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# An almost sure CLT for stretched polymers

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**Abstract**

We prove an almost sure central limit theorem (CLT) for spatial extension of stretched (meaning subject to a non-zero pulling force) polymers at very weak disorder in all dimensions $d+1 \geq 4$.

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## 1 Introduction and Results

Directed polymers in random media were introduced in [7] as an effective model of Ising interfaces in systems with random impurities. The precise mathematical formulation appeared in the seminal paper [9], which triggered a wave of subsequent investigations. The model of directed polymers can be described as follows. Let $\eta = (\eta_k)_{0\leq k\leq n}$ be a nearest-neighbour path on $\mathbb{Z}^d$ starting at $0$, and let $\gamma = (\gamma_k)_{0\leq k\leq n}$ with $\gamma_k = (k, \eta_k)$ be the corresponding directed path in $\mathbb{Z}^{d+1}$. Let also $\{V(x)\}_{x\in\mathbb{Z}^{d+1}}$ be a collection of i.i.d. random variables with finite exponential moments, whose joint law is denoted by $\mathbb{P}$. One is then interested in the behaviour of the path $\gamma$ under the random probability measure

$$\mu_n^\omega(\gamma) = (Z_{n;\beta}^\omega)^{-1} \exp\bigl(-\beta \sum_{k=1}^n V(\gamma_k)\bigr)\,(2d)^{-n},$$

where $\beta \geq 0$ is the inverse temperature. The behaviour of the path $\gamma$ is closely related to the behaviour of the partition function $Z_{n;\beta}^\omega$. Namely, one distinguishes between two regimes: the weak disorder regime, in which $\lim_{n\to\infty} Z_{n;\beta}^\omega/\mathbb{E}(Z_{n;\beta}^\omega) > 0$, $\mathbb{P}$-a.s., and the strong disorder regime, in which this limit is zero. It is known [2] that there is a sharp transition between these two regimes at an inverse temperature $\beta_c$ which is non-trivial when $d \geq 3$. In the weak disorder regime ($\beta < \beta_c$), the path $\gamma$ behaves diffusively, in that $\gamma_n$ satisfies a CLT. Diffusivity at sufficiently small values of $\beta$ was first established in [9]; this was extended to an almost-sure CLT in [1]; a CLT (in probability) valid in the whole weak disorder regime was then obtained in [2].

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In dimensions $d \geq 3$ the sequence $Z^{\omega}_{n;\beta}/\mathbb{E}(Z^{\omega}_{n;\beta})$ is bounded in $\mathbb{L}_2$ for all sufficiently small values of $\beta$. In such a situation local limit versions of the CLT, which hold in probability, were established in [16, 18].

In the case of directed polymers the disorder is always strong in dimensions $d = 1, 2$ [3, 14] and at sufficiently low temperatures. Concerning the (nondiffusive) behaviour in the strong disorder regime, we refer the reader to [4] and references therein.

In this work, we consider diffusive behaviour in dimensions $d+1 \geq 4$ for the related models of *stretched polymers*. The choice of notation $d+1$ indicates that stretched polymers on $\mathbb{Z}^{d+1}$ should be compared with directed polymers in $d$ dimensions. However, a stretched path $\gamma$ can be any nearest-neighbour path on $\mathbb{Z}^{d+1}$, which is permitted to bend and to return to particular vertices an arbitrary number of times. The disorder is modelled by a collection $\{V(x)\}_{x\in\mathbb{Z}^{d+1}}$ of i.i.d. non-negative random variables. Each visit of the path to a vertex $x$ exerts the price $\mathrm{e}^{-\beta V(x)}$. The *stretch* is introduced in one of the following two natural ways:

- The path $\gamma$ starts at $0$ and ends at a hyperplane at distance $n$ from $0$ and has arbitrary length. This is a model of crossing random walks in random potentials. In dimension $d + 1 = 2$, it presumably provides a better approximation to Ising interfaces in the presence of random impurities.
- The path $\gamma$ has a fixed length $n$, but it is subject to a drift, which can be interpreted physically as the effect of a force acting on the polymer's free end.

The precise model is described below. At this stage let us remark that models of stretched polymers have a richer morphology than models of directed polymers. Even the issue of ballistic behaviour for annealed models is non-trivial [10, 8, 13]. The issue of ballistic behaviour in the quenched case is still not resolved completely, and, in order to ensure ballisticity one needs to assume that the random potential $V$ is strictly positive in the crossing case, and that the applied drift is sufficiently large in the fixed length case. Both conditions are designed to ensure a somewhat massive nature of the model.

As in the directed case, the disorder is always strong [21] in low dimensions $d + 1 = 2, 3$ or at sufficiently low temperatures.

In the case of higher dimensions $d+1 \geq 4$, the existence of weak disorder on the level of equality between quenched and annealed free energies was established in [6, 20]. The case of high temperature discrete Wiener sausage with drift was addressed in [17].

In the crossing case, a CLT in probability was established in [11] in all dimensions $d + 1 \geq 4$ at sufficiently high temperatures.

The aim of the present paper is to establish an almost-sure CLT for the endpoint of the fixed-length version of the model of stretched polymers with non-zero drifts, also at sufficiently high temperatures and in all dimensions $d + 1 \geq 4$.

### 1.1 Class of Models

**Polymers.** For the purpose of this paper, a polymer $\gamma = (\gamma_0, \ldots, \gamma_n)$ is a nearest-neighbour trajectory on the integer lattice $\mathbb{Z}^{d+1}$. Unless stressed otherwise, $\gamma_0$ is always placed at the origin. The length of the polymer is $|\gamma| \triangleq n$ and its spatial extension is $X(\gamma) \triangleq \gamma_n - \gamma_0$. In the most general case, neither the length nor the spatial extension are fixed.

**Random Environment.** The random environment is a collection $\{V(x)\}_{x\in\mathbb{Z}^{d+1}}$ of non-degenerate non-negative i.i.d. random variables which are normalised by $0 \in \mathrm{supp}(V)$. There is no moment assumptions on $V$. The case of traps, $p_\infty \triangleq \mathbb{P}(V = \infty) > 0$, is not excluded, but then we shall assume that $p_\infty$ is small enough. In particular, we shall assume that $\mathbb{P}$-a.s. there is an infinite connected cluster $\mathrm{Cl}_\infty(V)$ of the set $\{x : V(x) < \infty\}$ in $\mathbb{Z}^{d+1}$. In fact, we shall assume more: Given $\mathbb{R}^{d+1} \ni h \neq 0$ and a number $\delta \in (0, \frac{1}{\sqrt{d+1}})$, define the positive cone

$$\mathcal{Y}_\delta^h \triangleq \left\{x \in \mathbb{R}^{d+1} \,:\, x \cdot h \geq \delta\, |x|\, |h|\right\}. \tag{1.1}$$

By construction, the cones $\mathcal{Y}_\delta^h$ always contain at least one lattice direction $\pm \mathsf{e}_i,\ i = 1, \ldots, d+1$. We assume that it is possible to choose $\delta$ in such a fashion that, for any $h$, the intersection $\mathrm{Cl}_\infty^{h,\delta}(V) \triangleq \mathrm{Cl}_\infty(V) \cap \mathcal{Y}_\delta^h$ contains ($\mathbb{P}$-a.s.) an infinite connected component. For the rest of the paper, we fix such a $\delta \in (0, \frac{1}{\sqrt{d+1}})$ and use the reduced notation $\mathcal{Y}^h$ and $\mathrm{Cl}_\infty^h(V)$ for the corresponding cones (1.1) and percolation clusters.

**Weights and Path Measures.** The reference measure $\mathsf{p}(\gamma) \triangleq (2(d+1))^{-|\gamma|}$ is given by simple random walk weights. The polymer weights we are going to consider are quantified by two parameters: the inverse temperature $\beta \geq 0$ and the external pulling force $h \in \mathbb{R}^{d+1}$.
The random quenched weights are given by

$$q_{h,\beta}^\omega(\gamma) \triangleq \exp\Big\{h \cdot X(\gamma) - \beta \sum_1^{|\gamma|} V(\gamma_i)\Big\} \mathsf{p}(\gamma). \tag{1.2}$$

The corresponding deterministic annealed weights are given by

$$q_{h,\beta}(\gamma) \triangleq \mathbb{E} q_{h,\beta}^\omega(\gamma) = \exp\left\{h \cdot X(\gamma) - \Phi_\beta(\gamma)\right\} \mathsf{p}(\gamma), \tag{1.3}$$

where $\Phi_\beta(\gamma) \triangleq \sum_x \phi_\beta\big(\ell_\gamma(x)\big)$, with $\ell_\gamma(x)$ denoting the local time (number of visits) of $\gamma$ at $x$, and

$$\phi_\beta(\ell) \triangleq -\log \mathbb{E} \mathrm{e}^{-\beta \ell V}. \tag{1.4}$$

Note that the annealed potential is positive, non-decreasing and attractive, in the sense that

$$0 < \phi_\beta(\ell) \leq \phi_\beta(\ell + m) \leq \phi_\beta(\ell) + \phi_\beta(m), \qquad \forall\, \ell, m \in \mathbb{N}. \tag{1.5}$$

In the sequel, we shall drop the index $\beta$ from the notation, and we shall drop the index $h$ whenever it equals zero. With this convention, the quenched partition functions are defined by

$$Q_n^\omega(x) \triangleq \sum_{\substack{X(\gamma)=x \\ |\gamma|=n}} q^\omega(\gamma), \quad Q_n^\omega(h) \triangleq \sum_{|\gamma|=n} q_h^\omega(\gamma) = \sum_x \mathrm{e}^{h\cdot x} Q_n^\omega(x), \tag{1.6}$$

and we use $Q_n(x) \triangleq \mathbb{E} Q_n^\omega(x)$ and $Q_n(h) \triangleq \mathbb{E} Q_n^\omega(h)$ to denote their annealed counterparts.

Finally, we define the corresponding quenched and annealed path measures by

$$\mathbb{Q}_{n,h}^\omega(\gamma) \triangleq \mathbb{1}_{\{|\gamma|=n\}} \frac{q_h^\omega(\gamma)}{Q_n^\omega(h)} \quad \text{and} \quad \mathbb{Q}_{n,h}(\gamma) \triangleq \mathbb{1}_{\{|\gamma|=n\}} \frac{q_h(\gamma)}{Q_n(h)}. \tag{1.7}$$

**Very Weak Disorder.** The notion of very weak disorder is technical and it depends on the strength $|h|$ of the pulling force , dimension $d \geq 3$ and the distribution of $V$. By Lemma 2.1 below, there exists a function $\zeta_d$ on $(0, \infty)$ such that a certain $\mathbb{L}_2$-estimate (2.4) holds if $\phi_\beta(1) < \zeta_d(|h|)$.

**Definition 1.1.** *The model of stretched polymers is in the regime of very weak disorder if $d \geq 3$ and*

$$\phi_\beta(1) < \zeta_d(|h|). \tag{1.8}$$

### 1.2 The Result

Fix $h \neq 0$. Then [19, 5, 10]

$$\lambda = \lambda(\beta, h) \triangleq \lim_{n\to\infty} \frac{1}{n} \log Q_n(h) \in (0, \infty), \tag{1.9}$$

for all sufficiently small $\beta$. The following two quantities play a central role in our limit theorems:

$$v = v(h, \beta) \triangleq \nabla\lambda(h), \qquad \Sigma \triangleq \mathrm{Hess}[\lambda](h).$$

If $\beta$ is sufficiently small then $v \neq 0$ and the matrix $\Sigma$ is positive definite and, moreover, $v$ and $\Sigma$ are the limiting spatial extension and, respectively, the diffusivity matrix for the annealed model. (Sections 4.1,4.2 in [10]). In Subsection 2.1 we recall further relevant facts about the annealed model.

