On convergence of general wavelet decompositions of nonstationary stochastic processes

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Abstract

The paper investigates uniform convergence of wavelet expansions of Gaussian random processes. The convergence is obtained under simple general conditions on processes and wavelets which can be easily verified. Applications of the developed technique are shown for several classes of stochastic processes. In particular, the main theorem is adjusted to the fractional Brownian motion case. New results on the rate of convergence of the wavelet expansions in the space $C([0, T])$ are also presented.

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1 Introduction

In the book [11] wavelet expansions of non-random functions bounded on $\mathbb{R}$ were studied in different spaces. However, developed deterministic methods may not be appropriate to investigate wavelet expansions of stochastic processes. For example, in the majority of cases, which are interesting from theoretical and practical application points of view, stochastic processes have almost surely unbounded sample paths on $\mathbb{R}$. It indicates the necessity of elaborating special stochastic techniques.

Recently, a considerable attention was given to the properties of the wavelet orthonormal series representation of random processes. More information on convergence of wavelet expansions of random processes in various spaces, references and numerous applications can be found in [3, 7, 14, 15, 16, 17, 18, 21, 20, 24]. Most known stochastic results concern the mean-square or almost sure convergence, but for various practical applications one needs to require uniform convergence. To give an

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adequate description of the performance of wavelet approximations in both cases, for points where the processes are relatively smooth and points where spikes occur, we can use the uniform distance instead of global integral L^p metrics. A more in depth discussion, further references and various applications in econometrics, simulations of stochastic processes and functional data analysis can be found in [6, 9, 10, 20, 23]. In his 2010 Szekeres Medal inauguration speech, an eminent leader in the field, Prof. P. Hall stated the development of uniform stochastic approximation methods as one of frontiers in modern functional data analysis.

Figures 1 and 2 illustrate some features of wavelet expansions of stochastic processes. Figure 1 presents a simulated realization of the Wiener process and its wavelet reconstructions by two sums with different numbers of terms. The figure has been generated by the R package *wmtsa* [22]. Besides providing a realization of the Wiener process and its wavelet reconstructions, we also plot corresponding reconstruction errors. Figure 2 shows maximum absolute reconstruction errors for 100 simulated realizations. To reconstruct each realization of the Wiener process two approximation sums (as in Figure 1) were used. We clearly see that empirical probabilities of obtaining large reconstruction errors become smaller if the number of terms in the wavelet expansions increases. Although this effect is expected, it has to be established theoretically in a stringent way for different classes of stochastic processes and wavelet bases. It is also important to obtain theoretical estimations of the rate of convergence for such stochastic wavelet expansions.
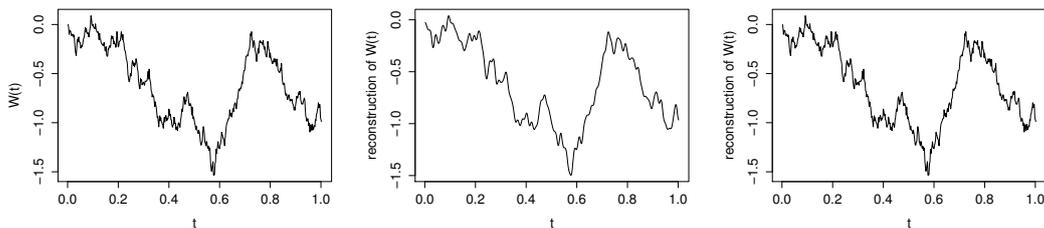

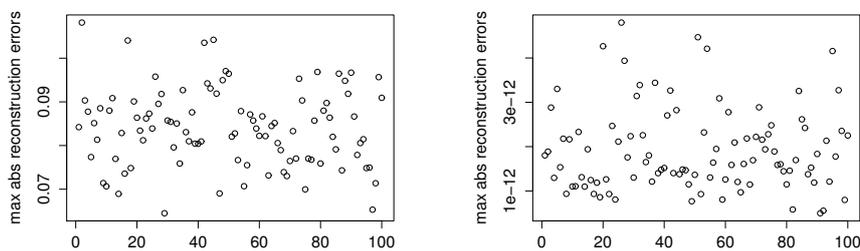Figure 1: Plots of the Wiener process and its wavelet reconstructions.

Figure 2: Plots of reconstruction errors for 100 simulated realizations.

In this paper we make an attempt to derive general results on stochastic uniform convergence which are valid for wavelet expansions of wide classes of stochastic processes. The paper deals with the most general class of such wavelet expansions in

comparison with particular cases considered by different authors, see, for example, [1, 3, 8, 13, 20]. Applications of the main theorem to special cases of practical importance (stationary processes, fractional Brownian motion, etc.) are demonstrated. We also prove the exponential rate of convergence of the wavelet expansions.

Throughout the paper, we impose minimal assumptions on the wavelet bases. The results are obtained under simple conditions which can be easily verified. The conditions are weaker than those in the former literature.

These are novel results on stochastic uniform convergence of general finite wavelet expansions of nonstationary random processes. The specifications of established results are also new (for example, for the case of stationary stochastic processes, compare [15, 16]).

Finally, it should be mentioned that the analysis of the rate of convergence gives a constructive algorithm for determining the number of terms in the wavelet expansions to ensure the uniform approximation of stochastic processes with given accuracy. It provides a practical way to obtain explicit bounds on the sharpness of finite wavelet series approximations.

The organization of the article is the following. In the second section we introduce the necessary background from wavelet theory and certain sufficient conditions for mean-square convergence of wavelet expansions in the space $L_2(\Omega)$. In §3 we formulate and discuss the main theorem on uniform convergence in probability of the wavelet expansions of Gaussian random processes. The next section contains the proof of the main theorem. Two applications of the developed technique are shown in section 4. In §5 the main theorem is adjusted to the fractional Brownian motion case. Lastly, we obtain the rate of convergence of the wavelet expansions in the space $C([0, T])$.

In what follows we use the symbol C to denote constants which are not important for our discussion. Moreover, the same symbol C may be used for different constants appearing in the same proof.

2 Wavelet representation of random processes

Let $\phi(x)$, $x \in \mathbb{R}$, be a function from the space $L_2(\mathbb{R})$ such that $\widehat{\phi}(0) \neq 0$ and $\widehat{\phi}(y)$ is continuous at 0, where

$$\widehat{\phi}(y) = \int_{\mathbb{R}} e^{-iyx} \phi(x) \, dx$$

is the Fourier transform of ϕ.

Suppose that the following assumption holds true:

$$\sum_{k \in \mathbb{Z}} |\widehat{\phi}(y + 2\pi k)|^2 = 1 \text{ (a.e.)}$$

There exists a function $m_0(x) \in L_2([0, 2\pi])$, such that $m_0(x)$ has the period 2π and

$$\widehat{\phi}(y) = m_0 \left(y/2 \right) \widehat{\phi} \left(y/2 \right) \text{ (a.e.)}$$

In this case the function $\phi(x)$ is called the f-wavelet.

Let $\psi(x)$ be the inverse Fourier transform of the function

$$\widehat{\psi}(y) = \overline{m_0 \left(\frac{y}{2} + \pi \right)} \cdot \exp \left\{ -i \frac{y}{2} \right\} \cdot \widehat{\phi} \left(\frac{y}{2} \right).$$

Then the function

$$\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iyx} \widehat{\psi}(y) \, dy$$

is called the m-wavelet.

Let
$$\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k), \quad \psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}, \tag{2.1}$$
where $\phi(x)$ and $\psi(x)$ are defined as above.

The conditions $\widehat{\phi}(0) \neq 0$ and $\widehat{\phi}(y)$ is continuous at 0 guarantee the completeness of system (2.1), see [12]. Then the family of functions $\{\phi_{0k}; \psi_{jk}, j \in \mathbb{N}_0, k \in \mathbb{Z}\}$, $\mathbb{N}_0 := \{0, 1, 2, ...\}$, is an orthonormal basis in $L_2(\mathbb{R})$ (see, for example, [11]. A necessary and sufficient condition on m_0 is given in [4, 5]).

Definition 2.1. *Let $\delta(x), x \in \mathbb{R}$, be a function from the space $L_2(\mathbb{R})$. If the system of functions $\{\delta(x - k), k \in \mathbb{Z}\}$ is an orthonormal system, then the function $\delta(x)$ is called a scaling function.*

Remark 2.2. *f-wavelets and m-wavelets are scaling functions.*

Definition 2.3. *Let $\delta(x), x \in \mathbb{R}$, be a scaling function. If there exists a bounded function $\Phi(x)$, $x \geq 0$, such that $\Phi(x)$ is a decreasing function, $|\delta(x)| \leq \Phi(|x|)$ a.e. on $\mathbb{R}$, and $\int_{\mathbb{R}} (\Phi(|x|))^\gamma \, dx < \infty$ for some $\gamma > 0$, then $\delta(x)$ satisfies assumption $S'(\gamma)$.*

Remark 2.4. *If assumption $S'(\gamma)$ is satisfied for some $\gamma > 0$ then the assumption is also true for all $\gamma_1 > \gamma$.*

Lemma 2.5. *If for some function $\Phi_\phi(\cdot)$ the f-wavelet ϕ satisfies assumption $S'(\gamma)$, then there exist a function $\Phi_\psi(\cdot)$ such that for the corresponding m-wavelet ψ assumption $S'(\gamma)$ holds true and $\Phi_\psi(x) \geq \Phi_\phi(x)$, for all $x \in \mathbb{R}$.*

Proof. The wavelets ϕ and ψ admit the following representations, see [11],
$$\phi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \phi(2x - k),$$
$$\psi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} \lambda_k \phi(2x - k), \tag{2.2}$$
where $h_k = \sqrt{2} \int_R \phi(u) \overline{\phi(2u - k)} du$, $\sum_{k \in \mathbb{Z}} |h_k|^2 < \infty$, and $\lambda_k = (-1)^{1-k} \overline{h_{1-k}}$.

