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The near-critical scaling window for directed polymers on disordered trees

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Abstract

We study a directed polymer model in a random environment on infinite binary trees. The model is characterized by a phase transition depending on the inverse temperature. We concentrate on the asymptotics of the partition function in the near-critical regime, where the inverse temperature is a small perturbation away from the critical one with the perturbation converging to zero as the system size grows large. Depending on the speed of convergence we observe very different asymptotic behavior. If the perturbation is small then we are inside the critical window and observe the same decay of the partition function as at the critical temperature. If the perturbation is slightly larger the near-critical scaling leads to a new range of asymptotic behaviors, which at the extremes match up with the already known rates for the sub- and super-critical regimes. We use our results to identify the size of the fluctuations of the typical energies under the critical Gibbs measure.

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1 Introduction and main results

1.1 Introduction

Polymers in a random environment are classical examples of models driven by an energy - entropy competition. In these models, the directed polymer corresponds to the path of a random walk on a lattice while the environment is a field of i.i.d. random variables. The path's interaction with the random environment is governed by an (inverse) temperature parameter β. As the temperature is decreased, the behavior changes from an entropy-dominated regime with a diffusively behaving polymer, to an energy dominated regime in which the polymers prefer regions where the environment is especially favorable. While the large temperature phase is fairly well understood, there are many open problems in the energy dominated regime (especially for general environments).

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Beginning with Derrida-Spohn [13] it was realized that changing the underlying space and studying directed polymers on trees allows the use of different techniques. Most notably, one can use the self-similarity of the graph to exactly compute several quantities. The basic model is the following: let T be an infinite binary tree and to each vertex $v \in T$ attach a random variable $\omega(v)$. The collection $\{\omega(v)\}_{v\in T}$ is assumed to be i.i.d., and throughout we assume that

$$e^{\lambda(\beta)} := \mathbb{E}\big[e^{\beta\omega}\big] < \infty \text{ for all } \beta \in \mathbb{R}.$$

Let o be the root of the tree and $|v|$ denote the generation of each vertex. If $|v| = n$ let $(o = v_0, v_1, \ldots, v_n)$ be the unique path of vertices from o to v. Since the path is unique we can refer to each polymer of length n by the last vertex. The interaction with the environment is described by introducing the Gibbs measure $\mu_n^{(\beta)}$ which assigns to each polymer v the probability

$$\mu_n^{(\beta)}(v) = \frac{1}{Z_n(\beta)} \exp\left\{-\beta H(v)\right\}, \tag{1.1}$$

where the energy $H(v)$ is defined by $H(v) = -\sum_{j=1}^{n} \omega(v_j)$, and the normalizing partition function at level n is given by

$$Z_n(\beta) := \sum_{|v|=n} \exp\Big\{ - \beta H(v)\Big\}.$$

Note that by interpreting the energies as spatial positions, one actually obtains a branching random walk (in our case with dyadic branching) and many results were first described in that language. Observe that $\mathbb{E}\left[Z_n(\beta)\right] = e^{n\lambda(\beta)+n\log 2}$, and in fact it is easy to see that

$$W_n(\beta) := Z_n(\beta)/\mathbb{E}Z_n(\beta)$$

is a positive martingale with respect to the filtration $\mathcal{W}_n := \sigma(\omega(v) : |v| \le n)$. Applying Kolmogorov's 0-1 law gives the usual dichotomy that exactly one of the events

$$\lim_{n\to\infty} W_n(\beta) > 0, \quad \lim_{n\to\infty} W_n(\beta) = 0$$

is of full probability. The β for which the limit is positive are said to be in the weak disorder regime; the remaining β are said to be in strong disorder. One of the main advantages of the tree is that there is a complete classification of weak and strong disorder: there exists a $\beta_c \ge 0$ such that the range $0 \le \beta < \beta_c$ is weak disorder and $\beta \ge \beta_c$ is strong disorder. Moreover, β_c is the unique non-negative solution to the equation

$$\lambda(\beta_c) + \log 2 = \beta_c \lambda'(\beta_c). \tag{1.2}$$

If no solution exists then $\beta_c = \infty$. See [16, 9] for proofs of this fact. We will assume throughout that $\beta_c < \infty$. At the critical inverse temperature β_c there is also a drastic change in the behavior of the free energy, as was first proved in a continuous time analogue in [13] and later in the tree case in [12]. The result is that

$$\varphi(\beta) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta) = \begin{cases} \lambda(\beta) + \log 2, & \beta \le \beta_c, \\ \frac{\beta}{\beta_c}\left(\lambda(\beta_c) + \log 2\right), & \beta > \beta_c. \end{cases} \tag{1.3}$$

Observe that the free energy varies continuously with β but starts growing linearly once $\beta > \beta_c$. Transferring this result to $W_n(\beta)$ combined with the convexity of λ gives that for $\beta > \beta_c$, $W_n(\beta)$ decays exponentially fast in n. Note that no statement is made about the decay of the martingale in the critical $\beta = \beta_c$ case, and for a long period of time the exact behavior was unknown. This problem was solved in the important work of Hu and Shi [14] where, among many other results, they prove that

$$W_n(\beta_c) = n^{-\frac{1}{2}+o(1)} \text{ a.s.} \tag{1.4}$$

In particular, this implies that even though $\beta = \beta_c$ is in the strong disorder regime, the partition function decays only polynomially fast rather than exponentially as for $\beta > \beta_c$.

1.2 Main results

The main goal of this paper is to probe the phase transition at β_c and to see, roughly speaking, "how far" it extends on either side of the critical temperature. More precisely, we consider the system at a temperature β_n depending on the system size (parametrized by n) and apply a near-critical scaling of $\beta_n \to \beta_c$ as $n \to \infty$. Our main result determines what types of asymptotics are exhibited for the different choices for scalings of β_n. This question was inspired by the recent work [4] on the lattice model in $1+1$ dimensions.

To formulate our results we introduce, for a polymer v in the nth generation, the normalized energy at criticality

$$V(v) = \beta_c \left(H(v) + n\lambda'(\beta_c) \right) = \beta_c H(v) + n(\lambda(\beta_c) + \log 2), \tag{1.5}$$

with the last equality coming from equation (1.2). Using this notation we have that

$$W_n := W_n(\beta_c) = \sum_{|v|=n} e^{-V(v)}. \tag{1.6}$$

For $\delta > 0$ we introduce the perturbed partition functions

$$W_n^{+,\delta} = \sum_{|v|=n} e^{-(1+n^{-\delta})V(v)}, \text{ and } W_n^{-,\delta} = \sum_{|v|=n} e^{-(1-n^{-\delta})V(v)}. \tag{1.7}$$

This perturbation of the energies corresponds to studying the model near the critical inverse temperature and is more convenient than taking $\beta_n \to \beta_c$ directly. The difference amounts to a deterministic factor which can be calculated explicitly.

The perturbed partition functions (1.7) will be our primary objects of study. We generally refer to $W_n^{\pm,\delta}$ as a either positive or negative perturbations, depending on the sign indicated. In our notation large (small) δ corresponds to small (large) perturbations, and we frequently refer to a perturbation as being large or small. We consider four different types of perturbations (small positive, small negative, large positive, and large negative) and our main results are on the asymptotic behavior of the corresponding partition functions. We show that the separation between small and large perturbations occurs at $\delta = 1/2$. If the perturbation is small, meaning that $\delta \geq 1/2$, then the perturbed partition function decays at the same rate as the unperturbed partition function W_n, see (1.4). This is true for both positive and negative perturbations, and the rate of decay does not depend on δ. However, if the perturbation is large, meaning $0 < \delta < 1/2$, then the asymptotics are different in the positive and negative cases, and the asymptotic rate has an explicit dependence on δ.

Theorem 1.1. *1. If $\delta \geq 1/2$, then in probability*

$$W_n^{\pm,\delta} = n^{-1/2+o(1)}.$$

2. If $0 < \delta < 1/2$, then, almost surely,

$$W_n^{-,\delta} = \exp\left\{ \frac{\beta_c^2 \lambda''(\beta_c)}{2} n^{1-2\delta}(1+o(1)) \right\}.$$

3. *If* $0 < \delta < 1/2$*, then in probability*

$$W_n^{+,\delta} = n^{2\delta - \frac{3}{2} + o(1)}.$$

There are two main features of the theorem that we call attention to. First, it clearly shows the existence of a critical scaling window described in terms of the δ parameter. The critical value of δ, by which we mean the point at which the perturbation switches from being influential to having no influence, is $\delta = 1/2$. The range $\delta \geq 1/2$ is what we call the critical window since the asymptotic behavior is as if the temperature were already at criticality. The range $0 < \delta < 1/2$ is what we call the near-critical window. In the critical window we see Hu-Shi asymptotics, while in the near-critical window we observe new behavior.

This new behavior inside the near-critical window is also of interest, in particular the non-trivial dependence on δ. The exponents $1 - 2\delta$ and $2\delta - 3/2$ in parts (ii) and (iii), respectively, may appear arbitrary at first but in fact show that there is a "smooth" crossover between what is already known for the sub- and super-critical regimes. To describe this crossover we introduce the random variables

$$W_{n,\gamma} = \sum_{|v|=n} e^{-\gamma V(v)}. \tag{1.8}$$

for $\gamma > 0$. Clearly $W_{n,1} = W_n$. For $\gamma < 1$ the martingale convergence of $W_n(\beta)$ for $\beta < \beta_c$ implies that

$$W_{n,\gamma} \sim W_\infty(\gamma) \exp\{c(\gamma)n\}$$

as $n \to \infty$, for some positive constant $c(\gamma)$ and $W_\infty(\gamma)$ a positive random variable. Hence as $\delta \downarrow 0$ we expect that $W_n^{-,\delta}$ should exhibit linear exponential growth, and the exponent $1 - 2\delta$ confirms this. Similarly, as $\delta \uparrow 1/2$ we should observe a transition from the exponential growth to the Hu-Shi polynomial decay (1.4). Our proofs are not strong enough to capture the transition to the polynomial behavior, but they do show that the exponential growth disappears.

For $\gamma > 1$ it was shown in [14, Theorem 1.4] that

$$W_{n,\gamma} = n^{-\frac{3}{2}\gamma + o(1)}$$

in probability. As $\gamma \downarrow 1$ there is a discontinuity in the decay exponent, with $n^{-3/2}$ appearing instead of the $n^{-1/2}$ in (1.4). Part (iii) of our theorem shows that the discontinuity is bridged by going through the near-critical window, and that there is a linear interpolation between the previously known exponents at the extremes.

This crossover behavior of exponents is not merely coincidental, but reflects a change in the underlying structure of the polymer measures. In the subcritical case $\beta < \beta_c$, it is known that the polymer measure $\mu_n^{(\beta)}$ chooses paths v whose energy $H(v)$ grows like $-\lambda'(\beta)n$ (up to first order). In particular, in the tree picture it means that exponentially many polymers contribute to the free energy, see for example [20].

In the supercritical case, [19] proves that the partition functions $W_{n,\gamma}$ in (1.8) with $\gamma > 1$ converges in law if normalized by $n^{-\frac{3}{2}\gamma}$. In [8] the limiting law is identified and used to show that the supercritical Gibbs measure converges to a purely atomic measure of Poisson-Dirichlet type. The convergence of the Gibbs measure for a continuous-time analogue was already described in [11] for generalized random energy models. However, more is known about the structure of the Gibbs measure. As pointed out in [6] for the case corresponding to branching Brownian motion, in the supercritical case the polymer measure is concentrated on those paths whose energy is within constant order from the minimal energy. The latter process of extremal particles was explicitly described in a recent break-through by [5] and [1] for branching Brownian motion and after that in [19] for branching random walks.