**Theorem A.** *Fix $h \neq 0$. Then, in the regime of very weak disorder, the following holds $\mathbb{P}$-a.s. on the event $\{0 \in \mathrm{Cl}_\infty(V)\}$:*

- *The limit*
$$\lim_{n\to\infty} \frac{Q_n^\omega(h)}{Q_n(h)} \tag{1.10}$$
*exists and is a strictly positive, square-integrable random variable.*
- *There exists a sequence $\{\epsilon_n\}$ with $\lim \epsilon_n = 0$, such that*
$$\sum_n \mathbb{Q}_{n,h}^\omega\Big(\big|\frac{X(\gamma)}{n} - v\big| > \epsilon_n\Big) < \infty. \tag{1.11}$$
- *For every $\alpha \in \mathbb{R}^{d+1}$,*
$$\lim_{n\to\infty} \mathbb{Q}_{n,h}^\omega\Big(\exp\big\{\frac{i\alpha}{\sqrt{n}}(X(\gamma) - nv)\big\}\Big) = \exp\big\{-\tfrac{1}{2}\Sigma\alpha\cdot\alpha\big\}. \tag{1.12}$$

We would like to stress that, in contrast to the case of directed polymers [2], our CLT does not pertain to the whole of the weak disorder region. The procedure of first fixing $h \neq 0$ and then going to $\beta > 0$ sufficiently small is essential. Furthermore, even in the regime we are working with, (1.12) falls short of the local CLT form of results as developed for directed polymers in [18]. These and related issues remain open in the context of stretched polymers.

Few remarks on the history of the problem: Flury [6] had established that under the conditions of Theorem A (and some additional moment assumptions of the potential $V$)

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{Q_n^\omega(h)}{Q_n(h)} = 0 \tag{1.13}$$

for on-axis exterior forces $h$. (1.13) was then extended to arbitrary directions $h \in \mathbb{R}^{d+1}$ by Zygouras [20]. In [6], the analysis was carried out directly in the canonical ensemble of polymers with fixed length $n$. In [20], the author derives results for the conjugate ensemble of the so-called crossing random walks.

Large deviations (LD) under both $\mathbb{Q}_{n,h}$ and $\mathbb{Q}_{n,h}^\omega$ were investigated in [19, 5]. The results therein imply that, under the conditions of Theorem A, the model is ballistic in the sense that the value of the quenched rate function at zero is strictly positive. However, [19, 5] do not imply a law of large numbers (LLN) even in the annealed case. In particular, these works do not contain information on the strict convexity of the corresponding rate functions. The issue of strict convexity for the annealed rate functions was settled in [10]. Therefore, (1.11) is a direct consequence of (1.13) and of the analysis of annealed canonical measures in [10].

The main new results of this work are (1.10) and (1.12). A version of Theorem A for the ensemble of crossing random walks appears in [11]. The length of crossing random walks is not fixed (only suppressed by an additional positive mass), and they are required to have their second endpoint on a distant hyperplane. In this way, crossing random walks in random potential are much more "martingale"-like than canonical random walks. Moreover, the canonical constraint of fixed length does not facilitate computations, to say the least. Finally, the CLT of [11] was only established in probability and not $\mathbb{P}$-a.s. Thus, although the techniques developed in [11] are useful here, they certainly do not imply the claims of Theorem A, and an alternative approach was required.

### 1.3 Irreducible Decomposition, Basic Ensembles and Basic Partition Functions

A polymer $\gamma = (\gamma_0, \ldots, \gamma_n)$ is said to be cone-confined if

$$\gamma \subset \left(\gamma_0 + \mathcal{Y}^h\right) \cap \left(\gamma_n - \mathcal{Y}^h\right). \tag{1.14}$$

A cone-confined polymer which cannot be represented as the concatenation of two (non-singleton) cone-confined polymers is said to be irreducible. We denote by $\mathcal{T}(x)$ the collection of all cone-confined paths leading from $0$ to $x$, and by $\mathcal{F}(x) \subset \mathcal{T}(x)$ the set of irreducible cone-confined paths. In the sequel we shall refer to $\mathcal{F}(x)$ and $\mathcal{T}(x)$ as to basic ensembles. The basic partition functions are defined by

$$t^\omega_{x,n} \triangleq \mathrm{e}^{-\lambda n} \sum_{\gamma \in \mathcal{T}(x)} \mathbb{1}_{\{|\gamma|=n\}} q^\omega_h(\gamma) \quad \text{and} \quad f^\omega_{x,n} \triangleq \mathrm{e}^{-\lambda n} \sum_{\gamma \in \mathcal{F}(x)} \mathbb{1}_{\{|\gamma|=n\}} q^\omega_h(\gamma). \tag{1.15}$$

We also set, accordingly, $t^\omega_n \triangleq \sum_x t^\omega_{x,n}$ and $f^\omega_n \triangleq \sum_x f^\omega_{x,n}$. The annealed counterparts of all these quantities are denoted by $\mathbf{t}_{x,n} \triangleq \mathbb{E} t^\omega_{x,n}$, $\mathbf{f}_{x,n} \triangleq \mathbb{E} f^\omega_{x,n}$, $\mathbf{t}_n \triangleq \mathbb{E} t^\omega_n$ and $\mathbf{f}_n \triangleq \mathbb{E} f^\omega_n$. As shown in Section 3.6 of [10], the collection $\{\mathbf{f}_{x,n}\}$ forms a probability distribution,

$$\sum_n \sum_x \mathbf{f}_{x,n} = \sum_n \mathbf{f}_n = 1,$$

with exponentially decaying tails:

$$\sum_{m \geq n} \mathbf{f}_m \triangleq \sum_{m \geq n} \sum_x \mathbf{f}_{x,m} \leq \mathrm{e}^{-\nu n}, \tag{1.16}$$

where $\nu = \nu(\beta, h) \to \infty$ as $\beta$ becomes large, and $\inf_{\beta \geq 0} \nu(\beta, h) > 0$, for all $h \neq 0$.

**Remark 1.2.** *Since by definition polymers are nearest neighbour paths, it always holds that* $\mathbf{t}_{x,n} = \mathbf{t}_{x,n} \mathbb{1}_{\{|x| \leq n\}}$.

As in [11, Subsections 2.7 and 3.5], the following statement about basic ensembles implies the claims (1.10) and (1.12) of Theorem A:

**Theorem B.** *Fix* $h \neq 0$. *Then, in the regime of very weak disorder, the following holds* $\mathbb{P}$*-a.s. on the event* $\left\{0 \in \mathrm{Cl}^h_\infty(V)\right\}$:

- *The limit*

$$s^\omega \triangleq \lim_{n \to \infty} \frac{t^\omega_n}{\mathbf{t}_n} \tag{1.17}$$

*exists and is a strictly positive, square-integrable random variable.*

- *For every* $\alpha \in \mathbb{R}^{d+1}$,

$$\lim_{n \to \infty} \frac{1}{t^\omega_n} \sum_x \exp\left\{\frac{i\alpha}{\sqrt{n}} \cdot (x - nv)\right\} t^\omega_{x,n} = \exp\left\{-\tfrac{1}{2} \Sigma\alpha \cdot \alpha\right\}. \tag{1.18}$$

For the rest of the paper, we shall focus on the proof of Theorem B.

## 2 Proof of Theorem B

To facilitate the exposition, we shall consider the case of on-axis external force $h = h\mathrm{e}_1$. The proof, however, readily applies for any non-zero $h \in \mathbb{R}^{d+1}$. By lattice symmetries, the mean displacement $v = \nabla\lambda(h)$ lies along the direction $\mathrm{e}_1$; $v = v\mathrm{e}_1$. As it was already mentioned in the beginning of Subsection 1.2, $v \neq 0$ whenever $\beta$ is small enough. We proceed assuming that both the drift and the speed are positive $h, v > 0$.

### 2.1 Three Main Inputs

The reduction to basic ensembles constitutes the central step of the Ornstein-Zernike theory. We rely on three facts: The first is the refined description of the annealed phase in the ballistic regime (which, in our regime, will always correspond to first fixing $h \neq 0$ and then choosing $\beta > 0$ small enough). Below, we shall summarize the required results from [10, 12]. The second is an $\mathbb{L}_2$-type estimate on overlaps which holds for all $\beta$ sufficiently small, and which could be understood as quantifying the notion of very weak disorder we employ here. The third is a maximal inequality for the so-called *mixingales*, due to McLeish. Unlike directed polymers, stretched polymers do not possess natural martingale structures, and McLeish's result happens to provide a convenient alternative framework.

**Ornstein-Zernike theory of annealed models.** Annealed asymptotics of $\mathbf{t}_n$ in the ballistic regime are not related to the strength of disorder and hold for all values of $\beta \geq 0$ and appropriately large drifts $h$ . In particular, for each $h \neq 0$ fixed, the annealed model is ballistic for all sufficiently small $\beta$. We refer to [10, Sections 4.1 and 4.2] and to [12, Section 4.2] for the proof of the following: Fix $h \neq 0$; then, for all $\beta > 0$ small enough, $\lambda(h) > 0$, $\nabla\lambda(h) \neq 0$ and $\mathrm{Hess}[\lambda](h)$ is positive definite. Furthermore, there exist a small complex neighbourhood $\mathcal{U} \subset \mathbb{C}^{d+1}$ of the origin, an analytic function $\mu$ (with $\mu(0) = 0$) on $\mathcal{U}$ and a non-vanishing analytic function $\kappa \neq 0$ on $\mathcal{U}$ such that:

$$\lim_{n\to\infty} \mathrm{e}^{-n\mu(z)}\mathbf{t}_n(z) \stackrel{\Delta}{=} \lim_{n\to\infty} \mathrm{e}^{-n\mu(z)} \sum_x \mathbf{t}_{x,n}\mathrm{e}^{z\cdot x} = \frac{1}{\kappa(z)}, \tag{2.1}$$

uniformly exponentially fast on $\mathcal{U}$. Note [12] (Section 4.2) that $\lambda(h+z) = \lambda(h) + \mu(z)$ for real $z$, and thus $v = \nabla\lambda(h) = \nabla\mu(0)$ and $\Sigma = \mathrm{Hess}[\lambda](h) = \mathrm{Hess}[\mu](0)$.

The annealed model satisfies a local LD upper bound: There exists $c = c(\beta, h) > 0$ such that, for all $x \in \mathcal{Y}^h$,

$$\mathbf{t}_{x,n} \leq \frac{1}{c\sqrt{n^{d+1}}} \exp\Big\{-c\frac{|x-nv|^2}{n}\Big\}. \tag{2.2}$$

In view of Remark 1.2 the above bound is trivial whenever $|x| > n$.

Finally, it is a straightforward consequence of (2.1) that the following annealed CLT holds:

$$\mathbf{S}_n\big(\frac{\alpha}{\sqrt{n}}\big) \stackrel{\Delta}{=} \sum_x \mathbf{t}_{x,n} \exp\big\{i\frac{\alpha}{\sqrt{n}}\cdot(x-nv)\big\} = \frac{1}{\kappa(0)} \exp\big\{-\tfrac{1}{2}\Sigma\,\alpha\cdot\alpha\big\}\big(1+O(n^{-1/2})\big), \tag{2.3}$$

with the second asymptotic equality holding uniformly in $\alpha$ on compact subsets of $\mathbb{R}^{d+1}$.

**An $\mathbb{L}_2$-estimate.** Fix an external force $h \neq 0$. We continue to employ notation $v = v(h, \beta)$. For a subset $A \subseteq \mathbb{Z}^{d+1}$, let $\mathcal{A}$ be the $\sigma$-algebra generated by $\{V(x)\}_{x\in A}$. We shall call such $\sigma$-algebras cylindrical.

