. One says that ϕ is superstable (SS) if there exist $A > 0$, $B \geq 0$ such that, for all $\gamma \in \Gamma$ such that $\gamma(\mathbb{R}^d) < \infty$ holds

$$\sum_{\{x,y\}\subset\gamma}\phi(x-y) \geq \sum_{r\in\mathbb{Z}^d} AN_r^2(\gamma) - BN_r(\gamma).$$

ϕ is called stable (S) if the above condition holds just for $A = 0$. One says that ϕ is regular (R) if ϕ is bounded below and there exists an $R > 0$ and a positive decreasing function φ on $[0,+\infty)$ such that $|\phi(x)| \leq \varphi(|x|)$ for all $x \in \mathbb{R}^d$ with $|x| \geq R$ and

$$\int_R^\infty t^{d-1}\varphi(t)dt < \infty. \tag{4.1}$$

For the notion of tempered Gibbs measure and the following theorem, see [23].

Theorem 4.1. *Let ϕ be (SS) and (R), then the set $\mathcal{G}_{temp}(z,E)$ of all tempered Gibbs measures is non-empty and for each measure from $\mathcal{G}_{temp}(z,E)$ all correlation functions exist and satisfy the so-called Ruelle bound, that is, there exists a constant $C_R > 0$ such that for all non-negative measurable functions φ holds that*

$$\int_\Gamma e^{\int_{\mathbb{R}^d} \ln(1+\varphi(x))\gamma(dx)}\mu(d\gamma) \leq e^{C_R \int_{\mathbb{R}^d} \varphi(x)dx}.$$

For a Gibbs measure that fulfills the Ruelle bound all (local) moments are finite and one can see quite easily that also $\gamma \mapsto \int_{\mathbb{R}^d} r(x,\gamma)dx$ is in $L^2(\Gamma,\mu)$, cf. e.g. [16]. Hence all assumptions of Subsection 3.1 and 3.2 are fulfilled. Hence, in the sequel, we will restrict ourself to Gibbs measures which fulfill a Ruelle bound.

4.2 Coercivity inequality

For Gibbs measures condition (3.10) takes the following form

Theorem 4.2. *Let μ be a Gibbs measure for a pair potential ϕ and activity z which fulfills a Ruelle bound. If for a.a. γ the kernel*

$$e^{-E(x,\gamma)}e^{-E(y,\gamma)}z(1-e^{-\phi(x-y)}) + (1-c)e^{-\frac{1}{2}E(x,\gamma)}e^{-\frac{1}{2}E(y,\gamma)}\delta(x-y) \tag{4.2}$$

is positive definite then the coercivity inequality (3.5) *for L with constant c holds for all $F \in \mathcal{F}\mathcal{C}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$.*

The following easy reformulation will become very fruitful later on. Using in (3.8) the function $e^{-\frac{1}{2}E(x,\gamma)}\psi(x)$ instead of ψ gives

Corollary 4.3. *Let μ be a Gibbs measure for a pair potential ϕ and activity z which fulfills a Ruelle bound. If for a.a. γ the kernel*

$$e^{-\frac{1}{2}E(x,\gamma)}e^{-\frac{1}{2}E(y,\gamma)}z(1-e^{-\phi(x-y)}) + (1-c)\delta(x-y) \tag{4.3}$$

is positive definite then the coercivity inequality (3.5) *for L with constant c holds for all $F \in \mathcal{F}\mathcal{C}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$.*

4.3 Potentials increasing the spectral gap

For the Poisson point process, i.e. the Gibbs measure for the potential $\phi = 0$, one has the spectral gap $c = 1$, which follows also directly from condition (4.2). In order to prove condition (4.3) for $c = 1$ it is obviously sufficient to prove non-negativity (for a.a. γ) of the expression for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{1}{2}E(x,\gamma)}e^{-\frac{1}{2}E(y,\gamma)}(1-e^{-\phi(x-y)})\psi(y)\psi(x)dxdy. \tag{4.4}$$

Considering this a bilinear form in $e^{-\frac{1}{2}E(x,\gamma)}\psi(x)$ and recalling that due to Ruelle bound and regularity the latter function is integrable, one is lead to the following sufficient condition

$$\int_{\mathbb{R}^d}(1-e^{-\phi(x)})\psi * \psi(x)dx \geq 0, \tag{4.5}$$

where $\psi * \psi$ denotes the convolution of ψ with ψ. Recalling the following definition

Definition 4.4. *A locally bounded measurable function $u : \mathbb{R}^d \mapsto \mathbb{C}$ is called positive definite if for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ holds*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)\psi * \psi(x)dx \geq 0$$

and $u(0) \leq 1$.