If $k \geq 0$, then

$$|h_k| \leq \sqrt{2} \int_{-\infty}^{\frac{k}{3}} \Phi_\phi(|u|) \Phi_\phi(|2u - k|) \, du + \sqrt{2} \int_{\frac{k}{3}}^{\infty} \Phi_\phi(u) \Phi_\phi(|2u - k|) \, du$$

$$\leq \sqrt{2}\Phi_\phi(k/3) \int_{-\infty}^{\frac{k}{3}} \Phi_\phi(|u|) \, du + \sqrt{2}\Phi_\phi(k/3) \int_{\frac{k}{3}}^{\infty} \Phi_\phi(|2u - k|) \, du$$

$$\leq \sqrt{2}\Phi(k/3) \left(\int_{\mathbb{R}} \Phi_\phi(|u|) \, du + \int_{\mathbb{R}} \Phi_\phi(|2u - k|) \, du \right) = C\Phi_\phi(k/3),$$

where
$$C := \frac{3}{\sqrt{2}} \int_{\mathbb{R}} \Phi_\phi(|u|) \, du.$$

Similarly, for $k \leq 0$ we get $|h_k| \leq C\Phi_\phi(|k|/3)$.

Thus, for all $k \in \mathbb{Z}$,
$$|h_k| \leq C \, \Phi_\phi(|k|/3). \tag{2.3}$$

Note that the series in the right-hand side of (2.2) converges in the $L_2(\mathbb{R})$-norm. Therefore, there exists a subsequence of partial sums which converges to $\psi(x)$ a.e. on $\mathbb{R}$. Thus, by (2.2) and (2.3) we obtain

$$|\psi(x)| \leq \sqrt{2}C \sum_{k \in \mathbb{Z}} \Phi_\phi(|2x - k|) \cdot \Phi_\phi\left(\frac{|1 - k|}{3}\right) =$$

$$= \sqrt{2}C \sum_{k \in \mathbb{Z}} \Phi_\phi \left(|2x - 1 - k|\right) \cdot \Phi_\phi \left(|k|/3\right) =: I\left(2x - 1\right) \tag{2.4}$$

a.e. on $\mathbb{R}$.

If $u := 2x - 1$, then for $u > 0$

$$I(u) \leq \sqrt{2}C \left(\sum_{|k| \leq \frac{3}{4}u} \Phi_\phi \left(|u - k|\right) \Phi_\phi \left(|k|/3\right) \right.$$

$$\left. + \sum_{|k| \geq \frac{3}{4}u} \Phi_\phi \left(|u - k|\right) \cdot \Phi_\phi \left(|k|/3\right) \right) =: \sqrt{2}C \left(A_1\left(u\right) + A_2\left(u\right)\right).$$

Notice also that

$$A_1(u) \leq \Phi_\phi \left(u/4\right) \cdot \sum_{|k| \leq \frac{3}{4}u} \Phi_\phi \left(|k|/3\right) \leq \Phi_\phi \left(u/4\right) \cdot \sum_{k \in \mathbb{Z}} \Phi_\phi \left(|k|/3\right),$$

$$A_2(u) \leq \Phi_\phi \left(u/4\right) \cdot \sum_{|k| \geq \frac{3}{4}u} \Phi_\phi \left(|u - k|\right) \leq \Phi \left(u/4\right) \cdot \sum_{k \in \mathbb{Z}} \Phi_\phi \left(|u - k|\right).$$

Therefore, for $u > 0$

$$I(u) \leq \sqrt{2}C\Phi_\phi \left(u/4\right) \left(\sum_{k \in \mathbb{Z}} \Phi_\phi \left(|u - k|\right) + \sum_{k \in \mathbb{Z}} \Phi_\phi \left(|k|/3\right) \right). \tag{2.5}$$

We are to prove that, $\sum_{k \in \mathbb{Z}} \Phi_\phi \left(|u - k|\right) + \sum_{k \in \mathbb{Z}} \Phi_\phi \left(|k|/3\right)$ is bounded.

Note that

$$\sum_{k \geq u+1} \Phi_\phi \left(k - u\right) \leq \sum_{k \geq u+1} \int_{k-1}^{k} \Phi_\phi \left(v - u\right) dv \leq \int_0^\infty \Phi_\phi \left(v\right) dv < \infty,$$

and, similarly,

$$\sum_{k \leq u-1} \Phi_\phi \left(u - k\right) \leq \int_0^\infty \Phi_\phi \left(v\right) dv.$$

Thus,

$$\sum_{k \in \mathbb{Z}} \Phi_\phi \left(|u - k|\right) \leq 2 \left(\Phi_\phi \left(0\right) + \int_0^\infty \Phi_\phi \left(v\right) dv \right) =: D_1.$$

Since

$$\sum_{k=1}^\infty \Phi_\phi \left(k/3\right) \leq \sum_{k=1}^\infty \int_{k-1}^{k} \Phi_\phi \left(v/3\right) dv = 3 \int_0^\infty \Phi_\phi(u) du,$$

it follows that

$$\sum_{k=1}^\infty \Phi_\phi \left(|k|/3\right) \leq \Phi_\phi \left(0\right) + 6 \int_0^\infty \Phi_\phi \left(u\right) du =: D_2.$$

Thus, we conclude that for $u > 0$

$$I(u) \leq \sqrt{2}C \left(D_1 + D_2\right) \Phi_\phi \left(u/4\right). \tag{2.6}$$

Since for $u < 0$

$$\Phi_\phi \left(|u + k|\right) = \Phi_\phi \left(|-u - k|\right) = \Phi_\phi \left(||u| - k|\right),$$

it follows that

$$I(u) \leq \sqrt{2}C \left(D_1 + D_2\right) \Phi_\phi \left(|u|/4\right) \tag{2.7}$$

for every $u \in \mathbb{R}$.

By (2.4), (2.5), (2.6), and (2.7),

$$|\psi(x)| \leq \tilde{C} \, \Phi_\phi \left(\left| \frac{2x-1}{4} \right| \right) \text{ a.e. on } \mathbb{R}. \tag{2.8}$$

The desired result follows from (2.8) if we chose

$$\Phi_\psi(x) := \begin{cases} \max(1, \tilde{C}), & x \in (0, 1/2), \\ \max(1, \tilde{C}) \cdot \max \left(\Phi_\phi(x), \Phi_\phi \left(\left| \frac{2x-1}{4} \right| \right) \right), & x \notin (0, 1/2). \end{cases}$$

$\square$

Motivated by Lemma 2.5, we will use the following assumption instead of two separate assumptions $S'(\gamma)$ for the f-wavelet ϕ and the m-wavelet ψ.

Assumption $S(\gamma)$. *For some function $\Phi(\cdot)$ and $\gamma > 0$ both the father and mother wavelets satisfy assumption $S'(\gamma)$.*

Let $\{\Omega, \mathcal{B}, \mathcal{P}\}$ be a standard probability space. Let $\mathbf{X}(t)$, $t \in \mathbb{R}$, be a random process such that $\mathbf{EX}(t) = 0$, $\mathbf{E}|\mathbf{X}(t)|^2 < \infty$.

If sample trajectories of this process are in the space $L_2(\mathbb{R})$ with probability one, then it is possible to obtain the representation (wavelet representation)

$$\mathbf{X}(t) = \sum_{k \in \mathbb{Z}} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \eta_{jk} \psi_{jk}(t), \tag{2.9}$$

where

$$\xi_{0k} = \int_{\mathbb{R}} \mathbf{X}(t) \overline{\phi_{0k}(t)} \, dt, \quad \eta_{jk} = \int_{\mathbb{R}} \mathbf{X}(t) \overline{\psi_{jk}(t)} \, dt. \tag{2.10}$$

The majority of random processes does not possess the required property. For example, sample paths of stationary processes are not in the space $L_2(\mathbb{R})$ (a.s.). However, in many cases it is possible to construct a representation of type (2.9) for $\mathbf{X}(t)$.

Consider the approximants of $\mathbf{X}(t)$ defined by

$$\mathbf{X}_{n, \mathbf{k}_n}(t) = \sum_{|k| \leq k_0'} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{n-1} \sum_{|k| \leq k_j} \eta_{jk} \psi_{jk}(t), \tag{2.11}$$

where $\mathbf{k}_n := (k_0', k_0, ..., k_{n-1})$.

Theorem 2.6 below guarantees the mean-square convergence of $\mathbf{X}_{n, \mathbf{k}_n}(t)$ to $\mathbf{X}(t)$ if $k_0' \to \infty$, $k_j \to \infty$, $j \in \mathbb{N}_0$, and $n \to \infty$. The latter means that we increase the number n of multiresolution analysis subspaces which are used to approximate $\mathbf{X}(t)$. For each multiresolution analysis subspace $j = 0', 0, 1, 2...$ the number k_j of its basis vectors, which are used in the approximation, increases too, as n tends to infinity. Thus, for each fixed k and j there is $n_0 \in \mathbb{N}_0$ that the terms $\xi_{0k} \phi_{0k}(t)$ and $\eta_{jk} \psi_{jk}(t)$ are included in all $\mathbf{X}_{n, \mathbf{k}_n}(t)$ for $n \geq n_0$ (i.e., each $\xi_{0k} \phi_{0k}(t)$ and $\eta_{jk} \psi_{jk}(t)$ can be absent only in the finite number of $\mathbf{X}_{n, \mathbf{k}_n}(t)$).