**Lemma 2.1.** *For any dimension* $d \geq 3$ *there exist a positive non-decreasing function* $\zeta_d$ *on* $(0, \infty)$ *and a number* $\rho < 1/12$ *such that the following holds: If* $\phi_\beta(1) < \zeta_d(|h|)$*, then there exist constants* $c_1, c_2 < \infty$ *such that the random weights* (1.15) *satisfy:*

$$\begin{aligned} &\left| \mathbb{E}\big[ t^\omega_{x,\ell} t^\omega_{x',\ell} \mathbb{E}(f^{\theta_x \omega}_{y,m} - \mathbf{f}_{y,m} \,|\, \mathcal{A}) \mathbb{E}(f^{\theta_{x'}\omega}_{y',m'} - \mathbf{f}_{y',m'} \,|\, \mathcal{A}) \big] \right| \\ &\quad \leq \frac{c_1 \mathrm{e}^{-c_2(m+m')}}{\ell^{d+1-\rho}} \exp\Big\{ -c_2 \Big( |x - x'| + \frac{|x - \ell v|^2}{\ell} + \frac{|x' - \ell v|^2}{\ell} \Big) \Big\}, \end{aligned} \tag{2.4}$$

*for all* $x, x', m, m', y, y', \ell$ *and all cylindrical* $\sigma$*-algebras* $\mathcal{A}$ *such that both* $t^\omega_{x,\ell}$ *and* $t^\omega_{x',\ell}$ *are* $\mathcal{A}$*-measurable.*

**Remark 2.2.** *The above bound is non-trivial only if both* $|x|, |x'| \leq \ell$ *(Remark 1.2). Also, there is nothing sacred about the condition* $\rho < 1/12$*. We just need* $\rho$ *to be sufficiently small. In fact,* (2.4) *holds with* $\rho = 0$*, although a proof of such statement would be a bit more involved.*

In spite of its technical appearance, (2.4) has a transparent intuitive meaning: For $\rho = 0$, the expressions on the right-hand side are just local limit bounds for a couple of independent annealed polymers with exponential penalty for disagreement at their end-points. The irreducible terms have exponential decay. In the very weak disorder regime, the interaction between polymers does not destroy these asymptotics. The proof of Lemma 2.1 is relegated to the concluding Section 4.

**McLeish's Maximal Inequality.** Let $Z_1, Z_2, \ldots$ be a sequence of zero-mean, square-integrable random variables. Let also $\{\mathcal{A}_k\}_{-\infty}^{\infty}$ be a filtration of $\sigma$-algebras. Suppose that we have chosen $\epsilon > 0$ and numbers $\psi_1, \psi_2, \ldots$ in such a way that

$$\mathbb{E}\big[\mathbb{E}(Z_\ell \,|\, \mathcal{A}_{\ell-k})^2\big] \leq \frac{\psi_\ell^2}{(1+k)^{1+\epsilon}} \quad \text{and} \quad \mathbb{E}\big[\big(Z_\ell - \mathbb{E}(Z_\ell \,|\, \mathcal{A}_{\ell+k})\big)^2\big] \leq \frac{\psi_\ell^2}{(1+k)^{1+\epsilon}} \tag{2.5}$$

for all $\ell = 1, 2, \ldots$ and $k \geq 0$. Then [15] there exists $K = K(\epsilon) < \infty$ such that, for all $n_1 \leq n_2$,

$$\mathbb{E}\Big[\max_{n_1 \leq r \leq n_2} \Big(\sum_{n_1}^{r} Z_\ell\Big)^2\Big] \leq K \sum_{n_1}^{n_2} \psi_\ell^2. \tag{2.6}$$

**Remark 2.3.** *In particular, if* $\sum_\ell \psi_\ell^2 < \infty$*, then* $\sum_\ell Z_\ell$ *converges* $\mathbb{P}$*-a.s. and in* $\mathbb{L}_2$*.*

In the sequel, we shall always work with the following filtration $\{\mathcal{A}_m\}$. Recall that we are discussing on-axis positive drifts $h = h\mathsf{e}_1$ which, for small $\beta$, give rise to on-axis limiting spatial extension $v = v\mathsf{e}_1$ with $v > 0$. At this stage, define the hyperplanes $\mathcal{H}_m^-$ and the corresponding $\sigma$-algebras $\mathcal{A}_m$ as

$$\mathcal{H}_m^- = \big\{x \in \mathbb{Z}^{d+1} \,:\, x \cdot \mathsf{e}_1 \leq m|v|\big\} \quad \text{and} \quad \mathcal{A}_m = \sigma\left\{V(x) \,:\, x \in \mathcal{H}_m^-\right\}. \tag{2.7}$$

**Notation for asymptotic relations.** The following notation is convenient, and we shall use it throughout the text: Given a (countable) set of indices $\mathcal{I}$ and two positive sequences $\{a_\alpha, b_\alpha\}_{\alpha \in \mathcal{I}}$, we say that $a_\alpha \lesssim b_\alpha$ if there exists a constant $c > 0$ such that $a_\alpha \leq c b_\alpha$ for all $\alpha \in \mathcal{I}$ . We shall use $a_\alpha \cong b_\alpha$ if both $a_\alpha \lesssim b_\alpha$ and $a_\alpha \gtrsim b_\alpha$ hold. For instance, for any $\epsilon > 0$ fixed,

$$\frac{\mathrm{e}^{-c_3 k^2/\ell}}{\ell^{(1+\epsilon)/2}} \lesssim \frac{1}{(1+k)^{1+\epsilon}}, \tag{2.8}$$

where the index set $\mathcal{I}$ is the set of pairs of integers $(k, \ell)$ with $k \geq 0$ and $\ell > 0$.

**Structure of upper bounds.** Our upper bounds are based on (2.8), (2.4) (applied with $\rho = \epsilon/2$) and on (2.6). Recall that $\rho < 1/12$, and hence $\epsilon < 1/6$.
In the sequel, we shall repeatedly derive variance bounds on quantities of the type $\sum_{\ell \leq n} Z_\ell^{(n)}$. The most general form of $Z_\ell^{(n)}$ we shall consider is

$$Z_\ell^{(n)} = \sum_x t_{x,\ell}^\omega \sum_{y,m} a_{x,\ell}^{(n)}(y,m) \left( f_{y,m}^{\theta_x \omega} - \mathbf{f}_{y,m} \right), \tag{2.9}$$

where $\left\{ a_{x,\ell}^{(n)}(y,m) \right\}$ are arrays of real or complex numbers. Assume that there exists another family of (non-negative) arrays $\left\{ \hat{a}_{x,\ell}^{(n)} \right\}$ and a number $\nu > 0$ such that

$$\mathrm{e}^{-c_2 m} \sum_{|y| \leq m} \left| a_{x,\ell}^{(n)}(y,m) \right| \lesssim \mathrm{e}^{-\nu m} \hat{a}_{x,\ell}^{(n)}, \tag{2.10}$$

where the constant $c_2$ is inherited from (2.4).

**Lemma 2.4.** *Set $\epsilon = \frac{\rho}{2}$, where $\rho$ is the power which shows up in* (2.4)*. Under assumption* (2.10)

$$\mathbb{E}\left[ \left( \mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell-k}) \right)^2 \right] \lesssim \frac{1}{\ell^{d+1-\epsilon/2}} \sum_{x \in \mathcal{H}_{\ell-k}^-} \mathrm{e}^{-c_2 \frac{|x-\ell v|^2}{\ell}} \left( \hat{a}_{x,\ell}^{(n)} \right)^2, \tag{2.11}$$

*and*

$$\mathbb{E}\left[ \left( Z_\ell^{(n)} - \mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell+k}) \right)^2 \right] \lesssim \frac{1}{\ell^{d+1-\epsilon/2}} \sum_x \mathrm{e}^{-c_2 \frac{|x-\ell v|^2}{\ell} - \nu \mathrm{d}_{\ell+k}(x)} \left( \hat{a}_{x,\ell}^{(n)} \right)^2. \tag{2.12}$$

*Above we introduced a provisional notation $\mathrm{d}_r(x) \triangleq (r\,|v| - \mathsf{e}_1 \cdot x) \vee 0$ for the distance from $x$ to $\mathcal{H}_r^+ = \mathbb{Z}^{d+1} \setminus \mathcal{H}_r^-$.*

*Proof.* Since $\mathbb{E}\left( t_{x,\ell}^\omega \left( f_{y,m}^{\theta_x\omega} - \mathbf{f}_{y,m} \right) | \, \mathcal{A}_{\ell-k} \right) = 0$ whever $x \notin \mathcal{H}_{\ell-k}^-$,

$$\mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell-k}) = \sum_{x \in \mathcal{H}_{\ell-k}^-} t_{x,\ell}^\omega \sum_{y,m} a_{x,\ell}^{(n)}(y,m)\, \mathbb{E}(f_{y,m}^{\theta_x\omega} - \mathbf{f}_{y,m} \,|\, \mathcal{A}_{\ell-k}). \tag{2.13}$$

Taking the expectation of the square of the latter expression and, for each $x, x'$, factorizing replicas using $|ab| \leq \frac{a^2+b^2}{2}$, one derives the first inequality (2.11) directly from (2.4) and (2.10).
Next,

$$\begin{aligned} Z_\ell^{(n)} - \mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell+k}) &= \sum_{x \in \mathcal{H}_{\ell+k}^+} t_{x,\ell}^\omega \sum_{y,m} a_{x,\ell}^{(n)}(y,m) \left( f_{y,m}^{\theta_x\omega} - \mathbf{f}_{y,m} \right) \\ &+ \sum_{x \in \mathcal{H}_{\ell+k}^-} t_{x,\ell}^\omega \sum_{z \in \mathcal{H}_{\ell+k}^+} \sum_m a_{x,\ell}^{(n)}(z-x,m) \left( f_{z-x,m}^{\theta_x\omega} - \mathbb{E}(f_{z-x,m}^{\theta_x\omega} \,|\, \mathcal{A}_{\ell+k}) \right). \end{aligned} \tag{2.14}$$

For any $x \in \mathcal{H}_{\ell+k}^+$, $\mathrm{d}_{\ell+k}(x) = 0$, and the first term in (2.14) has exactly the same structure as the right-hand side of (2.13). On the other hand, if $x \in \mathcal{H}_{\ell+k}^-$ and $z \in \mathcal{H}_{\ell+k}^+$, then, in view of Remark 1.2, $f_{z-x,m}^{\theta_x\omega}$ can be different from zero only if $m \geq \mathrm{d}_{\ell+k}(x)$ and $|z-x| \leq m$. Therefore, (2.12) is also a direct consequence of (2.4) and (2.10). ☐

The following is a useful corollary:

**Lemma 2.5.** *If $\hat{a}_{x,\ell}^{(n)} \lesssim \hat{a}_\ell^{(n)}$, then the bounds* (2.11) *and* (2.12) *reduce to*

$$\mathbb{E}\left[ \left( \mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell-k}) \right)^2 \right],\ \mathbb{E}\left[ \left( Z_\ell^{(n)} - \mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell+k}) \right)^2 \right] \lesssim \left( \hat{a}_\ell^{(n)} \right)^2 \frac{1}{\ell^{d/2-\epsilon}(1+k)^{1+\epsilon}}. \tag{2.15}$$