As $1-e^{-\phi}$ is bounded, condition (4.5) means that $f : x \mapsto 1-e^{-\phi}$ is a positive definite function.

Remark 4.5. *Note that the condition* (4.5) *does not depend on* z*.*

To show that this condition is not void, we now investigate if there exists any potential ϕ such that f is positive definite and ϕ fulfills the conditions guaranteeing the existence of a Gibbs measure, namely (SS) and (R).

Theorem 4.6. *Let f be a continuous positive definite function which is (R). Define*

$$\phi := -\ln(1-f). \tag{4.6}$$

Then ϕ fulfills (4.5) *and is (SS) and (R). For every Gibbs measure μ for the potential ϕ and* ***for any*** *activity z which fulfills a Ruelle bound the associated generator L of the Glauber dynamics fulfills a coercivity inequality for $c = 1$ and all $F \in \mathcal{FC}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$.*

Proof. Due to positive definiteness $|f(x)| \le f(0) \le 1$. Defining for $x \in [-1, 1]$ the function $h(x) = -\ln(1-x)$ one can write $\phi = h \circ f$. First, we show that ϕ is regular. Asf is regular there exists an $\tilde{R} > 0$ and a positive decreasing function φ on $[0, +\infty)$ which fulfills (4.1) and such that $|f(x)| \le \varphi(|x|)$ for all $x \in \mathbb{R}^d$ with $|x| \ge \tilde{R}$. Note that for $x \in [-1, 1/2]$ it holds that $|h(x)| \le 2x$. Choose an $R \ge \tilde{R}$ such that $\varphi(R) \le 1/2$. Then for all $x \in \mathbb{R}^d$ with $|x| \ge R$ it holds $|f(x)| \le 1/2$ and hence

$$|\phi(x)| \le 2f(x) \le 2\varphi(|x|),$$

which implies that ϕ is regular.

Second, we show that ϕ is superstable. One easily sees that $h(x) \ge x + 1\!\!1_{[f(0)/2,1]}(x)(-\ln(1-x)-x)$. Shorthanding $g(x) = -\ln(1-x)-x$ one obtains $\phi(x) \ge f(x) + 1\!\!1_{[f(0)/2,1]}(f(x))g(f(x))$. Hence, $\phi \ge \phi' + \phi''$ where $\phi' = f$ is a positive definite continuous function and ϕ'' is a continuous non-negative function positive in 0 with $\phi'' \le 1\!\!1_{[f(0)/2,1]}(f(x))g(f(x))$. Hence ϕ fulfills the assertions of Proposition 1.2 in [23] and thus the potential ϕ is a superstable. □

We now try to understand the structure of potentials fulfilling condition (4.5). For that let us recall the following definition

Definition 4.7. *A generalized function (distribution) $u \in \mathcal{D}(\mathbb{R}^d)$ is called positive definite if for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$*

$$\langle u, \tilde{\varphi} \star \varphi \rangle \ge 0 \tag{4.7}$$

holds, where $\tilde{\varphi}(x) := \overline{\varphi(-x)}$.

Proposition 4.8. *Let ϕ be a potential fulfilling condition* (4.5) *which is (S), (R), and lower semi-continuous at zero. Then it is of the from* (4.6) *and hence also (SS). Furthermore, ϕ is integrable, itself positive definite in the sense of generalized functions, and*

$$\limsup_{x \downarrow 0}(\phi(x) + 2\ln(x)) < \infty \tag{4.8}$$

Proof. Let us define $f := 1 - e^{-\phi}$ and show that the function f fulfills the conditions of Theorem 4.6. As ϕ is stable it is non-negative in 0 and hence $|f(0)| \le 1$. Furthermore, f is lower semi-continuous at zero. Due to the positive definiteness of f one has that f is continuous and $|f(x)| \le f(0) \le 1$. One obtains the representation (4.6) by inverting the definition of f. As in the proof of Theorem 4.6 one can check that f also fulfills (R). Then Theorem 4.6 implies that ϕ is also (SS).