Theorem 2.6. [18] *Let $\mathbf{X}(t), t \in \mathbb{R}$, be a random process such that $\mathbf{EX}(t) = 0$, $\mathbf{E}|\mathbf{X}(t)|^2 < \infty$ for all $t \in \mathbb{R}$, and its covariance function $R(t, s) := \mathbf{EX}(t)\overline{\mathbf{X}(s)}$ is continuous. Let the f-wavelet ϕ and the m-wavelet ψ be continuous functions which satisfy assumption $S(1)$. Let $c(x), x \in \mathbb{R}$, denote a non decreasing on $[0, \infty)$ even function with $c(0) > 0$.*

Suppose that there exists a function $A : (0, \infty) \to (0, \infty)$, and $a_0, x_0 \in (0, \infty)$ such that $c(ax) \leq c(x) \cdot A(a)$, for all $a \geq a_0$, $x \geq x_0$.

If

$$\int_{\mathbb{R}} c(x)\Phi(|x|)\,dx < \infty \quad \text{and} \quad |R(t,t)|^{1/2} \leq c(t),$$

then

1. $\mathbf{X}_{n,\mathbf{k}_n}(t) \in L_2(\Omega)$;
2. $\mathbf{X}_{n,\mathbf{k}_n}(t) \to \mathbf{X}(t)$ *in mean square when* $n \to \infty$, $k_0' \to \infty$, *and* $k_j \to \infty$ *for all* $j \in \mathbb{N}_0$.

3 Uniform convergence of wavelet expansions for Gaussian random processes

In this section we show that, under suitable conditions, the sequence $\mathbf{X}_{n,\mathbf{k}_n}(t)$ converges in probability in Banach space $C([0,T])$, $T > 0$, i.e.

$$P\left\{ \sup_{0 \leq t \leq T} |\mathbf{X}(t) - \mathbf{X}_{n,\mathbf{k}_n}(t)| > \varepsilon \right\} \to 0,$$

when $n \to \infty$, $k_0' \to \infty$, and $k_j \to \infty$ for all $j \in \mathbb{N}_0 := \{0, 1, ...\}$. More details on the general theory of random processes in the space $C([0,T])$ can be found in [2].

Theorem 3.1. [19] *Let $\mathbf{X}_n(t)$, $t \in [0,T]$, be a sequence of Gaussian stochastic processes such that all $\mathbf{X}_n(t)$ are separable in $([0,T], \rho)$, $\rho(t,s) = |t - s|$, and*

$$\sup_{n \geq 1} \sup_{|t-s| \leq h} \left(\mathbf{E}|\mathbf{X}_n(t) - \mathbf{X}_n(s)|^2 \right)^{1/2} \leq \sigma(h),$$

where $\sigma(h)$ is a function, which is monotone increasing in a neighborhood of the origin and $\sigma(h) \to 0$ when $h \to 0$.

Assume that for some $\varepsilon > 0$

$$\int_0^\varepsilon \sqrt{-\ln\left(\sigma^{(-1)}(u)\right)}\,du < \infty, \tag{3.1}$$

where $\sigma^{(-1)}(u)$ is the inverse function of $\sigma(u)$. If the random variables $\mathbf{X}_n(t)$ converge in probability to the random variable $\mathbf{X}(t)$ for all $t \in [0,T]$, then $\mathbf{X}_n(t)$ converges to $\mathbf{X}(t)$ in the space $C([0,T])$.

Remark 3.2. *For example, it is easy to check that assumption (3.1) holds true for*

$$\sigma(h) = \frac{C}{\ln^\beta \left(e^\alpha + \frac{1}{h} \right)} \quad \text{and} \quad \sigma(h) = Ch^\kappa,$$

when $C > 0$, $\beta > 1/2$, $\alpha > 0$, and $\kappa > 0$.

The following theorem is the main result of the paper.

Theorem 3.3. *Let a Gaussian process $\mathbf{X}(t)$, $t \in \mathbb{R}$, its covariance function, the f-wavelet ϕ, and the corresponding m-wavelet ψ satisfy the assumptions of Theorem 2.6.*
Suppose that

(i) *assumption $S(\gamma)$, $\gamma \in (0, 1)$, holds true for ϕ and ψ;*
(ii) *the integrals $\int_{\mathbb{R}} \ln^\alpha(1 + |u|)|\widehat{\psi}(u)|\,du$ and $\int_{\mathbb{R}} \ln^\alpha(1 + |u|)|\widehat{\phi}(u)|\,du$ converge for some $\alpha > 1/2(1 - \gamma)$;*

(iii) *there exist constants b_0 and c_j, $j \in \mathbb{N}_0$, such that for all integer k $\mathbf{E}|\xi_{0k}|^2 \leq b_0$, $\mathbf{E}|\eta_{jk}|^2 \leq c_j$, and*

$$\sum_{j=0}^{\infty} \sqrt{c_j} 2^{\frac{j}{2}} j^{\alpha(1-\gamma)} < \infty. \tag{3.2}$$

Then $\mathbf{X}_{n,\mathbf{k}_n}(t) \to \mathbf{X}(t)$ uniformly in probability on each interval $[0,T]$ when $n \to \infty$, $k_0' \to \infty$, and $k_j \to \infty$ for all $j \in \mathbb{N}_0$.

Remark 3.4. *If both wavelets ϕ and ψ have compact supports, then some assumptions of Theorem 3.3 are superfluous. In the following theorem we give an example by considering approximants of the form*

$$\mathbf{X}_n(t) := \sum_{k \in \mathbb{Z}} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{n-1} \sum_{k \in \mathbb{Z}} \eta_{jk} \psi_{jk}(t).$$

Theorem 3.5. *Let $\mathbf{X}(t)$, $t \in \mathbb{R}$, be a separable centered Gaussian random process such that its covariance function $R(t,s)$ is continuous. Let the f-wavelet ϕ and the corresponding m-wavelet ψ be continuous functions with compact supports and the integrals $\int_{\mathbb{R}} \ln^{\alpha}(1 + |u|)|\widehat{\psi}(u)|\, du$ and $\int_{\mathbb{R}} \ln^{\alpha}(1 + |u|)|\widehat{\phi}(u)|\, du$ converge for some $\alpha > 1/2(1-\gamma)$, $\gamma \in (0,1)$. If there exist constants c_j, $j \in \mathbb{N}_0$, such that $\mathbf{E}|\eta_{jk}|^2 \leq c_j$ for all $k \in \mathbb{Z}$, and assumption (3.2) is satisfied, then $\mathbf{X}_n(t) \to \mathbf{X}(t)$ uniformly in probability on each interval $[0,T]$ when $n \to \infty$.*

Proof. The assumptions of Theorem 2.6 and $S(\gamma), 0 < \gamma < 1$, are satisfied because ϕ and ψ have compact supports. Therefore, the desired result follows from Theorem 3.3. □

Remark 3.6. *For example, Daubechies wavelets satisfy the assumptions of Theorem 3.5.*

4 Proof of the main theorem

To prove Theorem 3.3 we need some auxiliary results.

Lemma 4.1. *If $\delta(x)$ is a scaling function satisfying assumption $S'(\gamma)$, then*

$$\sup_{x \in \mathbb{R}} S_{\gamma}(x) \leq 3\Phi^{\gamma}(0) + 4 \int_{1/2}^{\infty} \Phi^{\gamma}(t)\, dt, \tag{4.1}$$

where

$$S_{\gamma}(x) = \sum_{k \in \mathbb{Z}} |\delta(x - k)|^{\gamma}.$$

Proof. The lemma is a simple generalization of a result from [11].

Since $S_{\gamma}(x)$ is a periodic function with period 1, it is sufficient to prove (4.1) for $x \in [0,1]$.

Notice, that for $x \in [0,1]$ and integer $|k| \geq 2$ the inequality $|x - k| \geq |k|/2$ holds true. Hence, $\Phi(|x-k|) \leq \Phi(|k|/2)$ and

$$S_{\gamma}(x) \leq \Phi^{\gamma}(|x|) + \Phi^{\gamma}(|x+1|) + \Phi^{\gamma}(|x-1|) + \sum_{|k| \geq 2} \Phi^{\gamma}(|k|/2)$$

$$\leq 3\Phi^{\gamma}(0) + 2\sum_{k=2}^{\infty} \int_{k-1}^{k} \Phi^{\gamma}(t/2)\, dt = 3\Phi^{\gamma}(0) + 4\int_{\frac{1}{2}}^{\infty} \Phi^{\gamma}(t)\, dt.$$

□

Lemma 4.2. *Let $\hat{\delta}(x)$ denote the Fourier transform of the scaling function $\delta(x)$, $\delta_{jk}(x) :=$ $2^{\frac{j}{2}} \delta(2^j x - k)$, $j \in \mathbb{N}_0$, $k \in \mathbb{Z}$. If for some $\alpha > 0$*

$$\int_{\mathbb{R}} \ln^{\alpha}(1 + |u|) |\hat{\delta}(u)| \, du < \infty,$$

then for all $x, y \in \mathbb{R}$ and $k \in \mathbb{Z}$

$$|\delta_{0k}(x) - \delta_{0k}(y)| \le R_{0\alpha} \cdot \ln^{-\alpha} \left(e^{\alpha} + \frac{1}{|x - y|} \right),$$

$$|\delta_{jk}(x) - \delta_{jk}(y)| \le 2^{\frac{j}{2}} j^{\alpha} R_{1\alpha} \cdot \ln^{-\alpha} \left(e^{\alpha} + \frac{1}{|x - y|} \right), \quad j \in \mathbb{N}_0,$$

where $R_{0\alpha} := \frac{1}{\pi} \int_{\mathbb{R}} \ln^{\alpha} \left(e^{\alpha} + \frac{|u|}{2} \right) |\hat{\delta}(u)| \, du$ and $R_{1\alpha} := \frac{1}{\pi} \int_{\mathbb{R}} \ln^{\alpha} \left(e^{\alpha} + |u| + 1 \right) |\hat{\delta}(u)| \, du$.