*Proof.* Consider first the right-hand side of (2.11). Since $\sum_{x\in\mathcal{H}^-_{\ell-k}} \mathrm{e}^{-c_2\frac{|x-\ell v|^2}{\ell}} \lesssim \ell^{\frac{d+1}{2}}$, the non-trivial part is to check (2.15) for large values of $k$. In the latter case, we may assume that $|x - v\ell| > \frac{k|v|}{2}$ for all $x \in \mathcal{H}^-_{\ell-k}$. Consequently, the sum on the right-hand side of (2.11) is bounded above by

$$\begin{aligned} &\sum_{x\in\mathcal{H}^-_{\ell-k}} \mathrm{e}^{-c_2|x-v\ell|^2/\ell} \lesssim \int_{|y|>\frac{k|v|}{2}} \mathrm{e}^{-c_2|y|^2/\ell}\mathrm{d}y \\ &\cong \int_{\frac{k|v|}{2}}^{\infty} r^d \mathrm{e}^{-c_2 r^2/\ell}\mathrm{d}r \lesssim \ell^{(d+1)/2}\mathrm{e}^{-c_3k^2/\ell} \lesssim \frac{\ell^{d/2+1+\epsilon/2}}{(1+k)^{1+\epsilon}}, \end{aligned} \tag{2.16}$$

the last inequality being an application of (2.8). (2.15) follows.
Turning to the right-hand side of (2.12), we see that it remains to derive an upper bound on

$$\sum_{x\in\mathcal{H}^-_{\ell+k}} \mathrm{e}^{-c_2\frac{|x-\ell v|^2}{\ell} - \nu \mathrm{d}_{\ell+k}(x)} \lesssim \sum_{|y|>\frac{k|v|}{2}} \mathrm{e}^{-c_2\frac{|y|^2}{\ell}} + \sum_{|y|\leq\frac{k|v|}{2}} \mathrm{e}^{-\nu \mathrm{d}_k(y)}. \tag{2.17}$$

The first sum above is treated as in (2.16). On the other hand, the second sum is bounded above as $\lesssim \mathrm{e}^{-\nu' k}$, uniformly in all $k$ sufficiently large. Since $\mathrm{e}^{-\nu' k} \lesssim (1+k)^{-1-\epsilon}$, the bound (2.15) for $\mathbb{E}\big[\big(Z^{(n)}_\ell - \mathbb{E}(Z^{(n)}_\ell \,|\, \mathcal{A}_{\ell+k})\big)^2\big]$ follows as well. □

As an application of (2.15) we derive the following convergence result:

**Lemma 2.6.** *Assume that, for some $\nu' > 0$, the asymptotic bound* (2.15) *is, uniformly in $n$ and $\ell \leq n$, satisfied with $\hat{a}^{(n)}_\ell \lesssim \mathrm{e}^{-\nu'(n-\ell)}$. Then*

$$\lim_{n\to\infty} \sum_{\ell\leq n} Z^{(n)}_\ell = 0, \tag{2.18}$$

*$\mathbb{P}$-a.s. and in $\mathbb{L}_2$. In particular, assume that the asymptotic bound* (2.10) *is satisfied for an array $\{b^{(n)}_{x,\ell}(y,m)\}$ with some $\nu > 0$ and $\hat{b}^{(n)}_{x,\ell} \lesssim 1$. Then*

$$\lim_{n\to\infty} \sum_{\ell\leq n}\sum_x t^\omega_{x,\ell} \sum_{m>n-\ell}\sum_y b^{(n)}_{x,\ell}(y,m)\left(f^{\theta_x\omega}_{y,m} - \mathbf{f}_{y,m}\right) \triangleq \lim_{n\to\infty}\sum_{\ell\leq n} Z^{(n)}_\ell = 0, \tag{2.19}$$

*$\mathbb{P}$-a.s. and in $\mathbb{L}_2$.*

*Proof.* By (2.15),

$$\mathbb{E}\big[\big(\mathbb{E}(Z^{(n)}_\ell \,|\, \mathcal{A}_{\ell-k})\big)^2\big],\ \mathbb{E}\big[\big(Z^{(n)}_\ell - \mathbb{E}(Z^{(n)}_\ell \,|\, \mathcal{A}_{\ell+k})\big)^2\big] \lesssim \frac{\mathrm{e}^{-2\nu'(n-\ell)}}{\ell^{d/2-\epsilon}(1+k)^{1+\epsilon}} \tag{2.20}$$

Applying (2.6) for each $n = 1, 2, \dots$ (with $\psi^2_\ell = \big(\psi^{(n)}_\ell\big)^2 = \frac{\mathrm{e}^{-2\nu'(n-\ell)}}{\ell^{d/2-\epsilon}}$), we infer that

$$\mathbb{E}\big[\big(\sum_{\ell\leq n} Z^{(n)}_\ell\big)^2\big] \lesssim \sum_{\ell=1}^n \frac{\mathrm{e}^{-2\nu'(n-\ell)}}{\ell^{d/2-\epsilon}}.$$

Since $d \geq 3$ and $\epsilon < 1/2$, this implies that $\sum_n \mathbb{E}\big[\big(\sum_{\ell\leq n} Z^{(n)}_\ell\big)^2\big] < \infty$.
Consider now the left-hand side of (2.19). For each $\ell \leq n$, the $Z^{(n)}_\ell$-sum on the right-hand side of (2.19) can be rewritten in the form (2.9) with $a^{(n)}_{x,\ell}(y,m) = b^{(n)}_{x,\ell}(y,m)\mathbb{1}_{\{m>n-\ell\}}$. In this case, the inequality (2.10) is satisfied for the array $\left\{a^{(n)}_{x,\ell}(y,m)\right\}$ with any $\nu' < \nu/2$ and $\hat{a}^{(n)}_{x,\ell} \lesssim \mathrm{e}^{-\nu(n-\ell)/2} \triangleq \hat{a}^{(n)}_\ell$. □

### 2.2 Multi-Dimensional Renewal and Asymptotics of $t_n^\omega$

Let us turn to the quenched asymptotics of $t_n^\omega$. By construction,

$$t_{z,n}^\omega = \sum_{m=0}^{n-1}\sum_x t_{x,m}^\omega f_{z-x,n-m}^{\theta_x\omega} \ \text{ and } \ t_n^\omega = \sum_z t_{z,n}^\omega. \tag{2.21}$$

The claim (1.17) of Theorem B follows from:

**Theorem 2.7.** *Assume that* (2.4) *holds. Then,*

$$\lim_{n\to\infty} t_n^\omega = \frac{1}{\kappa}\Big(1+\sum_{x,y} t_x^\omega\left(f_{y-x}^{\theta_x\omega} - \mathbf{f}_{y-x}\right)\Big\} \triangleq \frac{1}{\kappa}s^\omega \in (0,\infty), \tag{2.22}$$

$\mathbb{P}$*-a.s. and in* $\mathbb{L}_2$ *on the event* $\left\{0\in \mathrm{Cl}_\infty^h(V)\right\}$.

*Proof.* Part of the proof appeared in Subsection 5.3 of the review paper [12]. We rely on an expansion similar to the one employed by Sinai [16] and rewrite (2.21) as (see the beginning of Section 5.3 of [12] for details)

$$t_{z,n}^\omega = \mathbf{t}_{z,n} + \sum_{\ell=0}^{n-1}\sum_{m=1}^{n-\ell}\sum_{r=0}^{n-\ell-m}\sum_{x,y} t_{x,\ell}^\omega\left(f_{y-x,m}^{\theta_x\omega} - \mathbf{f}_{y-x,m}\right)\mathbf{t}_{z-y,r}. \tag{2.23}$$

In this way, $t_n^\omega$ ((56) in Section 5.3 of [12]) can be represented as

$$t_n^\omega = \frac{1}{\kappa}s_n^\omega + \epsilon_n^\omega + \left(\mathbf{t}_n - \frac{1}{\kappa}\right) \tag{2.24}$$

where

$$s_n^\omega = 1 + \sum_{\ell\le n}\sum_x t_{x,\ell}^\omega\left(f^{\theta_x\omega} - 1\right), \tag{2.25}$$

and the correction term $\epsilon_n^\omega = -\epsilon_{n,1}^\omega + \epsilon_{n,2}^\omega$ is given by

$$\epsilon_n^\omega = -\frac{1}{\kappa}\sum_{\substack{\ell\le n\\ m>n-\ell}}\sum_x t_{x,\ell}^\omega\left(f_m^{\theta_x\omega} - \mathbf{f}_m\right) + \sum_{\ell+m+r=n}\sum_x t_{x,\ell}^\omega\left(f_m^{\theta_x\omega} - \mathbf{f}_m\right)\left(\mathbf{t}_r - \frac{1}{\kappa}\right). \tag{2.26}$$

By (2.1) $\mathbf{t}_n - \frac{1}{\kappa}$ tends to zero. We claim that, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} s_n^\omega = s^\omega \ \text{ and } \ \sum_n \mathbb{E}[(\epsilon_n^\omega)^2] < \infty. \tag{2.27}$$

**Convergence of $s_n^\omega$.** Following the discussion in Subsection 4.5 of [11], one readily verifies that $s^\omega > 0$ on the event $\left\{0\in \mathrm{Cl}_\infty^h(V)\right\}$. It remains to check (2.27).

Let us rewrite $s_n^\omega$ as

$$s_n^\omega - 1 = \sum_{\ell\le n}\sum_x t_{x,\ell}^\omega\left(f^{\theta_x\omega} - 1\right) \triangleq \sum_{\ell=0}^n Z_\ell. \tag{2.28}$$

The representation complies with (2.9) and (2.10) with $\hat{a}_{x,\ell}^{(n)} \lesssim 1$ and any positive $\nu < c_2$. Hence, by (2.15),

$$\mathbb{E}\left[\left(\mathbb{E}(Z_\ell\,|\,\mathcal{A}_{\ell-k})\right)^2\right],\ \mathbb{E}\left[\left(Z_\ell - \mathbb{E}(Z_\ell\,|\,\mathcal{A}_{\ell+k})\right)^2\right] \lesssim \frac{1}{\ell^{d/2-\epsilon}}\cdot\frac{1}{(1+k)^{1+\epsilon}}. \tag{2.29}$$

Since $d\ge 3$ and $\epsilon < 1/2$, Remark 2.3 applies and $\lim_{n\to\infty} s_n^\omega = 1 + \sum_0^\infty Z_\ell$ converges $\mathbb{P}$-a.s. and in $\mathbb{L}_2$ . ☐

**The $\epsilon_n^\omega$ term.** Let us turn now to the correction term $\epsilon_n^\omega$ in (2.26). The first summand to estimate is

$$\epsilon_{n,1}^\omega = \sum_{\ell \le n}\sum_x t_{x,\ell}^\omega \sum_{m>n-\ell}\left(f_m^{\theta_x\omega} - \mathbf{f}_m\right) \tag{2.30}$$

It tends to zero by Lemma 2.6. The second summand is

$$\epsilon_{n,2}^\omega = \sum_{\ell+m+r=n}\sum_x t_{x,\ell}^\omega \left(f_m^{\theta_x\omega} - \mathbf{f}_m\right)\left(\mathbf{t}_r - \frac{1}{\kappa}\right)$$

Since $\mathbf{t}_r - 1/\kappa$ is exponentially decaying in $r$, it is easy to see that (2.10) still holds with $a_{x,\ell}^{(n)} \lesssim \hat{a}_\ell^{(n)} \triangleq \mathrm{e}^{-c_4(n-\ell)}$, for any positive $\nu < c_2$ and some $c_4 = c_4(\beta) > 0$, and Lemma 2.6 applies. □

## 2.3 Quenched CLT

To facilitate notation set $\alpha_n = \alpha/\sqrt{n}$. For $r = 1, 2, \ldots$ define

$$\mathcal{S}_r^\omega(\alpha) \triangleq \sum_z t_{z,r}^\omega \mathrm{e}^{i\alpha\cdot(z-rv)}$$