Using that $1-\cos(x) \geq \frac{x^2}{2}$ for small enough x, f is non-negative, the positive definiteness and $f(0) \leq 1$, we obtain that there exists a constant $c > 0$ such that $1 - f(x) \geq c|x|^2$ for small enough x. Hence $\phi(x) \leq -2\ln(|x|) - \ln(c)$. As ϕ is bounded below and regular, it is integrable.

Writing again $\phi = h \circ f$, we note that $h(x) = -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ with radius of convergence 1. Approximate ϕ by the functions $\phi_\delta(x) := h \circ ((1-\delta)f(x))$ for $0 < \delta < 1$. Since $|(1-\delta)f(x)| < 1$ and h has a Taylor series with non-negative coefficients, for all $0 < \delta < 1$ the function ϕ_δ is positive definite, cf. e.g. [11, Proposition 3.5.17]. As h is monotone increasing $|\phi_\delta| \leq |\phi|$ and the latter function is integrable. Hence ϕ_δ is also positive definite in the sense of generalized functions. Since ϕ_δ converge pointwise to ϕ for $\delta \to 0$ uniformly bounded by ϕ, by Lebesgue's dominated convergence ϕ is also positive definite in the sense of generalized functions. □

4.4 Parameter dependence

A typical question in statistical mechanics is to study the behavior of the system under change of a parameters. In the previous subsection, we identify potentials which fulfill (4.5) for all z and hence will show no phase transition even for large z. To investigate the temperature dependence we reintroduce the inverse temperature $\beta > 0$ into our consideration, that is we consider instead of ϕ the potential $\beta\phi$. We consider ϕ as fix and vary β and z. The corresponding Papangelou intensity is $r(x,\gamma) = ze^{-\beta E(x)}$ and hence condition 4.5 takes the form

$$\int_{\mathbb{R}^d} (1 - e^{-\beta\phi(x)})\psi * \psi(x)dx \geq 0. \tag{4.9}$$

If (4.9) is positive for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ then we say that ϕ fulfills condition (4.9) for β. Note, that the condition is independent of the activity z.

Proposition 4.9. *Let ϕ be a potential which fulfills condition (4.5) for a $\bar{\beta} > 0$ and is (S), (R), and lower semi-continuous at zero. Then ϕ fulfills condition (4.5) for all $0 < \beta \leq \bar{\beta}$.*

Proof. Denote by $f := 1 - e^{-\bar{\beta}\phi}$ the function considered in condition (4.5), which is positive definite by assumption. One the one hand, it is easy to see that $f_\beta(x) := 1 - e^{-\beta\phi(x)}$ are also continuous and (R). One the other hand, $f_\beta(x) = 1 - (1-f(x))^{\beta/\bar{\beta}}$ has a power series expansion $f_\beta(x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n!}\beta/\bar{\beta}(\beta/\bar{\beta}-1)\dots(\beta/\bar{\beta}-n+1)(f(x))^n$ with radius of convergence 1. All the coefficients of the series are nonnegative, if $\beta/\bar{\beta} \leq 1$. Proceeding as in Proposition 4.8, one proves that f_β is the pointwise limit of positive definite functions. As f_β is itself bounded and a limit of positive definite functions, it is positive definite in the sense of functions. □

4.5 Examples

For concreteness we give a small collection of potentials which fulfills the condition of Theorem 4.6 to get a better feeling how such potentials may look like. Especially interesting is that among them are potentials, which have a non-trivial negative part.

$\phi(x)$	$f(x)$	Parameters
$-\ln(1-e^{-tx^2}\cos(ax))$,	$e^{-tx^2}\cos(ax)$,	$t>0, a\in\mathbb{R}$
$-\ln(1-e^{-t\lvert x\rvert}\cos(ax))$,	$e^{-t\lvert x\rvert}\cos(ax)$,	$t>0, a\in\mathbb{R}$
$-\ln\left(1-\frac{\cos(ax)}{1+\sigma^2x^2}\right)$,	$\frac{1}{1+\sigma^2x^2}\cos(ax)$,	$\sigma>0,\, a\in\mathbb{R}$
$-\ln(1-(1-\frac{\lvert x\rvert}{a})\mathbb{1}_{[-a,a]}(x)\cos(bx))$,	$(1-\frac{\lvert x\rvert}{a})\mathbb{1}_{[-a,a]}(x)\cos(bx)$,	$a>0, b\in\mathbb{R}$,

In all examples above one can exchange $\cos(ax)$ by $\frac{\sin(ax)}{ax}$.

