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Sub-ballistic random walk in Dirichlet environment*

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Abstract

We consider random walks in Dirichlet environment (RWDE) on $\mathbb{Z}^d$, for $d \geq 3$, in the sub-ballistic case. We associate to any parameter $(\alpha_1, \dots, \alpha_{2d})$ of the Dirichlet law a time-change to accelerate the walk. We prove that the continuous-time accelerated walk has an absolutely continuous invariant probability measure for the environment viewed from the particle. This allows to characterize directional transience for the initial RWDE. It solves as a corollary the problem of Kalikow's $0-1$ law in the Dirichlet case in any dimension. Furthermore, we find the polynomial order of the magnitude of the original walk's displacement.

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1 Introduction

The behaviour of random walks in random environment (RWRE) is fairly well understood in the case of dimension 1 (see Solomon ([16]), Kesten, Kozlov, Spitzer ([8]) and Sinaï([15])). In the multidimensional case, some results are available under ballisticity conditions (we refer to [20] and [2] for an overview of progress in this direction), or in the case of small perturbations. But some simple questions remain unanswered. For example, there is no general characterization of recurrence, Kalikow's $0-1$ law is known only for $d \leq 2$ ([21]).

Random walks in Dirichlet environment (RWDE) is the special case when the transition probabilities at each site are chosen as i.i.d. Dirichlet random variables. RWDE are interesting because of the analytical simplifications they offer, and because of their link with reinforced random walks. Indeed, the annealed law of a RWDE corresponds to the law of a linearly directed-edge reinforced random walk ([4], [11]). This model first appeared in [11] in relation with edge reinforced random walks on trees. It was then studied on $\mathbb{Z} \times G$ ([7]), and on $\mathbb{Z}^d$ ([5],[18],[12],[13],[14]).

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We are interested in RWDE on $\mathbb{Z}^d$ for $d \geq 3$. A condition on the weights ensures that the mean time spent in finite boxes is finite. Under this condition, it was proved ([13]) that there exists an invariant probability measure for the environment viewed from the particle, absolutely continuous with respect to the law of the environment. Using [14], this gives some criteria on ballisticity.

In this paper, we focus on the case when the condition on the weights is not satisfied. Then the mean time spent in finite boxes is infinite, and there is no absolutely continuous invariant probability measure ([13]). The law of large numbers gives a zero speed. To overcome this difficulty, we construct a time-change that accelerates the walk, such that the accelerated walk spends a finite mean time in finite boxes. An absolutely continuous invariant probability measure then exists. With ergodic results, it gives a characterization of the directional recurrence in the sub-ballistic case. As a corollary, it solves the problem of Kalikow's $0-1$ law in the Dirichlet case (the case $d=2$ has been treated in [21]).

Besides, in the directionally transient case, we show a law of large numbers with positive speed for our accelerated walk. This gives the polynomial order of the magnitude of the original walk's displacement, and could be a first step towards a limit theorem for the original RWDE.

2 Definitions and statement of the results

Let $(e_1,\dots,e_d)$ be the canonical basis of $\mathbb{Z}^d$, $d\geq 3$, and set $e_j=-e_{j-d}$, for $j\in[\![d+1,2d]\!]$. The set $\{e_1,\dots,e_{2d}\}$ is the set of unit vectors of $\mathbb{Z}^d$. We denote by $\|z\|=\sum_{i=1}^d |z_i|$ the L_1-norm of $z\in\mathbb{Z}^d$, and write $x\sim y$ if $\|y-x\|=1$. We consider the set of directed edges $E=\{(x,y)\in(\mathbb{Z}^d)^2, x\sim y\}$. Let Ω be the set of all possible environments on $\mathbb{Z}^d$:

$$\Omega=\{\omega=(\omega(x,y))_{x\sim y}\in]0,1]^E \text{ such that } \forall x\in\mathbb{Z}^d, \sum_{i=1}^{2d}\omega(x,x+e_i)=1\}.$$

For each $\omega\in\Omega$, we run a Markov chain Z_n on $\mathbb{Z}^d$ defined by the following transition probabilities : $\forall(x,y)\in\mathbb{Z}^d$, $\forall i\in[\![1,2d]\!]$,

$$P_x^\omega(Z_{n+1}=y+e_i|Z_n=y)=\omega(y,y+e_i).$$

We are interested in random iid Dirichlet environments. Given a family of positive weights $(\alpha_1,\dots,\alpha_{2d})$, a random iid Dirichlet environment is $\omega\in\Omega$ constructed by choosing independently at each site $x\in\mathbb{Z}^d$ the values of $(\omega(x,x+e_i))_{i\in[\![1,2d]\!]}$ according to a Dirichlet law with parameters $(\alpha_1,\dots,\alpha_{2d})$ that is with density :

$$\frac{\Gamma\left(\sum_{i=1}^{2d}\alpha_i\right)}{\prod_{i=1}^{2d}\Gamma(\alpha_i)}\left(\prod_{i=1}^{2d}x_i^{\alpha_i-1}\right)dx_1\dots dx_{2d-1}$$

on the simplex

$$\{(x_1,\dots,x_{2d})\in]0,1]^{2d},\sum_{i=1}^{2d}x_i=1\}.$$

Here Γ denotes the Gamma function $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt$, and $dx_1\dots dx_{2d-1}$ represents the image of the Lebesgue measure on $\mathbb{R}^{2d-1}$ by the application $(x_1,\dots,x_{2d-1})\to(x_1,\dots,x_{2d-1},1-x_1-\dots-x_{2d-1})$. Obviously, the law does not depend on the specific role of x_{2d}. We denote by $\mathbb{P}^{(\alpha)}$ the law obtained on Ω this way, by $\mathbb{E}^{(\alpha)}$ the expectation with respect to $\mathbb{P}^{(\alpha)}$, and by $\mathbb{P}_x^{(\alpha)}[.]=\mathbb{E}^{(\alpha)}[P_x^\omega(.)]$ the annealed law of the process starting at x.

In [13], it was proved that when

$$\kappa = 2\left(\sum_{i=1}^{2d}\alpha_i\right) - \max_{i=1,\dots,d}(\alpha_i+\alpha_{i+d}) > 1,$$

there exists an invariant probability measure for the environment viewed from the particle, absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$. This leads to a complete description of ballistic regimes and directional transience. However, when $\kappa \le 1$, such an invariant probability does not exist, and we only know that the walk is sub-ballistic. In this paper, we focus on the case $\kappa \le 1$. We prove the existence of an invariant probability measure for an accelerated walk. This allows to characterize recurrence in each direction for the initial walk.

Let $\sigma = (e^1,\dots,e^n)$ be a directed path. By directed path, we mean a sequence of directed edges e^i such that $\overline{e^i} = \underline{e^{i+1}}$ for all i ($\underline{e}$ and $\overline{e}$ are the head and tail of the edge e). We note $\omega_\sigma = \prod_{i=1}^n \omega(e^i)$. Let Λ be a finite connected set of vertices containing 0. Our accelerating function is :

$$\gamma^\omega(x) = \frac{1}{\sum \omega_\sigma}, \tag{2.1}$$

where the sum is on all σ finite simple (each vertex is visited at most once) paths starting from x, going out of $x+\Lambda$, and stopped just after exiting $x+\Lambda$. Let X_t be the continuous-time Markov chain whose jump rate from x to y is $\gamma^\omega(x)\omega(x,y)$, with $X_0=0$. Then $Z_n = X_{t_n}$, for $t_n = \sum_{k=1}^n \frac{1}{\gamma^\omega(Z_k)}E_k$, where the E_i are independent exponentially distributed random variables with rate parameters 1 : X_t is an accelerated version of the walk Z_n.

We note $(\tau_x)_{x\in\mathbb{Z}^d}$ the shift on the environment defined by : $\tau_x\omega(y,z) = \omega(x+y,x+z)$, and call process seen from the particle the process defined by $\overline{\omega}_t = \tau_{X_t}\omega$. Under $P_0^{\omega_0}$ ($\omega_0\in\Omega$), $\overline{\omega}_t$ is a Markov process on state space Ω, his generator R is given by

$$Rf(\omega) = \sum_{i=1}^{2d}\gamma^\omega(0)\omega(0,e_i)\left(f(\tau_{e_i}\omega) - f(\omega)\right),$$

for all bounded measurable functions f on Ω. Invariant probability measures absolutely continuous with respect to the law of the environment are a classical tool to study processes viewed from the particle. The following theorem provides one for our accelerated walk.

Theorem 2.1. *Let $d \ge 3$ and $\mathbb{P}^{(\alpha)}$ be the law of the Dirichlet environment for the weights $(\alpha_1,\dots,\alpha_{2d})$. Let $\kappa^\Lambda > 0$ be defined by*

$$\kappa^\Lambda = \min\{\sum_{e\in\partial_+(K)}\alpha_e\ ,\ K \text{ connected set of vertices }, 0\in K \text{ and } \partial\Lambda\cap K \neq \emptyset\}$$

where $\partial_+(K) = \{e\in E,\ \underline{e}\in K,\ \overline{e}\notin K\}$ and $\partial\Lambda = \{x\in\Lambda | \exists y\sim x \text{ such that } y\notin\Lambda\}$. If $\kappa^\Lambda > 1$, there exists a unique probability measure $\mathbb{Q}^{(\alpha)}$ on Ω that is absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$ and invariant for the generator R. Furthermore, $\frac{d\mathbb{Q}^{(\alpha)}}{d\mathbb{P}^{(\alpha)}}$ is in $L_p(\mathbb{P}^{(\alpha)})$ for all $1 \le p < \kappa^\Lambda$.

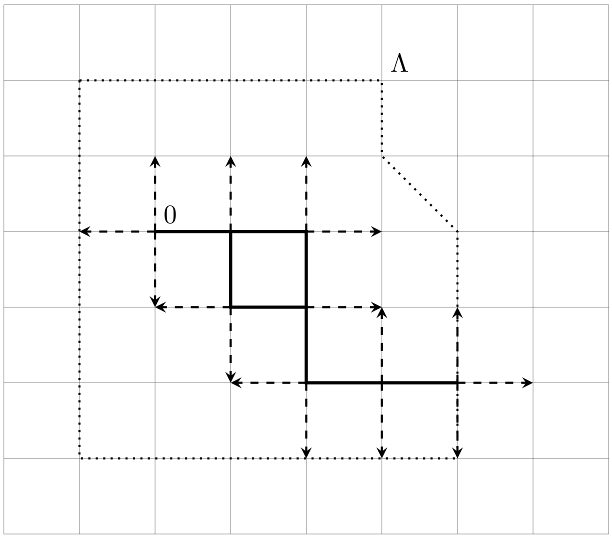


Figure : $\sum_{e\in\partial_+(K)} \alpha_e$ (dashed arrows) for an arbitrary K (thick lines).

Remark 2.2. *If Λ is a box of radius R_Λ, the formula is explicit :*

$$\kappa^\Lambda = \min_{i_0\in[\![1,d]\!]} \left(\alpha_{i_0} + \alpha_{i_0+d} + (R_\Lambda+1)\sum_{i\neq i_0}(\alpha_i+\alpha_{i+d}) \right).$$

Remark 2.3. *κ^Λ can be made as big as we want by taking the set Λ big enough. Then for each $(\alpha_1,\dots,\alpha_{2d})$, there exists an acceleration function such that the accelerated walk verifies theorem 2.1.*

Let $d_\alpha = \mathbb{E}_0^{(\alpha)}[Z_1] = \frac{1}{\sum_{i=1}^{2d}\alpha_i}\sum_{i=1}^{2d}\alpha_i e_i$ be the drift after the first jump.

Theorem 2.4. *Let $d\geq 3$,*

i) If $\kappa^\Lambda > 1$ and $d_\alpha = 0$, then

$$\lim_{t\to+\infty}\frac{X_t}{t} = 0,\ \mathbb{P}_0^{(\alpha)}\text{a.s.},$$

and $\forall i = 1\dots d$,

$$\liminf_{t\to+\infty} X_t\cdot e_i = -\infty,\ \limsup_{t\to+\infty} X_t\cdot e_i = +\infty,\ \mathbb{P}_0^{(\alpha)}\text{a.s.}.$$

ii) If $\kappa^\Lambda > 1$ and $d_\alpha \neq 0$, then $\exists v\neq 0$ such that

$$\lim_{t\to+\infty}\frac{X_t}{t} = v,\ \mathbb{P}_0^{(\alpha)}\text{a.s.},$$

and $\forall i = 1\dots d$ such that $d_\alpha\cdot e_i\neq 0$, we have

$$(d_\alpha\cdot e_i)(v\cdot e_i) > 0,$$

whereas if $d_\alpha\cdot e_i = 0$,

$$\liminf_{t\to+\infty} X_t\cdot e_i = -\infty,\ \limsup_{t\to+\infty} X_t\cdot e_i = +\infty,\ \mathbb{P}_0^{(\alpha)}a.s..$$

As $(X_t)_{t\in\mathbb{R}_+}$ and $(Z_n)_{n\in\mathbb{N}}$ go through exactly the same vertices in the same order, and as the two processes stay a finite time on each vertex without exploding, recurrence and transience for the original walk $Z_n \cdot e_i$ follow from those of $X_t \cdot e_i$.

Corollary 2.5. *Let $d \geq 3$, for $i = 1, \dots, d$*

i) If $d_\alpha \cdot e_i = 0$,

$$\liminf_{n\to+\infty} Z_n \cdot e_i = -\infty,\ \limsup_{n\to+\infty} Z_n \cdot e_i = +\infty,\ \mathbb{P}_0^{(\alpha)}\text{a.s.}.$$

ii) If $d_\alpha \cdot e_i > 0$,

$$\lim_{n\to+\infty} Z_n \cdot e_i = +\infty,\ \mathbb{P}_0^{(\alpha)}\text{a.s.}.$$

iii) If $d_\alpha \cdot e_i < 0$,

$$\lim_{n\to+\infty} Z_n \cdot e_i = -\infty,\ \mathbb{P}_0^{(\alpha)}\text{a.s.}.$$

The proof of theorem 2.4 allows besides to solve the problem of Kalikow's $0-1$ law in the Dirichlet case.