Proof. Since $\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixu} \hat{\delta}(u) \, du$, we have

$$|\delta_{jk}(x) - \delta_{jk}(y)| \le \frac{2^{\frac{j}{2}}}{2\pi} \int_{\mathbb{R}} \left| e^{ixu2^j} - e^{iyu2^j} \right| |\hat{\delta}(u)| \, du = \frac{2^{\frac{j}{2}}}{\pi} \int_{\mathbb{R}} \left| \sin \frac{(x - y)u2^j}{2} \right| |\hat{\delta}(u)| \, du. \quad (4.2)$$

Note that for $v \ne 0$ the following inequality holds:

$$\left| \sin \left(\frac{s}{v} \right) \right| \le \frac{|\ln (e^{\alpha} + |s|)|^{\alpha}}{|\ln (e^{\alpha} + |v|)|^{\alpha}}. \quad (4.3)$$

By (4.2) and (4.3) we obtain

$$|\delta_{jk}(x) - \delta_{jk}(y)| \le \frac{2^{\frac{j}{2}}}{\pi \ln^{\alpha} \left(e^{\alpha} + \frac{1}{|x-y|} \right)} \int_{\mathbb{R}} \ln^{\alpha} \left(e^{\alpha} + \frac{|u|2^j}{2} \right) |\hat{\delta}(u)| \, du.$$

The assertion of the lemma follows from this inequality. $\qquad \square$

Lemma 4.3. *If a scaling function $\delta(x)$ satisfies the assumptions of Lemmata 4.1 and 4.2, then for $\gamma \in (0, 1)$ and $\alpha > 0$:*

$$\sum_{k \in \mathbb{Z}} |\delta_{jk}(x) - \delta_{jk}(y)| \le 2^{\frac{j}{2}+1} j^{\alpha(1-\gamma)} R_1^{1-\gamma}(\alpha) \left(3\Phi^{\gamma}(0) + 4 \int_{1/2}^{\infty} \Phi^{\gamma}(t) \, dt \right)$$
$$\times \left(\ln \left(e^{\alpha} + \frac{1}{|x - y|} \right) \right)^{-\alpha(1-\gamma)}, \quad j \ne 0, \quad (4.4)$$

$$\sum_{k \in \mathbb{Z}} |\delta_{0k}(x) - \delta_{0k}(y)| \le 2 R_0^{1-\gamma}(\alpha) \left(3\Phi^{\gamma}(0) + 4 \int_{1/2}^{\infty} \Phi^{\gamma}(t) \, dt \right)$$
$$\times \left(\ln \left(e^{\alpha} + \frac{1}{|x - y|} \right) \right)^{-\alpha(1-\gamma)}. \quad (4.5)$$

Proof. By lemma 4.2 for $j \ge 1$ we obtain

$$\sum_{k \in \mathbb{Z}} |\delta_{jk}(x) - \delta_{jk}(y)| = \sum_{k \in \mathbb{Z}} |\delta_{jk}(x) - \delta_{jk}(y)|^{1-\gamma} \cdot |\delta_{jk}(x) - \delta_{jk}(y)|^{\gamma}$$
$$\le \left(\ln \left(e^{\alpha} + \frac{1}{|x - y|} \right) \right)^{-\alpha(1-\gamma)} \left(2^{\frac{j}{2}} j^{\alpha} R_1(\alpha) \right)^{1-\gamma} \cdot \sum_{k \in \mathbb{Z}} |\delta_{jk}(x) - \delta_{jk}(y)|^{\gamma}.$$

We now make use of the inequality $|a + b|^\alpha \leq q_\alpha \left(|a|^\alpha + |b|^\alpha\right)$, where

$$q_\alpha = \begin{cases} 1, & \alpha \leq 1, \\ 2^{\alpha - 1}, & \alpha > 1. \end{cases} \qquad (4.6)$$

By lemma 4.1 we get

$$
\begin{aligned}
\sum_{k \in \mathbb{Z}} |\delta_{jk}(x) - \delta_{jk}(y)|^\gamma &\leq 2 \sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\delta_{jk}(x)|^\gamma = 2^{\frac{j\gamma}{2}+1} \sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\delta(2^j x - k)|^\gamma \\
&\leq 2^{\frac{j\gamma}{2}+1} \left(3\Phi^\gamma(0) + 4 \int_{1/2}^\infty \Phi^\gamma(t)\, dt \right).
\end{aligned}
$$

Inequality (4.4) follows from this estimate. The proof of inequality (4.5) is similar.

$\square$

Now we are ready to prove Theorem 3.3.

Proof. In virtue of Theorem 3.1 and Remark 3.2 it is sufficient to prove that $\mathbf{X}_{n,\mathbf{k}_n}(t) \to \mathbf{X}(t)$ in mean square and

$$\left(\mathbf{E} \left| \mathbf{X}_{n,\mathbf{k}_n}(t) - \mathbf{X}_{n,\mathbf{k}_n}(s) \right|^2 \right)^{\frac{1}{2}} \leq \frac{C}{\ln^\beta \left(e^\alpha + \frac{1}{|t-s|} \right)}$$

for some $C > 0$, $\beta > 1/2$, and $\alpha > 0$.

Since $\gamma < 1$, assumption $S(1)$ holds true for $\phi(x)$ and $\psi(x)$. Hence, by Theorem 2.6 we get $\mathbf{X}_{n,\mathbf{k}_n}(t) \to \mathbf{X}(t)$ in mean square.

Note that (2.11) implies

$$
\begin{aligned}
\left(\mathbf{E} \left| \mathbf{X}_{n,\mathbf{k}_n}(t) - \mathbf{X}_{n,\mathbf{k}_n}(s) \right|^2 \right)^{\frac{1}{2}} &\leq \left(\mathbf{E} \left| \sum_{|k| \leq k_0'} \xi_{0k}(\phi_{0k}(t) - \phi_{0k}(s)) \right|^2 \right)^{1/2} \\
&\quad + \sum_{j=0}^{n-1} \left(\mathbf{E} \left| \sum_{|k| \leq k_j} \eta_{jk}(\psi_{jk}(t) - \psi_{jk}(s)) \right|^2 \right)^{1/2} =: \sqrt{S} + \sum_{j=0}^{n-1} \sqrt{S_j}.
\end{aligned}
$$

We will only show how to handle S_j. A similar approach can be used to deal with the remaining term S.

S_j can be bounded as follows

$$S_j = \sum_{|k| \leq k_j} \sum_{|l| \leq k_j} \mathbf{E}\eta_{jk}\overline{\eta_{jl}}(\psi_{jk}(t) - \psi_{jk}(s))(\psi_{jl}(t) - \psi_{jl}(s)) \leq c_j \left(\sum_{k \in \mathbb{Z}} |\psi_{jk}(t) - \psi_{jk}(s)| \right)^2.$$

By lemma 4.3

$$S_j \leq 2^j j^{2\alpha(1-\gamma)} c_j L^2 \left(\ln \left(e^\alpha + \frac{1}{|t-s|} \right) \right)^{-2\alpha(1-\gamma)},$$

where

$$L := 2R_1^{1-\gamma}(\alpha) \left(3\Phi^\gamma(0) + 4 \int_{1/2}^\infty \Phi^\gamma(t)\, dt \right).$$

Similarly,

$$S \leq b_0 L_1^2 \left(\ln \left(e^\alpha + \frac{1}{|t-s|} \right) \right)^{-2\alpha(1-\gamma)},$$

where

$$L_1 := 2R_0^{1-\gamma}(\alpha)\left(3\Phi^\gamma(0) + 4\int_{1/2}^\infty \Phi^\gamma(t)\,dt\right).$$

Therefore, we conclude that

$$\left(\mathbf{E}\,|\mathbf{X}_{n,\mathbf{k}_n}(t) - \mathbf{X}_{n,\mathbf{k}_n}(s)|^2\right)^{\frac{1}{2}} \le \frac{C}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{\alpha(1-\gamma)}},$$

where

$$C := \sqrt{b_0}L_1 + L\sum_{j=0}^\infty \sqrt{c_j}\,2^{\frac{j}{2}}j^{\alpha(1-\gamma)} < \infty \ \text{ and } \ \beta := \alpha(1-\gamma) > 1/2.$$

$\square$

5 Examples

In this section we consider some examples of wavelets and stochastic processes which satisfy assumption (3.2) of Theorem 3.3.

Example 5.1. *Let $\widehat{\psi}$ be a Lipschitz function of order $\varkappa > 0$ ($\varkappa = 1$ for the case $\sup_{u\in\mathbb{R}}|\widehat{\psi}'(u)| \le C$), i.e.*

$$|\widehat{\psi}(u) - \widehat{\psi}(v)| \le C|u - v|^\varkappa.$$

Assume that for the covariance function $R(t,s)$

$$\int_\mathbb{R}\int_\mathbb{R} |R(t,s)|\,dt\,ds < \infty$$

and

$$\int_\mathbb{R}\int_\mathbb{R} \left|\widehat{R}_2(z,w)\right|\cdot|z|^\varkappa\cdot|w|^\varkappa\,dz\,dw < \infty,$$

where

$$\widehat{R}_2(z,w) = \int_\mathbb{R}\int_\mathbb{R} R(u,v)\,e^{-izu}e^{-iwv}\,du\,dv.$$

Now we show that $\mathbf{E}|\eta_{jk}|^2 \le c_j$ for all $k \in \mathbb{Z}$ and find suitable upper bounds for c_j.