In the d-dimensional case we can give following examples:

$\phi(x)$	$f(x)$	Parameters
$-\ln(1-e^{-t\lvert x\rvert^2}\cos(a\cdot x))$	$e^{-t\lvert x\rvert^2}\cos(a\cdot x)$	$x\in\mathbb{R}^d, t>0, a\in\mathbb{R}^d$
$-\ln\left(1-e^{-t\lvert x\rvert^2}\prod_{j=1}^{d}\frac{\sin(a_jx_j)}{a_jx_j}\right)$	$e^{-t\lvert x\rvert^2}\prod_{j=1}^{d}\frac{\sin(a_jx_j)}{a_jx_j}$	$x\in\mathbb{R}^d, t>0$
$-\ln\left(1-(\frac{r}{\lvert x\rvert})^{n/2}J_{n/2}(r\lvert x\rvert)\right)$	$(\frac{r}{\lvert x\rvert})^{n/2}J_{n/2}(r\lvert x\rvert)$	$r\geq 0,\, n>2d-1$
$-\ln\left(1-\frac{2^{n/2}\Gamma(\frac{n+1}{2})}{\sqrt{\pi}}\cdot\frac{t}{(\lvert x\rvert^2+t^2)^{\frac{n+1}{2}}}\right)$	$\frac{2^{n/2}\Gamma(\frac{n+1}{2})}{\sqrt{\pi}}\cdot\frac{t}{(\lvert x\rvert^2+t^2)^{\frac{n+1}{2}}}$	$t>0,\, n>d-1$

where $J_{n/2}$ is the Bessel function of the first kind of order $n/2$. One can multiply f in any of the examples with factors of the form $\cos(a\cdot x)$ and $\prod_{j=1}^{d}\frac{\sin(a_jx_j)}{a_jx_j}$.

All these examples are constructed by choosing a positive definite function f and express $\phi(x)=-\ln(1-f(x))$.

4.6 High temperature and low densities

In the previous subsections, we considered potentials which give rise to a spectral gap at least as large as in the free case, that is the coercivity identity holds for $c=1$. Such potentials admit at most a logarithmic singularity at zero. In this subsection, we will derive a coercivity inequality for the sum of a potential from this special class and a general non-negative or hard-core potential. However, in this case our estimate works only for constants c in the coercivity inequality smaller than one and not longer independent of the activity z of the Gibbs measure. The bound of the constant c in (4.10) is similar to the formulas which define the usual high temperature low intensity regime and similar to the results obtained in [16] for measures with positive potentials.

Theorem 4.10. *Let ϕ_1 be (R) and assume that there exists a $D\geq 0$ such that for μ-a.a. γ holds that $E(x,\gamma)\geq -D$ and let ϕ_2 be a potential fulfilling the conditions of Theorem 4.6. Then for every Gibbs measure μ for the potential $\phi_1+\phi_2$ and the activity z, the associated generator L of the Glauber dynamics fulfills a coercivity inequality for the constant*

$$c=1-ze^D\int_{\mathbb{R}^d}dxe^{-\phi_2(x)}|1-e^{-\phi_1(x)}|. \tag{4.10}$$

Let us state two classes of potentials ϕ_1 which fulfill the condition in the previous theorem. If ϕ_1 is non-negative then the condition holds for $D=0$. If ϕ_1 has a hard core then ϕ_1 also fulfills the condition. There exist further potentials which fulfill this condition.