Corollary 2.6 (Kalikow's $0-1$ law in the Dirichlet case)**.** *Let $\mathbb{P}^{(\alpha)}$ be the law of the Dirichlet environment on $\mathbb{Z}^d$, $d \geq 1$, for the weights $(\alpha_1, \dots, \alpha_{2d})$, and Z_n the associated random walk in Dirichlet environment. Then for all $l \in \mathbb{R}^d \setminus \{0\}$, we are in one of the following cases :*

- $\liminf_{n\to+\infty} Z_n \cdot l = -\infty,\ \limsup_{n\to+\infty} Z_n \cdot l = +\infty,\ \mathbb{P}_0^{(\alpha)} a.s.$,
- $\lim_{n\to+\infty} Z_n \cdot l = -\infty,\ \mathbb{P}_0^{(\alpha)} a.s.$,
- $\lim_{n\to+\infty} Z_n \cdot l = +\infty,\ \mathbb{P}_0^{(\alpha)} a.s.$.

Remark 2.7. *Theorem 2.4 also gives the existence of a deterministic asymptotic direction $\mathbb{P}_0^{(\alpha)}$ a.s. when $d \geq 3$ and $d_\alpha \neq 0$. As I was finishing this article, Tournier informed me about the existence of a more general version of theorem 1 of [14]. Using this result instead of [14] in the proof of theorem 2.4 allows to show that the asymptotic direction is $\frac{d_\alpha}{|d_\alpha|}$, see [19] for details.*

In the transient sub-ballistic case, we also obtain the polynomial order of the magnitude of the walk's displacement :

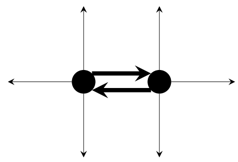
Theorem 2.8. *Let $d \geq 3$, $\mathbb{P}^{(\alpha)}$ be the law of the Dirichlet environment with parameters $(\alpha_1, \dots, \alpha_{2d})$ on $\mathbb{Z}^d$, and Z_n the associated random walk in Dirichlet environment. We suppose that $\kappa = 2\left(\sum_{i=1}^{2d} \alpha_i\right) - \max_{i=1,\dots,d}(\alpha_i + \alpha_{i+d}) \leq 1$. Let $l \in \{e_1, \dots, e_{2d}\}$ be such that $d_\alpha \cdot l \neq 0$. Then*

$$\lim_{n\to+\infty} \frac{\log(Z_n \cdot l)}{\log(n)} = \kappa \text{ in } \mathbb{P}^{(\alpha)}\text{-probability.}$$

Remark 2.9. *The directional transience shown in [19] should also enable to extend the results of theorem 2.4, corollary 2.5 and theorem 2.8 from $(e_i)_{i=1,\dots,2d}$ to any $l \in \mathbb{R}^d$.*

3 Outline of the proofs

All the difficulties compared to [13] come from the presence of some big traps in the sub-ballistic case. When the parameters $\alpha_1, \dots, \alpha_{2d}$ are small, the Dirichlet environment is very disordered. As the environment is not uniformly elliptic, it creates finite sets of edges where a lot of time is spent. The walk is trapped on those edges. The simplest example of trap is the case of two opposite edges with high probabilities :



To deal with the traps, we introduce an accelerated walk X_t, with the jump rates $\gamma^\omega(x)\omega(x,y)$ from x to y, which moves faster in strong traps. We choose the accelerating function $\gamma^\omega(x) = \frac{1}{\sum \omega_\sigma}$ defined in (2.1), where we sum on some paths exiting a box $x+\Lambda$. It "kills" all the traps of size smaller than $|\Lambda|$, and for Λ big enough the walk X_t becomes ballistic.

To prove theorems 2.1 and 2.4, we try to adapt the proofs of [13] to the accelerated walk X_t. First we need an invariant measure absolutely continuous with respect to $\mathbb{P}^{(\alpha)}$ for the process seen from the particle $\overline{\omega}_t$. For this we approximate $\mathbb{Z}^d$ by a periodic torus of large size N. As the torus is a finite graph, there exists an invariant probability $\tilde{\pi}_N^\omega$ for X_t, and $N^d \tilde{\pi}_N^\omega(0)\mathbb{P}_N^{(\alpha)}$ is an invariant probability for $\overline{\omega}_t$. We are then reduced to find a bound for $\mathbb{E}^{(\alpha)}\left[(N^d\tilde{\pi}_N^\omega(0))^p\right]$, uniformly in N, for $p < \kappa^\Lambda$.

This is bounded by $\mathbb{E}^{(\alpha)}\left[\prod_{x\in T_N}\left(\frac{\tilde{\pi}_N^\omega(0)}{\tilde{\pi}_N^\omega(x)}\right)^{p/N^d}\right]$ thanks to the arithmetico-geometrical inequality. We then use lemma 1 of [13] which tells that the time-reversed environment $\check{\omega}$ still follows i.i.d. Dirichlet laws of known parameters. We therefore rewrite the previous expectation as $\mathbb{E}^{(\alpha)}\left[\frac{\check{\omega}^{\check{\theta}_N}}{\omega^{\theta_N}}\left(\gamma^\omega(0)\right)^{\frac{p}{N^d}-p} \text{ ***}\right]$, where θ_N can be any positive function that follows some divergence conditions., and *** are extra-terms made precise in the proof (see expression 4.4) unnecessary for this heuristic approach.

At this point, Holder's inequality allows to bound this expectation by

$$\mathbb{E}^{(\alpha)}\left[\omega^{-q\theta_N}\left(\gamma^\omega(0)\right)^{\frac{pq}{N^d}-pq} \text{ ***}\right]^{\frac{1}{q}} \mathbb{E}^{(\alpha)}\left[\check{\omega}^{r\check{\theta}_N}\right]^{\frac{1}{r}}$$

where ω and $\check{\omega}$ follow i.i.d. Dirichlet laws of known parameters. This is where the acceleration factor comes useful : $\omega^{-q\theta_N}$ has finite expectation only when $\alpha(e) - q\theta_N(e) > 0$ for all edges e. The γ^ω factors allow to keep this integrability under weaker conditions.

A Taylor expansion then gives $\mathbb{E}^{(\alpha)}\left[(N^d\tilde{\pi}_N^\omega(0))^p\right] \leq \exp(c\sum\theta_N(e)^2)$. It only remains to show there exists θ_N of finite energy satisfying the divergence condition. For this, we use a theorem of the type max-flow min-cut, in a generalized version that also gives a bound on the norm of the flow.

It proves the existence of the invariant measure for the graph $\mathbb{Z}^d$. We conclude the proof of theorem 2.4 by proving its ergodicity and applying Birkhoff's ergodic theorem. The other results are corollaries.

4 Proof of theorem 2.1

We first give some definitions and notations. Let $G = (V,E)$ be an oriented graph. For $e \in E$, we note $\underline{e}$ the tail of the edge, and $\overline{e}$ his head, such that $e = (\underline{e}, \overline{e})$. The divergence operator is : div : $\mathbb{R}^E \to \mathbb{R}^V$ such that : $\forall x \in \mathbb{Z}^d$,

$$\mathrm{div}(\theta)(x) = \sum_{e\in E,\ \underline{e}=x} \theta(e) - \sum_{e\in E,\ \overline{e}=x} \theta(e).$$

For $N \in \mathbb{N}^*$, we set $T_N = (\mathbb{Z}/N\mathbb{Z})^d$ the d-dimensional torus of size N. We note $G_N = (T_N, E_N)$ the directed graph obtained by projection of $(\mathbb{Z}^d, E)$ on the torus T_N. Let Ω_N be the space of elliptic random environments on the torus :

$$\Omega_N = \{\omega = (\omega(x,y))_{x\sim y} \in]0,1]^{E_N} \text{ such that } \forall x \in T_N, \sum_{i=1}^{2d} \omega(x, x+e_i) = 1\}.$$

We denote by $\mathbb{P}_N^{(\alpha)}$ the law on the environment obtained by choosing independently for each $x \in T_N$ the exit probabilities of x according to a Dirichlet law with parameters $(\alpha_1, \dots, \alpha_{2d})$.

For $\omega \in \Omega_N$, we note π_N^ω the unique (because of ellipticity) invariant probability measure of Z_n^ω on the torus in the environment ω. Then $\left(\frac{\pi_N^\omega(x)}{\gamma^\omega(x)}\right)_{x\in T_N}$ is an invariant measure for X_t^ω on the torus in the environment ω, and

$$\tilde{\pi}_N^\omega(y) := \frac{\frac{\pi_N^\omega(y)}{\gamma^\omega(y)}}{\sum_{x\in T_N} \frac{\pi_N^\omega(x)}{\gamma^\omega(x)}}$$

is the associated invariant probability. Define

$$f_N(\omega) := N^d \tilde{\pi}_N^\omega(0) \text{ and } \mathbb{Q}_N^{(\alpha)} := f_N \mathbb{P}_N^{(\alpha)},$$

then, thanks to translation invariance, $\mathbb{Q}_N^{(\alpha)}$ is an invariant probability measure on Ω_N for the generator R of the accelerated process seen from the particle.

Proof. Let f be a bounded measurable function on Ω_N.

$$\begin{aligned}
&\int_{\Omega_N} Rf(\omega)\, d\mathbb{Q}_N^{(\alpha)}(\omega)\\
&= \int_{\Omega_N} \sum_{i=1}^{2d} \gamma^\omega(0)\omega(0,e_i)\left(f(\tau_{e_i}\omega) - f(\omega)\right)\, d\mathbb{Q}_N^{(\alpha)}(\omega)\\
&= \sum_{i=1}^{2d} \int_{\Omega_N} \gamma^\omega(0)\omega(0,e_i) f(\tau_{e_i}\omega) N^d \tilde{\pi}_N^\omega(0)\, d\mathbb{P}_N^{(\alpha)}(\omega) - \int_{\Omega_N} \gamma^\omega(0) f(\omega)\, d\mathbb{Q}_N^{(\alpha)}(\omega)
\end{aligned}$$

Using first translation invariance, then the definition of $\tilde{\pi}_N^\omega$ and the fact that π_N^ω is a probability measure, we get :

$$\begin{aligned}
&\sum_{i=1}^{2d} \int_{\Omega_N} \gamma^\omega(0)\omega(0,e_i) f(\tau_{e_i}\omega) N^d \tilde{\pi}_N^\omega(0)\, d\mathbb{P}_N^{(\alpha)}(\omega)\\
&= \sum_{i=1}^{2d} \int_{\Omega_N} \gamma^\omega(e_i)\omega(e_i,0) f(\omega) N^d \tilde{\pi}_N^\omega(e_i)\, d\mathbb{P}_N^{(\alpha)}(\omega)\\
&= \int_{\Omega_N} \sum_{i=1}^{2d} \omega(e_i,0) \frac{\pi_N^\omega(e_i)}{\sum_{x\in T_N} \frac{\pi_N^\omega(x)}{\gamma^\omega(x)}} f(\omega) N^d\, d\mathbb{P}_N^{(\alpha)}(\omega)\\
&= \int_{\Omega_N} \frac{\pi_N^\omega(0)}{\sum_{x\in T_N} \frac{\pi_N^\omega(x)}{\gamma^\omega(x)}} f(\omega) N^d\, d\mathbb{P}_N^{(\alpha)}(\omega)\\
&= \int_{\Omega_N} \gamma^\omega(0) f(\omega)\, d\mathbb{Q}_N^{(\alpha)}(\omega)
\end{aligned}$$

It gives $\int_{\Omega_N} Rf(\omega)\ d\mathbb{Q}_N^{(\alpha)}(\omega) = 0$, and proves that $\mathbb{Q}_N^{(\alpha)}$ is an invariant probability measure for R. $\square$

We can now reduce theorem 2.1 to the following lemma.

Lemma 4.1. $\forall p \in [1, \kappa^\Lambda[$,

$$\sup_{N\in\mathbb{N}} \parallel f_N \parallel_{L_p(\mathbb{P}_N^{(\alpha)})} < +\infty.$$

Once this lemma is proved, the proof of theorem 2.1 follows easily, we refer to [13], pages $5, 6$, where the situation is exactly the same, or to [2], pages 11 and $18, 19$.

Proof of lemma 4.1. This proof is divided in two main steps. First we introduce the "time-reversed environment" and prepare the application of the "time reversal invariance" (lemma 1 of [12], or proposition 1 of [14]). Then we apply this invariance, and use a lemma of the type "max-flow min-cut problem".