By Parseval's theorem,

$$\mathbf{E}\eta_{jk}\overline{\eta_{jl}} = \int_\mathbb{R}\int_\mathbb{R} \mathbf{E}\mathbf{X}(u)\overline{\mathbf{X}(v)}\,\overline{\psi_{jk}(u)}\psi_{jl}(v)\,du\,dv$$

$$= \frac{1}{(2\pi)^2}\int_\mathbb{R}\int_\mathbb{R} \widehat{R}_2(z,w)\,\overline{\widehat{\psi}_{jk}(z)}\,\widehat{\psi}_{jl}(w)\,dz\,dw.$$

Since

$$\widehat{\psi}_{jk}(z) = \frac{e^{-i\frac{k}{2^j}z}}{2^{j/2}}\cdot\widehat{\psi}\left(\frac{z}{2^j}\right),$$

it follows that

$$|\mathbf{E}\eta_{jk}\overline{\eta_{jl}}| \le \frac{1}{2^j(2\pi)^2}\int_\mathbb{R}\int_\mathbb{R}\left|\widehat{R}_2(z,w)\right|\cdot\left|\widehat{\psi}\left(\frac{z}{2^j}\right)\right|\cdot\left|\widehat{\psi}\left(\frac{w}{2^j}\right)\right|\,dz\,dw.$$

By properties of the m-wavelet ψ we have $\widehat{\psi}(0) = 0$. Therefore, using the Lipschitz conditions, we obtain

$$|\mathbf{E}\eta_{jk}\overline{\eta_{jl}}| \le \frac{C^2}{(2\pi)^2 2^{j(1+2\varkappa)}}\int_\mathbb{R}\int_\mathbb{R}\left|\widehat{R}_2(z,w)\right|\cdot|z|^\varkappa\cdot|w|^\varkappa\,dz\,dw.$$

This means that $\sqrt{c_j} \le C/2^{j/2(1+2\varkappa)}$ and assumption (3.2) holds.

In the following example we consider the case of stationary stochastic processes. This case was studied in detail by us in [15]. Note that assumptions in the example are much simpler than those used in [15].

Example 5.2. *Let* $\mathbf{X}(t)$ *be a centered short-memory stationary stochastic process and* $\widehat{\psi}$ *be a Lipschitz function of order* $\varkappa > 0$. *Assume that the covariance function* $R(t-s) := \mathbf{E}\mathbf{X}(t)\overline{\mathbf{X}(s)}$ *satisfies the following condition*

$$\int_{\mathbb{R}} \left| \widehat{R}(z) \right| \cdot |z|^{2\varkappa} \, dz < \infty.$$

By Parseval's theorem we deduce

$$|\mathbf{E}\eta_{jk}\overline{\eta_{jl}}| = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} R(u-v)\overline{\psi_{jk}(u)} \, du \, \psi_{jl}(v) \, dv \right| = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-ivz}}{2\pi} \widehat{R}(z)\overline{\widehat{\psi_{jk}}(z)} \, dz \psi_{jl}(v) \, dv \right|$$

$$\leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \widehat{R}(z) \right| \left| \overline{\widehat{\psi_{jk}}(z)} \, \widehat{\psi_{jl}}(z) \right| \, dz = \frac{1}{2^{j+1}\pi} \int_{\mathbb{R}} \left| \widehat{R}(z) \right| \cdot \left| \widehat{\psi}\left(\frac{z}{2^j}\right) \right| \cdot \left| \widehat{\psi}\left(\frac{z}{2^j}\right) \right| \, dz.$$

Thus, by the Lipschitz conditions, for all $k, l \in \mathbb{Z}$:

$$|\mathbf{E}\eta_{jk}\overline{\eta_{jl}}| \leq \frac{C}{\pi \, 2^{1+j(1+2\varkappa)}} \int_{\mathbb{R}} \left| \widehat{R}(z) \right| \cdot |z|^{2\varkappa} \, dz.$$

This means that $\sqrt{c_j} \leq C/2^{j/2(1+2\varkappa)}$ *and assumption (3.2) is satisfied.*

6 Application to fractional Brownian motion

In this section we show how to adjust the main theorem to the fractional Brownian motion case.

Let $W_\alpha(t)$, $t \in \mathbb{R}$, be a separable centered Gaussian random process such that $W_\alpha(-t) = W_\alpha(t)$ and its covariance function is

$$R(t,s) = \mathbf{E}W_\alpha(t)W_\alpha(s) = \frac{1}{2}\left(|t|^\alpha + |s|^\alpha - ||t| - |s||^\alpha\right), \ 0 < \alpha < 2. \tag{6.1}$$

Lemma 6.1. *If assumption* $S(\gamma)$, $0 < \gamma < 1$, *holds true and for some* $\alpha > 0$

$$c_\psi := \int_{\mathbb{R}} |u|^\alpha |\psi(u)| \, du < \infty,$$

then for the coefficients of the process $W_\alpha(t)$, *defined by (2.10),*

$$|\mathbf{E}\eta_{jk}\overline{\eta_{jl}}| \leq \frac{C}{2^{j(1+\alpha)}}$$

for all $k, l \in \mathbb{Z}$.

Proof. Since $|\mathbf{E}\eta_{jk}\overline{\eta_{jl}}| \leq \left(\mathbf{E}|\eta_{jk}|^2\right)^{\frac{1}{2}} \left(\mathbf{E}|\eta_{jl}|^2\right)^{\frac{1}{2}}$, it is sufficient to estimate $\mathbf{E}|\eta_{jk}|^2$.

By (2.1) and (6.1) we obtain

$$
\begin{aligned}
\mathbf{E}|\eta_{jk}|^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} R(t,s) \, \overline{\psi_{jk}(t)} \psi_{jk}(s) \, dtds = \frac{1}{2^j} \int_{\mathbb{R}} \int_{\mathbb{R}} R\left(\frac{u}{2^j}, \frac{v}{2^j}\right) \overline{\psi(u-k)} \psi(v-k) \, du \, dv \\
&= \frac{1}{2^j} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2}\left(\left|\frac{u}{2^j}\right|^\alpha + \left|\frac{v}{2^j}\right|^\alpha - \left|\left|\frac{u}{2^j}\right| - \left|\frac{v}{2^j}\right|\right|^\alpha\right) \overline{\psi(u-k)} \psi(v-k) \, du \, dv \\
&\leq \frac{1}{2^{1+j(1+\alpha)}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |u|^\alpha \, \overline{\psi(u-k)} \psi(v-k) \, du \, dv \right. \\
&\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} |v|^\alpha \, \overline{\psi(u-k)} \psi(v-k) \, du \, dv \\
&\quad + \left. \int_{\mathbb{R}} \int_{\mathbb{R}} ||u| - |v||^\alpha \, \overline{\psi(u-k)} \psi(v-k) \, du \, dv \right) =: \frac{z_1 + z_2 + z_3}{2^{1+j(1+\alpha)}}. \tag{6.2}
\end{aligned}
$$

It follows from assumption $S(\gamma)$ that

$$N := \int_{\mathbb{R}} |\psi(u)|\, du < \infty.$$

Hence, we get the estimate

$$
\begin{aligned}
z_1 &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |u|^\alpha\, |\psi(u-k)||\psi(v-k)|\, du\, dv = \int_{\mathbb{R}} |\psi(v)|\, dv \\
&\times \int_{\mathbb{R}} |u+k|^\alpha\, |\psi(u)|\, du \leq q_\alpha N \left(N|k|^\alpha + c_\psi\right) < \infty,
\end{aligned}
\tag{6.3}
$$

where q_α is defined by (4.6).

Using Fubini's theorem and $\int_{\mathbb{R}} \psi(u)\, du = 0$ we obtain

$$z_1 = \int_{\mathbb{R}} |u|^\alpha\, \overline{\psi(u-k)}\, du \int_{\mathbb{R}} \psi(v-k)\, dv = 0.$$

Similarly, $z_2 = 0$.

Finally, we estimate z_3 as follows

$$z_3 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} ||u+k| - |v+k||^\alpha\, |\psi(u)||\psi(v)|\, du\, dv.$$

By the reverse triangle inequality and (4.6) we obtain

$$||u+k| - |v+k||^\alpha \leq q_\alpha \left(|u|^\alpha + |v|^\alpha\right).$$

Hence,

$$|z_3| \leq q_\alpha\, 2^{\alpha+1} \int_{\mathbb{R}} |u|^\alpha\, |\psi(u)|\, du \int_{\mathbb{R}} |\psi(v)|\, dv = q_\alpha c_\psi\, 2^{\alpha+1} N, \tag{6.4}$$

which completes the proof of the lemma.

$\square$

In some case, for example, for the fractional Brownian motion the assumption $b_0 \geq |\mathbf{E}\xi_{0k}\overline{\xi_{0l}}|$ of Theorem 3.3 doesn't hold true. The following theorem gives the uniform convergence of wavelet expansions without this assumption.

Theorem 6.2. *Let a random process $\mathbf{X}(t)$, $t \in \mathbb{R}$, the f-wavelet ϕ, and the corresponding m-wavelet ψ satisfy the assumptions of Theorem 2.6 and assumptions* (i) *and* (ii) *of Theorem 3.3.*

Suppose that there exist

(iii') *constants c_j, $j \in \mathbb{N}_0$, such that $\mathbf{E}|\eta_{jk}|^2 \leq c_j$ for all $k \in \mathbb{Z}$ and (3.2) holds true;*

(iv) *some $\varepsilon > 0$ such that*

$$S := \mathbf{E} \left| \sum_{|k| < |k_0'|} \xi_{0k}(\phi_{0k}(t) - \phi_{0k}(s)) \right|^2 \leq \frac{C}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|} \right) \right)^{2\alpha(1-\gamma)}}, \quad \alpha > 1/2(1-\gamma),$$

if $|t - s| < \varepsilon$.