In the critical regime, [15] observe that the critical polymer measure converges, based on the result of [3] that identifies the limiting distribution of $n^{\frac{1}{2}} W_n$ as (a constant multiple of) the limit of the so-called derivative martingale. However, less is known about the structure of the Gibbs measure.

Our result about the perturbed partition function also sheds some light on the critical Gibbs measure. The fact that perturbations start showing an effect at $\delta = 1/2$ suggests that in the critical window the relevant energies are of order $V(v) \approx n^{1/2}$, and that subexponentially many particles contribute to the partition functions. Using Theorem 1.1 we easily obtain the following result on the order of the energy at criticality:

Theorem 1.2. *For any $\varepsilon, \varepsilon' > 0$, we have that in probability*

$$\mu_n^{(\beta_c)} \Big\{ |v| = n : n^{\frac{1}{2}-\varepsilon} \le V(v) \le n^{\frac{1}{2}+\varepsilon'} \Big\} \to 1 \, .$$

Proof of Theorem 1.2 assuming Theorem 1.1. Fix $\varepsilon > 0$ and observe that

$$\begin{aligned} \mu_n^{(\beta_c)} \Big\{ V(v) \le n^{\frac{1}{2}-\varepsilon} \Big\} &= \sum_{|v|=n} \frac{e^{-V(v)}}{W_n} \mathbb{1}_{\{V(v) \le n^{\frac{1}{2}-\varepsilon}\}} \\ &\le e^1 \sum_{|v|=n} \frac{e^{-(1+n^{-\frac{1}{2}+\varepsilon})V(v)}}{W_n} = e^1 \frac{W_n^{+,\frac{1}{2}-\varepsilon}}{W_n} \, . \end{aligned}$$

By [14] we have $W_n = n^{-1/2+o(1)}$ almost surely, and by Theorem 1.1 part (iii) we have that $W_n^{+,\frac{1}{2}-\varepsilon} = n^{-\frac{1}{2}-2\varepsilon+o(1)}$ in probability. Therefore the ratio above converges to zero in probability.

For the remaining bound fix $\varepsilon > 0$ and for $\delta = \frac{1}{2}(1-\varepsilon)$ consider

$$\begin{aligned} \mu_n^{(\beta_c)} \Big\{ V(v) \ge n^{\frac{1}{2}+\varepsilon} \Big\} &= \sum_{|v|=n} \frac{e^{-V(v)}}{W_n} \mathbb{1}_{\{V(v) \ge n^{\frac{1}{2}+\varepsilon}\}} \\ &\le e^{-n^{-\delta} n^{\frac{1}{2}+\varepsilon}} \sum_{|v|=n} \frac{e^{-(1-n^{-\delta})V(v)}}{W_n} = e^{-n^{-\delta} n^{\frac{1}{2}+\varepsilon}} \frac{W_n^{-,\delta}}{W_n} \, . \end{aligned}$$

Again, $W_n = n^{-\frac{1}{2}+o(1)}$ almost surely and by Theorem 1.1 part (ii) we have that

$$W_n^{-,\delta} \le \exp\left\{ \frac{\beta_c^2 \lambda''(\beta_c)}{2} n^{1-2\delta}(1+o(1)) \right\}$$

almost surely. Thus by our choice of δ the previous expression converges to zero in probability. □

The proofs also show that the typical behavior of a polymer is that the energy along its paths $(V(\xi_i))_{i=1}^n$ performs a random walk which stays positive. In the case of a large positive perturbation with $\delta < \frac{1}{2}$, we have to add the additional requirement that at the end $V(\xi_n)$ gets pushed down to an unusually low n^δ. In fact, this extends the intuition behind the proofs of [14] that the main contributing random walk in the supercritical case remains positive, but then has to take an unusually low value at the end.

Analogously to Theorem 1.2, we can describe the typical energy of a polymer in the near-critical regime. More precisely, we introduce the Gibbs measure at near-critical temperature by associating to each polymer v the probability

$$\mu_n^{(\beta_c,\pm,\delta)} = \frac{1}{W_n^{\pm,\delta}} \exp\{-(1 \pm n^{-\delta})V(v)\}.$$

Theorem 1.3. *Let* $\varepsilon, \varepsilon' > 0$.

(i) If $\delta \geq \frac{1}{2}$*, we have that in probability*

$$\mu_n^{(\beta_c,\pm,\delta)}\big\{|v| = n : n^{\frac{1}{2}-\varepsilon} \leq V(v) \leq n^{\frac{1}{2}+\varepsilon'}\big\} \to 1.$$

(ii) If $0 < \delta < \frac{1}{2}$*, we have that in probability*

$$\mu_n^{(\beta_c,-,\delta)}\big\{|v| = n : n^{1-\delta-\varepsilon} \leq V(v) \leq n^{1-\delta+\varepsilon'}\big\} \to 1.$$

(iii) If $0 < \delta < \frac{1}{2}$*, we have that in probability*

$$\mu_n^{(\beta_c,+,\delta)}\big\{|v| = n : n^{\delta-\varepsilon} \leq V(v) \leq n^{\delta+\varepsilon'}\big\} \to 1.$$

Proof. The proof follows as in the proof of Theorem 1.2 from the asymptotics of the normalizing constants $W_n^{\pm,\delta}$ in the Gibbs measure stated in Theorem 1.1.

(i) More precisely, along the lines of the proof of Theorem 1.2, one can show that if $\delta \geq \frac{1}{2}$, and $\varepsilon > 0$,

$$\mu^{(\beta_c,\pm,\delta)}\{V(v) \leq n^{\frac{1}{2}-\varepsilon}\} \leq e^{-(\pm n^{-\delta} - n^{-(\frac{1}{2}-\varepsilon)})n^{\frac{1}{2}-\varepsilon}} \frac{W_n^{+,\frac{1}{2}-\varepsilon}}{W_n^{\pm,\delta}},$$

which converges to 0 in probability by Theorem 1.1. Also,

$$\mu^{(\beta_c,\pm,\delta)}\{V(v) \geq n^{\frac{1}{2}+\varepsilon}\} \leq e^{-(\pm n^{-\delta} + n^{-\frac{1}{2}(1-\varepsilon)})n^{\frac{1}{2}+\varepsilon}} \frac{W_n^{-,\frac{1}{2}(1-\varepsilon)}}{W_n^{\pm,\delta}},$$

converges to 0 in probability.

(ii) For a negative perturbation with $\delta \in (0, \frac{1}{2})$, we find that for $\varepsilon \in (0, \frac{1}{2} - \delta)$,

$$\mu^{(\beta_c,-,\delta)}\{V(v) \leq n^{1-\delta-\varepsilon}\} \leq e^{(n^{-\delta} - n^{-(\delta+\varepsilon)})n^{1-\delta-\varepsilon}} \frac{W_n^{-,\delta+\varepsilon}}{W_n^{-,\delta}},$$

and moreover that for $\varepsilon \in (0, 2\delta)$,

$$\mu^{(\beta_c,-,\delta)}\{V(v) \geq n^{1-\delta+\varepsilon}\} \leq e^{(n^{-\delta} - n^{-(\delta-\frac{1}{2}\varepsilon)})n^{1-\delta+\varepsilon}} \frac{W_n^{-,\delta-\frac{1}{2}\varepsilon}}{W_n^{-,\delta}}.$$

Again the asymptotics in Theorem 1.1 (ii) show that the respective left-hand side converge to 0.

The proof of (iii) is very similar to the proof of (i) and is therefore omitted. □

To prove Theorem 1.1 we employ the standard technique of deriving the asymptotics of the partition functions from the asymptotics of its fractional moments. This is the strategy used in [14], and in our situation it is akin to computing the following asymptotics for the fractional moments of the perturbed partition functions:

Theorem 1.4. *Let* $\gamma \in (0, 1)$*. Then*

1. *for* $\delta \geq 1/2$ *we have* $\mathbb{E}\left[\left(W_n^{\pm,\delta}\right)^\gamma\right] = n^{-\gamma/2+o(1)}$,
2. *for* $0 < \delta < 1/2$ *we have* $\mathbb{E}\left[(W_n^{-,\delta})^\gamma\right] = \exp\left\{\frac{\gamma}{2} n^{1-2\delta}\beta_c^2\lambda''(\beta_c)(1 + o(1))\right\}$
3. *for* $0 < \delta < 1/2$ *we have* $\mathbb{E}\left[\left(W_n^{+,\delta}\right)^\gamma\right] = n^{(2\delta-\frac{3}{2})\gamma+o(1)}$.

In Appendix A we employ standard arguments to show that Theorem 1.1 is a corollary of Theorem 1.4, so the main focus of this paper is proving Theorem 1.4.

1.3 Organization and idea of the proofs

We give here a brief outline of our methods for proving Theorems 1.1 and 1.4. Before we concentrate on our proofs, we will comment on which parts of the asymptotics can be easily deduced from known results about the minimal energy (i.e. the minimal position of a branching random walk). We first recall that it was shown in [14, Thm 1.2] that

$$\limsup_{n\to\infty} \frac{1}{\log n} \inf_{|v|=n} V(v) = \frac{3}{2}\,, \quad \liminf_{n\to\infty} \frac{1}{\log n} \inf_{|v|=n} V(v) = \frac{1}{2}, \tag{1.9}$$

both almost surely.

Corollary 1.5. *For any negative perturbation, i.e. any $\delta > 0$,*

$$W_n^{-,\delta} \geq n^{-\frac{1}{2}+o(1)}, \quad \textit{almost surely},$$

for any positive perturbation

$$W_n^{+,\delta} \leq n^{-\frac{1}{2}+o(1)}, \quad \textit{almost surely},$$

and for any perturbation

$$\limsup_{n\to\infty} \frac{\log W_n^{\pm,\delta}}{\log n} \geq -\frac{1}{2}, \quad \textit{almost surely.}$$

Remark 1.6. *These bounds immediately prove that the lower bound for negative perturbations and the upper bound for positive perturbations in part (i) of Theorem 1.1 hold (and even in an almost sure sense). In fact our proofs will show that all lower bounds in Theorem 1.1 hold almost surely.*

Combining the second and third statement of Corollary 1.5 we also see that

$$\limsup_{n\to\infty} \frac{\log W_n^{+,\delta}}{\log n} = -\frac{1}{2}, \quad \textit{almost surely.}$$

Finally, note that for $0 < \delta < 1/2$, we can deduce, from the fact that the lower bounds in Theorem 1.1 hold almost surely and by extracting an almost surely convergent subsequence from the convergence in probability of part (iii) of Theorem 1.1, that

$$\liminf_{n\to\infty} \frac{\log W_n^{+,\delta}}{\log n} = 2\delta - \frac{3}{2}, \quad \textit{almost surely.}$$

Remark 1.7. *For negative perturbations the first statement of Corollary 1.5 completes the proof of the lower bound in part (i) of Theorem 1.1. Using this we do not need to prove the lower bound for the fractional moment of $W_n^{-,\delta}$ in the $\delta \geq 1/2$ case (i.e. part (i) of Theorem 1.4). However, we point out that the fractional moment is an easy corollary of the fractional moments of W_n [14, Thm. 1.5] and the asymptotics of $\inf_{|v|=n} V(v)$.*

Proof of Corollary 1.5. For any negative perturbation we have the lower bound

$$W_n^{-,\delta} = \sum_{|v|=n} e^{-(1-n^{-\delta})V(v)} \geq e^{n^{-\delta}\inf_{|u|=n} V(u)} \sum_{|v|=n} e^{-V(v)} \geq e^{\frac{1}{2}n^{-\delta}\log n(1+o(1))} W_n\,,$$

Since [14] implies $W_n = n^{-\frac{1}{2}+o(1)}$ almost surely, we immediately obtain that $W_n^{-,\delta} \geq n^{-1/2+o(1)}$ almost surely. Using the same idea we also obtain an upper bound for any positive perturbation, namely

$$W_n^{+,\delta} = \sum_{|v|=n} e^{-(1+n^{-\delta})V(v)} \leq e^{-\frac{1}{2}n^{-\delta}\log n(1+o(1))} W_n = n^{-1/2+o(1)},$$

where the last equality is again a consequence of the Hu-Shi asymptotics (1.4) for W_n.