Step 1 **:**

Let $(\omega(x,y))_{x\sim y}$ be in Ω_N. The time-reversed environment is defined by : $\forall (x,y) \in T_N^2$, $x \sim y$,

$$\check{\omega}(x,y) = \omega(y,x)\frac{\pi_N^\omega(y)}{\pi_N^\omega(x)}.$$

We know that : $\forall x \in T_N$,

$$\sum_{\underline{e}=x} \alpha(e) = \sum_{\overline{e}=x} \alpha(e) = \sum_{j=1}^{2d} \alpha_j,$$

then $\mathrm{div}(\alpha)(x) = 0$. We can therefore apply lemma 1 of [12] which gives : if $(\omega(x,y))$ is distributed according to $\mathbb{P}_N^{(\alpha)}$, then $(\check{\omega}(x,y))$ is distributed according to $\mathbb{P}_N^{(\check{\alpha})}$, where $\forall (x,y) \in E_N^2$,

$$\check{\alpha}(x,y) = \alpha(y,x).$$

Let p be a real, $1 < p < \kappa^\Lambda$. We have :

$$(f_N(\omega))^p = \left(N^d \tilde{\pi}_N^\omega(0)\right)^p.$$

Introducing the immediate fact that

$$1 = \sum_{x\in T_N} \tilde{\pi}_N^\omega(x),$$

it gives :

$$(f_N(\omega))^p = \left(\frac{\tilde{\pi}_N^\omega(0) \times N^d}{\sum\limits_{x\in T_N} \tilde{\pi}_N^\omega(x)}\right)^p$$

we can then use the arithmetico-geometric inequality :

$$\begin{aligned}(f_N(\omega))^p &\leq \prod_{x\in T_N} \left(\frac{\tilde{\pi}_N^\omega(0)}{\tilde{\pi}_N^\omega(x)}\right)^{\frac{p}{N^d}} \\ &= \prod_{x\in T_N} \left(\left(\frac{\pi_N^\omega(0)}{\pi_N^\omega(x)}\right)^{\frac{p}{N^d}} \left(\frac{\gamma^\omega(x)}{\gamma^\omega(0)}\right)^{\frac{p}{N^d}}\right)\end{aligned}$$

For N big enough and $x \neq 0$, the "$\gamma^\omega(x)$" term will have few effects on the integrability of $(f_N(\omega))^p$, as it is set to the small enough power $\frac{p}{N^d}$. On the contrary, the "$\gamma^\omega(0)$" term is set to the power $\frac{p}{N^d} - p$, and will therefore be small when 0 is in a trap : it is the key to the integrability computation for the accelerated walk. We thus need to give now some bounds for $\gamma^\omega(0)$, whereas $\gamma^\omega(x)^{\frac{p}{N^d}}$ will be dealt with later.

$$\begin{aligned}
(f_N(\omega))^p &\le \left(\frac{1}{\gamma^\omega(0)}\right)^{p-\frac{p}{N^d}} \prod_{x\in T_N}\left(\frac{\pi_N^\omega(0)}{\pi_N^\omega(x)}\right)^{\frac{p}{N^d}} \prod_{\substack{x\in T_N\\ x\neq 0}} (\gamma^\omega(x))^{\frac{p}{N^d}} \\
&= \left(\sum_{\sigma:0\to\Lambda^c}\omega_\sigma\right)^{p-\frac{p}{N^d}} \prod_{x\in T_N}\left(\frac{\pi_N^\omega(0)}{\pi_N^\omega(x)}\right)^{\frac{p}{N^d}} \prod_{\substack{x\in T_N\\ x\neq 0}} (\gamma^\omega(x))^{\frac{p}{N^d}} \\
&\le C \sum_{\sigma:0\to\Lambda^c}\left((\omega_\sigma)^{p-\frac{p}{N^d}} \prod_{x\in T_N}\left(\frac{\pi_N^\omega(0)}{\pi_N^\omega(x)}\right)^{\frac{p}{N^d}} \prod_{\substack{x\in T_N\\ x\neq 0}} (\gamma^\omega(x))^{\frac{p}{N^d}}\right) \qquad (4.1)
\end{aligned}$$

where $C = (\#\{\sigma : 0 \to \Lambda^c\})^{p-\frac{p}{N^d}}$ and the sums on σ correspond to the sums on simple paths.

Take $\theta_N^\sigma : E_N \to \mathbb{R}_+$, and define $\check{\theta}_N^\sigma$ by : $\forall x \sim y$, $\check{\theta}_N^\sigma(x,y) = \theta_N^\sigma(y,x)$. It is clear that

$$\frac{\check{\omega}^{\check{\theta}_N^\sigma}}{\omega^{\theta_N^\sigma}} = \pi_N^{\mathrm{div}(\theta_N^\sigma)} \qquad (4.2)$$

where by λ^β we mean $\prod_{e\in E_N}\lambda(e)^{\beta(e)}$ (resp. $\prod_{x\in T_N}\lambda(x)^{\beta(x)}$) for any couple of functions λ, β on E_N (resp. T_N). Therefore, if we choose for all $\sigma : 0 \to \Lambda^c$ simple path a $\theta_N^\sigma : E_N \to \mathbb{R}_+$ such that

$$\mathrm{div}(\theta_N^\sigma) = \frac{p}{N^d}\sum_{x\in T_N}(\delta_0 - \delta_x), \qquad (4.3)$$

(4.1) and (4.2) give us

$$f_N^p \le C \sum_{\sigma:0\to\Lambda^c}\left((\omega_\sigma)^{p-\frac{p}{N^d}} \frac{\check{\omega}^{\check{\theta}_N^\sigma}}{\omega^{\theta_N^\sigma}} \prod_{\substack{x\in T_N\\ x\neq 0}} (\gamma^\omega(x))^{\frac{p}{N^d}}\right).$$

Note that we choose a θ_N^σ for each term of the sum, they can be different as long as (4.3) is satisfied. As the sum is finite, we only have to show that for all $\sigma : 0 \to \Lambda^c$ simple path we can find $(\theta_N^\sigma)_{N\in\mathbb{N}}$ such that for all N, θ_N^σ satisfies (4.3) and :

$$\sup_{N\in\mathbb{N}} \mathbb{E}^{(\alpha)}\left[(\omega_\sigma)^{p-\frac{p}{N^d}} \frac{\check{\omega}^{\check{\theta}_N^\sigma}}{\omega^{\theta_N^\sigma}} \prod_{\substack{x\in T_N\\ x\neq 0}} (\gamma^\omega(x))^{\frac{p}{N^d}}\right] < \infty. \qquad (4.4)$$

Step 2 **:**

Take $p > 1$. We first construct for all $\sigma : 0 \to \Lambda^c$ simple path a sequence $(\theta_N^\sigma)_{N\in\mathbb{N}}$ that satisfies (4.3), and then we show that it satisfies (4.4) too.

Construction of $(\theta_N^\sigma)_{N\in\mathbb{N}}$**.** We want to use lemma 2 of [13], which is a result of type maw-flow min-cut (see for example [10], section 3.1, for a general description of the max-flow min-cut problem), with a L_2 bound. We first recall some definitions and notions on the matter. In the infinite graph $G = (\mathbb{Z}^d, E)$, a cut-set between $x \in \mathbb{Z}^d$ and ∞ is a subset S of E such that any infinite simple directed path (i.e. an infinite directed path that does not go twice through the same vertex) starting from x must necessarily go through one edge in S. A cut-set which is minimal for inclusion is necessarily of the form :

$$S = \partial_+(A) = \{e \in E,\ \underline{e} \in A,\ \overline{e} \in A^c\} \tag{4.5}$$

where A is a finite subset of $\mathbb{Z}^d$ containing x and such that any $y \in A$ can be reached by a directed path in A starting from x. Let $(c(e))_{e\in E}$ be a family of non-negative reals, called the capacities. The minimal cut-set sum between 0 and ∞ is defined by :

$$m((c(e))_{e\in E}) = \inf\{c(S),\ S \text{ a cut-set separating } 0 \text{ and } \infty\}$$

where $c(S) = \sum_{e\in S} c(e)$. Remark that the infimum can be taken only on minimal cut-sets, i.e. cut-sets of the form (4.5).

In our case, we set σ an arbitrary simple path from 0 to Λ^c. Set $N \in \mathbb{N}$, we define :

$$\alpha^{(\sigma)}(e) = \begin{cases} \alpha(e) + \kappa^\Lambda & \text{if } e \in \sigma \\ \alpha(e) & \text{otherwise} \end{cases}$$

Then $m((\alpha^{(\sigma)}(e))_{e\in E_N}) \geq \kappa^\Lambda$. Indeed :

- If some $e \in \sigma$ is in the min-cut, it is obvious.
- Otherwise, as $0 \in \sigma$ the min-cut is of the form $S = \partial_+(K)$ with $\sigma \subset K$ and K a finite connected set of vertices. The definition of κ^Λ in theorem 2.1 gives directly $m((\alpha^{(\sigma)}(e))_{e\in E_N}) \geq \kappa^\Lambda$.

Lemma 2 of [13] states :

Lemma 4.2 (Lemma 2 of [13]). *Let C' and C'' be two reals such that $0 < C' < C'' < \infty$. There exists a finite constant $c_1 > 0$ and an integer $N_0 > 0$ depending only on C', C'', d such that for all sequence $(c_e)_{e\in E}$ such that*

$$\forall e \in E, \quad C' < c_e < C'',$$

and for all integer $N > N_0$, there exists a function $\theta_N : E_N \to \mathbb{R}_+$ such that :

$$\text{div}(\theta_N) = m((c))\frac{1}{N^d}\sum_{x\in T_N}(\delta_0 - \delta_x),$$

$$\|\theta_N\|_2^2 = \sum_{e\in E_N}\theta_N(e)^2 < c_1,$$

and such that

$$\theta_N(e) \leq c(e), \quad \forall e \in E_N,$$

when we identify E_N with the edges of E such that $\underline{e} \in [-N/2, N/2[^d$.

Remark 4.3. *The first part of the lemma (without the L_2 bound) is an extension of the classical max-flow min-cut theorem on finite directed graphs. We add for all x a directed edge (x, δ), where δ is a new cemetery point, and give a capacity of $1/N^d$ to all those new edges. Then we apply the classical theorem with source 0 and sink δ to get the result. The second part of the lemma is more complicated (see [13] for the proof).*

With $c(e) = \frac{p}{\kappa^\Lambda}\alpha^{(\sigma)}(e)$, it gives here that for all $N \geq N_0$ there is a function θ_N^σ satisfying (4.3), such that $\theta_N^\sigma(e) \leq \frac{p}{\kappa^\Lambda}\alpha^{(\sigma)}(e)$ and such that $\|\theta_N^\sigma\|_2^2$ is uniformly bounded in N.

Preliminary computations about (4.4). Let q and r be positive reals such that $\frac{1}{r}+\frac{1}{q}=1$ and $pq<\kappa^{\Lambda}$. Using in a first time Hölder's inequality and then the time-reversed environment (lemma 1 of [12]), we obtain :

$$\mathbb{E}^{(\alpha)}\left[(\omega_\sigma)^{p-\frac{p}{N^d}}\frac{\check{\omega}^{\check{\theta}_N^\sigma}}{\omega^{\theta_N^\sigma}}\prod_{\substack{x\in T_N\\x\neq 0}}(\gamma^\omega(x))^{\frac{p}{N^d}}\right] \tag{4.6}$$

$$\leq \mathbb{E}^{(\alpha)}\left[(\omega_\sigma)^{pq-\frac{pq}{N^d}}\,\omega^{-q\theta_N^\sigma}\prod_{\substack{x\in T_N\\x\neq 0}}(\gamma^\omega(x))^{\frac{pq}{N^d}}\right]^{\frac{1}{q}}\mathbb{E}^{(\alpha)}\left[\check{\omega}^{r\check{\theta}_N^\sigma}\right]^{\frac{1}{r}}$$

$$= \mathbb{E}^{(\alpha)}\left[(\omega_\sigma)^{pq-\frac{pq}{N^d}}\,\omega^{-q\theta_N^\sigma}\prod_{\substack{x\in T_N\\x\neq 0}}(\gamma^\omega(x))^{\frac{pq}{N^d}}\right]^{\frac{1}{q}}\mathbb{E}^{(\check{\alpha})}\left[\omega^{r\check{\theta}_N^\sigma}\right]^{\frac{1}{r}} \tag{4.7}$$

Define $\alpha(x)=\sum_{\underline{e}=x}\alpha(e)$, $\theta_N^\sigma(x)=\sum_{\underline{e}=x}\theta_N^\sigma(e)$. For all $x\in T_N$, we have $\alpha(x)=\check{\alpha}(x)=\sum_{i=0}^{2d}\alpha_i$, we thus note $\alpha_0=\sum_{i=0}^{2d}\alpha_i$. In order to simplify notations, we note $d\lambda_\Omega=\prod_{e\in\tilde{E}_N}d\omega(e)$, where we obtain $\tilde{E}_N$ from E_N by removing for each x one arbitrary edge leaving x. We can now compute the first expectation in (4.7) :

$$\mathbb{E}^{(\alpha)}\left[(\omega_\sigma)^{pq-\frac{pq}{N^d}}\,\omega^{-q\theta_N^\sigma}\prod_{\substack{x\in T_N\\x\neq 0}}(\gamma^\omega(x))^{\frac{pq}{N^d}}\right]$$

$$=\int(\omega_\sigma)^{pq-\frac{pq}{N^d}}\left(\prod_{e\in E_N}\omega(e)^{\alpha(e)-1-q\theta_N^\sigma(e)}\right)\prod_{\substack{x\in T_N\\x\neq 0}}(\gamma^\omega(x))^{\frac{pq}{N^d}}\frac{\prod_{x\in T_N}\Gamma(\alpha_x)}{\prod_{e\in E_N}\Gamma(\alpha_e)}d\lambda_\Omega$$

$$=\int(\omega_\sigma)^{pq-\frac{pq}{N^d}}\left(\prod_{e\in E_N}\omega(e)^{\alpha(e)-1-q\theta_N^\sigma(e)}\right)\frac{\prod_{x\in T_N}\Gamma(\alpha_x)}{\prod_{\substack{x\in T_N\\x\neq 0}}\left(\sum_{\sigma:x\to(x+\Lambda)^c}\omega_\sigma\right)^{\frac{pq}{N^d}}\prod_{e\in E_N}\Gamma(\alpha_e)}d\lambda_\Omega$$

$$\mathbb{E}^{(\alpha)}\left[(\omega_\sigma)^{pq-\frac{pq}{N^d}}\,\omega^{-q\theta_N^\sigma}\prod_{x\in T_N}(\gamma^\omega(x))^{\frac{pq}{N^d}}\right]$$

$$\leq\int\frac{(\omega_\sigma)^{pq-\frac{pq}{N^d}}\left(\prod_{e\in E_N}\omega(e)^{\alpha(e)-1-q\theta_N^\sigma(e)}\right)\left(\prod_{x\in T_N}\Gamma(\alpha_x)\right)}{\left(\prod_{x\in T_N,\,x\neq 0}\omega_{\sigma_x}^{\frac{pq}{N^d}}\right)\left(\prod_{e\in E_N}\Gamma(\alpha_e)\right)}d\lambda_\Omega$$

where σ_x is an arbitrarily chosen simple path in the preceding sum. Then

$$\mathbb{E}^{(\alpha)}\left[(\omega_\sigma)^{pq-\frac{pq}{N^d}}\,\omega^{-q\theta_N^\sigma}\prod_{x\in T_N}(\gamma^\omega(x))^{\frac{pq}{N^d}}\right]\leq\frac{\prod_{x\in T_N}\Gamma(\alpha_0)\prod_{e\in E_N}\Gamma\left(\beta^\sigma(e)-q\theta_N^\sigma(e)\right)}{\prod_{e\in E_N}\Gamma(\alpha_e)\prod_{x\in T_N}\Gamma\left(\beta^\sigma(x)-q\theta_N^\sigma(x)\right)}$$

with

$$\beta^\sigma(x) = \sum_{x=\underline{e}} \beta^\sigma(e)$$

and

$$\beta^\sigma(e) = \alpha(e) + pq\left(1 - \frac{1}{N^d}\right)\mathbb{1}_{e\in\sigma} - \sum_{x\in T_N, x\neq 0} \frac{pq}{N^d}\mathbb{1}_{e\in\sigma_x}.$$

As Λ is finite, an edge can be in only a finite number of σ_x. We have then for all e, $\sum_{x\in T_N, x\neq 0} \frac{pq}{N^d}\mathbb{1}_{e\in\sigma_x} < +\infty$. This proves that β^σ is well defined and takes only finite values.