Proof. The main idea is to apply condition 4.3 directly. In order to prove positive definiteness of the kernel (4.3) one has to prove non-negativity of the following expression for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \psi(x)\psi(y) \left[e^{-\frac{1}{2}E(x,\gamma)} e^{-\frac{1}{2}E(y,\gamma)} z(1-e^{-\phi(x-y)}) + (1-c)\delta(x-y)\right] \tag{4.11}$$

Rewriting

$$1 - e^{-\phi} = 1 - e^{-\phi_2} + e^{-\phi_2}(1 - e^{-\phi_1}).$$

the first part of (4.3) takes the form

$$e^{-\frac{1}{2}E(x,\gamma)} e^{-\frac{1}{2}E(y,\gamma)} z(1-e^{-\phi_2(x-y)}) + e^{-\frac{1}{2}E(x,\gamma)} e^{-\frac{1}{2}E(y,\gamma)} z e^{-\phi_2(x-y)}(1-e^{-\phi_1(x-y)})$$

As in the beginning of Subsection 4.3 the first summand is a positive definite due to the assumptions on ϕ_2. The second summand can be bounded as follows

$$\begin{aligned} &\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \psi(x) e^{-\frac{1}{2}E(x,\gamma)} \psi(y) e^{-\frac{1}{2}E(y,\gamma)} z e^{-\phi_2(x-y)}(1-e^{-\phi_1(x-y)}) \\ \geq \quad & -z \int_{\mathbb{R}^d} dx e^{-\phi_2(x)} |1-e^{-\phi_1(x)}| \int_{\mathbb{R}^d} dy |\psi(x+y)| e^{-\frac{1}{2}E(x+y,\gamma)} |\psi(y)| e^{-\frac{1}{2}E(y,\gamma)} \end{aligned}$$

Applying Cauchy-Schwarz inequality to the last factor one obtains

$$\begin{aligned} &\int_{\mathbb{R}^d} dy |\psi(x+y)| e^{-\frac{1}{2}E(x+y,\gamma)} |\psi(y)| e^{-\frac{1}{2}E(y,\gamma)} \\ \leq \quad & \int_{\mathbb{R}^d} dy\; e^D \psi^2(y). \end{aligned}$$

Summarizing (4.11) can be bounded below by

$$\int_{\mathbb{R}^d} dy \left[-z e^D \int_{\mathbb{R}^d} dx e^{-\phi_2(x)} |1 - e^{-\phi_1(x)}| + (1-c)\right] \psi^2(y) \tag{4.12}$$

which is non-negative if and only if the bracket is non-negative. □

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A Appendix

We will give more details for the calculations mentioned in Subsection 3.1 and 3.2.

The dual operator to $(L, \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma))$ is difficult to describe. To our knowledge neither an explicit formula for the domain nor for the dual operator is known. Define for each bounded measurable subset Λ of $\mathbb{R}^d$ the following localized version of the generator L

$$(L_\Lambda F)(\gamma) := \int_\Lambda \gamma(dx)\,(D_x^- F)(\gamma) - \int_\Lambda r(x,\gamma)(D_x^+ F)(\gamma) dx \qquad \mu\text{-a.e.}$$

If $G \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ with cylinder-support in Λ, that is, $\gamma \mapsto G(\gamma)$ only depends on γ_Λ, then $LG = L_\Lambda G$. We can extend the action of L in the following generalized sense: Let

F be a bounded measurable function, then we consider as the action of L the collection of the functions $(L_\Lambda F)_\Lambda$. Then for all $G \in \mathcal{FC}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$, it holds $\int_\Gamma L_\Lambda G(\gamma)F(\gamma)\mu(d\gamma) = \int_\Gamma G(\gamma)L_\Lambda F(\gamma)\mu(d\gamma)$ for each Λ such that the cylinder support of G is inside of Λ. This generalized sense of the action of L is so weak that it generalizes the action of any self-adjoint extension of L. The derivation of the expression fo the "carré du champ" and "second carré du champ" holds already in this generalized sense. Hence the derived formulas should hold for any F from any self-adjoint extension. However, this is not sufficient to prove spectral gap.

A.1 Carré du champ

For $F, G \in \mathcal{FC}_{\mathrm{b}}(\mathcal{C}_0(\mathbb{R}^d), \Gamma)$ the "carré du champ" corresponding to L in the generalized sense is defined as the collection of

$$\Box_\Lambda(F,G) := \frac{1}{2}(L_\Lambda(FG) - FL_\Lambda G - GL_\Lambda F). \tag{1.1}$$