Finally, we can always obtain a lower bound by only keeping the minimizing particle in the sum defining the partition function, so that

$$W_n^{\pm,\delta} \geq e^{-(1-n^{-\delta})\inf_{|v|=n} V(v)}.$$

Now, the $\liminf$ asymptotics (1.9) of $\inf_{|v|=n} V(v)$ yield the lower bound on the $\limsup$ asymptotics for $W_n^{\pm,\delta}$. □

The rest of the paper is focused on proving Theorem 1.4. In several papers on branching processes the *spine method* is the main technique used to understand asymptotics of the process. The first step is to enlarge the probability space by identifying a special ray, the "spine", in the tree. The second step involves constructing a size-biased probability measure that is tilted towards environments and rays for which the normalized energy $\{V(\xi_i)\}_{i=1}^n$ is typical along the chosen ray ξ. Precise definitions and properties of the construction are reviewed in Section 2.

The main purpose of this construction is that one can deduce the asymptotics of the partition function from the behavior of the normalized energies on the spine $\{V(\xi_i)\}_{i=1}^n$. Moreover, under this tilted measure these normalized energies are in distribution equal to a mean zero random walk. The problem is thus broken into two smaller pieces: first showing that the fractional moments can be estimated by some functional of a simple random walk, and then using random walk methods to estimate the functional.

We explain this strategy in more detail in the case of small and large positive perturbations. Our aim is to show that, in a rough sense, the perturbed partition function $W_n^{\cdot,\delta}$ decays like the inverse of

$$g(n) = \begin{cases} n^{1/2} & \text{if } \delta \geq \frac{1}{2}, \text{ any perturbation,} \\ n^{\frac{3}{2}-2\delta} & \text{if } \delta \in (0, \frac{1}{2}), \text{ positive perturbation.} \end{cases}$$

Following the philosophy of the spine method, we can reduce a fractional moment to a functional of a random walk and we eventually show that for $s \in (0,1)$,

$$\mathbb{E}[(g(n)W_n^{\pm,\delta})^{1-s}] \approx \mathbb{E}[(g(n) \star (S_n^+)^\alpha) e^{\mp n^{-\delta} S_n} \mathbb{1}_{\{\min_j S_j \geq 0\}}], \tag{1.10}$$

where S_n is a mean zero random walk with exponential moments, $\star$ is maximum $\vee$ or minimum $\wedge$ (depending on whether we consider an upper or lower bound), X^+ denotes $\max\{X, 0\}$ for a real-valued random variable X and $\alpha > 0$ is a free parameter. If our choice of parameters is correct, then the right hand side should be essentially constant (and the dependency on s is hidden in constants).

At this point we can fully notice the effect of the perturbation. If $\delta \geq \frac{1}{2}$, i.e. if the perturbation is small, the term $e^{-n^{-\delta} S_n}$ is negligible. Hence, the dominating behavior is that of a random walk conditioned to be positive so that the end point fluctuates around $n^{\frac{1}{2}}$. However, if we are in the case of a positive large perturbation the $e^{-n^{-\delta} S_n}$ factor starts to push the random walk down at the end, so that the dominating contributions come from random walks that stay positive but end up at a scale n^δ at time n. In particular, we see that if we choose the parameter α as

$$\alpha := \alpha(\delta) = \begin{cases} 1 & \text{if } \delta \geq \frac{1}{2}, \text{ any perturbation,} \\ \frac{3}{2\delta} - 2 & \text{if } \delta \in (0, \frac{1}{2}), \text{ positive perturbation.} \end{cases}$$

then, under the dominating behavior in (1.10), the random walk satisfies $(S_n^+)^\alpha \approx g(n)$.

We emphasize that the strategy behind our proofs is highly motivated by the use of fractional moments and the spine methods in [14]. However, their proofs cannot be translated directly to deal with a perturbation of the partition function. Moreover, in order to be able to concentrate on the new difficulties, we focus exclusively on the case of a binary tree instead of general Galton-Watson trees. The binary tree model also appears naturally as a toy model for polymers.

The *organization* of our paper is as follows: in Section 2 we give a brief review of the spine method. In Section 3, we deal with the simplest case of a fractional moment bound for a large negative perturbation, which is part (ii) of Theorem 1.4. Since we only show less refined asymptotics, we can use simpler methods. In the remainder of the paper, we carry out the above strategy for all small and large negative perturbations. In Section 4, we show that we obtain an upper bound on the fractional moments in terms of a random walk as in (1.10), while in Section 5 we show the corresponding lower bound. To complete the proof of the fractional moment estimates, Theorem 1.4, we analyze in Section 6 the random walk functional on the right hand side of (1.10) using a coupling argument with a Brownian motion. Appendix A shows how to deduce Theorem 1.1 from Theorem 1.4.

Notation: Throughout the paper, we will use generic constants $c, C > 0$, whose values may change from line to line. If it is essential, we will indicate their dependence on parameters.

2 Spine Method

Recall the weight function $V : T \to \mathbb{R}$ defined by (1.5) and the expression (1.6) for W_n. Let **SpinedTrees** $= \{(\mathcal{V}, \xi) : \mathcal{V} = (V(v) : v \in T), \xi \in \partial T\}$ be the space of weights on the vertices of T with a marked spine ξ. Let $\mathcal{F}_n = \sigma(V(v), |v| \leq n; \xi_i, i \leq n)$ be the filtration giving all the information on the weights and spine up to level n, and recall that $\mathcal{W}_n = \sigma(\omega(v) : |v| \leq n)$. Extend $\mathbb{P}$ to a probability measure on **SpinedTrees** such that the $V(v)$ variables have the distribution defined by (1.5) with all of the ω being i.i.d. and ξ chosen uniformly from ∂T. Let $\mathbb{Q}$ be the probability measure on **SpinedTrees** defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_n}(\mathcal{V}, \xi) = e^{-V(\xi_n) + n \log 2}. \tag{2.1}$$

It is easy to check that the latter expression is an $\mathcal{F}_n$-martingale under $\mathbb{P}$, and hence $\mathbb{Q}$ extends to a measure on all of **SpinedTrees**. A straightforward computation shows that conditional on the weights $\mathcal{V}$ up to level n (i.e. on $\mathcal{W}_n$), the distribution of ξ_n is given by

$$\mathbb{Q}\left(\xi_n = v \mid \mathcal{W}_n\right) = \frac{e^{-V(v)}}{W_n}. \tag{2.2}$$

Comparing (2.1) and (2.2) with (1.5) we see that the measure $\mathbb{Q}$ is tilted towards elements of **SpinedTrees**, for which the Gibbs measure is large. Note also that $\mathbb{Q}$ restricted to $\mathcal{W}_n$ has Radon-Nikodým derivative W_n. Moreover, under $\mathbb{Q}$ the sequence $V(\xi_n)$ turns out to be a random walk with mean zero increments. This is proved in a number of different sources (see [20, 14], for example) but we recall the basic facts here. For each $n \geq 1$ let b_n be the sibling vertex of ξ_n. Define the σ-algebras $\mathcal{G}_n, \mathcal{G}_n^*$ by

$$\mathcal{G}_n := \sigma(V(\xi_i), \xi_i; i \leq n) \quad \text{and} \quad \mathcal{G}_n^* := \sigma(V(\xi_i), V(b_i), \xi_i; i \leq n).$$

Further, let $(S_n, n \geq 0)$ be a random walk with $S_0 = 0$ whose independent increments have the $\mathbb{Q}$-distribution of $V(\xi_1)$. Then there is the following well-known set of results:

Proposition 2.1. *Under the measure* $\mathbb{Q}$*,*

1. *the process* $(V(\xi_n))_{n \geq 0}$ *has the same distribution as the random walk* $(S_n)_{n \geq 0}$*,*

2. *for any measurable function* $F : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}_{\mathbb{Q}}\left[F(S_1)\right] = 2\mathbb{E}\left[F(V_0)e^{-V_0}\right]$$

where $-V_0 = \beta_c\omega - \lambda(\beta_c) - \log 2$*,*

3. *the random variables* $(V(\xi_n) - V(\xi_{n-1}), V(b_n) - V(\xi_{n-1}))$ *are i.i.d. and distributed as* (S_1, V_0)*,*
4. *conditionally on* $\mathcal{G}_n^*$ *the weights* $V(v) - V(b_k)$ *on the subtree* $T(b_k)$ *rooted at* b_k *are independent of* $V(b_k)$ *(and independent for each subtree) and have the same distribution as under the original measure* $\mathbb{P}$*.*

Choosing $F(x) = x$ in (ii) and using the relation (1.2) gives that $\mathbb{E}_{\mathbb{Q}}[S_1] = 0$. Hence S_n is a mean zero random walk by parts (i) and (iii).

3 Large negative perturbations

Using the spine method we prove part (ii) of Theorem 1.4, which is the fractional moments for a large negative perturbation. Combined with the results of Appendix A this completes the proof of part (ii) of Theorem 1.1.

Theorem 3.1 (Theorem 1.4, part (ii))**.** *For any* $\delta \in (0, \frac{1}{2})$ *and* $s \in (0,1)$*, we have that*

$$\mathbb{E}[(W_n^{-,\delta})^{1-s}] = \exp\{\tfrac{1}{2}(1-s)n^{1-2\delta}\beta_c^2\lambda''(\beta_c)(1+o(1))\}.$$

Proof. We first record a standard computation, where we recall the definition of V in (1.5) and compute for any $0 \le k \le n$,

$$\begin{aligned}
\mathbb{E}\Big[\sum_{|v|=k} e^{-(1-n^{-\delta})V(v)}\Big] &= \sum_{|v|=k} \mathbb{E}[e^{(1-n^{-\delta})(\beta_c \sum_{j=1}^k \omega(v_j) - k(\lambda(\beta_c)+\log 2))}] \\
&= 2^k\, \mathbb{E}[e^{(1-n^{-\delta})\beta_c\omega}]^k e^{-k(1-n^{-\delta})(\lambda(\beta_c)+\log 2)} \\
&= \exp\left\{k\big(\lambda((1-n^{-\delta})\beta_c) - \lambda(\beta_c) + n^{-\delta}\beta_c\lambda'(\beta_c)\big)\right\} \\
&= \exp\{k(\tfrac{1}{2}n^{-2\delta}\beta_c^2\lambda''(\beta_c) + O(n^{-3\delta}))\},
\end{aligned} \tag{3.1}$$

where in the penultimate step we used the definition of β_c in (1.2) and a Taylor expansion.

In particular, taking $k = n$ we immediately obtain the *upper bound* on the fractional moments by using Jensen's inequality to estimate that for any $s \in (0,1)$,

$$\mathbb{E}\big[(W_n^{-,\delta})^{1-s}\big] \le \mathbb{E}[W_n^{-,\delta}]^{1-s} = \exp\{\tfrac{1}{2}(1-s)n^{1-2\delta}\beta_c^2\lambda''(\beta_c) + O(n^{1-3\delta})\}\,,$$

the last equality following from the calculation in (3.1).