The second expectation in (4.7) is easy to compute :

$$\mathbb{E}^{(\check{\alpha})}(\omega^{r\check{\theta}^\sigma_N}) = \frac{\prod\limits_{e\in E_N} \Gamma(\alpha(e) + r\theta^\sigma_N(e)) \prod\limits_{x\in T_N} \Gamma(\alpha_0)}{\prod\limits_{x\in T_N} \Gamma(\alpha_0 + r\check{\theta}^\sigma_N(x)) \prod\limits_{e\in E_N} \Gamma(\alpha(e))}.$$

We did not check that the previous expressions are well defined : we need to prove that for the given θ^σ_N, the arguments of the Gamma functions are positive. As it is a bit tedious, we delay this checking to the next point in the proof.

We now have that $\mathbb{E}^{(\alpha)}\left[(\omega_\sigma)^{p-\frac{p}{N^d}} \frac{\check{\omega}^{\check{\theta}^\sigma_N}}{\omega^{\theta^\sigma_N}} \prod_{x\in T_N} (\gamma^\omega(x))^{\frac{p}{N^d}}\right]$ is smaller than :

$$\left(\frac{\prod\limits_x \Gamma(\alpha_0) \prod\limits_e \Gamma\left(\beta^\sigma(e) - q\theta^\sigma_N(e)\right)}{\prod\limits_e \Gamma(\alpha_e) \prod\limits_x \Gamma\left(\beta^\sigma(x) - q\theta^\sigma_N(x)\right)}\right)^{\frac{1}{q}} \left(\frac{\prod\limits_e \Gamma(\alpha(e) + r\theta^\sigma_N(e)) \prod\limits_x \Gamma(\alpha_0)}{\prod\limits_x \Gamma(\alpha_0 + r\check{\theta}^\sigma_N(x)) \prod\limits_e \Gamma(\alpha(e))}\right)^{\frac{1}{r}}$$

We are reduced to prove that $\forall \sigma : 0 \to \Lambda^c$ simple path,

$$\sup_{N\in\mathbb{N}} \left(\frac{\prod\limits_{x\in T_N} \Gamma(\alpha_0)}{\prod\limits_{e\in E_N} \Gamma(\alpha_e)} \left(\frac{\prod\limits_{e\in E_N} \Gamma\left(\beta^\sigma(e) - q\theta^\sigma_N(e)\right)}{\prod\limits_{x\in T_N} \Gamma\left(\beta^\sigma(x) - q\theta^\sigma_N(x)\right)} \right)^{\frac{1}{q}} \left(\frac{\prod\limits_{e\in E_N} \Gamma(\alpha(e) + r\theta^\sigma_N(e))}{\prod\limits_{x\in T_N} \Gamma(\alpha_0 + r\check{\theta}^\sigma_N(x))} \right)^{\frac{1}{r}} \right) < +\infty \tag{4.8}$$

Checking that the previous Gamma functions were well defined. As for all $e \in E_N$ $\alpha(e) > 0$ and $\theta^\sigma_N(e) \geq 0$, the result is straightforward except for $\Gamma\left(\beta^\sigma(e) - q\theta^\sigma_N(e)\right)$ and $\Gamma\left(\beta^\sigma(x) - q\theta^\sigma_N(x)\right)$. By construction of θ^σ_N, we know that $\beta^\sigma(e) - q\theta^\sigma_N(e) \geq \beta^\sigma(e) - \frac{pq}{\kappa^\Lambda}\alpha^{(\sigma)}(e)$. Then we just have to check the positivity of this second expression. Take $e \in E_N$:

- If $e \in \sigma$, then
$$\begin{aligned}\beta^\sigma(e) - \frac{pq}{\kappa^\Lambda}\alpha^{(\sigma)}(e) &= \alpha(e) - \frac{pq}{\kappa^\Lambda}(\alpha(e) + \kappa^\Lambda) + pq - \frac{pq}{N^d} - \sum_{\substack{x\in T_N\\ x\neq 0}} \frac{pq}{N^d}\mathbb{1}_{e\in\sigma_x} \\ &= \alpha(e)\left(1 - \frac{pq}{\kappa^\Lambda}\right) - \frac{pq}{N^d}\Big(1 + \sum_{\substack{x\in T_N\\ x\neq 0}} \mathbb{1}_{e\in\sigma_x}\Big)\end{aligned}$$
As we assumed $pq < \kappa^\Lambda$ and $\kappa^\Lambda > 1$, $\alpha(e)\left(1 - \frac{pq}{\kappa^\Lambda}\right) > 0$. The second term can be made as small as needed by choosing N big enough. Then $\beta^\sigma(e) - \frac{pq}{\kappa^\Lambda}\alpha^{(\sigma)}(e) > 0$ for N big enough.
- If $e \notin \sigma$, then
$$\begin{aligned}\beta^\sigma(e) - \frac{pq}{\kappa^\Lambda}\alpha^{(\sigma)}(e) &= \alpha(e) - \frac{pq}{\kappa^\Lambda}\alpha(e) - \sum_{\substack{x\in T_N\\ x\neq 0}} \frac{pq}{N^d}\mathbb{1}_{e\in\sigma_x} \\ &\geq \alpha(e)\left(1 - \frac{pq}{\kappa^\Lambda}\right) - (\sharp\Lambda)\frac{pq}{N^d}\end{aligned}$$

As before, by choosing N big enough we make sure that it is positive. Remark that for N big enough, $\min_{i=1\ldots 2d}\alpha_i\left(1-\frac{pq}{\kappa^\Lambda}\right)-(\sharp\Lambda)\frac{pq}{N^d}$ is also positive, and it is a uniform lower bound of $\beta^\sigma(e)-q\theta_N^\sigma(e)$, for all $e\notin\sigma$.

Proof of (4.8). As σ is a finite path, the above tells us that there exists $\varepsilon>0$ such that :

$$\forall e\in\sigma,\ \varepsilon\leq\beta^\sigma(e)-\frac{pq}{\kappa^\Lambda}\alpha^{(\sigma)}(e)=\alpha(e)\left(1-\frac{pq}{\kappa^\Lambda}\right)-\frac{pq}{N^d}\Big(1+\sum_{\substack{x\in T_N\\x\neq 0}}\mathbb{1}_{e\in\sigma_x}\Big)\leq\alpha(e),$$

and the same is true for $\alpha(x)$ by summing on e. Define :

$$A_1^\sigma=\left(\frac{\prod\limits_{x\in e\in\sigma}\Gamma(\alpha_0)}{\prod\limits_{e\in\sigma}\Gamma(\alpha(e))}\frac{\prod\limits_{e\in\sigma}\sup_{[\varepsilon,\max_i\alpha_i]}\Gamma(s)}{\prod\limits_{x\in e\in\sigma}\inf_{[\varepsilon,\max_i\alpha_i]}\Gamma(s)}\right)^{\frac{1}{q}}$$

$$A_2^\sigma=\left(\frac{\prod\limits_{x\in e\in\sigma}\Gamma(\alpha_0)}{\prod\limits_{e\in\sigma}\Gamma(\alpha(e))}\frac{\prod\limits_{e\in\sigma}\sup_{[\alpha(e),\alpha(e)(1+r)+r\kappa^\Lambda]}\Gamma(s)}{\prod\limits_{x\in e\in\sigma}\inf_{[\alpha(e),\alpha(e)(1+r)+r\kappa^\Lambda]}\Gamma(s)}\right)^{\frac{1}{r}}$$

We have then, for any fixed σ :

$$\begin{aligned}
&\frac{\prod\limits_{x\in T_N}\Gamma(\alpha_0)}{\prod\limits_{e\in E_N}\Gamma(\alpha_e)}\left(\frac{\prod\limits_{e\in E_N}\Gamma\left(\beta^\sigma(e)-q\theta_N^\sigma(e)\right)}{\prod\limits_{x\in T_N}\Gamma\left(\beta^\sigma(x)-q\theta_N^\sigma(x)\right)}\right)^{\frac{1}{q}}\left(\frac{\prod\limits_{e\in E_N}\Gamma(\alpha(e)+r\theta_N^\sigma(e))}{\prod\limits_{x\in T_N}\Gamma(\alpha_0+r\check{\theta}_N^\sigma(x))}\right)^{\frac{1}{r}}\\
&\leq A_1^\sigma A_2^\sigma\frac{\prod\limits_{x\notin\sigma}\Gamma(\alpha_0)}{\prod\limits_{e\notin\sigma}\Gamma(\alpha_e)}\left(\frac{\prod\limits_{e\notin\sigma}\Gamma\left(\beta^\sigma(e)-q\theta_N^\sigma(e)\right)}{\prod\limits_{x\notin\sigma}\Gamma\left(\beta^\sigma(x)-q\theta_N^\sigma(x)\right)}\right)^{\frac{1}{q}}\left(\frac{\prod\limits_{e\notin\sigma}\Gamma(\alpha(e)+r\theta_N^\sigma(e))}{\prod\limits_{x\notin\sigma}\Gamma(\alpha_0+r\check{\theta}_N^\sigma(x))}\right)^{\frac{1}{r}}\\
&\leq A_1^\sigma A_2^\sigma\exp\left(\sum_{\substack{e\in E_N\\e\notin\sigma}}\nu\left(\alpha(e),\theta_N^\sigma(e),\beta^\sigma(e)\right)-\sum_{\substack{x\in T_N\\x\notin\sigma}}\tilde{\nu}\left(\alpha_0,\theta_N^\sigma(x),\beta^\sigma(x)\right)\right)
\end{aligned}$$

with :

$$\nu\left(\alpha(e),\theta_N^\sigma(e),\beta^\sigma(e)\right)=\frac{1}{r}\ln\Gamma(\alpha(e)+r\theta_N^\sigma(e))+\frac{1}{q}\ln\Gamma(\beta^\sigma(e)-q\theta_N^\sigma(e))-\ln\Gamma(\alpha(e))$$

$$\tilde{\nu}\left(\alpha_0,\theta_N^\sigma(x),\beta^\sigma(x)\right)=\frac{1}{r}\ln\Gamma(\alpha_0+r\theta_N^\sigma(x)+\frac{pr}{N^d})+\frac{1}{q}\ln\Gamma(\beta^\sigma(x)-q\theta_N^\sigma(x))-\ln\Gamma(\alpha_0)$$

(the $\frac{pr}{N^d}$ comes from the fact that $\forall x\neq 0,\quad \theta_N^\sigma(x)-\check{\theta}_N^\sigma(x)=\mathrm{div}(\theta_N^\sigma)(x)=-\frac{p}{N^d}$). We set $\underline{\alpha}=\min_{i\in[\![1,2d]\!]}\alpha_i$ and $\overline{\alpha}=\max_{i\in[\![1,2d]\!]}\alpha_i$. Then $\forall e\in E_N,\quad \underline{\alpha}\leq\alpha(e)\leq\overline{\alpha}$, $\forall e\notin\sigma\quad q\theta_N^\sigma(e)\leq\frac{pq}{\kappa^\Lambda}\alpha(e)$ and $pq<\kappa^\Lambda$. Taylor's inequality gives : $\forall e\notin\sigma,\forall x\notin\sigma$,

$$\begin{cases}\left|\nu\left(\alpha(e),\theta_N^\sigma(e),\beta^\sigma(e)\right)\right|\leq C_1\left(\theta_N^\sigma(e)^2+\frac{pq}{N^d}\right)\\ \left|\tilde{\nu}\left(\alpha_0,\theta_N^\sigma(x),\beta^\sigma(x)\right)\right|\leq C_2\left(\theta_N^\sigma(x)^2+\frac{p}{N^d}+\frac{pq}{N^d}\right)\end{cases}$$

with C_1 and C_2 positive constants. Then we can find a constant $C_3>0$ independent of $N\geq N_0$ such that :

$$\begin{aligned}
&\frac{\prod\limits_{x\in T_N}\Gamma(\alpha_0)}{\prod\limits_{e\in E_N}\Gamma(\alpha_e)}\left(\frac{\prod\limits_{e\in E_N}\Gamma\left(\beta^\sigma(e)-q\theta_N^\sigma(e)\right)}{\prod\limits_{x\in T_N}\Gamma\left(\beta^\sigma(x)-q\theta_N^\sigma(x)\right)}\right)^{\frac{1}{q}}\left(\frac{\prod\limits_{e\in E_N}\Gamma(\alpha(e)+r\theta_N^\sigma(e))}{\prod\limits_{x\in T_N}\Gamma(\alpha_0+r\check{\theta}_N^\sigma(x))}\right)^{\frac{1}{r}}\\
&\leq\exp\left(C_3\left(\sum_{e\in E_N}\theta_N^\sigma(e)^2+\sum_{x\in T_N}\theta_N^\sigma(x)^2\right)\right).
\end{aligned}$$

As seen before with the L_2 bound part of lemma 2 of [13], this is bounded by a finite constant independent of N. It follows that the supremum on N is finite too. This concludes the argument for any fixed σ and proves (4.8). This proves the lemma.