We are studying $\mathcal{S}_n^\omega(\alpha_n)$. The asymptotics of $\mathbf{S}_n(\alpha_n) = \mathbb{E}\mathcal{S}_n^\omega(\alpha_n)$ is given in (2.3). Using (2.23),

$$\mathcal{S}_n^\omega(\alpha_n) = \mathbf{S}_n(\alpha_n) + \sum_{\ell+m+r=n}\sum_{x,y,z} t_{x,\ell}^\omega\left(f_{y-x,m}^{\alpha_x\omega} - \mathbf{f}_{y-x,m}\right)\mathbf{t}_{z-y,r}\mathrm{e}^{i(z-nv)\cdot\alpha_n}. \tag{2.31}$$

Define

$$g_m^\omega(\alpha) = \sum_y \mathrm{e}^{i(y-mv)\cdot\alpha}\left(f_{y,m}^\omega - \mathbf{f}_{y,m}\right). \tag{2.32}$$

Note that $g_m^\omega(0) = f_m^\omega - \mathbf{f}_m$ and that $g_m^\omega(\alpha) - g_m^\omega(0) = \sum_y\left(\mathrm{e}^{i(y-mv)\cdot\alpha} - 1\right)\left(f_{y,m}^\omega - \mathbf{f}_{y,m}\right)$. We can rewrite (2.31) as

$$\mathcal{S}_n^\omega(\alpha_n) = \mathbf{S}_n(\alpha_n) + \sum_{\ell+m+r=n}\mathbf{S}_r(\alpha_n)\sum_x t_{x,\ell}^\omega \mathrm{e}^{i(x-\ell v)\cdot\alpha_n} g_m^{\theta_x\omega}(\alpha_n). \tag{2.33}$$

Expanding terms in the products $\mathbf{S}_r(\alpha_n)\mathrm{e}^{i(x-\ell v)\cdot\alpha_n} g_m^{\theta_x\omega}(\alpha_n)$ as

$$\mathbf{S}_r(\alpha_n) = \mathbf{S}_n(\alpha_n) + \left(\mathbf{S}_r(\alpha_n) - \mathbf{S}_n(\alpha_n)\right)$$

and, accordingly,

$$\mathrm{e}^{i(x-\ell v)\cdot\alpha_n} = 1 + \left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_n} - 1\right),\quad g_m^\omega(\alpha_n) = g_m^\omega(0) + \left(g_m^\omega(\alpha_n) - g_m^\omega(0)\right),$$

we rewrite (2.33) as:

$$\begin{aligned}
\mathcal{S}_n^\omega(\alpha_n) &= \mathbf{S}_n(\alpha_n)\Big(1 + \sum_{\ell+m\le n}\sum_x t_{x,\ell}^\omega\left(f_m^{\theta_x\omega} - \mathbf{f}_m\right)\Big)\\
&\quad + \mathbf{S}_n(\alpha_n)\sum_{\ell+m\le n}\sum_x t_{x,\ell}^\omega\left(g_m^{\theta_x\omega}(\alpha_n) - g_m^{\theta_x\omega}(0)\right)\\
&\quad + \sum_{\ell+m+r=n}\left(\mathbf{S}_r(\alpha_n) - \mathbf{S}_n(\alpha_n)\right)\sum_x t_{x,\ell}^\omega\left(f_m^{\theta_x\omega} - \mathbf{f}_m\right)\\
&\quad + \mathbf{S}_n(\alpha_n)\sum_{\ell+m\le n}\sum_x t_{x,\ell}^\omega\left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_n} - 1\right)\left(f_m^{\theta_x\omega} - \mathbf{f}_m\right)\\
&\quad + \text{cross-terms}\\
&\triangleq \mathbf{S}_n(\alpha_n)\Big(1 + \sum_{\ell+m\le n}\sum_x t_{x,\ell}^\omega\left(f_m^{\theta_x\omega} - \mathbf{f}_m\right)\Big) + \sum_{i=1}^3 \eta_{n,i}^\omega + \text{cross-terms.}
\end{aligned} \tag{2.34}$$

By Theorem 2.7 the sequence of random factors of $\mathbf{S}_n(\alpha)$ tend to $s^\omega$. The cross terms are of lower order and we shall briefly discuss them at the end of the present section. The crux of the matter is to prove:

**Theorem 2.8.** *For every $\alpha \in \mathbb{R}^d$ the correction terms $\eta^\omega_{n,i}$ in (2.34) satisfy :*

$$\text{For } i = 1,2,3 \quad \lim_{n\to\infty} \eta^\omega_{n,i} = 0 \quad \mathbb{P}\text{-a.s. and in } \mathbb{L}_2(\Omega). \tag{2.35}$$

Once (2.35) is established, we readily infer from (2.1), (2.3) and (1.17) that

$$\lim_{n\to\infty} \frac{\mathcal{S}^\omega_n(\alpha/\sqrt{n})}{t^\omega_n} = \exp\left\{-\tfrac{1}{2}\Sigma\,\alpha\cdot\alpha\right\}, \tag{2.36}$$

$\mathbb{P}$-a.s. on the event $\left\{0 \in \mathrm{Cl}^h_\infty(V)\right\}$ for every $\alpha \in \mathbb{R}^{d+1}$ fixed. This is precisely (1.18) of Theorem B.

## 3 Correction Terms

In this Section, we prove (2.35). The correction terms $\eta^\omega_{n,i}$; $i = 1,2,3$, will be treated separately. Recall that we are working with $\epsilon < 1/6$ such that (2.4) holds with $\rho = \epsilon/2$.

**The $\eta^\omega_{n,1}$ term .** Consider

$$\frac{\eta^\omega_{n,1}}{\mathbf{S}_n(\alpha_n)} = \sum_{\ell\le n}\sum_x t^\omega_{x,\ell} \sum_{m\le n-\ell}\sum_y \left(\mathrm{e}^{i(y-mv)\cdot\alpha_n} - 1\right)\left(f^{\theta_x\omega}_{y,m} - \mathbf{f}_{y,m}\right)$$

By Lemma 2.6, the constraint $m \le n-\ell$ might be removed, and we need to prove the convergence to zero of

$$\hat\eta^\omega_{n,1} \triangleq \sum_{\ell\le n}\sum_x t^\omega_{x,\ell}\sum_{m,y} a^{(n)}_{x,\ell}(y,m)\left(f^{\theta_x\omega}_{y,m} - \mathbf{f}_{y,m}\right) \triangleq \sum_{\ell\le n} Z^{(n)}_\ell. \tag{3.1}$$

with $a^{(n)}_{x,\ell}(y,m) = \left(\mathrm{e}^{i(y-mv)\cdot\alpha_n} - 1\right)$.

**Lemma 3.1.** *In the very weak disorder regime,*

$$\lim_{n\to\infty} \hat\eta^\omega_{n,1} = 0, \tag{3.2}$$

*$\mathbb{P}$-a.s. and in $\mathbb{L}_2$ for each $\alpha \in \mathbb{R}^d$ fixed.*

*Proof of Lemma 3.1.* For $a^{(n)}_{x,\ell}(y,m)$ as above, (2.10) is satisfied with $\hat a^{(n)}_{x,\ell} \lesssim \hat a^{(n)}_\ell \triangleq 1/\sqrt{n}$ and any $\nu < c_2$. By (2.15) of Lemma 2.5,

$$\mathbb{E}\left[\left(\mathbb{E}(Z^{(n)}_\ell \,|\, \mathcal{A}_{\ell-k})\right)^2\right],\ \mathbb{E}\left[\left(Z^{(n)}_\ell - \mathbb{E}(Z^{(n)}_\ell \,|\, \mathcal{A}_{\ell+k})\right)^2\right] \lesssim \frac{1}{n\ell^{d/2-\epsilon}}\cdot\frac{1}{(1+k)^{1+\epsilon}}. \tag{3.3}$$

By (2.6), $\mathbb{V}\mathrm{ar}\left(\hat\eta^\omega_{n,1}\right) \lesssim 1/n$. Consequently, the lacunary sequence $\left\{\hat\eta^\omega_{n^{1+\delta},1}\right\}$ converges to zero $\mathbb{P}$-a.s. and in $\mathbb{L}_2$ for any $\delta > 0$.

It remains to choose $\delta > 0$ appropriately and to control fluctuations of $\hat\eta^\omega_{\cdot,1}$ on the intervals of the form $[N,\dots,N+R]$ with

$$N \cong n^{1+\delta} \quad\text{and}\quad R \cong (1+n)^{1+\delta} - n^{1+\delta} \cong n^\delta. \tag{3.4}$$

Now,

$$\hat\eta^\omega_{N+r,1} - \hat\eta^\omega_{N,1} = \sum_{\ell\le N}\left(Z^{(N+r)}_\ell - Z^{(N)}_\ell\right) + \sum_{\ell=N+1}^{N+r} Z^{(N+r)}_\ell. \tag{3.5}$$

We should not worry about the second term above: (2.6) can still be applied to bound $\mathbb{V}\mathrm{ar}\big(\sum_{\ell=N+1}^{N+r} Z_\ell^{(N+r)}\big)$ for each $r$ fixed. By (3.3) and the union bound,

$$\mathbb{E}\Big[\max_{r\leq R}\big(\sum_{\ell=N+1}^{N+r} Z_\ell^{(N+r)}\big)^2\Big] \leq \sum_{r=1}^{R}\mathbb{E}\Big[\big(\sum_{\ell=N+1}^{N+r} Z_\ell^{(N+r)}\big)^2\Big] \lesssim \frac{R}{N^{d/2-\epsilon}} \cong \frac{1}{n^{(1+\delta)\left(\frac{d}{2}-\epsilon-\frac{\delta}{1+\delta}\right)}}.$$

The right-hand side above is summable (in $n$) by our choice (3.4) whenever $\frac{d}{2}-\epsilon-\frac{\delta}{1+\delta} > 1$. Since $d\geq 3$ and $\epsilon < 1/2$, there are feasible choices of $\delta > 0$ to ensure the latter.
As for the first term in (3.5), note that for $\ell \leq N$,

$$a_{x,\ell}^{(N+r)}(y,m) - a_{x,\ell}^{(N)}(y,m) = \left(\mathrm{e}^{i(y-mv)\cdot\alpha_{N+r}} - \mathrm{e}^{i(y-mv)\cdot\alpha_N}\right) \triangleq b_{x,\ell}^{(N,r)}(y,m). \tag{3.6}$$

The array $\big\{b_{x,\ell}^{(N,r)}(y,m)\big\}$ satisfies (2.10) with $\hat{b}_{x,\ell}^{(N,r)} \lesssim \hat{b}_\ell^{(N,r)} \triangleq r/N^{3/2}$ and any $\nu < c_2$. By (2.15), (2.6) and the union bound,

$$\mathbb{E}\Big[\max_{r\leq R}\Big(\sum_{\ell\leq N}\big(Z_\ell^{(N+r)} - Z_\ell^{(N)}\big)\Big)^2\Big] \lesssim \frac{R^3}{N^3}.$$

By our choice (3.4), $\frac{R^3}{N^3} \cong n^{-3}$ for any choice of $\delta > 0$, and, consequently, the right-hand side above is summable. □

**The $\eta_{n,2}^\omega$ term.** By (2.1),

$$\frac{\mathbf{S}_r(\alpha_n)}{\mathbf{S}_n(\alpha_n)} = \frac{\mathbf{t}_r(i\alpha_n)\mathrm{e}^{-irv\cdot\alpha_n}}{\mathbf{t}_n(i\alpha_n)\mathrm{e}^{-inv\cdot\alpha_n}} = \mathrm{e}^{(r-n)(\mu(i\alpha_n)-iv\cdot\alpha_n)}\left(1+o(\mathrm{e}^{-c_4 r})\right). \tag{3.7}$$