We now prove the *lower bound.* Fix $s \in (0,1)$ and observe that with the notation for the spine technique as introduced in Section 2,

$$\frac{W_n^{-,\delta}}{W_n} = \sum_{|v|=n} e^{n^{-\delta}V(v)}\mathbb{Q}\left(\xi_n = v|\mathcal{W}_n\right) = \mathbb{E}_{\mathbb{Q}}[e^{n^{-\delta}V(\xi_n)}|\mathcal{W}_n].$$

Then the fractional moment can be written as

$$\mathbb{E}[(W_n^{-,\delta})^{1-s}] = \mathbb{E}_{\mathbb{Q}}\Big[\frac{W_n^{-,\delta}}{W_n}(W_n^{-,\delta})^{-s}\Big] = \mathbb{E}_{\mathbb{Q}}\big[e^{n^{-\delta}V(\xi_n)}(W_n^{-,\delta})^{-s}\big],$$

By conditioning on the weights on the spine $\mathcal{G}_n$ and applying Jensen's inequality we obtain a lower bound of

$$\begin{aligned}
\mathbb{E}[(W_n^{-,\delta})^{1-s}] &= \mathbb{E}_{\mathbb{Q}}\big[e^{n^{-\delta}V(\xi_n)}\mathbb{E}[(W_n^{-,\delta})^{-s}|\mathcal{G}_n]\big] \\
&\ge \mathbb{E}_{\mathbb{Q}}\big[e^{n^{-\delta}V(\xi_n)}\mathbb{E}[(W_n^{-,\delta})^{s}|\mathcal{G}_n]^{-1}\big].
\end{aligned} \tag{3.2}$$

We now decompose the tree along its spine to write

$$W_n^{-,\delta} = e^{-(1-n^{-\delta})V(\xi_n)} + \sum_{i=1}^{n} e^{-(1-n^{-\delta})V(\xi_{i-1})} e^{-(1-n^{-\delta})(V(b_i)-V(\xi_{i-1}))} \sum_{v\in T_{n-i}(b_i)} e^{-(1-n^{-\delta})(V(v)-V(b_i))},$$

where we recall that b_i denotes the sibling of ξ_i in the tree and moreover $T_k(b_i)$ denotes the vertices in the kth generation of the tree rooted at b_i. Using Proposition 2.1 and the subadditivity inequality $(\sum_i a_i)^s \le \sum a_i^s$ for $a_i \ge 0$, we can show that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[W_n^{-,\delta}|\mathcal{G}_n]^s &\le e^{-s(1-n^{-\delta})V(\xi_n)} \\ &\quad + \sum_{i=1}^{n} e^{-s(1-n^{-\delta})V(\xi_{i-1})}\mathbb{E}[e^{-(1-n^{-\delta})V_0}]^s \mathbb{E}\Big[\sum_{|v|=n-j} e^{-(1-n^{-\delta})V(v)}\Big]^s \\ &\le e^{\frac{1}{2}sn^{1-2\delta}\beta_c^2\lambda''(\beta_c)+O(n^{1-3\delta})}\sum_{i=0}^{n} e^{-s(1-n^{-\delta})V(\xi_i)}. \end{aligned} \tag{3.3}$$

The last inequality uses that $\mathbb{E}[e^{-V_0}]^s = 2^{-s}$ (see Proposition 2.1, part (ii), for the definition of V_0) and finally the calculation in (3.1). Combining these last two estimates (3.2) and (3.3), we conclude that

$$\mathbb{E}[(W_n^{-,\delta})^{1-s}] \ge e^{-\frac{1}{2}sn^{1-2\delta}\beta_c^2\lambda''(\beta_c)+O(n^{1-3\delta})}\mathbb{E}_{\mathbb{Q}}\Big[\frac{e^{n^{-\delta}V(\xi_n)}}{\sum_{i=0}^{n} e^{-s(1-n^{-\delta})V(\xi_i)}}\Big]. \tag{3.4}$$

Denoting by $(S_i)_{i\ge 0}$ the random walk introduced in Proposition 2.1, we can rewrite the expectation on the right hand side as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\Big[\frac{e^{n^{-\delta}V(\xi_n)}}{\sum_{i=0}^{n} e^{-s(1-n^{-\delta})V(\xi_i)}}\Big] &= \mathbb{E}_{\mathbb{Q}}\Big[\frac{e^{n^{-\delta}S_n}}{\sum_{i=0}^{n} e^{-s(1-n^{-\delta})S_i}}\Big] \ge \frac{1}{n+1}\mathbb{E}_{\mathbb{Q}}\big[e^{n^{-\delta}S_n}\mathbb{1}_{\{\min_{i=1}^n S_i\ge 0\}}\big] \\ &\ge \frac{1}{n+1}\mathbb{E}_{\mathbb{Q}}\big[e^{n^{-\delta}S_n}\big]\mathbb{Q}\big\{\min_{i=1,\ldots,n} S_i \ge 0\big\},\end{aligned}$$

where we used the FKG inequality noticing that $(x_i)_{i=1}^n \mapsto \mathbb{1}\{\min_{i=1,\ldots,n}\sum_{j=1}^i x_i \ge 0\}$ and $(x_i)_{i=1}^n \mapsto e^{n^{-\delta}\sum_{j=1}^n x_j}$ are both increasing functions. For more details of the FKG inequality in a similar context see e.g. [7, Section 2.2]. To complete the proof, we note that by (3.1) we can calculate the first moment as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\big[e^{n^{-\delta}S_n}\big] &= \mathbb{E}_{\mathbb{Q}}\big[e^{n^{-\delta}V(\xi_n)}\big] = \mathbb{E}_{\mathbb{Q}}\big[\sum_{|v|=n} e^{n^{-\delta}V(v)}\mathbb{Q}\{\xi_n = v|\mathcal{W}_n\}\big] \\ &= \mathbb{E}[W_n^{-,\delta}] = \exp\big\{\tfrac{1}{2}\beta_c^2\lambda''(\beta_c)n^{1-2\delta} + O(n^{1-3\delta})\big\},\end{aligned}$$

and for the second term we have that $\mathbb{Q}\{\min_{i=1,\ldots,n} S_i \ge 0\} = n^{-\frac{1}{2}+o(1)}$ by a standard random walk computation. Hence the latter is negligible compared to the first term and from (3.4) we can deduce the required lower bound. $\square$

4 Upper bounds

In this section we find an upper bound on the fractional moments for all positive perturbations and small negative perturbations. The method we use works for all three types of perturbations simultaneously. To unify the argument we write $Y_n = W_n^{\cdot,\delta}$, where

$\cdot$ is either $+$ or $-$ depending on whether we are considering a positive or negative perturbation. Define the growth function

$$g(n) = \begin{cases} n^{1/2} & \text{if } Y_n = W_n^{\pm,\delta}, \delta \geq \frac{1}{2}, \\ n^{\frac{3}{2}-2\delta} & \text{if } Y_n = W_n^{+,\delta}, \delta \in (0, \frac{1}{2}), \end{cases} \tag{4.1}$$

and also let $\gamma_n = \pm n^{-\delta}$ depending on which perturbation is under consideration.

We start by defining the auxiliary quantity $\overline{Y}_n$, which gives an upper bound as follows:

$$g(n)Y_n \leq \sum_{|v|=n} (g(n) \vee V^+(v)^\alpha) e^{-(1+\gamma_n)V(v)} =: \overline{Y}_n.$$

Here $\alpha = \alpha(\delta)$ is chosen as

$$\alpha(\delta) = \begin{cases} 1 & \text{if } Y_n = W_n^{\pm,\delta}, \delta \geq \frac{1}{2}, \\ \frac{3}{2\delta} - 2 & \text{if } Y_n = W_n^{+,\delta}, \delta \in (0, \frac{1}{2}). \end{cases}$$

This reasoning behind this particular choice of α is discussed in Section 1.3.

Proposition 4.1. *For all $s \in (0,1)$ there exists a constant $\kappa_0 > 0$ such that for all $\kappa \geq \kappa_0$*

$$\mathbb{E}[\overline{Y}_n^{1-s}] \leq 1 + \mathbb{E}_{\mathbb{Q}}[(g(n) \vee S_n^\alpha) e^{-\gamma_n S_n} \mathbb{1}_A] + o(1)$$

as $n \to \infty$, where A is the event

$$A = \left\{ \min_{0 \leq j \leq n} S_j \geq -\kappa \log n, S_n \geq 0 \right\}.$$

Proof. First note that it is sufficient to prove the proposition for s small, since if it holds for small s then it also holds for all larger $s \in (0,1)$. Indeed, by Hölder's inequality we have that for $s' > s$

$$\mathbb{E}[\overline{Y}_n^{1-s'}] \leq \mathbb{E}[\overline{Y}_n^{1-s}]^{\frac{1-s'}{1-s}} \leq 1 + \mathbb{E}[\overline{Y}_n^{1-s}],$$

for n sufficiently large, where we used that $\frac{1-s'}{1-s} < 1$.

Now observe that $\overline{Y}_n$ can be rewritten as

$$\overline{Y}_n = W_n \mathbb{E}_{\mathbb{Q}} \left[(g(n) \vee V^+(\xi_n)^\alpha) e^{-\gamma_n V(\xi_n)} \middle| \mathcal{W}_n \right],$$

and then using the spine techniques of Section 2 we obtain that

$$\mathbb{E}\big[\overline{Y}_n^{1-s}\big] = \mathbb{E}_{\mathbb{Q}}\big[\overline{Y}_n^{-s} (g(n) \vee V^+(\xi_n)^\alpha) e^{-\gamma_n V(\xi_n)}\big]. \tag{4.2}$$

As in the proof by [14], the main idea is to show that the relevant contributions to $\overline{Y}_n$ only come from the spine particle ξ_n.

We first notice that we can concentrate on the event $\overline{Y}_n \geq 1$ (on the complement $\overline{Y}_n^{1-s}$ is bounded by 1). Now define $\underline{V}(\xi_n) = \inf_{i=1,\ldots,n} V(\xi_i)$. Fix $\kappa > 0$ and let

$$E := \{(\mathcal{V}, \xi) \in \textbf{SpinedTrees} : \underline{V}(\xi_n) \geq -\kappa \log n, V(\xi_n) \geq 0\},$$

and notice that we can write $E^c = F_1 \cup F_2$ where

$$F_1 = \{(\mathcal{V}, \xi) : \underline{V}(\xi_n) < -\kappa \log n\} \quad \text{and} \quad F_2 = \{(\mathcal{V}, \xi) : V(\xi_n) < 0, \underline{V}(\xi_n) \geq -\kappa \log n\}.$$

We will show that $\mathbb{E}[\overline{Y}_n^{1-s} \mathbb{1}_{\overline{Y}_n \geq 1} \mathbb{1}_{F_i}] \to 0$ as $n \to \infty$, for $i = 1, 2$, so that by equation (4.2) we will have

$$\begin{aligned} \mathbb{E}\big[\overline{Y}_n^{1-s}\big] &\leq 1 + \mathbb{E}[\overline{Y}_n^{1-s} \mathbb{1}_{\overline{Y}_n \geq 1}] \\ &\leq 1 + \mathbb{E}_{\mathbb{Q}}\big[\overline{Y}_n^{-s} (g(n) \vee V^+(\xi_n)^\alpha) e^{-\gamma_n V(\xi_n)} \mathbb{1}_{\overline{Y}_n \geq 1} \mathbb{1}_E\big] + o(1). \end{aligned}$$

This will prove the lemma once we recall that, by Proposition 2.1, $(S_i)_{i=1}^n$ is a random walk which has the same $\mathbb{Q}$-distribution as the weights $(V(\xi_i))_{i=1}^n$ along the spine.