□

5 Proof of theorem 2.4 and corollary 2.6

To obtain results on the initial random walk Z_n, we need some estimates on our acceleration function γ^ω. In particular, we will need the following lemma :

Lemma 5.1. *Let Λ be an arbitrary finite connected set of vertices containing 0. For all $x \in \mathbb{Z}^d$ and $s < \kappa$,*

$$\mathbb{E}^{(\alpha)}\left((\gamma^\omega(x))^s\right) < +\infty.$$

As its proof is quite computational, we defer it to the appendix. Remark that it is nevertheless quite easy to get a weaker bound : $\gamma^\omega(0) = \frac{1}{\sum \omega_\sigma} \le \frac{1}{\omega_{\sigma_1}}$, where the sum is on all σ finite simple paths from 0 to Λ^c, and where σ_1 is the path from 0 to Λ^c going only through edges $(ne_1, (n+1)e_1)$. Then $\mathbb{E}^{(\alpha)}\left(\gamma^\omega(0)^\lambda\right) \le \mathbb{E}^{(\alpha)}\left(\frac{1}{\omega_{\sigma_1}}^\lambda\right) < +\infty$ for all $\lambda < \alpha_1$. This weaker bound suffices for the proof of theorem 2.4, but lemma 5.1 is fully needed in the proof of theorem 2.8. Also remark that this lemma is the only result involving γ^ω where we do not need $\kappa^\Lambda > 1$. This is of little importance as the assumption $\kappa^\Lambda > 1$ is needed otherwise in the proof of theorem 2.8.

Theorem 2.4 is based on classical results on ergodic stationary sequences, see [3] pages $342 - 344$. We need another preliminary lemma.

Lemma 5.2. *$\mathbb{Q}^{(\alpha)}$ is ergodic and equivalent to $\mathbb{P}^{(\alpha)}$. Set $\Delta_i = X_i - X_{i-1}$, $i \in \mathbb{N}$, then Δ_i is stationary and ergodic under $\mathbb{Q}^{(\alpha)}[P_0^\omega(.)]$.*

Proof. The proof of the first point is easily adapted from chapter 2 of [2], by replacing the discrete process by the continuous process : we use the continuous martingales convergence theorems, and the continuous version of Birkhoff's theorem (see for example [9], pages $9 - 11$).

For the second point, as $\mathbb{Q}^{(\alpha)}$ is an invariant probability for $\overline{\omega}_t$, it is straightforward that Δ_i is stationary. It remains to prove ergodicity. Set $A \subset \left(\mathbb{Z}^d\right)^{\mathbb{N}}$ a measurable set such that $\forall t$, $\theta_t^{-1}(A) = A$ with θ_t the time-shift. We note

$$r(x,\omega) = P_x^\omega((\Delta_i \in A)) \text{ and } r(\omega) = r(0,\omega).$$

We have $\forall \omega \in \Omega$,

$$\lim_{n\to\infty} r(X_n,\omega) = \mathbb{1}_A((\Delta_i)),\ P_x^\omega \text{ a.s.}. \tag{5.1}$$

Indeed, setting $\mathcal{F}_n = \sigma((X_t)_{t\le n})$ gives :

$$P_x^\omega\left((\Delta_i) \in A|\mathcal{F}_n\right) = P_x^\omega\left((\Delta_{i+n}) \in A|\mathcal{F}_n\right) = P_{X_n}^\omega\left((\Delta_i) \in A\right) = r(X_n,\omega),$$

then $r(X_n,\omega)$ is a (closed) bounded martingale and we have the wanted limit (5.1) by a.s. convergence, as $\mathbb{1}_A((\Delta_i))$ is $\mathcal{F}_\infty$-measurable. Remark that $r(X_n,\omega) = r(\overline{\omega}_n)$. The application of Birkhoff's ergodic theorem ([3], page 337) for the time-shift of size 1 gives

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n r(X_k,\omega) = \mathbb{E}^{\mathbb{Q}^{(\alpha)}}(r(\omega))\ ,\ P_0^\omega \text{a.s.}.$$

Comparing with (5.1), it implies that $\mathbb{E}^{\mathbb{Q}^{(\alpha)}}(r(\omega)) \in \{0,1\}$.

□

Lemma 5.3. *Let $D(l,n) = \max_{t\in[0,1]} |(X_{n+t} - X_n)\cdot l|$ be the maximum distance travelled by the walk in direction $l = e_1,\ldots,e_{2d}$, during a time $[n,n+1]$, $n\in\mathbb{N}$. Choose R_Λ such that Λ is included in the ball $B(0,R_\Lambda)$. Then*

$$\mathbb{P}^{(\alpha)}\left(D(l,n)\geq 2kR_\Lambda\right)\leq \frac{C_1^k}{\Gamma(C_2 k)}$$

where C_1 and C_2 are positive constants depending only on the parameters $(\alpha_1,\ldots,\alpha_{2d})$.

Proof. Let N be the number of visits of 0 before exiting Λ. The random variable N follows a geometric law of parameter $p_N := \frac{1}{G^{\omega,\Lambda}(0,0)}$ the inverse of the Green function killed at the exit time of Λ. We note T the total time spent on 0 before exiting Λ, $T = \frac{1}{\gamma^\omega(0)}\sum_{i=1}^N E_i$, where the E_i are independent exponential random variables of parameter 1. Set $\varepsilon>0$.

$$P^\omega\left(T\leq\varepsilon|N\right) = P^\omega\left(\sum_{i=1}^N E_i\leq \gamma^\omega(0)\varepsilon|N\right) = e^{-\gamma^\omega(0)\varepsilon}\sum_{k=N}^{+\infty}\frac{(\gamma^\omega(0)\varepsilon)^k}{k!}.$$

Then

$$\begin{aligned}
P^\omega\left(T\leq\varepsilon\right) &= e^{-\gamma^\omega(0)\varepsilon}E^\omega\left(\sum_{k=N}^{+\infty}\frac{(\gamma^\omega(0)\varepsilon)^k}{k!}\right)\\
&= e^{-\gamma^\omega(0)\varepsilon}\sum_{n=1}^{+\infty}\sum_{k=n}^{+\infty}\frac{(\gamma^\omega(0)\varepsilon)^k}{k!}p_N\left(1-p_N\right)^{n-1}\\
&= e^{-\gamma^\omega(0)\varepsilon}\sum_{k=1}^{+\infty}\sum_{n=1}^{k}\frac{(\gamma^\omega(0)\varepsilon)^k}{k!}p_N\left(1-p_N\right)^{n-1}\\
&= e^{-\gamma^\omega(0)\varepsilon}\sum_{k=1}^{+\infty}\frac{(\gamma^\omega(0)\varepsilon)^k}{k!}p_N\frac{1-(1-p_N)^k}{p_N}\\
&= 1-e^{-p_N\gamma^\omega(0)\varepsilon} = 1-e^{-\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}\varepsilon}
\end{aligned}$$

For all $a>0$, let $0<\lambda<\kappa$,

$$\begin{aligned}
&\mathbb{P}^{(\alpha)}\left(T\leq\varepsilon\right)\\
&= \mathbb{E}^{(\alpha)}\left(1-e^{-\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}\varepsilon}\right)\\
&= \mathbb{E}^{(\alpha)}\left((1-e^{-\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}\varepsilon})\mathbb{1}_{\{\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}\geq a\}}\right)+\mathbb{E}^{(\alpha)}\left((1-e^{-\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}\varepsilon})\mathbb{1}_{\{\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}< a\}}\right)\\
&\leq \mathbb{P}^{(\alpha)}\left(\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}\geq a\right)+\mathbb{E}^{(\alpha)}\left(\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}\varepsilon\mathbb{1}_{\{\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}< a\}}\right)\\
&\leq \mathbb{P}^{(\alpha)}\left(\gamma^\omega(0)\geq a\right)+\mathbb{E}^{(\alpha)}\left(\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}\varepsilon\mathbb{1}_{\{\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}< a\}}\right)\\
&\leq \frac{\mathbb{E}^{(\alpha)}\left(\gamma^\omega(0)^\lambda\right)}{a^\lambda}+a\varepsilon\mathbb{P}^{(\alpha)}\left(\frac{\gamma^\omega(0)}{G^{\omega,\Lambda}(0,0)}<a\right)
\end{aligned}$$

As $\lambda<\kappa$, lemma 5.1 gives :

$$\mathbb{P}^{(\alpha)}\left(T\leq\varepsilon\right)\leq\frac{C}{a^\lambda}+a\varepsilon$$

with C a positive constant independent of a. Then for $a=\varepsilon^{-\frac{1}{\lambda+1}}$ we have :

$$\mathbb{P}^{(\alpha)}\left(T\leq\varepsilon\right)\leq (C+1)\varepsilon^{\frac{\lambda}{\lambda+1}}.$$

If $D(l,n) \geq 2kR_\Lambda$, the walk went through at least k distinct sets $X_t + \Lambda$ of empty intersection. The time spent in such a set is bigger than the time spent on one point in the set, and those times are independent in disjoint sets (because the environments in the sets are independent). We get (for $T_1, \dots, T_k$ i.i.d. of same law as T) :

$$\begin{aligned}
&\mathbb{P}^{(\alpha)}\left(D(l,n) \geq 2kR_\Lambda\right) \\
&\leq \mathbb{P}^{(\alpha)}\left(\exists \varepsilon_1, \dots, \varepsilon_k \text{ such that } \sum_{i=1}^k \varepsilon_i \leq 1 \text{ and } T_1 = \varepsilon_1, \dots, T_k = \varepsilon_k\right) \\
&\leq \int_{\sum \varepsilon_i \leq 1} (C+1)^k \left(\frac{\lambda}{\lambda+1}\right)^k \prod_{i=1}^k \varepsilon_i^{\frac{\lambda}{\lambda+1}-1} d\varepsilon_1 \dots d\varepsilon_k \\
&= (C+1)^k \left(\frac{\lambda}{\lambda+1}\right)^k \frac{\Gamma(\frac{\lambda}{\lambda+1})^k}{\Gamma(k\frac{\lambda}{\lambda+1}+1)} \\
&\leq \frac{((C+1)\Gamma(\frac{\lambda}{\lambda+1}+1))^k}{\Gamma(k\frac{\lambda}{\lambda+1})}
\end{aligned}$$

This concludes the proof of the lemma.

□

We can now prove theorem 2.4 : lemma 5.2 allows to use Birkhoff's ergodic theorem to the Δ_i and get the result for discrete times, lemma 5.3 extends this result to the continuous walk.

Proof of theorem 2.4. Lemma 5.2 gives that the sequence $(\Delta_i)_{i\in\mathbb{N}}$ is stationary and ergodic under $\mathbb{Q}^{(\alpha)}\,(P_0^\omega(.))$. We apply Birkhoff's ergodic theorem to the Δ_i to get a law of large numbers :

$$\frac{X_k}{k} \rightarrow_{k\to\infty,\, k\in\mathbb{N}} \mathbb{E}^{\mathbb{Q}^{(\alpha)}}\left[E_0^\omega(X_1)\right]\ ,\ \mathbb{Q}_0^{(\alpha)} \text{ a.s. and thus } \mathbb{P}_0^{(\alpha)} \text{ a.s. .}$$

If $d_\alpha \cdot e_i = 0$, the symmetry of the law of the environment gives $\mathbb{E}^{\mathbb{Q}^{(\alpha)}}\left[E_0^\omega(X_1)\right] \cdot e_i = 0$. Then $\frac{X_k}{k} \to 0$ when $d_\alpha = 0$. Furthermore theorem 6.3.2 of [3] gives that the processes X_k is directionally recurrent when $d_\alpha \cdot e_i = 0$. As X_t stays only a finite time on each vertex before the next jump, directional recurrence for $(X_k)_{k\in\mathbb{N}}$ implies directional recurrence for $(X_t)_{t\in\mathbb{R}_+}$ (the probability to come back to 0 after a finite time is 1).

For $l \in \mathbb{R}^d$, we note $A_l = \{X_{t_k} \cdot l \to \infty\}$, where $(t_k)_{k\in\mathbb{N}}$ are the jump times. If $l \neq 0$ and if $\mathbb{P}_0^{(\alpha)}(A_l) > 0$, Kalikow's $0-1$ law ([6], [21] proposition 3) gives $\mathbb{P}_0^{(\alpha)}(A_l \cup A_{-l}) = 1$. Suppose that $d_\alpha \cdot e_i > 0$ then ([14]) $\mathbb{P}_0^{(\alpha)}(A_{e_i}) > 0$, this implies that $(X_{t_k} \cdot e_i)_{k\in\mathbb{N}}$ visits 0 a finite number of times $\mathbb{Q}_0^{(\alpha)}$ a.s.. Then $(X_k \cdot e_i)_{k\in\mathbb{N}}$ visits 0 a finite number of times $\mathbb{Q}_0^{(\alpha)}$ a.s. (as X_t stays only a finite time on each vertex). Theorem 6.3.2 of [3] and Birkhoff's ergodic theorem give then : $E^{\mathbb{Q}^{(\alpha)}}(E^\omega(X_1)) \cdot e_i > 0$.