Then $\mathbf{X}_{n,\mathbf{k}_n}(t) \to \mathbf{X}(t)$ uniformly in probability on each interval $[0, T]$ when $n \to \infty$, $k_0' \to \infty$, and $k_j \to \infty$ for all $j \in \mathbb{N}_0$.

Proof. The assertion of the theorem follows from the proof of Theorem 3.3. $\square$

Now, under some mild additional conditions on the f-wavelet ϕ, we show that estimate (iv) holds true in the fractional Brownian motion case.

Lemma 6.3. *Let the f-wavelet ϕ satisfy the assumptions of Theorem* 2.6*,*

- $\widehat{\phi}(z) \to 0$ *and* $\widehat{\phi}'(z) \to 0$*, when* $z \to \pm\infty$;
- *the integrals* $\int_{\mathbb{R}} |u|^{\alpha} |\phi(u)|$, $\int_{\mathbb{R}} |\widehat{\phi}'(u)| \, du$ *and* $\int_{\mathbb{R}} \ln^{\alpha}(1 + |u|)|\widehat{\phi}^{(i)}(u)| \, du$, $i = 0, 1, 2$, *converge for some* $\alpha > 1/2(1 - \gamma)$.

Then assumption (iv) *of Theorem* 6.2 *holds true for* $W_{\alpha}(t)$.

Proof. S can be bounded as follows

$$S \le \sum_{|k| \le k_0} \sum_{|l| \le k_0} |\mathbf{E}\xi_{0k}\overline{\xi_{0l}}||\phi_{0k}(t) - \phi_{0k}(s)||\phi_{0l}(t) - \phi_{0l}(s)| \,.$$

Since $\left|\mathbf{E}\xi_{0k}\overline{\xi_{0l}}\right| \le \left(\mathbf{E}|\xi_{0k}|^2\right)^{\frac{1}{2}} \left(\mathbf{E}|\xi_{0l}|^2\right)^{\frac{1}{2}}$, it is sufficient to estimate $\mathbf{E}|\xi_{0k}|^2$.

Similarly to (6.2) we get

$$\begin{aligned}
\mathbf{E}|\xi_{0k}|^2 &\le \int_{\mathbb{R}} \int_{\mathbb{R}} |u|^{\alpha} \overline{\phi(u - k)}\phi(v - k) \, du \, dv + \int_{\mathbb{R}} \int_{\mathbb{R}} |v|^{\alpha} \overline{\phi(u - k)}\phi(v - k) \, du \, dv \\
&+ \int_{\mathbb{R}} \int_{\mathbb{R}} ||u| - |v||^{\alpha} \overline{\phi(u - k)}\phi(v - k) \, du \, dv.
\end{aligned}$$

Analogously to (6.3) and (6.4) we obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u|^{\alpha} \left|\overline{\phi(u - k)}\phi(v - k)\right| du \, dv \le q_{\alpha}M \left(M|k|^{\alpha} + c_{\phi}\right),$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} ||u| - |v||^{\alpha} \left|\overline{\phi(u - k)}\phi(v - k)\right| du \, dv \le q_{\alpha}c_{\phi}M \, 2^{\alpha+1},$$

where

$$M := \int_{\mathbb{R}} |\phi(u)| \, du < \infty, \; c_{\phi} := \int_{\mathbb{R}} |u|^{\alpha}|\phi(u)| \, du < \infty.$$

Then

$$\begin{aligned}
\left|\mathbf{E}\xi_{0k}\overline{\xi_{0l}}\right| &\le 2q_{\alpha} \, M \left(M|k|^{\alpha} + c_{\phi}(2^{\alpha} + 1)\right)^{\frac{1}{2}} \left(M|l|^{\alpha} + c_{\phi}(2^{\alpha} + 1)\right)^{\frac{1}{2}} \\
&\le 2q_{\alpha} \, M \left(M|k|^{\frac{\alpha}{2}} + \sqrt{c_{\phi}2^{\alpha+1}}\right) \left(M|l|^{\frac{\alpha}{2}} + \sqrt{c_{\phi}2^{\alpha+1}}\right).
\end{aligned}$$

To estimate $|\phi_{0k}(t) - \phi_{0k}(s)|$, we use the representations

$$\widehat{\phi}_{0k}(z) = e^{-ikz} \cdot \widehat{\phi}(z) \,,$$

$$\phi_{0k}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itz} e^{-ikz} \widehat{\phi}(z) \, dz.$$

Repeatedly using integration by parts and the assumptions of the lemma, we obtain that for $k \ne 0$:

$$\begin{aligned}
|\phi_{0k}(t) - \phi_{0k}(s)| &= \left|\frac{1}{2ik\pi} \int_{\mathbb{R}} \left(e^{itz} - e^{isz}\right) \widehat{\phi}(z) \, d(e^{-ikz})\right| \\
&= \frac{1}{2\pi k^2} \left|\int_{\mathbb{R}} \left[i \left(te^{itz} - se^{isz}\right) \widehat{\phi}(z) + \left(e^{itz} - e^{isz}\right) \widehat{\phi}'(z)\right] d(e^{-ikz})\right| \\
&= \frac{1}{2\pi k^2} \left|\int_{\mathbb{R}} \left[-\left(t^2 e^{itz} - s^2 e^{isz}\right) \widehat{\phi}(z) + 2i \left(te^{itz} - se^{isz}\right) \widehat{\phi}'(z)\right. \right. \\
&\left. \left. + \left(e^{itz} - e^{isz}\right) \widehat{\phi}''(z)\right] e^{-ikz} \, dz\right| \le \frac{1}{2\pi k^2} \left(\int_{\mathbb{R}} \left|t^2 e^{itz} - s^2 e^{isz}\right| |\widehat{\phi}(z)| \, dz\right. \\
&\left. + 2 \int_{\mathbb{R}} \left|te^{itz} - se^{isz}\right| |\widehat{\phi}'(z)| \, dz + \int_{\mathbb{R}} \left|e^{itz} - e^{isz}\right| |\widehat{\phi}''(z)| \, dz\right).
\end{aligned} \tag{6.5}$$

By inequalities (8) and (12) given in [15] we get

$$
\begin{aligned}
|te^{itz} - se^{isz}| &\leq |t - s| + t|e^{itz} - e^{isz}| \\
&\leq \frac{c_{\alpha,T}}{\ln^\alpha\left(e^\alpha + \frac{1}{|t-s|}\right)} + 2T\left(\frac{\ln\left(e^\alpha + \frac{|z|}{2}\right)}{\ln\left(e^\alpha + \frac{1}{|t-s|}\right)}\right)^\alpha,
\end{aligned}
$$

$$
\begin{aligned}
|t^2 e^{itz} - s^2 e^{isz}| &\leq |t^2 - s^2| + t^2|e^{itz} - e^{isz}| \\
&\leq \frac{\tilde{c}_{\alpha,T}}{\ln^\alpha\left(e^\alpha + \frac{1}{|t-s|}\right)} + 2T^2\left(\frac{\ln\left(e^\alpha + \frac{|z|}{2}\right)}{\ln\left(e^\alpha + \frac{1}{|t-s|}\right)}\right)^\alpha,
\end{aligned}
$$

where $c_{\alpha,T}$ and $\tilde{c}_{\alpha,T}$ are constants which do not depend on t, s and z.

Applying these inequalities to (6.5) we obtain

$$
|\phi_{0k}(t) - \phi_{0k}(s)| \leq \frac{B_{\phi,\alpha,T}}{k^2 \ln^\alpha\left(e^\alpha + \frac{1}{|t-s|}\right)}, \quad k \neq 0,
$$

where

$$
\begin{aligned}
B_{\phi,\alpha,T} :=\ & \frac{1}{2\pi}\left(\tilde{c}_{\alpha,T}\int_{\mathbb{R}}|\widehat{\phi}(z)|\,dz + 2T^2\int_{\mathbb{R}}\ln^\alpha\left(e^\alpha + \frac{|z|}{2}\right)|\widehat{\phi}(z)|\,dz\right. \\
&+\ 2c_{\alpha,T}\int_{\mathbb{R}}|\widehat{\phi'}(z)|\,dz + 4T^2\int_{\mathbb{R}}\ln^\alpha\left(e^\alpha + \frac{|z|}{2}\right)|\widehat{\phi'}(z)|\,dz \\
&+\ 2\int_{\mathbb{R}}\ln^\alpha\left(e^\alpha + \frac{|z|}{2}\right)|\widehat{\phi''}(z)|\,dz\bigg).
\end{aligned}
$$

If $k = 0$, then

$$
|\phi_{00}(t) - \phi_{00}(s)| \leq \frac{1}{2\pi}\int_{\mathbb{R}}\left|e^{itz} - e^{isz}\right||\widehat{\phi}(z)|\,dz \leq \frac{B_{0,\alpha,T}}{\ln^\alpha\left(e^\alpha + \frac{1}{|t-s|}\right)},
$$

where

$$
B_{0,\alpha,T} := \frac{1}{\pi}\int_{\mathbb{R}}\ln^\alpha\left(e^\alpha + \frac{|z|}{2}\right)|\widehat{\phi}(z)|\,dz.
$$

Consequently, we can estimate S as follows

$$
S \leq \frac{C}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{2\alpha(1-\gamma)}}, \quad \alpha > 1/2(1-\gamma),
$$

where

$$
C := 2q_\alpha M\left(\sqrt{c_\phi 2^{\alpha+1}}\,B_{0,\alpha,T} + B_{\phi,\alpha,T}\sum_{\substack{|k|\,\leq\,k_0 \\ k\,\neq\,0}}\frac{M|k|^{\frac{\alpha}{2}} + \sqrt{c_\phi 2^{\alpha+1}}}{k^2}\right)^2.
$$

$\square$

Theorem 6.4. *If the assumptions of Lemmata 6.1, 6.3, and assumptions (i) and (ii) of Theorem 3.3 are satisfied, then the wavelet expansions of the fractional Brownian motion uniformly converge to $W_\alpha(t)$.*

7 Convergence rate in the space $C[0,T]$

Returning now to the general case introduced in Theorem 3.3, let us investigate what happens when the number of terms in the approximants (2.11) becomes large.