Step 1. We will show that

$$\mathbb{E}\left[\overline{Y}_n^{1-s}\mathbb{1}_{F_1}\right] = \mathbb{E}_{\mathbb{Q}}\left[\overline{Y}_n^{-s}(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{F_1}\right] \to 0$$

as $n\to\infty$. Let $\underline{\xi}_n$ be the last element of $\xi_1,\xi_2,\dots,\xi_n$ such that $\underline{V}(\xi_n)=V(\underline{\xi}_n)$. Let b be the child of $\underline{\xi}_n$ that is not on the spine if $|\underline{\xi}_n|<n$, otherwise set $b=\xi_n=\underline{\xi}_n$. On F_1, we can estimate $\overline{Y}_n$ from below by

$$\overline{Y}_n \geq g(n)e^{-(1+\gamma_n)V(b)}e^{-(1+\gamma_n)\inf_{v\in T_{n-|b|}(b)}V(v)-V(b)},$$

where we recall that, for $k\in\mathbb{N}$, $T_k(b)$ denotes the vertices in the kth generation of the tree rooted at b. Thus,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\big[\overline{Y}_n^{-s}(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{F_1}\big]\\
&\leq \mathbb{E}_{\mathbb{Q}}\big[(g(n)\vee V^+(\xi_n)^\alpha)g(n)^{-s}e^{s(1+\gamma_n)V(b)-\gamma_n V(\xi_n)}e^{s(1+\gamma_n)\inf_{v\in T_{n-|b|}(b)}V(v)-V(b)}\mathbb{1}_{F_1}\big]\\
&\leq C\mathbb{E}_{\mathbb{Q}}\big[(g(n)\vee V^+(\xi_n)^\alpha)g(n)^{-s}e^{s(1+\gamma_n)V(\underline{\xi}_n)-\gamma_n V(\xi_n)}\\
&\qquad\qquad\qquad\times\max\Big\{1,\sup_{k=0,\dots,n}\mathbb{E}\big[e^{s(1+\gamma_n)\inf_{v\in T_{n-k}}V(v)}\big]\Big\}\mathbb{1}_{F_1}\big],
\end{aligned}$$

where in the last step we took expectation conditionally on the weights on the spine, and we twice used that the weights that are not on the spine are independent and their distribution is not affected by the change of measure, see Proposition 2.1. Now, by [14, Prop. 5.1], there is a $s_0\in(0,1)$ such that for all $s\leq s_0$ and any $\varepsilon>0$, there exists $C=C(s)>0$ such that

$$\mathbb{E}\big[\exp\{s\inf_{|v|=n}V(v)\}\big]\leq Cn^{(3+\varepsilon)s/2}.$$

In fact, the result in [14] is stated for some fixed s_0, but by Hölder's inequality it immediately translates to all smaller $s\leq s_0$. Substituting back into the above display (and noting that we can absorb the $(1+\gamma_n)$ in front of the infimum into the ε), we obtain

$$\begin{aligned}
&E_{\mathbb{Q}}\big[\overline{Y}_n^{-s}(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{F_1}\big]\\
&\qquad\leq Cn^{(3+\varepsilon)s/2}\mathbb{E}_{\mathbb{Q}}\big[(g(n)\vee V^+(\xi_n)^\alpha)g(n)^{-s}e^{s(1+\gamma_n)\underline{V}(\xi_n)-\gamma_n V(\xi_n)}\mathbb{1}_{F_1}\big].
\end{aligned}\tag{4.3}$$

From (4.3), in the case that $\gamma_n\geq 0$, we use that $V(\xi_n)\geq\underline{V}(\xi_n)$ and that $\underline{V}(\xi_n)<-\kappa\log n$ on the event F_1 to obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\big[\overline{Y}_n^{-s}(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{F_1}\big]\\
&\qquad\leq Cn^{(3+\varepsilon)s/2}g(n)^{-s}n^{-\kappa(s(1+\gamma_n)-\gamma_n)}\mathbb{E}_{\mathbb{Q}}[(g(n)\vee V^+(\xi_n)^\alpha)\mathbb{1}_{F_1}].
\end{aligned}\tag{4.4}$$

Since $V(\xi_n)$ has the same distribution as S_n, the latter expectation can be bounded as

$$\mathbb{E}_{\mathbb{Q}}[(g(n)\vee(S_n^+)^\alpha)]\leq(g(n)\vee n^{\alpha/2})\mathbb{E}_{\mathbb{Q}}\big[\max\{1,\big(n^{-\frac12}S_n^+\big)^\alpha\}\big],$$

where by the central limit theorem the latter expectation converges to a constant (where we note that the increments of S_n have exponential moments). Consequently, by the choice of α, the right hand side of (4.4) is $o(1)$, provided we choose $\kappa\geq\kappa_0$, where κ_0 has to be chosen large enough.

In the case that $\gamma_n\leq 0$, we obtain an upper bound on (4.3) of

$$Cg(n)^{-s}n^{(3+\varepsilon)s/2}n^{-\kappa s+o(1)}\mathbb{E}_{\mathbb{Q}}[(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{F_1}].\tag{4.5}$$

We can further bound the expectation using Cauchy-Schwarz to obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{F_1}] &\le \mathbb{E}_{\mathbb{Q}}[(g(n)\vee V^+(\xi_n)^\alpha)^2]^{1/2}\mathbb{E}_{\mathbb{Q}}[e^{-2\gamma_n V(\xi_n)}]^{1/2}\\ &\le C(g(n)\vee n^{\alpha/2})\mathbb{E}_{\mathbb{Q}}[e^{-2\gamma_n V(\xi_n)}]^{1/2}.\end{aligned}$$

Since we are considering $\gamma_n \le 0$ we have that $\delta \ge 1/2$, and since $V(\xi_n)$ has the distribution of S_n (which is a mean zero random walk with exponential moments), it follows that the expectation in the latter expression is of constant order. Hence (4.5) is of order $o(1)$, again if $\kappa \ge \kappa_0$ for some suitably chosen κ_0.

Step 2. We now show that

$$\mathbb{E}[\overline{Y}_n^{1-s}\mathbb{1}_{\overline{Y}_n\ge 1}\mathbb{1}_{F_2}] = \mathbb{E}_{\mathbb{Q}}[\overline{Y}_n^{-s}(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{\overline{Y}_n\ge 1}\mathbb{1}_{F_2}] \to 0$$

as $n\to\infty$. We upper bound the latter expression by

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\big[\overline{Y}_n^{-s}(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{\overline{Y}_n\ge 1}\mathbb{1}_{F_2}\big] &\le \mathbb{E}_{\mathbb{Q}}\big[(g(n)\vee V^+(\xi_n)^\alpha)e^{-\gamma_n V(\xi_n)}\mathbb{1}_{F_2}\big]\\ &\le g(n)e^{\kappa n^{-\delta}\log n}\mathbb{Q}(F_2)\,,\end{aligned}$$

where we used that if $\gamma_n \ge 0$, then we can bound $V(\xi_n)\ge V(\underline{\xi}_n)\ge -\kappa\log n$, and if $\gamma_n\le 0$, then $-\gamma_n V(\xi_n)\le 0$. However, using that $V(\xi_i)$ is a mean zero random walk we may upper bound $\mathbb{Q}(F_2)$ by $Cn^{-3/2}(\log n)^3$ for some $C=C(\kappa)$, which corresponds to the probability that a random walk comes back to zero at time n on the event that it stays positive, see e.g. [2, Lemma A.1]. Thus

$$g(n)\mathbb{Q}(F_2)\le Cn^{\frac{3}{2}-2\delta}n^{-\frac{3}{2}}(\log n)^3 = o(1),$$

which completes the proof. □

5 Lower Bounds

The goal of this section is to find a lower bound on the fractional moment $\mathbb{E}[(W_n^{\pm,\delta})^{1-s}]$ in terms of an expression that only involves a (non-trivial) functional of a random walk. By Remark 1.7 and Section 3 we already have the required bounds for negative perturbations, hence it suffices to consider only positive perturbations.

Let $g(n)$ be as in (4.1) and let $\gamma_n = n^{-\delta}$. We lower bound $g(n)W_n^{+,\delta}$ by

$$g(n)W_n^{+,\delta}\ge\sum_{|v|=n}(g(n)\wedge V^+(v)^\alpha)e^{-(1+n^{-\delta})V(v)} =: \underline{Y}_n,$$

where $\alpha=\alpha(\delta)$ is chosen as

$$\alpha(\delta)=\begin{cases}1 & \text{if } \delta\ge\frac{1}{2},\\ \frac{3}{2\delta}-2 & \text{if } \delta\in(0,\frac{1}{2}).\end{cases}$$

Proposition 5.1. *For any $s\in(0,1)$, there exist constants $\kappa^*=\kappa^*(s)$ and $\gamma(s)$ such that for $n_0=\lceil\kappa^*(\log n)^2\rceil$,*

$$\mathbb{E}[\underline{Y}_n^{1-s}]\ge\frac{1}{n_0^{\gamma(s)}}\mathbb{E}_{\mathbb{Q}}[(g(n-n_0)\wedge(S^+_{n-n_0})^\alpha)e^{-(n-n_0)^{-\delta}S_{n-n_0}}\mathbb{1}_{\{\min_{j\le n-n_0}S_j\ge 0\}}].$$

The proof of the proposition splits into two lemmas. We first estimate the fractional moments of $\underline{Y}_n$ with an expression that only involves the weights along the spine.

Lemma 5.2. *Let $\kappa>0$ and define $n_0=\lceil(\kappa\log n)^2\rceil$. For any $s\in(0,1)$ there exists a constant $c>0$ and $\gamma(s)>0$ such that for all n sufficiently large,*

$$\mathbb{E}[\underline{Y}_n^{1-s}]\ge c\,\mathbb{E}_{\mathbb{Q}}\Big[\frac{(g(n)\wedge V^+(\xi_n)^\alpha)e^{-n^{-\delta}V(\xi_n)}}{(\log n)^{\gamma(s)}\sum_{j=0}^{n_0-1}e^{-s(1+n^{-\delta})V(\xi_j)}+g(n)\sum_{j=n_0}^{n}e^{-sV(\xi_j)}}\Big]$$

Proof. Recall that $\mathcal{G}_n = \sigma(V(\xi_i), \xi_i, i \le n)$ is the filtration containing all the information about the spine and its weights up to generation n. Then, we note that

$$\begin{aligned}\mathbb{E}\left[\underline{Y}_n^{1-s}\right] &= \mathbb{E}\left[(\underline{Y}_n)^{-s} W_n \mathbb{E}_{\mathbb{Q}}\left[(g(n) \wedge V^+(\xi_n)^\alpha) e^{-n^{-\delta} V(\xi_n)} | \mathcal{W}_n\right]\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[(\underline{Y}_n)^{-s}(g(n) \wedge V^+(\xi_n)^\alpha) e^{-n^{-\delta} V(\xi_n)}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[(\underline{Y}_n)^{-s} | \mathcal{G}_n\right](g(n) \wedge V^+(\xi_n)^\alpha) e^{-n^{-\delta} V(\xi_n)}\right]. \qquad (5.1)\end{aligned}$$

We first use Jensen's inequality to estimate $\mathbb{E}_{\mathbb{Q}}[\underline{Y}_n^{-s} | \mathcal{G}_n] \ge \mathbb{E}_{\mathbb{Q}}[\underline{Y}_n^s | \mathcal{G}_n]^{-1}$, and then estimate the latter by grouping the terms in $\underline{Y}_n$ according to the generation at which they first deviate from the spine. This gives us the expression

$$\begin{aligned}\underline{Y}_n = \sum_{j=1}^n e^{-(1+n^{-\delta})V(b_j)} \sum_{v \in T_{n-j}(b_j)} (g(n) \wedge V^+(v)^\alpha) e^{-(1+n^{-\delta})(V(v)-V(b_j))} \\ + (g(n) \wedge V^+(\xi_n)^\alpha) e^{-(1+n^{-\delta})V(\xi_n)}, \qquad (5.2)\end{aligned}$$

where b_j is the sibling of ξ_j in the tree and $T_k(b_j)$ denotes the vertices in the kth generation of the tree rooted at b_j. Call the summands on the right hand side U_n^j:

$$U_n^j := e^{-(1+\gamma_n)V(b_j)} \sum_{v \in T_{n-j}(b_j)} (g(n) \wedge V^+(v)^\alpha) e^{-(1+n^{-\delta})(V(v)-V(b_j))}. \qquad (5.3)$$