Set $\phi(\alpha) = iv\cdot\alpha - \mu(i\alpha)$. The function $\phi$ is defined in a neighbourhood of the origin and it is of quadratic growth there. By Lemma 2.6, the residual term $\mathrm{o}\left(\mathrm{e}^{-c_4 r}\right)$ is negligible. Next, for $\ell \leq n$ the coefficients

$$a_{x,\ell}^{(n)}(y,m) = \mathrm{e}^{(m+\ell)\phi(\alpha_n)} - 1 \tag{3.8}$$

satisfy (2.10) with $\hat{a}_{x,\ell}^{(n)} \lesssim \hat{a}_\ell^{(n)} \triangleq \ell/n$ and any $\nu < c_2$. Consequently, (2.19) enables to lift the restriction $m \leq n-\ell$. Therefore, we need to prove convergence to zero of

$$\hat{\eta}_{n,2}^\omega = \sum_{\ell\leq n}\sum_x t_{x,\ell}^\omega \sum_m \left(\mathrm{e}^{(m+\ell)\phi(\alpha_n)} - 1\right)\left(f_m^{\theta_x\omega} - \mathbf{f}_m\right) \triangleq \sum_{\ell\leq n} Z_\ell^{(n)}. \tag{3.9}$$

**Lemma 3.2.** *In the very weak disorder regime*

$$\lim_{n\to\infty}\hat{\eta}_{n,2}^\omega = 0, \tag{3.10}$$

*$\mathbb{P}$-a.s. and in $\mathbb{L}_2$ for each $\alpha\in\mathbb{R}^d$ fixed.*

*Proof of Lemma 3.2.* Recall that for in $a_{x,\ell}^{(n)}(y,m)$ defined in (3.8) the asymptotic bound (2.10) is satisfied with $\hat{a}_{x,\ell}^{(n)} \lesssim \ell/n$ uniformly in $n$, $\ell \leq n$ and $x$. By (2.15) and (2.6),

$$\mathbb{V}\mathrm{ar}\left(\hat{\eta}_{n,2}^\omega\right) \lesssim \frac{1}{n^2}\sum_{\ell\leq n}\frac{1}{\ell^{d/2-2-\epsilon}} \cong \frac{1}{n^{\frac{d}{2}-1-\epsilon}\wedge n^2}, \tag{3.11}$$

which already implies the claim of Lemma 3.2 in dimensions $d\geq 5$. We shall continue discussion for the most difficult case of $d = 3$. (3.11) implies that

$$\mathbb{E}\big[\sum_n\left(\hat{\eta}_{n^{2+\delta},2}^\omega\right)^2\big] < \infty \Rightarrow \lim_{n\to\infty}\hat{\eta}_{n^{2+\delta},2}^\omega = 0 \ \ \mathbb{P}-\text{a.s. and in } \mathbb{L}_2, \tag{3.12}$$

whenever

$$(2+\delta)\left(\tfrac{1}{2}-\epsilon\right) > 1 \quad \text{, that is,} \quad \delta > \frac{4\epsilon}{1-2\epsilon}. \tag{3.13}$$

Since $\epsilon < 1/6$, there are choices of $\delta \in (0,1)$ which comply with (3.13). We need to control fluctuations of $\hat{\eta}^{\omega}_{N+r,2} - \hat{\eta}^{\omega}_{N,2}$ on the intervals of the form $[N,\dots,N+R]$, where

$$N \cong n^{2+\delta} \ \text{ and } \ R \cong (n+1)^{2+\delta} - n^{2+\delta} \cong n^{1+\delta}. \tag{3.14}$$

Consider the following decomposition:

$$\begin{aligned}\hat{\eta}^{\omega}_{N+r,2} - \hat{\eta}^{\omega}_{N,2} &= \sum_{\ell\le N}\sum_x t^{\omega}_{x,\ell}\sum_m \left(\mathrm{e}^{(m+\ell)\phi(\alpha_{N+r})} - \mathrm{e}^{(m+\ell)\phi(\alpha_N)}\right)\left(f^{\theta_x\omega}_m - \mathbf{f}_m\right) + \sum_{\ell=N+1}^{N+r} Z^{(N+r)}_{\ell} \\ &= \sum_{\ell\le N+r}\sum_x t^{\omega}_{x,\ell}\sum_m \left(\mathrm{e}^{(m+\ell)\phi(\alpha_{N+r})} - \mathrm{e}^{(m+\ell)\phi(\alpha_N)}\right)\left(f^{\theta_x\omega}_m - \mathbf{f}_m\right) + \sum_{\ell=N+1}^{N+r} Z^{(N)}_{\ell}.\end{aligned} \tag{3.15}$$

The terms $Z^{(N)}_{\ell}$ in the second sum above were defined in (3.9) and they do not depend on $r$. By (2.6), we are entitled to control its maximum on the interval $[N,\dots,N+R]$:

$$\begin{aligned}\mathbb{E}\Big[\max_{r\le R}\Big(\sum_{\ell=N+1}^{N+r} Z^{(N)}_{\ell}\Big)^2\Big] &\lesssim \frac{1}{N^2}\sum_{\ell=N+1}^{N+R}\frac{\ell^2}{\ell^{3/2-\epsilon}} \cong \frac{1}{N^2}\left\{(N+R)^{3/2+\epsilon} - N^{3/2+\epsilon}\right\} \\ &\cong \frac{R}{N^{3/2-\epsilon}} \cong \frac{n^{1+\delta}}{n^{3(1+\delta/2)-\epsilon(2+\delta)}} \cong \frac{1}{n^{2+\frac{\delta}{2}-\epsilon(2+\delta)}} \triangleq a_n,\end{aligned} \tag{3.16}$$

by our choice of parameters (3.14).

For each $r \le R$ the first term in (3.15) corresponds to the following choice of coefficients in the representation (2.9): $a^{(N+r)}_{x,\ell}(y,m) = \left(\mathrm{e}^{(m+\ell)\phi(\alpha_{N+r})} - \mathrm{e}^{(m+\ell)\phi(\alpha_N)}\right)$. Thus, (2.10) is satisfied with $\hat{a}^{(N+r)}_{x,\ell} \lesssim \hat{a}^{(N+r)}_{\ell} \triangleq \ell r/N^2$ and any $\nu < c_2$. By the very same (2.15) and (2.6), we infer that, for any $r \le R$,

$$\mathbb{V}\mathrm{ar}\Big(\sum_{\ell=1}^{N+r}\sum_x t^{\omega}_{x,\ell}\sum_m\left(\mathrm{e}^{(m+\ell)\phi(\alpha_{N+r})} - \mathrm{e}^{(m+\ell)\phi(\alpha_N)}\right)\Big) \lesssim \frac{r^2}{N^4}\sum_{\ell=1}^{N+r}\ell^{\frac{1}{2}+\epsilon} \lesssim \frac{R^2}{N^{5/2-\epsilon}}. \tag{3.17}$$

Hence, by the union bound and our choice of parameters (3.14),

$$\begin{aligned}&\mathbb{E}\Big[\max_{r\le R}\Big(\sum_{\ell=1}^{N+r}\sum_x t^{\omega}_{x,\ell}\sum_m\big(\mathrm{e}^{(m+\ell)\phi(\alpha_{N+r})} - \mathrm{e}^{(m+\ell)\phi(\alpha_N)}\big)\Big)^2\Big] \\ &\quad\lesssim \frac{R^3}{N^{5/2-\epsilon}} \cong \frac{1}{n^{2-\frac{\delta}{2}-\epsilon(2+\delta)}} \triangleq b_n.\end{aligned} \tag{3.18}$$

Since $\epsilon < 1/6$, the inequality $\frac{\delta}{2} + (2+\delta)\epsilon < 1$ holds for any choice of $\delta \le 1$. Therefore, any such choice ensures that $\sum_n (a_n + b_n) < \infty$, which implies that

$$\lim_{n\to\infty}\max_{n^{2+\delta}\le r<(n+1)^{2+\delta}}\left|\hat{\eta}^{\omega}_{r,1} - \hat{\eta}^{\omega}_{n^{2+\delta},1}\right| = 0,$$

$\mathbb{P}$-a.s and in $\mathbb{L}_2$. The proof of Lemma 3.2 is completed. □

**The $\eta_{n,3}^{\omega}$ term.** This is the most difficult term, and, at this stage, we need to rely on Lemma 2.4 rather than on Lemma 2.5. Recall that

$$\frac{\eta_{n,3}^{\omega}}{\mathbf{S}_n(\alpha_n)} = \sum_{\ell\le n}\sum_x t_{x,\ell}^{\omega}\left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_n}-1\right)\sum_{m\le n-\ell}\left(f_m^{\theta_x\omega}-\mathbf{f}_m\right).$$

By Lemma 2.6, we may remove the constraint $m\le n-\ell$. Define, therefore,

$$Z_\ell^{(n)} = \sum_x t_{x,\ell}^{\omega}\left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_n}-1\right)\left(f^{\theta_x\omega}-1\right) \quad \text{and} \quad \hat{\eta}_{n,3}^{\omega} = \sum_{\ell\le n} Z_\ell^{(n)}. \tag{3.19}$$

We need to prove:

**Lemma 3.3.** *In the very weak disorder regime,*

$$\lim_{n\to\infty}\hat{\eta}_{n,3}^{\omega} = 0, \tag{3.20}$$

$\mathbb{P}$-*a.s. and in* $\mathbb{L}_2$ *for each* $\alpha\in\mathbb{R}^d$ *fixed.*

*Proof of Lemma 3.3.* For $Z_\ell^{(n)}$ defined in (3.19), the bound (2.10) is satisfied with

$$\hat{a}_{x,\ell}^{(n)} = \left|\mathrm{e}^{i(x-\ell v)\cdot\alpha_n}-1\right| \cong \frac{|x-\ell v|}{\sqrt{n}}\wedge 1, \tag{3.21}$$

for any $\nu < c_2$. Applying (2.11), we infer that

$$\mathbb{E}\big[\big(\mathbb{E}(Z_\ell^{(n)}\,|\,\mathcal{A}_{\ell-k})\big)^2\big] \lesssim \frac{1}{\ell^{d+1-\epsilon/2}}\sum_{x\in\mathcal{H}_{\ell-k}^-}\mathrm{e}^{-c_2|x-\ell v|^2/\ell}\big(\frac{|x-\ell v|^2}{n}\wedge 1\big). \tag{3.22}$$

As in the derivation of (2.15), we may assume that $k$ is sufficiently large, so that, in particular, $|x-\ell v|\ge\frac{k|v|}{2}$ for all $x\in\mathcal{H}_{\ell-k}^-$. In the latter case, the sum on the right-hand side of (3.22) is bounded above by

$$\begin{aligned} &\lesssim \int_{|y|>\frac{k|v|}{2}}\mathrm{e}^{-c_2|y|^2/\ell}\left(\frac{|y|^2}{n}\wedge 1\right)\mathrm{d}y = \int_{\frac{k|v|}{2}}^{\infty} r^d\mathrm{e}^{-c_2r^2/\ell}\left(\frac{r^2}{n}\wedge 1\right)\mathrm{d}r \\ &\cong \ell^{(d+1)/2}\int_{\frac{k|v|}{2\sqrt{\ell}}}^{\infty} t^d\mathrm{e}^{-c_2t^2}\left(\frac{t^2\ell}{n}\wedge 1\right)\mathrm{d}t \triangleq \ell^{(d+1)/2}I_n(\ell,k). \end{aligned} \tag{3.23}$$

We shall repeatedly rely on (2.8). There are two cases to consider:
CASE 1. If $\frac{k|v|}{2}\le\sqrt{n}$, then

$$\begin{aligned} I_n(\ell,k) &= \frac{\ell}{n}\int_{\frac{k|v|}{2\sqrt{\ell}}}^{\sqrt{n/\ell}} t^{d+2}\mathrm{e}^{-c_2t^2}\mathrm{d}t + \int_{\sqrt{n/\ell}}^{\infty} t^d\mathrm{e}^{-c_2t^2}\mathrm{d}t \lesssim \frac{\ell}{n}\mathrm{e}^{-c_2\frac{(k|v|)^2}{8\ell}} + \mathrm{e}^{-c_2\frac{n}{2\ell}} \\ &\lesssim \frac{\ell}{n}\mathrm{e}^{-c_2\frac{(k|v|)^2}{8\ell}} \lesssim \frac{1}{n}\cdot\frac{\ell^{3/2+\epsilon/2}}{(1+k)^{1+\epsilon}}. \end{aligned} \tag{3.24}$$

CASE 2. If $\frac{k|v|}{2}>\sqrt{n}$, then

$$I_n(\ell,k) = \int_{\frac{k|v|}{2\sqrt{\ell}}}^{\infty} t^d\mathrm{e}^{-c_2t^2}\mathrm{d}t \lesssim \mathrm{e}^{-c_2\frac{(k|v|)^2}{8\ell}} \lesssim \frac{\ell}{n}\mathrm{e}^{-c_2\frac{(k|v|)^2}{16\ell}} \lesssim \frac{1}{n}\cdot\frac{\ell^{3/2+\epsilon/2}}{(1+k)^{1+\epsilon}} \tag{3.25}$$

as well.