First we specify an estimate for the supremum of Gaussian processes.

Definition 7.1. [2, §3.2] *A set $\mathcal{Q} \subset \mathcal{S} \subset \mathbb{R}$ is called an ε-net in the set $\mathcal{S}$ with respect to the semimetric ρ if for any point $x \in \mathcal{S}$ there exists at least one point $y \in \mathcal{Q}$ such that $\rho(x,y) \le \varepsilon$.*

Definition 7.2. [2, §3.2] *Let*

$$H_\rho(\mathcal{S}, \varepsilon) := \begin{cases} \ln(N_\rho(\mathcal{S}, \varepsilon)), & \text{if } N_\rho(\mathcal{S}, \varepsilon) < +\infty; \\ +\infty, & \text{if } N_\rho(\mathcal{S}, \varepsilon) = +\infty, \end{cases}$$

where $N_\rho(\mathcal{S}, \varepsilon)$ is the number of point in a minimal ε-net in the set $\mathcal{S}$.

The function $H_\rho(\mathcal{S}, \varepsilon)$, $\varepsilon > 0$, is called the metric entropy of the set $\mathcal{S}$.

Lemma 7.3. [2, (4.10)] *Let $\mathbf{Y}(t)$, $t \in [0,T]$ be a separable Gaussian random process,*

$$\varepsilon_0 := \sup_{0 \le t \le T} \left(E|\mathbf{Y}(t)|^2 \right)^{1/2} < \infty \,,$$

$$I(\varepsilon_0) := \frac{1}{\sqrt{2}} \int_0^{\varepsilon_0} \sqrt{H(\varepsilon)} \, d\varepsilon < \infty \,, \tag{7.1}$$

where $H(\varepsilon)$ is the metric entropy of the space $([0,T], \rho)$, $\rho(t,s) = (\mathbf{E}|\mathbf{Y}(t) - \mathbf{Y}(s)|^2)^{1/2}$.

Then

$$P \left\{ \sup_{0 \le t \le T} |\mathbf{Y}(t)| > u \right\} \le 2 \exp \left\{ -\frac{\left(u - \sqrt{8uI(\varepsilon_0)} \right)^2}{2\varepsilon_0^2} \right\} \,,$$

where $u > 8I(\varepsilon_0)$.

Assume that there exists a nonnegative monotone nondecreasing in some neighborhood of the origin function $\sigma(\varepsilon)$, $\varepsilon > 0$, such that $\sigma(\varepsilon) \to 0$ when $\varepsilon \to 0$ and

$$\sup_{\substack{|t-s| \le \varepsilon \\ t, s \in [0,T]}} \left(\mathbf{E}|\mathbf{Y}(t) - \mathbf{Y}(s)|^2 \right)^{1/2} \le \sigma(\varepsilon) \,. \tag{7.2}$$

Lemma 7.4. [16] *If*

$$\sigma(\varepsilon) = \frac{C}{\ln^\beta \left(e^\alpha + \frac{1}{\varepsilon} \right)}, \ \beta > 1/2, \ \alpha > 0 \,, \tag{7.3}$$

then (7.1) *holds true and*

$$I(\varepsilon_0) \le \delta(\varepsilon_0) := \frac{\gamma}{\sqrt{2}} \left(\sqrt{\ln(T+1)} + \left(1 - \frac{1}{2\beta} \right)^{-1} \left(\frac{C}{\gamma} \right)^{\frac{1}{2\beta}} \right) \,,$$

where $\gamma := \min \left(\varepsilon_0, \sigma \left(\frac{T}{2} \right) \right)$.

Lemma 7.5. *If a scaling function $\delta(x)$ satisfies assumption $S'(\gamma)$, then*

$$\sup_{|x| \le T} \sum_{|k| \ge k_1} |\delta(x-k)|^\gamma \le \int_{k_1 - T - 1}^{\infty} \Phi^\gamma(t) \, dt + \int_{k_1 - 1}^{\infty} \Phi^\gamma(t) \, dt$$

for $k_1 \ge T + 1$.

Proof. Since

$$\sum_{|k|\geq k_1} |\delta(x-k)|^\gamma \leq \sum_{k\geq k_1} \left(\Phi^\gamma(|x+k|) + \Phi^\gamma(|x-k|)\right) =: z_{k_1}(x),$$

$z_{k_1}(x)$ is an even function.

Then the assertion of the theorem follows from

$$\sup_{|x|\leq T} \sum_{|k|\geq k_1} |\delta(x-k)|^\gamma \leq \sup_{0\leq x\leq T} z_{k_1}(x) \leq \sum_{k\geq k_1} \left(\Phi^\gamma(k-T) + \Phi^\gamma(k)\right)$$

$$\leq \sum_{k\geq k_1} \left(\int_{k-1}^k \Phi^\gamma(t-T)\,dt + \int_{k-1}^k \Phi^\gamma(t)\,dt\right) \leq \int_{k_1-T-1}^\infty \Phi^\gamma(t)\,dt + \int_{k_1-1}^\infty \Phi^\gamma(t)\,dt.$$

$\square$

Now we formulate the main result of this section.

Theorem 7.6. *Let a separable Gaussian random process $\mathbf{X}(t)$, $t\in[0,T]$, the f-wavelet ϕ, and the corresponding m-wavelet ψ satisfy the assumptions of Theorem 3.3.*
Then

$$P\left\{\sup_{0\leq t\leq T} |\mathbf{X}(t) - \mathbf{X}_{n,\mathbf{k}_n}(t)| > u\right\} \leq 2\exp\left\{-\frac{(u - \sqrt{8u\delta(\varepsilon_{\mathbf{k}_n})})^2}{2\varepsilon_{\mathbf{k}_n}^2}\right\},$$

where $u > 8\delta(\varepsilon_{\mathbf{k}_n})$ and the decreasing sequence $\varepsilon_{\mathbf{k}_n}$ is defined by (7.5) in the proof of the theorem.

Proof. Let us verify that $\mathbf{Y}(t) := \mathbf{X}(t) - \mathbf{X}_{n,\mathbf{k}_n}(t)$ satisfies (7.2) with $\sigma(\varepsilon)$ given by (7.3).
First, we observe that

$$\left(\mathbf{E}|\mathbf{Y}(t) - \mathbf{Y}(s)|^2\right)^{1/2} = \left(\mathbf{E}\left|\sum_{|k|>k_0'} \xi_{0k}(\phi_{0k}(t) - \phi_{0k}(s))\right.\right.$$

$$\left.\left.+ \sum_{j=0}^{n-1}\sum_{|k|>k_j} \eta_{jk}(\psi_{jk}(t) - \psi_{jk}(s)) + \sum_{j=n}^\infty \sum_{k\in\mathbb{Z}} \eta_{jk}(\psi_{jk}(t) - \psi_{jk}(s))\right|^2\right)^{1/2}$$

$$\leq \left(\mathbf{E}\left|\sum_{|k|>k_0'} \xi_{0k}(\phi_{0k}(t) - \phi_{0k}(s))\right|^2\right)^{1/2} + \sum_{j=0}^{n-1}\left(\mathbf{E}\left|\sum_{|k|>k_j} \eta_{jk}(\psi_{jk}(t) - \psi_{jk}(s))\right|^2\right)^{1/2}$$

$$+ \sum_{j=n}^\infty \left(\mathbf{E}\left|\sum_{k\in\mathbb{Z}} \eta_{jk}(\psi_{jk}(t) - \psi_{jk}(s))\right|^2\right)^{1/2} := \sqrt{S'} + \sum_{j=0}^{n-1}\sqrt{S_j'} + \sum_{j=n}^\infty \sqrt{R_j'}.$$

We will only show how to handle S_j'. A similar approach can be used to deal with the remaining terms S' and R_j'.

By Lemmata 4.1 and 4.2 we get

$$
\begin{aligned}
S_j' &\leq \sum_{|k|>k_j} \sum_{|l|>k_j} |\mathbf{E}\eta_{jk}\overline{\eta_{jl}}||\psi_{jk}(t)-\psi_{jk}(s)||\psi_{jl}(t)-\psi_{jl}(s)| \\
&\leq c_j \left(\sum_{|k|>k_j} |\psi_{jk}(t)-\psi_{jk}(s)|^\gamma |\psi_{jk}(t)-\psi_{jk}(s)|^{1-\gamma} \right)^2 \\
&\leq c_j \frac{\left(2^{j/2} j^\alpha R_{1\alpha}\right)^{2(1-\gamma)}}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{2\alpha(1-\gamma)}} \left(\sum_{|k|>k_j} |\psi_{jk}(t)-\psi_{jk}(s)|^\gamma \right)^2 \\
&\leq c_j \frac{\left(2^{j/2} j^\alpha R_{1\alpha}\right)^{2(1-\gamma)}}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{2\alpha(1-\gamma)}} \left(2 \sup_{|t|\leq T} \sum_{|k|>k_j} |\psi_{jk}(t)|^\gamma \right)^2 \\
&\leq \frac{c_j 2^{j+2} \left(j^\alpha R_{1\alpha}\right)^{2(1-\gamma)}}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{2\alpha(1-\gamma)}} \left(3\Phi^\gamma(0) + 4 \int_{1/2}^\infty \Phi^\gamma(t)\,dt \right)^2 ,
\end{aligned}
$$

where $R_{1\alpha} = \frac{1}{\pi} \int_{\mathbb{R}} \ln^\alpha\left(e^\alpha + |u| + 1\right) |\widehat{\psi}(u)|\,du$.