Then by the subadditivity inequality $(\sum_i a_i)^s \le \sum_i a_i^s$ for $a_i \ge 0$, we have

$$\mathbb{E}_{\mathbb{Q}}\left[\underline{Y}_n^s | \mathcal{G}_n\right] \le \sum_{j=1}^n \mathbb{E}_{\mathbb{Q}}\left[(U_n^j)^s | \mathcal{G}_n\right] + g(n)^s e^{-s(1+n^{-\delta})V(\xi_n)}.$$

We now proceed to upper bound the expectation terms. First observe that

$$\begin{aligned}U_n^j &\le \sum_{v \in T_{n-j}(b_j)} g(n) \mathbb{1}_{\{V(v)>0\}} e^{-(1+n^{-\delta})V(v)} \\ &\le \sum_{v \in T_{n-j}(b_j)} g(n) e^{-V(v)} = g(n) e^{-V(\xi_{j-1})} e^{-(V(b_j)-V(\xi_{j-1}))} \sum_{v \in T_{n-j}(b_j)} e^{-(V(v)-V(b_j))}\end{aligned}$$

from which, using Proposition 2.1, we get the simple inequality

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[(U_n^j)^s | \mathcal{G}_n\right] &\le \mathbb{E}_{\mathbb{Q}}\left[U_n^j | \mathcal{G}_n\right]^s \\ &\le C g(n)^s e^{-sV(\xi_{j-1})} \mathbb{E}\Big[\sum_{|v|=n-j} e^{-V(v)}\Big]^s \le C g(n) e^{-sV(\xi_{j-1})}.\end{aligned}$$

The first and last inequalities both use that $s \in (0,1)$. We only use this bound for $j > n_0$. In the case $j \le n_0$ we replace the minimum in (5.3) by $g(n)$ and use parts (iii) and (iv) of Proposition 2.1 to get the following upper bound:

$$\mathbb{E}_{\mathbb{Q}}\left[(U_n^j)^s | \mathcal{G}_n\right] \le C g(n)^s e^{-s(1+n^{-\delta})V(\xi_{j-1})} \mathbb{E}\Big[\Big(\sum_{|v|=n-j} e^{-(1+n^{-\delta})V(v)}\Big)^s\Big].$$

We claim the expectation term is further bounded above as follows:

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{|v|=n-j} e^{-(1+n^{-\delta})V(v)}\Big)^s\Big] \\ &\qquad \le \mathbb{E}\Big[e^{s((n-j)^{-\delta}-n^{-\delta}) \sup_{|v|=n-j} V(v)} \Big(\sum_{|v|=n-j} e^{-(1+(n-j)^{-\delta})V(v)}\Big)^s\Big] \\ &\qquad \le (1+o(1)) \mathbb{E}\big[\big(W_{n-j}^{+,\delta}\big)^s\big]. \qquad (5.4)\end{aligned}$$

We temporarily delay the proof of (5.4). Assuming it is true and combining it with the previous display we obtain that for $j \le n_0$,

$$\mathbb{E}_{\mathbb{Q}}[(U_n^j)^s|\mathcal{G}_n] \le Ce^{-s(1+n^{-\delta})V(\xi_{j-1})}g(n)^s\mathbb{E}\big[\big(W_{n-j}^{+,\delta}\big)^s\big] \le Ce^{-s(1+n^{-\delta})V(\xi_{j-1})}\log n.$$

The logarithmic factor in the last inequality is from Propositions 4.1 and 6.1, which are proved independently in Sections 4 and 6, respectively. The estimates on $\mathbb{E}_{\mathbb{Q}}[(U_n^j)^s|\mathcal{G}_n]$ (for $j \le n_0$ and $j > n_0$) combined with the spine decomposition (5.2) yield the statement of the lemma.

Finally, it remains to prove the claim (5.4). First, note that by a standard application of Chebychev, for any vertex v with $|v| = n$, and any $\ell \ge 0$,

$$\begin{aligned}\mathbb{P}\big\{\sup_{|v|=n} V(v) \ge \ell n\big\} &\le 2^n\mathbb{P}\{V(v) \ge \ell n\} \le 2^n e^{-\ell n}\mathbb{E}[e^{V(v)}]\\ &= 2^n e^{-\ell n}\mathbb{E}[e^{V(v_1)}]^n \le e^{(\ell_0-\ell)n},\end{aligned} \tag{5.5}$$

where we define $\ell_0 = \lceil \log 2 + \log \mathbb{E}[\exp\{V(v_1)\}]\rceil$. For $j \le n_0 = \lceil(\kappa\log n)^2\rceil$, noting that $((n-j)^{-\delta} - n^{-\delta})$ is of order $n^{-1-\delta}(\log n)^2$, we therefore obtain

$$\begin{aligned}&\mathbb{E}\Big[e^{s((n-j)^{-\delta}-n^{-\delta})\sup_{|v|=n-j}V(v)}\Big(\sum_{|v|=n-j} e^{-(1+(n-j)^{-\delta})V(v)}\Big)^s\Big]\\ &\quad\le e^{n^{-\delta+o(1)}(\ell_0+1)}\mathbb{E}[(W_{n-j}^{+,\delta})^s\mathbb{1}_{\{\sup_{|v|=n-j}V(v)\le(\ell_0+1)(n-j)\}}]\\ &\qquad+\mathbb{E}\Big[e^{pn^{-\delta-1+o(1)}\sup_{|v|=n-j}V(v)}\mathbb{1}_{\{\sup_{|v|=n-j}V(v)\ge(\ell_0+1)(n-j)\}}\Big]^{1/p}\,\mathbb{E}[(W_{n-j}^{+,\delta})^{sq}]^{1/q},\end{aligned} \tag{5.6}$$

where in the last step we used Hölder's inequality with conjugates $p, q \ge 1$ such that $sq < 1$. Now, the first summand on the right hand side is of order $(1 + o(1))\mathbb{E}[(W_n^{+,\delta})^s]$, so that it only remains to consider the second term, which we can bound using (5.5),

$$\begin{aligned}&\mathbb{E}\Big[e^{pn^{-\delta-1+o(1)}\sup_{|v|=n-j}V(v)}\mathbb{1}_{\{\sup_{|v|=n-j}V(v)\ge(\ell_0+1)(n-j)\}}\Big]^{1/p}\\ &\quad\le\sum_{i\ge1} e^{n^{-\delta+o(1)}(\ell_0+i+1)}\mathbb{P}\big\{\sup_{|v|=n-j}V(v)\ge(\ell_0+i)(n-j)\big\}^{1/p}\\ &\quad\le\sum_{i\ge1} e^{n^{-\delta+o(1)}(\ell_0+i+1)}e^{-\frac{1}{p}i(n-j)} \le Ce^{-n\frac{1}{p}(1+o(1))}.\end{aligned}$$

Hence, we obtain from (5.6) that

$$\begin{aligned}&\mathbb{E}\Big[e^{s((n-j)^{-\delta}-n^{-\delta})\sup_{|v|=n-j}V(v)}\Big(\sum_{|v|=n-j} e^{-(1+(n-j)^{-\delta})V(v)}\Big)^s\Big]\\ &\qquad\qquad\le(1+o(1))\mathbb{E}[(W_n^{+,\delta})^s] + C\mathbb{E}[(W_n^{+,\delta})^{sq}]^{1/q}e^{-\frac{1}{p}n(1+o(1))}.\end{aligned}$$

The second term is exponentially small by the upper bound in Theorem 1.4 (which is proved independently in Sections 4 and 6). This proves (5.4). □

In the next lemma, we simplify the lower bound in Lemma 5.2 by substituting in a suitable strategy for the weights on the spine. Recall that these weights are in distribution equal to the random walk $(S_n)_{n\ge0}$, see Section 2. In particular, the next lemma is simply a statement about functionals of a random walk.

Lemma 5.3. *Let $(S_n)_{n\ge0}$ be a centered random walk started at 0. For any $\gamma > 0$, $s \in (0,1)$, $\kappa \ge \kappa_0 := \frac{3}{s}$ and $n_0 = \lceil(\kappa\log n)^2\rceil$, there exists a constant $c = c(\kappa) > 0$ such* *that*

$$\mathbb{E}_{\mathbb{Q}}\Big[\frac{(g(n)\wedge (S_n^+)^\alpha)e^{-n^{-\delta}S_n}}{(\log n)^\gamma \sum_{j=0}^{n_0-1} e^{-s(1+n^{-\delta})S_j} + g(n)\sum_{j=n_0}^n e^{-sS_j}}\Big] \tag{5.7}$$
$$\geq \frac{c}{\kappa^3(\log n)^{3+\gamma}}\mathbb{E}_{\mathbb{Q}}\big[(g(n-n_0)\wedge (S^+_{n-n_0})^\alpha)e^{-(n-n_0)^{-\delta}S_{n-n_0}}\mathbb{1}_{\{\underline{S}_{n-n_0}\geq -\kappa\log(n-n_0)\}}\big],$$

where $\underline{S}_n = \min_{i=1,\dots,n} S_i$.

Proof. We formulate an event which gives a suitable strategy for the random walk to achieve the lower bound. Namely, define the event

$$E = \left\{\begin{array}{ll} S_j \geq 0 & \text{for all } j = 0,\dots,n_0 \\ 2\sqrt{n_0} \leq S_{n_0} \leq n_0 & \\ S_j \geq \sqrt{n_0} & \text{for all } j = n_0+1,\dots,n \end{array}\right\}$$

where we recall that $n_0 = \lceil(\kappa\log n)^2\rceil$. Note that on the event E, we can estimate the denominator on the left hand side of (5.7) as follows: for any $n_0 = \lceil(\kappa\log n)^2\rceil$ with $\kappa \geq \kappa_0 := \frac{3}{s}$,

$$\begin{aligned}(\log n)^\gamma \sum_{j=0}^{n_0-1} e^{-s(1+n^{-\delta})S_j} + g(n)\sum_{j=n_0+1}^{n} e^{-sS_j} &\leq n_0(\log n)^\gamma + g(n)\sum_{j=n_0+1}^{n} e^{-s\sqrt{n_0}} \\ &\leq \lceil(\kappa\log n)^2\rceil(\log n)^\gamma + ng(n)e^{-s\kappa\log n} \\ &\leq \lceil(\kappa\log n)^2\rceil(\log n)^\gamma + n^{\frac{5}{2}}n^{-s\kappa_0} \leq \kappa^2(\log n)^{2+\gamma}(1+o(1))\end{aligned}$$

where we used that $g(n) \leq n^{\frac{3}{2}}$ for all $\delta > 0$, and our choice of κ_0 ensures that the second term is of order $o(1)$. Thus, by introducing the event E, we obtain the following lower bound

$$\mathbb{E}_{\mathbb{Q}}\Big[\frac{(g(n)\wedge (S_n^+)^\alpha)e^{-n^{-\delta}S_n}}{(\log n)^\gamma \sum_{j=0}^{n_0-1} e^{-s(1+n^{-\delta})S_j} + g(n)\sum_{j=n_0}^n e^{-sS_j}}\Big]$$
$$\geq \frac{1+o(1)}{\kappa^2(\log n)^{2+\gamma}}\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_E\,(g(n)\wedge (S_n^+)^\alpha)e^{-n^{-\delta}S_n}\Big].$$

Using first that on the event E, $S_{n_0} \leq n_0 = \lceil(\kappa\log n)^2\rceil$ and invoking the Markov property at time n_0, the expectation in the above right hand side can be bounded by