Let us split the generator L_Λ into its death and birth part

$$L_\Lambda^- F(\gamma) := \sum_{x\in\gamma_\Lambda} D_x^- F(\gamma), \qquad L_\Lambda^+ F(\gamma) := \int_\Lambda r(x,\gamma) D_x^+ F(\gamma) dx, \tag{1.2}$$

such that $L_\Lambda = L_\Lambda^- - L_\Lambda^+$. Due to linearity one obtains that $\Box_\Lambda(F,G) = \Box_\Lambda^-(F,G) + \Box_\Lambda^+(F,G)$, where $\Box_\Lambda^-$ and $-\Box_\Lambda^+$ are the "carré du champ" corresponding to the death and birth parts

$$\Box_\Lambda^-(F,G) := \frac{1}{2}\int_\Lambda \gamma(dx) D_x^- F(\gamma) D_x^- G(\gamma), \quad \Box_\Lambda^+(F,G) := \frac{1}{2}\int_\Lambda r(x,\gamma) D_x^+ F(\gamma) D_x^+ G(\gamma) dx.$$

The generalized version of the "second carré du champ"of $\Box_{2,\Lambda_1,\Lambda_2}$ is given by

$$2\Box_{2,\Lambda_1,\Lambda_2}(F,F) := L_{\Lambda_2}\Box_{\Lambda_1}(F,F) - 2\Box_{\Lambda_1}(F, L_{\Lambda_2}F). \tag{1.3}$$

The splitting in birth and death part allows us to split $\Box_{2,\Lambda_1,\Lambda_2}$ correspondingly in the following way:

$$\begin{aligned} 2\Box_{2,\Lambda_1,\Lambda_2}(F,F) \;=\; & \left(L_{\Lambda_2}^-\Box_{\Lambda_1}^-(F,F) - 2\Box_{\Lambda_1}^-(F, L_{\Lambda_2}^- F)\right) \\ & - \left(L_{\Lambda_2}^+\Box_{\Lambda_1}^+(F,F) - 2\Box_{\Lambda_1}^+(F, L_{\Lambda_2}^+ F)\right) \\ & + \left(L_{\Lambda_2}^-\Box_{\Lambda_1}^+(F,F) - 2\Box_{\Lambda_1}^+(F, L_{\Lambda_2}^- F)\right) \\ & - \left(L_{\Lambda_2}^+\Box_{\Lambda_1}^-(F,F) - 2\Box_{\Lambda_2}^-(F, L_{\Lambda_1}^+ F)\right). \end{aligned} \tag{1.4}$$

All brackets will be calculated separately using the following product rules type formulas

Lemma A.1. *If $H : \mathbb{R}^d \times \Gamma_{\mathrm{temp}} \to \mathbb{R}$ is locally bounded and for fixed $\gamma \in \Gamma_{\mathrm{temp}}$ the function $x \mapsto H_x(\gamma)$ has compact support, then*

$$D_x^+ \sum_{y\in\gamma} H_y(\gamma) = \sum_{y\in\gamma} D_x^+ H_y(\gamma) - H_x(\gamma+\delta_x) \tag{1.5}$$

$$D_x^- \sum_{y\in\gamma} H_y(\gamma) = \sum_{y\in\gamma-\delta_x} D_x^- H_y(\gamma) - H_x(\gamma) \tag{1.6}$$

$$D_x^+\left(\int_\Lambda r(y,\gamma)H_y(\gamma)dy\right) = \int_\Lambda r(y,\gamma)D_x^+ H_y(\gamma)dy + \int_\Lambda D_x^+ r(y,\gamma)H_y(\gamma+\delta_x)dy, \tag{1.7}$$

$$D_x^-\left(\int_\Lambda r(y,\gamma)H_y(\gamma)dy\right) = \int_\Lambda r(y,\gamma)D_x^- H_y(\gamma)dy + \int_\Lambda D_x^- r(y,\gamma)H_y(\gamma-\delta_x)dy. \tag{1.8}$$

Computing the first summand of (1.4) we obtain

$$L^-_{\Lambda_2}\square^-_{\Lambda_1}(F,F)(\gamma)-2\square^-_{\Lambda_1}(L^-_{\Lambda_2}F,F)(\gamma)=\frac{1}{2}\sum_{\substack{x\in\gamma_{\Lambda_2}\\ y\in\gamma_{\Lambda_1}:x\neq y}}\left(D^-_xD^-_yF\right)^2(\gamma)-\square^-_{\Lambda_2}(F,F)(\gamma)+2\square^-_{\Lambda_1}(F,F)(\gamma),$$