Hence

$$
\sum_{j=0}^{n-1} \sqrt{S_j'} \leq \frac{B_\alpha}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{\alpha(1-\gamma)}}, \tag{7.4}
$$

where

$$
B_\alpha := \sum_{j=0}^\infty \sqrt{c_j} 2^{(j+2)/2} \left(j^\alpha R_{1\alpha}\right)^{1-\gamma} \left(3\Phi^\gamma(0) + 4 \int_{1/2}^\infty \Phi^\gamma(t)\,dt \right) < \infty .
$$

From Lemma 4.3, it follows that

$$
\sum_{j=n}^\infty \sqrt{R_j'} \leq \frac{L \cdot \sum_{j=n}^\infty \sqrt{c_j} 2^{\frac{j}{2}} j^{\alpha(1-\gamma)}}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{\alpha(1-\gamma)}},
$$

where

$$
L := 2 R_1^{1-\gamma}(\alpha) \left(3\Phi^\gamma(0) + 4 \int_{1/2}^\infty \Phi^\gamma(t)\,dt \right) .
$$

Similarly to (7.4) by lemma 7.4 we obtain

$$
\sqrt{S'} \leq \frac{b_0 R_{0\alpha}^{1-\gamma}}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{\alpha(1-\gamma)}} \left(\int_{k_0'-T-1}^\infty \Phi^\gamma(t)\,dt + \int_{k_0'-1}^\infty \Phi^\gamma(t)\,dt \right),
$$

where $R_{0\alpha} = \frac{1}{\pi} \int_{\mathbb{R}} \ln^\alpha\left(e^\alpha + \frac{|u|}{2}\right) |\widehat{\phi}(u)|\,du$ and $k_0' \geq T+1$.

Thus,

$$
\begin{aligned}
\left(\mathbf{E}|\mathbf{Y}(t)-\mathbf{Y}(s)|^2\right)^{1/2} &= \left(\mathbf{E}\left|(\mathbf{X}(t)-\mathbf{X}_{n,\mathbf{k}_n}(t)) - (\mathbf{X}(s)-\mathbf{X}_{n,\mathbf{k}_n}(s))\right|^2\right)^{1/2} \\
&\leq \frac{C}{\left(\ln\left(e^\alpha + \frac{1}{|t-s|}\right)\right)^{\alpha(1-\gamma)}} =: \sigma(|t-s|), \quad \alpha(1-\gamma) > 1/2,
\end{aligned}
$$

where

$$C := L \cdot \sum_{j=n}^{\infty} \sqrt{c_j} 2^{\frac{j}{2}} j^{\alpha(1-\gamma)} + B_\alpha + b_0 R_{0\alpha}^{1-\gamma} \left(\int_{k_0'-T-1}^{\infty} \Phi^\gamma(t) \, dt + \int_{k_0'-1}^{\infty} \Phi^\gamma(t) \, dt \right).$$

Then by lemmata 7.3 and 7.4

$$P \left\{ \sup_{0 \le t \le T} |\mathbf{X}(t) - \mathbf{X}_{n,\mathbf{k}_n}(t)| > u \right\} \le 2 \exp \left\{ -\frac{(u - \sqrt{8uI(\tilde{\varepsilon}_{\mathbf{k}_n})})^2}{2\tilde{\varepsilon}_{\mathbf{k}_n}^2} \right\},$$

where

$$\tilde{\varepsilon}_{\mathbf{k}_n} = \sup_{0 \le t \le T} \left(\mathbf{E} |\mathbf{X}(t) - \mathbf{X}_{n,\mathbf{k}_n}(t)|^2 \right)^{1/2}.$$

The proof will be completed by investigating $\tilde{\varepsilon}_{\mathbf{k}_n}$ as a function of $\mathbf{k}_n$.

Note that

$$\left(\mathbf{E} |\mathbf{X}(t) - \mathbf{X}_{n,\mathbf{k}_n}(t)|^2 \right)^{1/2} \le \left(\mathbf{E} \left| \sum_{|k|>k_0'} \xi_{0k} \phi_{0k}(t) \right|^2 \right)^{1/2} +$$

$$+ \sum_{j=0}^{n-1} \left(\mathbf{E} \left| \sum_{|k|>k_j} \eta_{jk} \psi_{jk}(t) \right|^2 \right)^{1/2} + \sum_{j=n}^{\infty} \left(\mathbf{E} \left| \sum_{k \in \mathbb{Z}} \eta_{jk} \psi_{jk}(t) \right|^2 \right)^{1/2}.$$

Let $J := \min \left\{ n, \min\{j \in \mathbb{N}_0 : k_j < 2^j T + 1\} \right\}$. Notice that $J \to \infty$, when $n \to \infty$ and $k_j \to \infty$ for all $j \in \mathbb{N}_0$.

By Lemmata 4.2 and 7.5 for $k_0' \ge T + 1$

$$\tilde{\varepsilon}_{\mathbf{k}_n} \quad \le \quad b_0 \left(\int_{k_0'-T-1}^{\infty} \Phi^\gamma(t) \, dt + \int_{k_0'-1}^{\infty} \Phi^\gamma(t) \, dt \right) + \sum_{j=0}^{J-1} \sqrt{c_j} 2^{\frac{j}{2}} \left(\int_{k_j - 2^j T - 1}^{\infty} \Phi^\gamma(t) \, dt \right.$$

$$+ \quad \left. \int_{k_j-1}^{\infty} \Phi^\gamma(t) \, dt \right) + \sum_{j=J}^{\infty} \sqrt{c_j} 2^{\frac{j}{2}} \left(3\Phi^\gamma(0) + 4 \int_{1/2}^{\infty} \Phi^\gamma(t) \, dt \right) =: \varepsilon_{\mathbf{k}_n}. \qquad (7.5)$$

The choice of J and assumption (3.2) imply that $\varepsilon_{\mathbf{k}_n} \to 0$, when $n \to \infty$, $k_0' \to \infty$, and $k_j \to \infty$ for all $j \in \mathbb{N}_0$.

It is worth noticing that $\delta(\cdot)$ is an increasing function, $I(\varepsilon_0) \le \delta(\varepsilon_0)$ for any ε_0, and $\tilde{\varepsilon}_{\mathbf{k}_n} \le \varepsilon_{\mathbf{k}_n}$. Hence, for $u > 8\delta(\varepsilon_{\mathbf{k}_n})$ we get

$$\exp \left\{ -\frac{\left(u - \sqrt{8uI(\tilde{\varepsilon}_{\mathbf{k}_n})} \right)^2}{2\tilde{\varepsilon}_{\mathbf{k}_n}^2} \right\} \le \exp \left\{ -\frac{\left(u - \sqrt{8u\delta(\varepsilon_{\mathbf{k}_n})} \right)^2}{2\varepsilon_{\mathbf{k}_n}^2} \right\}.$$

Finally, an application of Lemmata 7.3 and 7.4 completes the proof. $\qquad \square$

Remark 7.7. *Note, that $\varepsilon_{\mathbf{k}_n} \to 0$, if and only if n, k_0', and all $k_j, j \ge 0$, approach infinity.*

Remark 7.8. *If $\varepsilon_{\mathbf{k}_n} \to 0$ then $\delta(\varepsilon_{\mathbf{k}_n}) \to 0$. Therefore the convergence in the theorem is exponential with the rate bounded by $2 \exp \left\{ -\dfrac{const}{\varepsilon_{\mathbf{k}_n}^2} \right\}$, where one can choose $const \approx u^2$ for $\delta(\varepsilon_{\mathbf{k}_n}) \ll u$.*

Remark 7.9. *In the theorem we only require that k_0', and all $k_j, j \geq 0$, approach infinity. If we narrow our general class of wavelet expansions $\mathbf{X}_{n,\mathbf{k}_n}(t)$ by specifying rates of growth of the sequences $\mathbf{k}_n$ we can enlarge classes of wavelets bases and random processes in the theorem and obtain explicit rates of convergence by specifying $\varepsilon_{\mathbf{k}_n}$.*

For instance, consider the examples in Section 5. It was shown that $\sqrt{c_j} \leq C/2^{j/2(1+2\varkappa)}$. Let the sequence $\mathbf{k}_n := (k_0', k_0, ..., k_{n-1})$ be chosen so that $\int_{k^*}^{\infty} \Phi^{\gamma}(t)\,dt \leq C/2^{J\varkappa}$ for $k^* := \min(k_0' - T, k_0 - T, k_1 - 2T, ..., k_{J-1} - 2^{J-1}T) - 1$. Then, by (7.5) we get $\varepsilon_{\mathbf{k}_n} \leq C/2^{J\varkappa}$.

Remark 7.10. *Lemma 7.4 and formula (7.5) provide simple expressions to computer $\varepsilon_{\mathbf{k}_n}$ and $\delta(\varepsilon_{\mathbf{k}_n})$. It allows specifying Theorem 7.6 for various stochastic processes and wavelets.*

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