We now consider the limit for the continuous-time walk. For $t > 0$, we set $k = \lfloor t \rfloor$. Then for all $i = 1, \dots, 2d$,

$$X_k \cdot e_i - D(e_i, k) \leq X_t \cdot e_i \leq X_k \cdot e_i + D(e_i, k).$$

Then

$$\frac{X_k \cdot e_i}{k-1} - \frac{D(e_i,k)}{k-1} \leq \frac{X_t \cdot e_i}{t} \leq \frac{X_k \cdot e_i}{k} + \frac{D(e_i,k)}{k}.$$

Lemma 5.3 gives : for $\varepsilon > 0$,

$$\sum_{k=1}^{+\infty} \mathbb{P}^{(\alpha)}\left(\left|\frac{D(e_i,k)}{k}\right| \geq \varepsilon\right) \leq \sum_{k=1}^{+\infty} \frac{C_1^{\frac{k\varepsilon}{2R_\Lambda}}}{\Gamma\left(\frac{C_2 k\varepsilon}{2R_\Lambda}\right)} < +\infty.$$

Then by Borel-Cantelli's lemma, $\frac{D(e_i,k)}{k} \to_{t\to+\infty} 0$, $\mathbb{P}_0^{(\alpha)}$ a.s. . It gives

$$\lim_{t\to+\infty} \frac{X_t}{t} = \lim_{k\to+\infty} \frac{X_k}{k} = \mathbb{E}^{\mathbb{Q}^{(\alpha)}}\left[E_0^\omega(X_1)\right] \ , \ \mathbb{P}_0^{(\alpha)} \text{ a.s. .}$$

This gives the directional transience in the case $d_\alpha \cdot e_i > 0$, and finishes the proof.

□

Proof of corollary 2.6. We prove as in the proof of theorem 2.4 that

$$\lim_{t\to+\infty} \frac{X_t}{t} = \lim_{k\to+\infty} \frac{X_k}{k} = \mathbb{E}^{\mathbb{Q}^{(\alpha)}}\left[E_0^\omega(X_1)\right] \ , \ \mathbb{P}_0^{(\alpha)} \text{ a.s. .} \tag{5.2}$$

We still note $A_l = \{X_{t_k} \cdot l \to \infty, \ k \in N\}$. Suppose that $\mathbb{P}_0^{(\alpha)}(A_l) > 0$. Then $\mathbb{P}_0^{(\alpha)}(A_l \cup A_{-l}) = 1$ ([6], [21] proposition 3). It allows to find a finite interval I of $\mathbb{R}$, of positive measure, containing 0 and such that $(X_{t_k} \cdot l)_{k\in\mathbb{N}}$ goes a finite number of times in I, $\mathbb{P}_0^{(\alpha)}$ a.s. and thus $\mathbb{Q}_0^{(\alpha)}$ a.s.. As before, it implies that $(X_k \cdot l)_{k\in\mathbb{N}}$ goes a finite number of times in I, $\mathbb{Q}_0^{(\alpha)}$ a.s.. We can then apply the theorem of [1] to $(X_k)_{k\in\mathbb{Z}}$ (obtained via the extension of $(X_t)_{t\in\mathbb{R}_+}$ to $t \in \mathbb{R}$) to get $\mathbb{E}^{\mathbb{Q}^{(\alpha)}}\left[E_0^\omega(X_1 \cdot l)\right] \neq 0$. We then deduce from (5.2) that : $X_t \cdot l \to_{t\to\infty} +\infty$ $\mathbb{P}_0^{(\alpha)}$ a.s. if $\mathbb{E}^{\mathbb{Q}^{(\alpha)}}\left[E_0^\omega(X_1 \cdot l)\right] > 0$, $X_t \cdot l \to_{t\to\infty} -\infty$ $\mathbb{P}_0^{(\alpha)}$ a.s. else-wise.

As $(X_t)_{t\in\mathbb{R}_+}$ and $(Z_n)_{n\in\mathbb{N}}$ go through exactly the same vertices in the same order, and as the two processes stay a finite time on each vertex, without exploding (see lemma 5.3), recurrence and transience for $Z_n \cdot l$ follows from those of $X_t \cdot l$. This gives as a consequence Kalikow's $0-1$ law in the $d \geq 3$ Dirichlet case.

The $0-1$ law is true in the general case of random walks in random environments for $d = 1$ and $d = 2$ (see respectively Solomon ([16]) and Zerner and Merkl ([21])), it concludes the proof.

□

6 Proof of theorem 2.8

To prove the result, we need a preliminary theorem on the polynomial order of the hitting times of the walk.

Theorem 6.1. *Let $d \geq 3$, $\mathbb{P}^{(\alpha)}$ be the law of the Dirichlet environment with parameters $(\alpha_1, \ldots, \alpha_{2d})$ on $\mathbb{Z}^d$, and Z_n the associated random walk in Dirichlet environment. We suppose that $\kappa = 2\left(\sum_{i=1}^{2d} \alpha_i\right) - \max_{i=1,\ldots,d}(\alpha_i + \alpha_{i+d}) \leq 1$. Let $l \in \{e_1, \ldots, e_{2d}\}$ be such that $d_\alpha \cdot l \neq 0$. Let $T_n^{l,Z} = \inf_i\{i \in \mathbb{N} | Z_i \cdot l \geq n\}$ be the hitting time of the level n in direction l, for the non-accelerated walk Z. Then :*

$$\lim_{n\to+\infty} \frac{\log(T_n^{l,Z})}{\log(n)} = \frac{1}{\kappa} \text{ in } \mathbb{P}^{(\alpha)}\text{-probability.}$$

The proof of this preliminary theorem consists of two bounds. The upper bound is a consequence of theorem 2.4 and therefore needs an accelerated walk with $\kappa^\Lambda > 1$. The lower bound does not make use of an accelerated walk at all.

Proof. **Upper bound**

For Λ such that $\kappa^\Lambda > 1$, we define $A(t) = \int_0^t \gamma^\omega(X_s)ds$. Then $X_{A^{-1}(t)}$ is the continuous-time Markov chain whose jump rate from x to y is $\omega(x,y)$. This Markov chain has asymptotically the same behaviour as Z_n, then we only have to prove that

$$\lim_{n\to+\infty} \frac{\log(A(T_n^{l,X}))}{\log(n)} \leq \frac{1}{\kappa},$$

with $T_n^{l,X} = \inf_t\{t \in \mathbb{R}_+ | X_t \cdot l \geq n\}$.

Set $0 < \alpha < \kappa$, and take β such that $\alpha < \beta < \kappa$. Using first Markov's inequality and then the inequality $(\sum_{i=1}^j \lambda_i)^\varepsilon \leq \sum_{i=1}^j (\lambda_i)^\varepsilon$ for $\varepsilon < 1$ gives :

$$\begin{aligned}
\mathbb{P}^{(\alpha)}\left(\frac{A(t)}{t^{\frac{1}{\alpha}}} \geq x\right) &\leq \frac{1}{x^\beta t^{\frac{\beta}{\alpha}}} \mathbb{E}\left(\left(\int_0^t \gamma^\omega(X_s)ds\right)^\beta\right) \\
&\leq \frac{1}{x^\beta t^{\frac{\beta}{\alpha}}} \mathbb{E}\left(\sum_{i=1}^{\lceil t\rceil}\left(\int_{i-1}^i \gamma^\omega(X_s)ds\right)^\beta\right) \\
&= \frac{1}{x^\beta t^{\frac{\beta}{\alpha}}} \sum_{i=1}^{\lceil t\rceil} \mathbb{E}\left(\left(\int_{i-1}^i \gamma^\omega(X_s)ds\right)^\beta\right)
\end{aligned}$$

where $\lceil t\rceil$ represents the upper integer part of t. Let $D_i = \max_{l\in\{e_1,\dots,e_{2d}\}}(D(l,i))$ (cf. lemma 5.3 for the definition of $D(l,i)$). Splitting the expectation depending on the value of D_i gives :

$$\begin{aligned}
\mathbb{E}\left(\left(\int_{i-1}^i \gamma^\omega(X_s)ds\right)^\beta\right) &= \mathbb{E}\left(\sum_{k=0}^{+\infty} \mathbb{1}_{\{D_i=k\}}\left(\int_{i-1}^i \gamma^\omega(X_s)ds\right)^\beta\right) \\
&\leq \sum_{k=0}^{+\infty} \mathbb{E}\left(\mathbb{1}_{\{D_i=k\}}\left(\sum_{x\in B(X_i,k)} \gamma^\omega(x)\right)^\beta\right) \\
&\leq \sum_{k=0}^{+\infty} \mathbb{E}\left(\mathbb{1}^p_{\{D_i=k\}}\right)^{\frac{1}{p}} \mathbb{E}\left(\left(\sum_{x\in B(X_i,k)} \gamma^\omega(x)\right)^{q\beta}\right)^{\frac{1}{q}}
\end{aligned}$$

where $B(X_i,k) = \{x \in \mathbb{Z}^d | \max_{j=1,\dots,2d} |(X_i - x)\cdot e_j| \leq k\}$, and $\frac{1}{p} + \frac{1}{q} = 1$. For the last Hölder's inequality, we chose $q > 1$ such that $q\beta < \kappa$.

In the following, c, C_1 and C_2 will be finite constants, that can change from line to line. As $\mathbb{P}(D_i = k) \leq \mathbb{P}(D_i \geq k) \leq \frac{C_1^k}{\Gamma(C_2 k)}$ by lemma 5.3 and $q\beta < 1$ we get :

$$\begin{aligned}
\mathbb{E}\left(\left(\int_{i-1}^i \gamma^\omega(X_s)ds\right)^\beta\right) &\leq \sum_{k=0}^{+\infty} \frac{C_1^{\frac{k}{p}}}{\Gamma(C_2 k)^{\frac{1}{p}}} \mathbb{E}\left(\left(\sum_{x\in B(X_i,k)} \gamma^\omega(x)\right)^{q\beta}\right)^{\frac{1}{q}} \\
&\leq \sum_{k=0}^{+\infty} c\frac{C_1^k}{\Gamma(C_2 k)^{\frac{1}{p}}} \sum_{x\in B(X_i,k)} \mathbb{E}\left(\gamma^\omega(x)^{q\beta}\right)^{\frac{1}{q}}
\end{aligned}$$

As the Dirichlet laws are iid, the value of the expectation is independent of x. Lemma

5.1 then gives a uniform finite bound for all x.

$$\begin{aligned}
\mathbb{P}^{(\alpha)}\left(\frac{A(t)}{t^{\frac{1}{\alpha}}} \geq x\right) &\leq \frac{1}{x^{\beta} t^{\frac{\beta}{\alpha}}} \sum_{i=1}^{\lceil t \rceil} \sum_{k=0}^{+\infty} c \frac{C_1^k}{\Gamma(C_2 k)^{\frac{1}{p}}} \sum_{x \in B(X_i,k)} 1 \\
&= \frac{1}{x^{\beta} t^{\frac{\beta}{\alpha}}} \sum_{i=1}^{\lceil t \rceil} \sum_{k=0}^{+\infty} c \frac{C_1^k}{\Gamma(C_2 k)^{\frac{1}{p}}} (2k+1)^d \\
&\leq \frac{\lceil t \rceil}{x^{\beta} t^{\frac{\beta}{\alpha}}} \sum_{k=0}^{+\infty} c k^d \frac{C_1^k}{\Gamma(C_2 k)^{\frac{1}{p}}} \\
&\leq c \frac{t^{1-\frac{\beta}{\alpha}}}{x^{\beta}}
\end{aligned}$$

As $\beta > \alpha$, it implies that $\frac{A(t)}{t^{\frac{1}{\alpha}}} \to_{t \to +\infty} 0$ in $\mathbb{P}^{(\alpha)}$-probability, for all $\alpha < \kappa$. Then, $\frac{A(T_n^{l,X})}{(T_n^{l,X})^{\frac{1}{\alpha}}} \to_{t \to +\infty} 0$ in $\mathbb{P}^{(\alpha)}$-probability. As we chose $\kappa^{\Lambda} > 1$, we can apply theorem 2.4 that gives

$$\lim_{t \to +\infty} \frac{X_t \cdot l}{t} = v \cdot l \neq 0, \ \mathbb{P}_0^{(\alpha)} \text{a.s..}$$

Then $\frac{X_{T_n^{l,X}} \cdot l}{T_n^{l,X}} \to v \cdot l$ and $T_n^{l,X} \sim \frac{n}{v \cdot l}$. It implies that $\frac{A(T_n^{l,X})}{n^{\frac{1}{\alpha}}} \sim \frac{A(T_n^{l,X})}{(T_n^{l,X})^{\frac{1}{\alpha}}} (v \cdot l)^{\frac{1}{\alpha}} \to_{n \to +\infty} 0$ in $\mathbb{P}^{(\alpha)}$-probability, for all $\alpha < \kappa$.

It gives $\lim_{n \to +\infty} \frac{\log(A(T_n^{l,X}))}{\log(n)} \leq \frac{1}{\kappa}$ and concludes the proof of the upper bound.

Lower bound

This proof follows the lines of the proof of proposition 12 in [18]. As $\kappa \leq 1$, we can assume that $\alpha_1 + \alpha_{-1} \geq 2\sum_{j=1}^{2d} \alpha_j - 1$. We prove that, for every $l \in \{e_1, \ldots, e_{2d}\}$, for every $\alpha > \kappa$, $\frac{T_{2n}^{l,Z}}{n^{\frac{1}{\alpha}}} \to_{n \to \infty} +\infty$ $\mathbb{P}^{(\alpha)}$ a.s.. The same being true for $(T_{2n+1}^{l,Z})$, this is sufficient to conclude.