whereas for the second summand we may derive the following expression

$$\begin{aligned}
&L^+_{\Lambda_2}\square^+_{\Lambda_1}(F)(\gamma)-2\square^+_{\Lambda_1}(F,L^+_{\Lambda_2}F)(\gamma)=-\frac{1}{2}\int_{\Lambda_1}\int_{\Lambda_2}r(x,\gamma)r(y,\gamma+\delta_x)(D^+_xD^+_yF)^2(\gamma)dxdy\\
&+\int_{\Lambda_1}\int_{\Lambda_2}\left(r(y,\gamma)D^+_yr(x,\cdot)(\gamma)-r(x,\gamma)D^+_xr(y,\cdot)(\gamma)\right)(D^+_xD^+_yF)(\gamma)D^+_yF(\gamma)dxdy\\
&+\frac{1}{2}\int_{\Lambda_1}\int_{\Lambda_2}r(x,\gamma)D^+_xr(y,\cdot)(\gamma)(D^+_yF)^2(\gamma)dxdy\\
&-\int_{\Lambda_1}\int_{\Lambda_2}r(y,\gamma)D^+_yr(x,\cdot)(\gamma)D^+_xF(\gamma)D^+_yF(\gamma)dxdy.
\end{aligned}$$

Finally, calculating the mixed terms in (1.4), we obtain

$$\begin{aligned}
(L^-_{\Lambda_2}\square^+_{\Lambda_1}(F)-2\square^+_{\Lambda_1}(F,L^-_{\Lambda_2}F) &= \frac{1}{2}\sum_{x\in\gamma_{\Lambda_2}}\int_{\Lambda_1}r(y,\gamma)(D^-_xD^+_yF)^2(\gamma)dy \qquad (1.9)\\
&\quad+\frac{1}{2}\sum_{x\in\gamma_{\Lambda_2}}\int_{\Lambda_1}D^-_xr(y,\cdot)(\gamma)\left[D^-_x(D^+_yF)^2(\gamma)+(D^+_yF)^2(\gamma)\right]dy\\
&\quad+\int_{\Lambda_1}r(y,\gamma)(D^+_yF)^2(\gamma)dy\\
-L^+_{\Lambda_2}\square^-_{\Lambda_1}(F)+2\square^-_{\Lambda_1}(F,L^+_{\Lambda_2}F))(\gamma) &= \frac{1}{2}\sum_{y\in\gamma_{\Lambda_1}}\int_{\Lambda_2}r(x,\gamma)(D^-_yD^+_xF)^2(\gamma)dx \qquad (1.10)\\
&\quad+\frac{1}{2}\int_{\Lambda_2}r(y,\gamma)(D^+_xF)^2(\gamma)dx\\
&\quad+\sum_{y\in\gamma_{\Lambda_1}}\int_{\Lambda_2}D^-_yr(x,\cdot)D^-_yF(\gamma)\left[D^-_yD^+_xF(\gamma)+D^+_xF(\gamma)\right]dx
\end{aligned}$$

Summarizing, adding all four parts we gain the following expression for $\square_2$

$$
\begin{aligned}
&\square_2(F,F)(\gamma) \qquad (1.11)\\
&= \frac{1}{4}\sum_{\substack{x\in\gamma_{\Lambda_2}\\ y\in\gamma_{\Lambda_1}:x\neq y}} \left(D_x^- D_y^- F\right)^2(\gamma) - \frac{1}{2}\square^-_{\Lambda_2}(F,F)(\gamma) + \square^-_{\Lambda_1}(F,F)(\gamma)\\
&+\frac{1}{4}\int_{\Lambda_1}\int_{\Lambda_2} r(x,\gamma)r(y,\gamma+\delta_x)(D_x^+D_y^+F)^2(\gamma)dxdy\\
&-\frac{1}{2}\int_{\Lambda_1}\int_{\Lambda_2} \left(r(y,\gamma)D_y^+ r(x,\cdot)(\gamma) - r(x,\gamma)D_x^+ r(y,\cdot)(\gamma)\right)(D_x^+D_y^+F)(\gamma)D_y^+F(\gamma)dxdy\\
&-\frac{1}{4}\int_{\Lambda_1}\int_{\Lambda_2} r(x,\gamma)D_x^+ r(y,\cdot)(\gamma)(D_y^+F)^2(\gamma)dxdy\\
&+\frac{1}{2}\int_{\Lambda_1}\int_{\Lambda_2} r(y,\gamma)D_y^+ r(x,\cdot)(\gamma)D_x^+F(\gamma)D_y^+F(\gamma)dxdy\\
&+\frac{1}{4}\sum_{x\in\gamma_{\Lambda_2}}\int_{\Lambda_1} r(y,\gamma)(D_x^-D_y^+F)^2(\gamma)dy + \square^+_{\Lambda_1}(F,F)(\gamma)\\
&+\frac{1}{4}\sum_{x\in\gamma_{\Lambda_2}}\int_{\Lambda_1} D_x^- r(y,\cdot)(\gamma)\left[D_x^-(D_y^+F)^2(\gamma) + (D_y^+F)^2(\gamma)\right]dy\\
&+\frac{1}{4}\sum_{y\in\gamma_{\Lambda_1}}\int_{\Lambda_2} r(x,\gamma)(D_y^-D_x^+F)^2(\gamma)dx + \frac{1}{2}\square^+_{\Lambda_2}(F,F)(\gamma)\\
&+\frac{1}{2}\sum_{y\in\gamma_{\Lambda_1}}\int_{\Lambda_1} D_y^- r(x,\cdot)D_y^-F(\gamma)\left[D_y^-D_x^+F(\gamma) + D_x^+F(\gamma)\right]dx
\end{aligned}
$$