As a result (again we restrict attention to the most difficult case $d = 3$):

$$\mathbb{E}\big[\big(\mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell-k})\big)^2\big] \lesssim \frac{1}{(1+k)^{1+\epsilon}} \cdot \frac{1}{n} \cdot \frac{1}{\ell^{1/2-\epsilon}}. \tag{3.26}$$

Let us turn to the bound (2.12) on $\mathbb{E}\big[\big(Z_\ell^{(n)} - \mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell+k})\big)^2\big]$. As before, we apply it with $\hat{a}_{x,\ell}^{(n)} = \frac{|x-\ell v|}{\sqrt{n}} \wedge 1$. We need to estimate

$$\frac{1}{\ell^{d+1-\epsilon/2}} \sum_{x\in\mathcal{H}_{\ell+k}^-} \mathrm{e}^{-c_2|x-\ell v|^2/\ell - \nu \mathrm{d}_{\ell+k}(x)} \big(\frac{|x-\ell v|^2}{n} \wedge 1\big). \tag{3.27}$$

Proceeding as in (2.17) and noting that $\frac{k^2}{n}\mathrm{e}^{-\nu' k} \lesssim \frac{1}{n}\mathrm{e}^{-\nu' k/2}$, we infer that the upper bound (3.26) holds for $\mathbb{E}\big[\big(Z_\ell^{(n)} - \mathbb{E}(Z_\ell^{(n)} \,|\, \mathcal{A}_{\ell+k})\big)^2\big]$ as well.
In particular, by (2.6), $\mathbb{V}\mathrm{ar}\left(\hat{\eta}_{n,3}^\omega\right) \lesssim n^{-1/2+\epsilon}$ and, as in the case of $\hat{\eta}_{n,2}^\omega$, we infer that there is $\mathbb{P}$-a.s. and $\mathbb{L}_2$ convergence to zero along lacunary subsequences $\left\{n^{2+\delta}\right\}$, whenever $\delta$ satisfies (3.13). Hence, again as in the case of $\hat{\eta}_{n,2}^\omega$, we need to control the fluctuations $\hat{\eta}_{N+r,3}^\omega - \hat{\eta}_{N,3}^\omega$ over intervals of the form (3.14). As in (3.15), we make use of the decomposition

$$\begin{aligned} \hat{\eta}_{N+r,3}^\omega - \hat{\eta}_{N,3}^\omega &= \sum_{\ell=1}^{N+r} \sum_x t_{x,\ell}^\omega \left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_{N+r}} - \mathrm{e}^{i(x-\ell v)\cdot\alpha_N}\right)\left(f^{\theta_x\omega} - 1\right) + \sum_{\ell=N+1}^{N+r} Z_\ell^{(N)} \\ &\triangleq \sum_{\ell=1}^{N+r} Z_\ell^{(N,r)} + \sum_{\ell=N+1}^{N+r} Z_\ell^{(N)} \end{aligned} \tag{3.28}$$

We continue to work with $d = 3$. For each $r = 1, \dots, R$ fixed the bound (2.10) is satisfied with

$$\left|\mathrm{e}^{i(x-\ell v)\cdot\alpha_{N+r}} - \mathrm{e}^{i(x-\ell v)\cdot\alpha_N}\right| \lesssim \frac{R\,|x-\ell v|}{N^{3/2}} \wedge 1 \triangleq \hat{a}_{x,\ell}^{(N)}. \tag{3.29}$$

The expression for $\hat{a}_{x,\ell}^{(N)}$ in (3.29) is similar to (3.21) with $1/\sqrt{n}$ being replaced by the higher order term $R/N^{3/2}$. Literally repeating the derivation of (3.26) we infer that for each $r = 1, \dots, R$ fixed random variables $Z_\ell^{(N,r)}$ in (3.28) satisfy:

$$\mathbb{E}\big[\big(\mathbb{E}(Z_\ell^{(N,r)} \,|\, \mathcal{A}_{\ell-k})\big)^2\big], \mathbb{E}\big[\big(Z_\ell^{(N,r)} - \mathbb{E}(Z_\ell^{(N,r)} \,|\, \mathcal{A}_{\ell+k})\big)^2\big] \lesssim \frac{1}{(1+k)^{1+\epsilon}} \cdot \frac{R^2}{N^3} \cdot \frac{1}{\ell^{1/2-\epsilon}}. \tag{3.30}$$

Applying (2.6) we conclude: For $N$ and $R$ in the range (3.14), and for any $r = 1, \dots, R$ fixed, the variance of the first term on the right hand side of (3.28) is uniformly bounded above ($d = 3$) by

$$\lesssim \frac{R^2}{N^3} \cdot N^{1/2+\epsilon} = \frac{R^2}{N^{5/2-\epsilon}}.$$

As in the case of (3.18), the union bound suffices.

Finally, using (3.26) as an input for (2.6), we conclude that

$$\mathbb{E}\big[\big(\max_{r\leq R} \sum_{\ell=N+1}^{N+r} Z_\ell^{(N)}\big)^2\big] \lesssim \frac{1}{N} \sum_{\ell=N+1}^{N+R} \frac{1}{\ell^{1/2-\epsilon}} \cong \frac{(N+R)^{1/2+\epsilon} - N^{1/2+\epsilon}}{N} \cong \frac{R}{N^{3/2-\epsilon}}.$$

The right-most term above is summable in $N$ for any choice of $\delta \in (0,1)$ in (3.14). In particular, it is summable if $\delta$ complies with (3.13). Consequently, for such choices of $\delta$,

$$\lim_{n\to\infty} \max_{n^{2+\delta} \leq r < (n+1)^{2+\delta}} \left|\hat{\eta}_{r,3}^\omega - \hat{\eta}_{n^{2+\delta},3}^\omega\right| = 0,$$

$\mathbb{P}$-a.s. and in $\mathbb{L}_2$, and we are home. □

**Cross-terms in** (2.34)**.** Since our treatment of the correction terms $\eta^{\omega}_{n,i}$ was based either on the estimates (2.10) on the absolute value of the coefficients $a^{(n)}_{x,\ell}(y,m)$ which appear in (3.1), (3.9) and (3.19), or on estimates on the absolute value of differences $a^{(N+r)}_{x,\ell}(y,m) - a^{(N)}_{x,\ell}(y,m)$ between these coefficients, which show up in the decompositions (3.5), (3.15) and, respectively, (3.28), we readily infer that (2.35) of Theorem 2.8 carry over to cross-terms in (2.34). For instance, let us consider the (2-3) cross term

$$\mathbf{S}_n(\alpha_n)\sum_{\ell\le n}\sum_{x,m} t^{\omega}_{x,\ell}\left(\mathrm{e}^{(m+\ell)\phi(\alpha_n)}-1\right)\left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_n}-1\right)\left(f^{\theta_x\omega}_m-\mathbf{f}_m\right)\triangleq\sum_{\ell\le n} Z^{(n)}_{\ell}.$$

In terms of (2.9), we are working with coefficients

$$a^{(n)}_{x,\ell}(y,m)=a^{(n)}_{x,\ell}(m)=\left(\mathrm{e}^{(m+\ell)\phi(\alpha_n)}-1\right)\left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_n}-1\right).$$

In particular, (3.11) carries over, and we infer convergence to zero along the lacunary sequence $n^{2+\delta}$. In order to study the fluctuations of $\sum_{\ell\le n} Z^{(n)}_{\ell}$ on the intervals $[N,\dots,N+R]$, one, as was done in (3.15), employs the decomposition

$$\begin{aligned}&\sum_{1}^{N+r} Z^{(N+r)}_{\ell}-\sum_{1}^{N} Z^{(N)}_{\ell}\\&\quad=\sum_{\ell\le N+r}\sum_{x,m} t^{\omega}_{x,\ell}\big(a^{(N+r)}_{x,\ell}(m)-a^{(N)}_{x,\ell}(m)\big)\left(f^{\theta_x\omega}_m-\mathbf{f}_m\right)+\sum_{\ell=N+1}^{N+r} Z^{(N)}_{\ell}\end{aligned}\tag{3.31}$$

Since (3.11) holds, the second term on the right-hand side of (3.31) is worked out exactly as in (3.18). On the other hand,

$$\begin{aligned}a^{(N+r)}_{x,\ell}(m)-a^{(N)}_{x,\ell}(m)&=\left(\mathrm{e}^{(m+\ell)\phi(\alpha_{N+r})}-\mathrm{e}^{(m+\ell)\phi(\alpha_N)}\right)\left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_{N+r}}-1\right)\\&\quad+\left(1-\mathrm{e}^{(m+\ell)\phi(\alpha_N)}\right)\left(\mathrm{e}^{i(x-\ell v)\cdot\alpha_N}-\mathrm{e}^{i(x-\ell v)\cdot\alpha_{N+r}}\right)\end{aligned}\tag{3.32}$$

In view of Remark 1.2, we restrict attention to $\ell\le N+R$, which implies that $|\ell\phi(\alpha_N)|\lesssim 1$ uniformly in all the situations in question. Hence both terms above can be worked out exactly as in the cases of, respectively, the corrections $\eta^{\omega}_{n,2}$ and $\eta^{\omega}_{n,3}$.

# 4 Proof of the $\mathbb{L}_2$ estimate (2.4)

A variant of our target estimate (2.4) was proved in [11, Proposition 3.1] and we shall follow a similar line of reasoning and, eventually, rely on upper bounds derived in the latter paper.

## 4.1 Preliminaries

For $u,v\in\mathbb{Z}^{d+1}$ and $m\in\mathbb{N}$, we set

$$t^{\omega}_{u,v,m}\triangleq t^{\theta_u\omega}_{v-u,m},\quad f^{\omega}_{u,v,m}\triangleq f^{\theta_u\omega}_{v-u,m},\quad \mathbf{t}_{u,v,m}\triangleq\mathbb{E}(t^{\omega}_{u,v,m}),\quad \mathbf{f}_{u,v,m}\triangleq\mathbb{E}(f^{\omega}_{u,v,m}),$$

and

$$D(u,v)\triangleq(u+\mathcal{Y}^h)\cap(v-\mathcal{Y}^h)\,\cap\,\mathbb{Z}^{d+1}.$$

Moreover, we write $\mathcal{T}(u,v;n)$ for the set of all cone-confined paths $\gamma\subseteq D(u,v)$ of length $n$ leading from $u$ to $v$, and $\mathcal{F}(u,v;n)$ for the corresponding subset of irreducible paths.