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_E\,(g(n)\wedge (S_n^+)^\alpha)e^{-n^{-\delta}S_n}\Big] \\ &\quad\geq \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_E\,(g(n)\wedge ((S_n-S_{n_0})^+)^\alpha)e^{-n^{-\delta}(S_n-S_{n_0})-n^{-\delta}\lceil(\kappa\log n)^2\rceil}\Big] \\ &\quad\geq \mathbb{Q}\{\underline{S}_{n_0}\geq 0; \sqrt{2n_0}\leq S_{n_0}\leq n_0\} \\ &\qquad\times \mathbb{E}_{\mathbb{Q}}\big[(g(n-n_0)\wedge (S^+_{n-n_0})^\alpha)e^{-(n-n_0)^{\delta}S_{n-n_0}}\mathbb{1}_{\{\underline{S}_{n-n_0}\geq -\kappa\log(n-n_0)\}}\big].\end{aligned}$$

To complete the proof we only need to show that the first term of the last line is bounded below. We have

$$\begin{aligned}\mathbb{Q}\{\underline{S}_{n_0}\geq 0; \sqrt{2n_0}\leq S_{n_0}\leq n_0\} &= \mathbb{Q}\{\sqrt{2n_0}\leq S_{n_0}\leq n_0 \,|\, \underline{S}_{n_0}\geq 0\}\mathbb{Q}\{\underline{S}_{n_0}\geq 0\} \\ &= c\,\mathbb{Q}\{\sqrt{2n_0}\leq S_{n_0}\leq n_0 \,|\, \underline{S}_{n_0}\geq 0\}\, n_0^{-\frac{1}{2}}(1+o(1)),\end{aligned}$$

where we used a standard random walk computation, see e.g. [18, Thm. A]. Moreover, $\mathbb{Q}\{\sqrt{2n_0} \leq S_{n_0} \leq n_0 \,|\, \underline{S}_{n_0} \geq 0\}$ converges to a constant not depending on κ, since the (diffusively rescaled) random walk conditioned to stay positive converges to the Brownian meander, see e.g. [10]. □

The proof of Proposition 5.1 follows by combining Lemmas 5.2 and 5.3.

6 Evaluating the random walk expression

In this section, we evaluate the functionals of a simple random walk, which we have encountered in the proofs of the upper and lower bounds respectively. These are sufficiently similar to be treated by the same techniques.

Recall that

$$g(n) = \begin{cases} n^{\frac{1}{2}} & \text{if } \gamma_n = \pm n^{-\delta}, \delta \geq \frac{1}{2}, \\ n^{\frac{3}{2}-2\delta} & \text{if } \gamma_n = n^{-\delta}, \delta \in (0, \frac{1}{2}). \end{cases}$$

Moreover, $\alpha = \alpha(\delta) \geq 1$ is defined as

$$\alpha(\delta) = \begin{cases} 1 & \text{if } \delta \geq \frac{1}{2}, \\ \frac{3}{2\delta} - 2 & \text{if } \delta \in (0, \frac{1}{2}). \end{cases}$$

Also, recall that $(S_n)_{n\geq 0}$ is a centered random walk whose increments have all exponential moments. Denote by $\underline{S}_n = \min_{1\leq j\leq n} S_j$.

Proposition 6.1. *Suppose either $\delta > 0$ and $\gamma_n = n^{-\delta}$ or otherwise $\delta \geq \frac{1}{2}$ and $\gamma_n = -n^{-\delta}$. Then, there exists a constant κ_0 depending only on α and the distribution of S_1 such that for any $\kappa \geq \kappa_0$, there exist constants c, C such that*

$$c \leq \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{\underline{S}_n \geq -\kappa \log n; S_n \geq 0\}}(g(n) \star (S_n^+)^\alpha) e^{-\gamma_n S_n}\Big] \leq C \log n,$$

where $\star$ is either $\wedge$ or $\vee$.

We will prove this proposition in two steps. First, in Lemma 6.2, we will show that we can replace the functional of a random walk by an equivalent functional of a Brownian motion. Here, we will use the coupling of a random walk with a Brownian motion due to Komlós-Major-Tusnády. Finally, we can evaluate that expression which is only a functional of the end point of the Brownian motion and its maximum using the explicit formula of the their joint density, see Lemma 6.3.

In what follows we let $(B_t)_{t\geq 0}$ denote a standard Brownian motion started at the origin and $\mathbb{E}_0$ denote expectation with respect to this Brownian motion.

Lemma 6.2. *Let $\sigma^2 = \mathrm{Var}(S_1)$. Under the assumptions of Proposition 6.1, there exists $\kappa_0 > 0$ such that for any $\kappa \geq \kappa_0$ there exist constants $c, C > 0$ (depending only on δ and the distribution of S_1) such that*

$$\begin{aligned} &c\,\mathbb{E}_0[(g(n) \star (B_n^+)^\alpha) e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\underline{\kappa} \log n, B_n \geq 0\}}] + O(n^{-(1\wedge 2\delta)+o(1)}) \\ &\quad \leq \mathbb{E}_{\mathbb{Q}}[(g(n) \star (S_n^+)^\alpha) e^{-\gamma_n S_n} \mathbb{1}_{\underline{S}_n \geq -\kappa \log n, S_n \geq 0}] \\ &\quad\quad \leq C\,\mathbb{E}_0[(g(n) \star (B_n^+)^\alpha) e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n, B_n \geq 0\}}] + O(n^{-(1\wedge 2\delta)+o(1)}), \end{aligned}$$

where $\overline{\kappa}, \underline{\kappa} \geq 0$ are some suitable constants (depending on κ and the distribution of S_1) and $\underline{B}_n = \min_{0\leq t\leq n} B_t$.

Proof. Let $(S_t)_{t\geq 0}$ denote the piecewise constant approximation of $(S_n)_{n\geq 0}$, defined by $S_t = S_{\lfloor t \rfloor}$. Since the increments of the random walk have exponential moments, the Komlós-Major-Tusnády theory [17] provides a coupling of $(S_t)_{t\in[0,n]}$ and a standard Brownian motion $(B_t)_{t\in[0,n]}$ such that for any $\rho > 0$, there exists a constant $c_\rho > 0$ (depending on ρ and the distribution of S_1) satisfying

$$\mathbb{P}\Big\{\sup_{s\in[0,n]} |S_s - \sigma B_s| \geq c_\rho \log n\Big\} \leq n^{-\rho}.$$

This is an easy extension of the original result, see e.g. the proof of Thm. 2.6. in [7]. Denote by $E = \{\sup_{s\in[0,n]} |S_s - \sigma B_s| \leq c_\rho \log n\}$. It will be convenient to choose $\rho = 4\alpha$

and especially for the lower bound set $\kappa_0 = \frac{c_\rho}{\sigma}$. From now on we will assume that $\kappa \geq \kappa_0$.

Step 1. *Upper bound on the event* E. On the event E the coupling works well and we can replace $(S_t)_{t\in[0,n]}$ by $(B_t)_{t\in[0,n]}$ in the following sense

$$\begin{aligned}&(g(n) \star (S_n^+)^\alpha) e^{-\gamma_n S_n} \mathbb{1}_{\{\underline{S}_n \geq -\kappa \log n, S_n \geq 0\}} \\ &\quad \leq (g(n) \star (\sigma B_n^+ + c_\rho \log n)^\alpha) e^{-\gamma_n \sigma B_n + c_\rho |\gamma_n| \log n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n, \sigma B_n \geq -c_\rho \log n\}},\end{aligned}$$

where $\overline{\kappa} := \frac{\kappa + c_\rho}{\sigma}$. Now, note that $|\gamma_n| \log n \to 0$ as $n \to \infty$, so that we can bound $e^{c_\rho |\gamma_n| \log n}$ by a constant and further we can bound the sum $(B_n^+ + c_\rho \log n)^\alpha \leq 2^\alpha (B_n^+ \vee c_\rho \log n)$. Hence, we find that

$$\begin{aligned}&\mathbb{E}\big[(g(n) \star (S_n^+)^\alpha) e^{-\gamma_n S_n} \mathbb{1}_{\{\underline{S}_n \geq -\kappa \log n, S_n \geq 0\}} \mathbb{1}_E\big] \\ &\quad \leq C 2^\alpha \mathbb{E}\big[(g(n) \star ((\sigma B_n^+)^\alpha \vee (c_\rho \log n)^\alpha)) e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n, \sigma B_n \geq -c_\rho \log n\}}\big]\end{aligned} \tag{6.1}$$

Now if $\star = \vee$, then $g(n) \star ((\sigma B_n^+)^\alpha \vee (c_\rho \log n)^\alpha) = g(n) \star (\sigma B_n^+)^\alpha$ and on the other hand if $\star = \wedge$, we have to estimate

$$\begin{aligned}&\mathbb{E}[(g(n) \wedge ((\sigma B_n^+)^\alpha \vee (c_\rho \log n)^\alpha)) e^{-\gamma_n B_n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n,\, \sigma B_n \geq -c_\rho \log n\}}] \\ &\quad \leq \mathbb{E}[(g(n) \wedge (\sigma B_n^+)^\alpha) e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n,\, \sigma B_n \geq c_\rho \log n\}}] \\ &\qquad + \mathbb{E}[(c_\rho \log n)^\alpha e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n, -c_\rho \log n \leq \sigma B_n \leq c_\rho \log n\}}]\end{aligned}$$

We now claim that the second summand in the previous display is of order $o(n^{-1})$. Indeed,

$$\begin{aligned}&\mathbb{E}[(c_\rho \log n)^\alpha e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n, -c_\rho \log n \leq \sigma B_n \leq c_\rho \log n\}}] \\ &\quad \leq (c_\rho \log n)^\alpha e^{|\gamma_n| c_\rho \log n} \mathbb{P}\{\underline{B}_n \geq -\overline{\kappa} \log n, -c_\rho \log n \leq \sigma B_n \leq c_\rho \log n\} \\ &\quad \leq C n^{-\frac{3}{2} + o(1)}\end{aligned}$$

where the last bound follows from a standard Brownian calculation using for example the explicit density of maximum and final position (see e.g. the proof of Lemma 6.3).

Hence, we can summarize the two possible choices for $\star$ and conclude from (6.1) that

$$\begin{aligned}&\mathbb{E}\big[(g(n) \star (S_n^+)^\alpha) e^{-\gamma_n S_n} \mathbb{1}_{\{\underline{S}_n \geq -\kappa \log n, S_n \geq 0\}} \mathbb{1}_E\big] \\ &\quad \leq C\, \mathbb{E}[(g(n) \star (\sigma B_n^+)^\alpha) e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n,\, \sigma B_n \geq -c_\rho \log n\}}] + O(n^{-\frac{3}{2} + o(1)}).\end{aligned}$$

This is almost of the right form for the main term in the statement of the lemma (where the σ in front of B_n^+ can be absorbed into the constants). Thus it remains to show that we can replace the indicator $\sigma B_n \geq -c_\rho \log n$ by that of $B_n \geq 0$ to obtain the correct upper bound on the event E.

Here, it suffices to show that the following expression is of order $O(n^{-2(\frac{1}{2} \wedge \delta) + o(1)})$,

$$\begin{aligned}&\mathbb{E}[(g(n) \star (\sigma B_n^+)^\alpha) e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\log} n,\, -c_\rho \log n \leq \sigma B_n \leq 0\}}] \\ &\quad \leq \mathbb{E}[g(n) e^{|\gamma_n| c_\rho \log n} \mathbb{1}_{\{\underline{B}_n \geq -\overline{\kappa} \log n,\, -c_\rho \log n \leq \sigma B_n \leq 0\}}] \\ &\quad \leq C g(n) \mathbb{P}\{\underline{B}_n \geq -\overline{\kappa} \log n,\, -c_\rho \log n \leq \sigma B_n \leq 0\} \\ &\quad \leq C g(n) n^{-\frac{3}{2} + o(1)},\end{aligned}$$

where the last step follows from a standard Brownian calculation. However, if $|\delta| \geq \frac{1}{2}$, then $g(n) = n^{\frac{1}{2}}$, so that the latter expression is of order $n^{-1+o(1)}$, whereas if $\delta \in (0, \frac{1}{2})$, then $g(n) = n^{\frac{3}{2}-2\delta}$, so that the expression is of order $n^{-2\delta+o(1)}$ as claimed. This last step completes the proof of the upper bound on the event E.