Set $l \in \{e_1, \ldots, e_{2d}\}$. We introduce the exit times

$$\Theta_0 = \inf\{n \in \mathbb{N} | Z_n \notin \{Z_0, Z_0 + e_1\}\}$$

(with a minus sign instead of the plus if $l = -e_1$), and for $k \geq 1$, $\Theta_k = \Theta_0 \circ \tau_{T_{2k}^{l,Z}}$ (where τ is the time-shift). We use the convention that $\Theta_k = \infty$ if $T_{2k}^{l,Z} = \infty$. The only dependence between the times Θ_k is that $\Theta_j = \infty$ implies $\Theta_k = \infty$ for all $k \geq j$. The "2" in $T_{2k}^{l,Z}$ causes indeed Θ_k to depend only on $\{x \in \mathbb{Z}^d | x \cdot l \in \{2k, 2k+1\}\}$ which are disjoint parts of the environment.

For $t_0, \ldots, t_k \in N$, one has, using the Markov property at time $T_{2k}^{l,Z}$, the independence and the translation invariance of $\mathbb{P}^{(\alpha)}$:

$$\begin{aligned}
\mathbb{P}_0^{(\alpha)}(\Theta_0 = t_0, \ldots, \Theta_k = t_k) &= \mathbb{P}_0^{(\alpha)}\left(\Theta_0 = t_0, \ldots, \Theta_{k-1} = t_{k-1}, \Theta_k = t_k, T_{2k}^{l,Z} < \infty\right) \\
&\leq \mathbb{P}_0^{(\alpha)}(\Theta_0 = t_0, \ldots, \Theta_{k-1} = t_{k-1}) \, \mathbb{P}_0^{(\alpha)}(\Theta_0 = t_k) \\
&\leq \cdots \leq \mathbb{P}_0^{(\alpha)}(\Theta_0 = t_0) \ldots \mathbb{P}_0^{(\alpha)}(\Theta_0 = t_{k-1}) \, \mathbb{P}_0^{(\alpha)}(\Theta_0 = t_k) \\
&= \mathbb{P}^{(\alpha)}\left(\hat{\Theta}_0 = t_0, \ldots, \hat{\Theta}_k = t_k\right)
\end{aligned}$$

where, under $\mathbb{P}^{(\alpha)}$, the random variables $\hat{\Theta}_k$ are independent and have the same distribution as Θ_0. From this, we deduce that for all $A \subset \mathbb{N}^{\mathbb{N}}$,

$$\mathbb{P}_0^{(\alpha)}((\Theta_k) \in A) \leq \mathbb{P}^{(\alpha)}\left((\hat{\Theta}_k) \in A\right).$$

In particular, for $\alpha > \kappa$,

$$\mathbb{P}_0^{(\alpha)}\left(\liminf_k \frac{\Theta_0 + \cdots + \Theta_{k-1}}{k^{\frac{1}{\alpha}}} < \infty\right) \leq \mathbb{P}^{(\alpha)}\left(\liminf_k \frac{\hat{\Theta}_0 + \cdots + \hat{\Theta}_{k-1}}{k^{\frac{1}{\alpha}}} < \infty\right). \quad (6.1)$$

In order to bound this probability, we compute the tail of the distribution of Θ_0 using Stirling's formula :

$$\begin{aligned}\mathbb{P}_0^{(\alpha)}(\Theta_0 \geq n) &= \mathbb{E}\left(\omega(0,e_1)^{\lceil \frac{n}{2}\rceil}\omega(e_1,0)^{\lfloor \frac{n}{2}\rfloor}\right)\\ &= \frac{\Gamma(\alpha_0)^2}{\Gamma(\alpha_1)\Gamma(\alpha_{-1})}\frac{\Gamma(\alpha_1+\lceil\frac{n}{2}\rceil)\Gamma(\alpha_{-1}+\lfloor\frac{n}{2}\rfloor)}{\Gamma(\alpha_0+\lceil\frac{n}{2}\rceil)\Gamma(\alpha_0+\lfloor\frac{n}{2}\rfloor)}\\ &\sim_{n\to\infty} cn^{\alpha_1+\alpha_{-1}-2\alpha_0} = cn^{-\kappa}\end{aligned}$$

with c a constant. We can then use the limit theorem for stable laws (see for example [3]) that gives :

$$\frac{\hat{\Theta}_0 + \cdots + \hat{\Theta}_{k-1}}{k^{\frac{1}{\kappa}}} \Rightarrow Y$$

where Y has a non-degenerate distribution. Then for $\alpha > \kappa$, $\frac{\hat{\Theta}_0+\cdots+\hat{\Theta}_{k-1}}{k^{\frac{1}{\alpha}}} \to \infty$. (6.1) then gives $\mathbb{P}_0^{(\alpha)}\left(\liminf_k \frac{\Theta_0+\cdots+\Theta_{k-1}}{k^{\frac{1}{\alpha}}} < \infty\right) = 0$.

As $T_{2k}^{l,Z} \geq \Theta_0 + \cdots + \Theta_{k-1}$, it gives $\frac{T_{2n}^{l,Z}}{n^{\frac{1}{\alpha}}} \to_{n\to\infty} +\infty$ $\mathbb{P}^{(\alpha)}$ a.s. as wanted, for all $\alpha > \kappa$. It gives $\lim_{n\to+\infty} \frac{\log(T_n^{l,Z})}{\log(n)} \geq \frac{1}{\kappa}$ and concludes the proof of the lower bound.

□

Using an inversion argument, we can now prove theorem 2.8.

Proof of theorem 2.8. We note $\overline{Z_n} = \max_{i\leq n} Z_i \cdot l$. As $\overline{Z_n} \geq m \Leftrightarrow T_m^{l,Z} \leq n$, theorem 6.1 gives that for any $\varepsilon > 0$ we have, for n big enough,

$$n^{\kappa-\varepsilon} \leq \overline{Z_n} \leq n^{\kappa+\varepsilon} \text{ in } \mathbb{P}^{(\alpha)}\text{-probability.}$$

As $Z_n \cdot l$ is transient, we can introduce renewal times τ_i for the direction l (see [17] or [20] p71 for a detailed construction) such that $\tau_i < +\infty$ $\mathbb{P}^{(\alpha)}$ a.s., for all i. Then

$$0 \leq \overline{Z_n} - Z_n \cdot l \leq \max_{i=0,\ldots,n-1}(Z_{\tau_{i+1}} - Z_{\tau_i}) \cdot l \ \text{ for } n \geq \tau_1.$$

When the walk $Z_n \cdot l$ discovers a new vertex in direction l, there is a positive probability that this vertex will be the next Z_{τ_i}. As the vertices have i.i.d. exit probabilities under $\mathbb{P}^{(\alpha)}$, this probability is independent of the newly discovered vertex, and is independent of the path that lead to this vertex. Then $(Z_{\tau_{i+1}} - Z_{\tau_i}) \cdot l$ follows a geometric law of parameter $\mathbb{P}^{(\alpha)}(Z_0 = Z_{\tau_1})$, for all $i \in \mathbb{N}$. This means that we can find C and c two positive constants such that for all n, $\mathbb{P}^{(\alpha)}\left((Z_{\tau_{i+1}} - Z_{\tau_i}) \cdot l \geq n\right) \leq Ce^{-cn}$.

Borel Cantelli's lemma then gives that, for n big enough,

$$\max_{i=0,\ldots,n-1}(Z_{\tau_{i+1}} - Z_{\tau_i}) \cdot l \leq (\log n)^2 \quad \mathbb{P}^{(\alpha)} \text{ a.s..}$$

As $\tau_1 < \infty$, it gives

$$n^{\kappa-\varepsilon} \leq Z_n \cdot l \leq n^{\kappa+\varepsilon} \text{ in } \mathbb{P}^{(\alpha)}\text{-probability.}$$

Taking the limit $\varepsilon \to 0$ gives $\lim_{n\to+\infty} \frac{\log(Z_n \cdot l)}{\log(n)} = \kappa$ and concludes the proof.

□

A Appendix : Proof of lemma 5.1

The proof that follows is largely inspired by the article [18] by Tournier. His result can however not be directly applied here, as $\gamma^\omega(x) \geq G^{\omega,\Lambda}(x,x)$, and some of the paths he considered are not necessarily simple paths. To adapt the proof to our case, we need an additional assumption on the graph (some symmetry property for the edges), which simplifies the proof (the construction of the set $C(\omega)$ is quite shorter).

To prove the result, we consider the case of finite directed graphs with a cemetery vertex. A vertex δ is said to be a cemetery vertex when no edge exits δ, and every vertex is connected to δ through a directed path. We furthermore suppose that the graphs have no multiple edges, no elementary loop (consisting of one edge starting and ending at the same point), and that if $(x,y) \in E$ and $y \neq \delta$, then $(y,x) \in E$.

We need a definition of $\gamma^\omega(x)$ for those graphs. Let $G = (V \cup \{\delta\}, E)$ be a finite directed graph, $(\alpha(e))_{e\in E}$ be a family of positive real numbers, $\mathbb{P}^{(\alpha)}$ be the corresponding Dirichlet distribution, and (Z_n) the associated random walk in Dirichlet environment. We need the following stopping times : the hitting times

$$H_x = \inf\{n \geq 0 | Z_n = x\}$$

and

$$\tilde{H}_x = \inf\{n \geq 1 | Z_n = x\}$$

for $x \in G$, the exit time

$$T_A = \inf\{n \geq 0 | Z_n \notin A\}$$

for $A \subset V$, and the time of the first loop

$$L = \inf\{n \geq 1 | \exists n_0 < n \text{ such that } Z_n = Z_{n_0}\}.$$

For x in such a G, we define :

$$\gamma^\omega(x) = \frac{1}{P_x^\omega(H_\delta < \tilde{H}_x \wedge L)} = \frac{1}{\sum\limits_{\sigma : x \to \delta} \omega_\sigma}.$$

where we sum on simple paths from x to δ. In the following, we denote by 0 an arbitrary fixed vertex in G. We use the notations $\underline{A} = \{\underline{e} | e \in A\}$ and $\overline{A} = \{\overline{e} | e \in A\}$ for $A \subset E$, and we call strongly connected a subset A of E such that for all $x, y \in \overline{A} \cup \underline{A}$, there is a path in A from x to y. Remark that if A is strongly connected, then $\overline{A} = \underline{A}$.

For the new function γ^ω on G, we get the following result

Theorem A.1. *Let $G = (V \cup \{\delta\}, E)$ be a finite directed graph, where δ is a cemetery vertex. We furthermore suppose that G has no multiple edges, no elementary loop, and that if $(x,y) \in E$ and $y \neq \delta$, then $(y,x) \in E$. Let $(\alpha(e))_{e\in E}$ be a family of positive real numbers, and $\mathbb{P}^{(\alpha)}$ be the corresponding Dirichlet distribution. Let $0 \in V$. There exist $c, C, r > 0$ such that, for t large enough,*

$$\mathbb{P}^{(\alpha)}(\gamma^\omega(0) > t) \leq C \frac{(\ln t)^r}{t^{\min_A \beta_A}}$$

where the minimum is taken over all strongly connected subsets A of E such that $0 \in \underline{A}$, and $\beta_A = \sum_{e \in \partial_+ \underline{A}} \alpha(e)$, (we recall that $\partial_+(K) = \{e \in E,\ \underline{e} \in K,\ \overline{e} \notin K\}$).

In $\mathbb{Z}^d$, we can identify Λ^c (where Λ is the subset involved in the construction of γ^ω) with a cemetery vertex δ. We obtain a graph where the two definitions of γ^ω coincide, and that verifies the hypothesis of theorem A.1. Among the strongly connected subsets A of edges such that $\underline{A}$ contains a given x, the ones minimizing the "exit sum" β_A are made of only two edges $(x, x+e_i)$ and $(x+e_i, x)$, $i \in [\![1, 2d]\!]$. Then $\min_A \beta_A = \kappa = 2\left(\sum_{i=1}^{2d} \alpha_i\right) - \max_{i=1,\dots,d}(\alpha_i + \alpha_{i+d})$. It proves lemma 5.1.

Proof of theorem A.1. This proof is based on the proof of the "upper bound" in [18]. We need lower bounds on the probability to reach δ by a simple path. We construct a random subset $C(\omega)$ where a weaker ellipticity condition holds. Quotienting by this subset allows to get a lower bound for the equivalent of $P_0^\omega(H_\delta < \tilde{H}_0 \wedge L)$ in the quotient graph. Proceeding by induction then allows to conclude.

We proceed by induction on the number of edges of G. More precisely, we prove :

Proposition A.2. *Let $n \in \mathbb{N}^*$. Let $G = (V \cup \{\delta\}, E)$ be a directed graph possessing at most n edges, and such that every vertex is connected to δ by a directed path. We furthermore suppose that G has no multiple edges, no elementary loop, and that if $(x, y) \in E$ and $y \neq \delta$, then $(y, x) \in E$. Let $(\alpha(e))_{e \in E}$ be positive real numbers. Then, for every vertex $0 \in V$, there exist real numbers $C, r > 0$ such that, for small $\varepsilon > 0$,*

$$\mathbb{P}^{(\alpha)}\left(P_0^\omega(H_\delta < \tilde{H}_0 \wedge L) \leq \varepsilon\right) \leq C\varepsilon^\beta(-\ln\varepsilon)^r$$

where $\beta = \min\{\beta_A | A$ is a strongly connected subset of V and $0 \in \underline{A}\}$.