Observe first that, by definition, $f^{\omega}_{u,v,m}$ is $\sigma\left\{V(x)\,:\,x\in D(u,v)\right\}$-measurable. In particular, if $D(x,y)\cap D(x',y')=\varnothing$, then

$$\mathbb{E}\big[t^{\omega}_{x,\ell}\,t^{\omega}_{x',\ell}\,\mathbb{E}(f^{\omega}_{x,y,m}-\mathbf{f}_{x,y,m}\,|\,\mathcal{A})\,\mathbb{E}(f^{\omega}_{x',y',m'}-\mathbf{f}_{x',y',m'}\,|\,\mathcal{A})\big]=0.\tag{4.1}$$

Indeed, in that case, either $D(x,y) \cap (x' - \mathcal{Y}^h) = \varnothing$, or $D(x',y') \cap (x - \mathcal{Y}^h) = \varnothing$. For definiteness, let us assume the latter. We can then conclude that the random variable $f^\omega_{x',y',m'}$ is independent of $t^\omega_{x,\ell}\, t^\omega_{x',\ell}\, \mathbb{E}(f^\omega_{x,y,m} - \mathbf{f}_{x,y,m} \,|\, \mathcal{A})$. The same is thus also true of $\mathbb{E}(f^\omega_{x',y',m'} - \mathbf{f}_{x',y',m'} \,|\, \mathcal{A})$, and the claim follows, since the latter has mean zero.

A second observation is that, for any $A \subseteq \mathbb{Z}^{d+1}$ and the corresponding cylindrical $\sigma$-algebra $\mathcal{A} = \sigma\{V(z) \,:\, z \in A\}$,

$$\mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\, \mathbb{E}(f^\omega_{x,y,m} \,|\, \mathcal{A})\, \mathbb{E}(f^\omega_{x',y',m'} \,|\, \mathcal{A})\big] \le \mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\, f^\omega_{x,y,m}\, f^\omega_{x',y',m'}\big]. \tag{4.2}$$

Indeed, define $g = (\lambda + \log(2d+2))(2\ell + m + m') - h \cdot (y + y')$ and let $\Sigma^*$ be the sum over all the paths $\gamma \in \mathcal{T}(0,x;\ell), \eta \in \mathcal{F}(x,y;m), \gamma' \in \mathcal{T}(0,x';\ell)$ and $\eta' \in \mathcal{F}(x',y';m')$. Then the attractivity property (1.5) implies that

$$\begin{aligned} &\mathrm{e}^g\, \mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\, \mathbb{E}(f^\omega_{x,y,m} \,|\, \mathcal{A})\, \mathbb{E}(f^\omega_{x',y',m'} \,|\, \mathcal{A})\big] \\ &\quad = \sum\nolimits^* \prod_{u \in A} \mathrm{e}^{-\phi_\beta(\ell_\gamma(u) + \ell_{\gamma'}(u) + \ell_\eta(u) + \ell_{\eta'}(u))} \prod_{v \notin A} \mathrm{e}^{-\phi_\beta(\ell_\eta(v)) - \phi_\beta(\ell_{\eta'}(v)) - \phi_\beta(\ell_\gamma(v) + \ell_{\gamma'}(v))} \\ &\quad \le \sum\nolimits^* \prod_{u \in \mathbb{Z}^{d+1}} \mathrm{e}^{-\phi_\beta(\ell_\gamma(u) + \ell_{\gamma'}(u) + \ell_\eta(u) + \ell_{\eta'}(u))} = \mathrm{e}^g\, \mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\, f^\omega_{x,y,m}\, f^\omega_{x',y',m'}\big]. \end{aligned}$$

Note that (4.2) implies, in particular, that

$$\begin{aligned} &\big|\mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\, \mathbb{E}(f^\omega_{x,y,m} - \mathbf{f}_{x,y,m} \,|\, \mathcal{A})\, \mathbb{E}(f^\omega_{x',y',m'} - \mathbf{f}_{x',y',m'} \,|\, \mathcal{A})\big]\big| \\ &\qquad\qquad \le 2\, \mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\, f^\omega_{x,y,m}\, f^\omega_{x',y',m'}\big]\, \mathbb{1}_{\{D(x,y) \cap D(x',y') \neq \varnothing\}}. \end{aligned} \tag{4.3}$$

### 4.2 Getting rid of the last irreducible steps

For several paths $\gamma_1, \dots, \gamma_k$, define

$$\Phi_\beta(\gamma_1, \dots, \gamma_k) = \sum_{u \in \mathbb{Z}^{d+1}} \phi_\beta\Big(\sum_1^k \ell_{\gamma_i}(u)\Big).$$

Applying (1.5) once more, we see that

$$\{\Phi_\beta(\gamma, \gamma', \eta, \eta') - \Phi_\beta(\gamma, \gamma')\} + \{(m + m')\phi_\beta(1) - \Phi_\beta(\eta) - \Phi_\beta(\eta')\} \ge 0, \tag{4.4}$$

uniformly in all paths $\gamma \in \mathcal{T}(0,x;\ell)$, $\gamma' \in \mathcal{T}(0,x';\ell)$, $\eta \in \mathcal{F}(x,y;m)$ and $\eta' \in \mathcal{F}(x',y';m')$. This implies that

$$\sum_{\substack{\eta \in \mathcal{F}(x,y;m), \\ \eta' \in \mathcal{F}(x',y';m')}} \mathrm{e}^{-\Phi_\beta(\gamma, \gamma', \eta, \eta')} \le \mathrm{e}^{\phi_\beta(1)(m+m')}\, \mathrm{e}^{-\Phi_\beta(\gamma, \gamma')} \sum_{\substack{\eta \in \mathcal{F}(x,y;m), \\ \eta' \in \mathcal{F}(x',y';m')}} \mathrm{e}^{-\Phi_\beta(\eta) - \Phi_\beta(\eta')}.$$

Since $\lim_{\beta \to 0} \phi_\beta(1) = -\log(1 - p_\infty)$, it follows from (4.3) and (1.16) that, in the very weak disorder regime,

$$\begin{aligned} &\big|\mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\, \mathbb{E}(f^\omega_{x,y,m} - \mathbf{f}_{x,y,m} \,|\, \mathcal{A})\, \mathbb{E}(f^\omega_{x',y',m'} - \mathbf{f}_{x',y',m'} \,|\, \mathcal{A})\big]\big| \\ &\qquad\qquad \le 2\mathrm{e}^{\phi_\beta(1)(m+m')}\, \mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\big]\, \mathbb{1}_{\{D(x,y) \cap D(x',y') \neq \varnothing\}}\, \mathbf{f}_{x,y,m} \mathbf{f}_{x',y',m'} \\ &\qquad\qquad \le 2\mathrm{e}^{-(\nu - \phi_\beta(1))(m+m')}\, \mathbb{1}_{\{D(x,y) \cap D(x',y') \neq \varnothing\}}\, \mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\big] \\ &\qquad\qquad \le 2\mathrm{e}^{-(\nu/4)(m+m') - (\nu/4)(|x - x'|)}\, \mathbb{E}\big[t^\omega_{x,\ell}\, t^\omega_{x',\ell}\big]. \end{aligned}$$

Indeed, in order for the event $D(x,y) \cap D(x',y') \neq \varnothing$ to occur, it is necessary that $|x - x'| \le m \vee m' \le m + m'$.

### 4.3 Weakly interacting random walks

There remains to prove that

$$\mathbb{E}\big[t^{\omega}_{x,\ell}t^{\omega}_{x',\ell}\big] \le \frac{c_1}{\ell^{d+1-\rho}}\exp\Big\{-c_2\frac{|x-\ell v|^2}{\ell} - c_2\frac{|x'-\ell v|^2}{\ell}\Big\}. \tag{4.5}$$

Let us denote by $\mathbb{P}_{\rm RW}$ the law of the random walk on $\mathbb{Z}^{d+1}\times\mathbb{N}$, whose increments have law $(\mathbf{f}_{x,n})_{x\in\mathbb{Z}^{d+1},n\in\mathbb{N}}$. We shall denote its (random) trajectory by the couple $(\mathbf{X},\mathbf{L})$, with $\mathbf{X}=(X_0=0,X_1,X_2,\ldots)$ and $\mathbf{L}=(L_0=0,L_1,L_2,\ldots)$. With these notations, we can write

$$\mathbb{E}\big[t^{\omega}_{x,\ell}\big] = \mathbf{t}_{x,\ell} = \mathbb{P}_{\rm RW}(\exists k:\,(X_k,L_k)=(x,\ell)).$$

In general, the left-hand side of (4.5) does not allow for a similar expression. Notice, however, that the attractivity property implies the lower bound

$$\mathbb{E}\big[t^{\omega}_{x,\ell}\,t^{\omega}_{x',\ell}\big] \ge \mathbf{t}_{x,\ell}\mathbf{t}_{x',\ell} = \mathbb{P}^{\otimes}_{\rm RW}\big(\exists k,k':\,(X_k,L_k)=(x,\ell),(X'_{k'},L'_{k'})=(x',\ell)\big),$$

where $\mathbb{P}^{\otimes}_{\rm RW}$ denotes the law of a couple of independent random walks $(\mathbf{X},\mathbf{L})$ and $(\mathbf{X}',\mathbf{L}')$ as above. It is important to observe that (4.5) would be an immediate consequence of the local limit theorem for random walks if its left-hand side was replaced by $\mathbf{t}_{x,\ell}\mathbf{t}_{x',\ell}$. To prove (4.5), we thus have to prove that, in the very weak disorder regime, this local limit behaviour is not destroyed by the effective attractive interaction between the two paths resulting from averaging $t^{\omega}_{x,\ell}t^{\omega}_{x',\ell}$ over the disorder.

To facilitate the notation, define the events $R_{x,\ell}=\{\exists k:\,(X_k,L_k)=(x,\ell)\}$ and $R_{x',\ell}=\{k':\,(X'_k,L_k)=(x',\ell)\}$. Then, by the very same attractivity property of the potential,

$$\mathbb{E}\big[t^{\omega}_{x,\ell}\,t^{\omega}_{x',\ell}\big] \le \mathbb{E}^{\otimes}_{\rm RW}\big[\mathrm{e}^{\Delta_\beta(\mathbf{X},\mathbf{X}')}\mathbb{1}_{R_{x,\ell}}\mathbb{1}_{R_{x',\ell}}\big], \tag{4.6}$$

where

$$\Delta_\beta(\mathbf{X},\mathbf{X}') = \log\Big\{\sum_{\substack{\gamma\sim\mathbf{X}\\ \gamma'\sim\mathbf{X}'}} \mathrm{e}^{\Phi_\beta(\gamma)+\Phi_\beta(\gamma')-\Phi_\beta(\gamma,\gamma')}\Big\}. \tag{4.7}$$

Fix $\rho<1/12$. Formula (4.13 ) in [11] implies that

$$\mathbb{E}^{\otimes}_{\rm RW}\mathrm{e}^{\frac{d+1}{\rho}\Delta_\beta(\mathbf{X},\mathbf{X}')} \lesssim 1$$

in the very weak disorder regime. Therefore, by Hölder inequality,

$$\mathbb{E}\big[t^{\omega}_{x,\ell}\,t^{\omega}_{x',\ell}\big] \lesssim (\mathbf{t}_{x,\ell}\mathbf{t}_{x',\ell})^{\frac{d+1-\rho}{d+1}}\ .$$

The target (4.5) follows now by (2.2). □

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