To prove Theorem 3.1, it remains to recognize that the third line in (1.11) is zero because of the following general property of Papangelou intensities

Lemma A.2. *For $\mu\otimes dx$-a.a. (γ,x) holds that*

$$r(x,\gamma)D_x^+ r(y,\cdot)(\gamma)dxdy = r(y,\gamma)D_y^+ r(x,\cdot)(\gamma)dydx$$

Proof. As the above equality has to be interpreted a.s. it is sufficient to show that the following expression is invariant under the interchange of x and y for any cylinder function H. This is obvious after the following rewriting

$$
\begin{aligned}
&\int_\Gamma\int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} D_x^+ r(y,\cdot)(\gamma)H(\gamma+\delta_x+\delta_y,x,y)dydx\mu(d\gamma)\\
&= \int_\Gamma\int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} r(y,\gamma)H(\gamma+\delta_x+\delta_y,x,y)dydx\mu(d\gamma)\\
&\quad -\int_\Gamma\sum_{\substack{x,y\in\gamma\\ x\neq y}} H(\gamma,x,y)\mu(d\gamma)
\end{aligned}
$$

□

A.2 Expectation of fourth order terms

As mentioned before the representation given in Theorem 3.1 was chosen such that the expectations of the first three of the fourth order terms coincides. One easily computes using the Papangelou density, cf. (2.1) that for all $F \in \mathcal{FC}_b(\mathcal{C}_0(\mathbb{R}^d),\Gamma)$ holds

$$\begin{aligned}&\int_\Gamma \sum_{y\in\gamma}\int_{\mathbb{R}^d} r(x,\gamma)\left(D_x^+D_y^-F\right)^2(\gamma)dx\mu(d\gamma)\\ &= \int_\Gamma \sum_{x\in\gamma}\sum_{y\in\gamma-\delta_x}\left(D_x^+D_y^-F\right)^2(\gamma-\delta_x)\mu(d\gamma)\\ &= \int_\Gamma \sum_{x\in\gamma}\sum_{y\in\gamma-\delta_x}\left(D_x^-D_y^-F\right)^2(\gamma)\mu(d\gamma).\end{aligned}$$

and indeed in Subsection 3.1 the second fourth order term in the fourth line of (3.3) was arranged in such a form that holds

$$\int_\Gamma\int_{\mathbb{R}^d} r(x,\gamma)\int_{\mathbb{R}^d} r(y,\gamma+\delta_x)(D_x^+D_y^+F)^2(\gamma)dydx\mu(d\gamma) = \int_\Gamma\sum_{y\in\gamma}\sum_{x\in\gamma-\delta_y}(D_x^-D_y^-F)^2(\gamma)\mu(d\gamma) \tag{1.12}$$

Note that the second line in (A.1.9) actually contains an additional fourth order term. Though both fourth order terms in (A.1.9) could be estimate jointly by zero, this is not what has been done here because the second fourth order term is used to cancel some terms in Subsection 3.2. We have no explanation why this is advantageous to do.

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