As $\gamma^\omega(0) = \frac{1}{P_0^\omega(H_\delta < \tilde{H}_0 \wedge L)}$, this proposition suffices to prove the result. The following is devoted to its proof.

Initialization : if $|E| = 1$, the only edge links 0 to δ, then $P_0^\omega(H_\delta < \tilde{H}_0 \wedge L) = 1$ and the property is true.

If $|E| = 2$, the only possible edges link 0 to δ, and another vertex x to δ, then $P_0^\omega(H_\delta < \tilde{H}_0 \wedge L) = 1$ and the property is true.

Let $n \in \mathbb{N}^*$. We suppose the induction hypothesis to be true at rank n. Let $G = (V \cup \{\delta\}, E)$ be a directed graph with $n+1$ edges, and such that every vertex is connected to δ by a directed path. We furthermore suppose that G has no multiple edges, no elementary loop, and that if $(x, y) \in E$ and $y \neq \delta$, then $(y, x) \in E$. Let $(\alpha(e))_{e \in E}$ be positive real numbers. To get a "weak ellipticity condition", we introduce the random subset $C(\omega)$ of E constructed as follows :

Construction of $C(\omega)$. Let $\omega \in \Omega$. Let x be chosen for $\omega(0, x)$ to be a maximizer on all $\omega(0, y)$, $y \sim 0$. If $x \neq \delta$, we set

$$C(\omega) = \{(0, x); (x, 0)\}.$$

If $x = \delta$, we set $C(\omega) = \{(0, \delta)\}$. Remark that $C(\omega)$ is well defined as soon as x is uniquely defined, which means almost surely, as there is always a directed path heading to δ.

The support of the distribution of $\omega \to C(\omega)$ writes as a disjoint union $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_\delta$ depending whether $x = \delta$ or not. For $C \in \mathcal{C}$, we define the event

$$\mathcal{E}_C = \{C(\omega) = C\}.$$

As $\mathcal{C}$ is finite, it is sufficient to prove the upper bound separately on all events $\mathcal{E}_C$. If $C \in \mathcal{C}_\delta$, on $\mathcal{E}_C$, $P_0^\omega(H_\delta < \tilde{H}_0 \wedge L) \geq P_0^\omega(Z_1 = \delta) \geq \frac{1}{|E|}$ by construction of $C(\omega)$. Then we have for small $\varepsilon > 0$:

$$\mathbb{P}^{(\alpha)}\left(P_0^\omega(H_\delta < \tilde{H}_0 \wedge L) \leq \varepsilon, \mathcal{E}_C\right) = 0$$

In the following, we will therefore work on $\mathcal{E}_C$, when $C \in \mathcal{C}_0$ (ie when $x \neq \delta$). In this case, C is strongly connected.

Quotienting procedure.

Definition A.3. *If A is a strongly connected subset of edges of a graph $G = (V, E)$, the quotient graph of G obtained by contracting $A \subset E$ to the vertex $\tilde{a}$ is the graph $\tilde{G}$ deduced from G by deleting the edges of A, replacing all the vertices of $\underline{A}$ by one new vertex $\tilde{a}$, and modifying the endpoints of the edges of $E \setminus A$ accordingly. Thus the set of edges of $\tilde{G}$ is naturally in bijection with $E \setminus A$ and can be thought of as a subset of E.*

In our case, we consider the quotient graph $\tilde{G}$ obtained by contracting $C(\omega)$, which is a strongly connected subset of E, to a new vertex $\tilde{0}$. We need to define the associated quotient environment $\tilde{\omega} \in \tilde{\Omega}$. For every edge in $\tilde{E}$, if $e \notin \partial_+\underline{C}$ then $\tilde{\omega}(e) = \omega(e)$, and if $e \in \partial_+\underline{C}$, $\tilde{\omega}(e) = \frac{\omega(e)}{\Sigma}$, where $\Sigma = \sum_{e\in\partial_+\underline{C}} \omega(e)$.

This environment allows us to bound $\gamma^\omega(0)$ using the similar quantity in $\tilde{G}$. Notice that, from 0, one way for the walk to reach δ without coming back to 0 and without making loops consists in exiting C without coming back to 0, and then reaching δ without coming back to $\underline{C}$ (0 or x) and without making loops. Then, for $\omega \in \mathcal{E}_C$,

$$\begin{aligned}
&P_0^\omega(H_\delta < \tilde{H}_0 \wedge L)\\
&\geq P_0^\omega(H_\delta < \tilde{H}_{\underline{C}} \wedge L) + P_0^\omega(Z_1 = x, H_\delta < 1 + (\tilde{H}_{\underline{C}} \wedge L)\circ\tau_1)\\
&= P_0^\omega(H_\delta < \tilde{H}_{\underline{C}} \wedge L) + P_0^\omega(Z_1 = x) P_x^\omega(H_\delta < \tilde{H}_{\underline{C}} \wedge L)\\
&\geq P_0^\omega(H_\delta < \tilde{H}_{\underline{C}} \wedge L) + \frac{1}{|E|} P_x^\omega(H_\delta < \tilde{H}_{\underline{C}} \wedge L)\\
&\geq \frac{1}{|E|}\Big(P_0^\omega(H_\delta < \tilde{H}_{\underline{C}} \wedge L) + P_x^\omega(H_\delta < \tilde{H}_{\underline{C}} \wedge L)\Big)\\
&= \frac{1}{|E|}\Sigma P_{\tilde{0}}^{\tilde{\omega}}(H_\delta < \tilde{H}_{\tilde{0}} \wedge L)
\end{aligned}$$

where we used the Markov property, the construction of C, $\frac{1}{|E|} \leq 1$, and the definition of the quotient. Finally, we have

$$\mathbb{P}^{(\alpha)}\Big(P_0^\omega(H_\delta < \tilde{H}_0 \wedge L) \leq \varepsilon, \mathcal{E}_C\Big) \leq \mathbb{P}^{(\alpha)}\Big(\Sigma P_{\tilde{0}}^{\tilde{\omega}}(H_\delta < \tilde{H}_{\tilde{0}} \wedge L) \leq |E|\varepsilon, \mathcal{E}_C\Big). \tag{A.1}$$

Back to Dirichlet environment. Under $\mathbb{P}^{(\alpha)}$, $\tilde{\omega}$ does not follow a Dirichlet distribution because of the normalization. But we can reduce to the Dirichlet situation with the following lemma (which is a particular case of lemma 9 in [18]).

Lemma A.4. *Let $(\omega_i^{(0)})_{1\leq i\leq n_0}, (\omega_i^{(x)})_{1\leq i\leq n_x}$ be the exit probabilities out of 0 and x for $\omega \in \Omega$, they are independent random variables following Dirichlet laws of respective parameters $(\alpha_i^{(0)})_{1\leq i\leq n_0}, (\alpha_i^{(x)})_{1\leq i\leq n_x}$. Let $\Sigma = \sum_{e\in\partial_+\underline{C}} \omega(e)$ and $\beta_C = \sum_{e\in\partial_+\underline{C}} \alpha(e)$. There exists positive constants c, c' such that, for every $\varepsilon > 0$,*

$$\mathbb{P}^{(\alpha)}\Big(\Sigma P_{\tilde{0}}^{\tilde{\omega}}(H_\delta < \tilde{H}_{\tilde{0}} \wedge L) \leq \varepsilon\Big) \leq c\tilde{\mathbb{P}}^{(\alpha)}\Big(\tilde{\Sigma} P_{\tilde{0}}^{\omega}(H_\delta < \tilde{H}_{\tilde{0}} \wedge L) \leq \varepsilon\Big),$$

where $\tilde{\mathbb{P}}^{(\alpha)}$ is the Dirichlet distribution of parameter $(\alpha(e))_{e\in\tilde{E}}$ on $\tilde{\Omega}$, ω is the canonical random variable on $\tilde{\Omega}$, and, under $\tilde{\mathbb{P}}^{(\alpha)}$, $\tilde{\Sigma}$ is a positive bounded random variable independent of ω and such that, for all $\varepsilon > 0$, $\tilde{\mathbb{P}}^{(\alpha)}(\tilde{\Sigma} \leq \varepsilon) \leq c'\varepsilon^{\beta_C}$.

Remark that the symmetry property we imposed on the edges is important here : if there was no edge from x to 0, the probability for a walk in $\tilde{G}$ to exit $\tilde{0}$ through one of the edges exiting x in G would necessarily be bigger than $\frac{1}{2}$. Then asymptotically, it could not be bounded by Dirichlet variables.

This lemma and (A.1) give :

$$\begin{aligned}
\mathbb{P}^{(\alpha)}\Big(P_0^\omega(H_\delta < \tilde{H}_0 \wedge L) \leq \varepsilon, \mathcal{E}_C\Big) &\leq \mathbb{P}^{(\alpha)}\Big(\Sigma P_{\tilde{0}}^{\tilde{\omega}}(H_\delta < \tilde{H}_{\tilde{0}} \wedge L) \leq |E|\varepsilon, \mathcal{E}_C\Big) \\
&\leq \mathbb{P}^{(\alpha)}\Big(\Sigma P_{\tilde{0}}^{\tilde{\omega}}(H_\delta < \tilde{H}_{\tilde{0}} \wedge L) \leq |E|\varepsilon\Big)\\
&\leq c\tilde{\mathbb{P}}^{(\alpha)}\Big(\tilde{\Sigma} P_{\tilde{0}}^{\omega}(H_\delta < \tilde{H}_{\tilde{0}} \wedge L) \leq |E|\varepsilon\Big).
\end{aligned} \tag{A.2}$$

Induction. Inequality (A.2) relates the same quantities in G and $\tilde{G}$, allowing to complete the induction argument.

The edges in C do not appear in $\tilde{G}$ any more : $\tilde{G}$ has $n-2$ edges. In order to apply the induction hypothesis, we need to check that each vertex is connected to δ. This results directly from the same property for G. If $(x,y)\in\tilde{E}$ and $y\neq\delta$, then $(x,y)\notin C(\omega)$ and $(y,x)\notin C(\omega)$. As only the edges of $C(\omega)$ disappeared, then $(y,x)\in\tilde{E}$. $\tilde{G}$ has no elementary loop. Indeed G has none, and the quotienting only merges the vertices of $\underline{C}$, whose joining edges are those of C, deleted in the construction. It only remains to prove that $\tilde{G}$ has no multiple edges. It is not necessarily the case (quotienting may have created multiple edges), but it is possible to reduce to this case, using the additivity property of the Dirichlet distribution.

The induction hypothesis applied to $\tilde{G}$ and $\tilde{0}$ then gives, for small $\varepsilon>0$,

$$\tilde{\mathbb{P}}^{(\alpha)}\left(P_{\tilde{0}}^{\omega}(H_\delta<\tilde{H}_{\tilde{0}}\wedge L)\leq\varepsilon\right)\leq c''\varepsilon^{\tilde{\beta}}(-\ln\varepsilon)^r, \tag{A.3}$$

where $c''>0$, $r>0$ and $\tilde{\beta}$ is the exponent "β" from the statement of the induction hypothesis corresponding to the graph $\tilde{G}$.

This inequality, associated with (A.2) and the following simple lemma (also see [18] for the proof of the lemma) then allows to carry out the induction :

Lemma A.5. *If X and Y are independent positive bounded random variables such that, for some real numbers $\alpha_X,\alpha_Y,r>0$,*

- *there exists $C>0$ such that $P(X<\varepsilon)\leq C\varepsilon^{\alpha_X}$ for all $\varepsilon>0$ (or equivalently for small ε);*
- *there exists $C'>0$ such that $P(Y<\varepsilon)\leq C'\varepsilon^{\alpha_Y}(-\ln\varepsilon)^r$ for small $\varepsilon>0$;*

then there exists a constant $C''>0$ such that, for small $\varepsilon>0$,

$$P(XY\leq\varepsilon)\leq C''\varepsilon^{\alpha_X\wedge\alpha_Y}(-\ln\varepsilon)^{r+1}$$

(and $r+1$ can be replaced by r if $\alpha_X\neq\alpha_Y$).

We get from this lemma, (A.2) and (A.3) some constants $c,r>0$ such that, for small $\varepsilon>0$,

$$\mathbb{P}^{(\alpha)}\left(P_0^{\omega}(H_\delta<\tilde{H}_0\wedge L)\leq\varepsilon,\mathcal{E}_C\right)\leq c\varepsilon^{\beta_C\wedge\tilde{\beta}}(-\ln\varepsilon)^{r+1}.$$

It remains to prove that $\tilde{\beta}\geq\beta$, where β is the exponent defined in the induction hypothesis relative to G and 0. Let $\tilde{A}$ be a strongly connected subset of $\tilde{E}$ such that $\tilde{0}\in\underline{\tilde{A}}$. Set $A=\tilde{A}\cup C\subset E$. In view of the definition of $\tilde{E}$, every edge exiting $\tilde{A}$ corresponds to an edge exiting A, and vice-versa (the only edges deleted in the quotient procedure are those of C). Thus, recalling that the weights of the edges are preserved in the quotient, $\beta_{\tilde{A}}=\beta_A$. Moreover, $\tilde{0}\in\underline{A}$ and A is strongly connected, so that $\beta_A\geq\beta$. As a consequence, $\tilde{\beta}\geq\beta$ as announced.

Then $\beta_C\wedge\tilde{\beta}\geq\beta_C\wedge\beta=\beta$ because C is strongly connected, and $0\in\underline{C}$. It gives, for small $\varepsilon>0$:

$$\mathbb{P}^{(\alpha)}\left(P_0^{\omega}(H_\delta<\tilde{H}_0\wedge L)\leq\varepsilon,\mathcal{E}_C\right)\leq c\varepsilon^{\beta}(-\ln\varepsilon)^{r+1}.$$

Summing on all events $\mathcal{E}_C, C\in\mathcal{C}$ concludes the induction and the proof.

□

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