Step 2. *Upper bound on the event E^c.* In this scenario, we can estimate using Cauchy-Schwarz

$$\mathbb{E}\big[(g(n) \star (S_n^+)^\alpha) e^{-\gamma_n S_n} \mathbb{1}_{\{\underline{S}_n \geq -\kappa \log n, S_n \geq 0\}} \mathbb{1}_{E^c}\big] \\ \leq \mathbb{E}\big[(g(n) \vee (S_n^+)^\alpha)^2 e^{2n^{-\frac{1}{2}} S_n}\big]^{\frac{1}{2}} \mathbb{P}(E^c)^{\frac{1}{2}},$$

where we also used in the last step that if $\gamma_n = -n^\delta$, we only consider the case $\delta \geq \frac{1}{2}$ so that $-\gamma_n S_n \leq n^{-\frac{1}{2}} S_n$, while if $\gamma_n \geq 0$ this bound holds trivially since $S_n \geq 0$. Using that $g(n) \leq n^{\frac{\alpha}{2}}$, we have that this expression can be bounded from above by

$$n^\alpha\, \mathbb{E}\big[\big[(1 \vee (n^{-\frac{1}{2}} S_n^+)^\alpha)^2 e^{2n^{-\frac{1}{2}} S_n}\big]^{\frac{1}{2}} \mathbb{P}(E^c)^{\frac{1}{2}} \tag{6.2}$$

Now, we combine the weak convergence of $n^{-\frac{1}{2}} S_n$ to σB_1 with a standard uniform integrability bound that follows easily from S_1 having exponential moments to deduce that

$$\mathbb{E}\big[\big[(1 \vee (n^{-\frac{1}{2}} S_n^+)^\alpha)^2 e^{2n^{-\frac{1}{2}} S_n}\big] \to \mathbb{E}\Big[(1 \vee (\sigma B_1^+)^\alpha)^2 e^{2\sigma B_1}\Big].$$

Hence, if we combine this observation with the estimate $\mathbb{P}(E^c) \leq n^{-\rho}$ we have the following bound on (6.2):

$$n^\alpha\, \mathbb{E}\big[\big[(1 \vee (n^{-\frac{1}{2}} S_n^+)^\alpha)^2 e^{2n^{-\frac{1}{2}} S_n}\big]^{\frac{1}{2}} \mathbb{P}(E^c)^{\frac{1}{2}} \leq C n^{\alpha - \frac{1}{2}\rho}.$$

Since we chose $\rho = 4\alpha$ the latter expression is of order $n^{-\alpha} \leq n^{-1}$ (since $\alpha \geq 1$) as claimed.

A *lower bound* simply follows by interchanging the roles of random walk and Brownian motion and replacing standard Brownian calculations by standard random walk calculations, see e.g. [2, Lemma A.1]. Moreover, we then need to replace the role of $\overline{\kappa}$ by κ and that of κ by a suitable $\underline{\kappa}$. In particular, we will choose $\underline{\kappa} := \sigma\kappa - c_\rho$, which is non-negative if $\kappa \geq \kappa_0 := c_\rho/\sigma$. ☐

Lemma 6.3. *Under the assumptions of Proposition 6.1, for any $\kappa \geq 0$, and all n sufficiently large,*

$$c \leq \mathbb{E}_0\Big[(g(n) \star B_n^\alpha) e^{-\gamma_n \sigma B_n} \mathbb{1}_{\{\underline{B}_n \geq -\kappa \log n, B_n \geq 0\}}\Big] \leq C \log n.$$

Proof. We use the explicit formula for the joint density of B_t and its running maximum $\overline{B}_t = \sup_{0 \leq s \leq t} B_s$, see e.g. [21, Thm. 3.7.3], which states that $(B_t, \overline{B}_t)$ has for fixed $t > 0$ a joint density with respect to 2-dimensional Lebesgue measure given by

$$f(x, m) = \frac{2(2m - x)}{t\sqrt{2\pi t}} e^{-\frac{(2m-x)^2}{2t}}, \quad \text{for } x \leq m, m > 0.$$

Thus, we can explicitly calculate the functional of the Brownian motion and its minimum by first reflecting the Brownian motion as

$$\begin{aligned} &\mathbb{E}_0\Big[(g(t) \star B_t^\alpha) e^{-\gamma_t \sigma B_t} \mathbb{1}_{\{\underline{B}_t \geq -\kappa \log t, B_t \geq 0\}}\Big] \\ &\quad = \mathbb{E}_0[(g(t) \star (-B_t)^\alpha) e^{\gamma_t \sigma B_t} \mathbb{1}_{\{\overline{B}_t \leq \kappa \log t, B_t \leq 0\}}] \\ &\quad = \frac{2}{\sqrt{2\pi}} t^{-\frac{3}{2}} \int_0^{\kappa \log t} \int_{-\infty}^0 (g(t) \star (-x)^\alpha) e^{\gamma_t \sigma x} (2m - x) e^{-\frac{(2m-x)^2}{2t}}\, dx\, dm\,. \end{aligned} \tag{6.3}$$

We will show lower and upper bound separately and also distinguish the case of a large or a small perturbation.

Upper bound in the case $|\gamma_t| \le t^{\frac{1}{2}}$. In this case, $g(t) = t^{\frac{1}{2}}$ and $\alpha = 1$, so we can bound the expression in (6.3) by

$$Ct^{-\frac{3}{2}}(\kappa \log t) \int_0^\infty (t^{\frac{1}{2}} \vee x) e^{t^{\frac{1}{2}}\sigma x} (2\kappa \log t + x) e^{-\frac{x^2}{2t}} \, dx \\ \le C(\log t)\big(1 + \tfrac{2\kappa \log t}{t^{\frac{1}{2}}}\big) \int_0^\infty (1 \vee x)^2 e^{\sigma x - \frac{x^2}{2}} \, dx \,,$$

which is bounded by $C \log t$ for some (different to above) constant C.

Upper bound in the case $\gamma_t = t^{-\delta}$ *for* $\delta \in (0, \frac{1}{2})$. Note that here we have defined $g(t) = t^{\frac{3}{2}-2\delta}$ and that α is chosen so that $t^{\delta\alpha} = g(t)$, therefore we can bound (6.3) by

$$Ct^{-\frac{3}{2}}(\kappa \log t) \int_0^\infty (g(t) \vee x^\alpha)(2\kappa \log t + x) e^{-t^{-\delta} x} dx \\ \le Ct^{-\frac{3}{2}+2\delta} g(t) (\log t)\big(1 + \tfrac{2\kappa \log t}{t^\delta}\big) \int_0^\infty (1 \vee x^\alpha)^2 e^{-x} dx \,,$$

so that by our choice of $g(t)$, the latter is bounded by $C \log t$.

Lower bound in the case $|\gamma_t| \le t^{\frac{1}{2}}$. Here, we have chosen $g(t) = t^{\frac{1}{2}}$ and $\alpha = 1$. We can lower bound the expression in (6.3) by

$$ct^{-\frac{3}{2}} \int_0^\infty (g(t) \wedge x) x e^{-\sigma t^{-\frac{1}{2}} x} e^{-\frac{1}{2} t^{-\frac{1}{2}} (2\kappa \log t + x)^2} dx \\ \ge c \int_0^\infty (1 \wedge x) x e^{-\sigma x} e^{-2\kappa t^{-\frac{1}{2}} \log t - x^2} dx,$$

where we used the inequality $(x+y)^2 \le 2(x^2 + y^2)$. This is expression is bounded from below by an absolute constant.

Lower bound in the case $\gamma_t = t^{-\delta}, \delta \in (0, \frac{1}{2})$. Here, we have defined $g(t) = t^{\frac{3}{2}-2\delta}$ and α is chosen so that $t^{\delta\alpha} = g(t)$. Then, we can similarly to above find a lower bound on the integral in (6.3)

$$ct^{-\frac{3}{2}} \int_0^\infty (g(t) \wedge x^\alpha) x e^{-\sigma t^{-\delta} x} e^{-\frac{1}{2} t^{-\frac{1}{2}} (2\kappa \log t + x)^2} dx \\ \ge c \int_0^\infty (1 \wedge x^\alpha) x e^{-\sigma x} e^{-t^{\delta - \frac{1}{2}} (2\kappa \log t)^2 - t^{\delta - \frac{1}{2}} x} dx,$$

which, by dominated convergence, is bounded below by an absolute constant. □

Proof of Proposition 6.1. The proof now follows by combining the previous two Lemmas 6.2 and 6.3. □

A Fractional Moment Bounds to Asymptotics

In this appendix we show how the fractional moment bounds obtained in Theorem 1.4 imply the asymptotics in the main Theorem 1.1. The arguments are fairly standard and in a variation are also used in [14].

Lemma A.1 (Upper bounds). *Write $W_n^{\cdot,\delta} = W_n^{\pm,\delta}$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $|a_n| \to \infty$ as $n \to \infty$, and suppose that for every $\gamma \in (0,1)$ we have*

$$\mathbb{E}[(W_n^{\cdot,\delta})^\gamma] = e^{\gamma a_n (1+o(1))}.$$

Then $W_n^{\cdot,\delta} \le e^{a_n(1+o(1))}$ in probability, as $n \to \infty$. Moreover, if for any $\varepsilon > 0$, $\sum_{n\ge1} e^{-\varepsilon|a_n|} < \infty$, then $W_n^{\cdot,\delta} \le e^{a_n(1+o(1))}$ almost surely.

Remark A.2. *This lemma shows how to deduce the upper bounds in Theorem 1.1 from Theorem 1.4. We take*

$$a_n = \begin{cases} -\frac{1}{2}\log n & \text{for } \delta \geq 1/2, \\ \frac{1}{2}n^{1-2\delta}\beta_c^2\lambda''(\beta_c) & \text{for } W_n^{\cdot,\delta} = W_n^{-,\delta}, 0 < \delta < 1/2, \\ (2\delta - \frac{3}{2})\log n & \text{for } W_n^{\cdot,\delta} = W_n^{+,\delta}, 0 < \delta < 1/2. \end{cases}$$

In particular, the lemma shows that in the case of large, negative perturbations the upper bound holds almost surely.

Proof. Fix $\epsilon > 0$ and let $\gamma \in (0,1)$. Then by assumption $\mathbb{E}[(W_n^{\cdot,\delta})^\gamma] \leq e^{\gamma a_n + \epsilon\gamma|a_n|/2}$ for all n sufficiently large. Then by Chebyshev's inequality

$$\begin{aligned} \mathbb{P}(W_n^{\cdot,\delta} > e^{a_n + \epsilon|a_n|}) &\leq e^{-\gamma a_n - \epsilon\gamma|a_n|}\mathbb{E}[(W_n^{\cdot,\delta})^\gamma] \\ &\leq e^{-\epsilon\gamma|a_n|/2} \end{aligned}$$

Thus by the assumption on the $|a_n|$ we have $W_n^{\cdot,\delta} \leq e^{a_n(1+o(1))}$ in probability. The second part of the statement follows from Borel-Cantelli. □

Lemma A.3 (Lower bounds)**.** *Let $W_n^{\cdot,\delta} = W_n^{\pm,\delta}$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with $(\log n)^{\frac{1}{2}} \ll |a_n| \ll n$ such that $n \mapsto |a_n|$ is increasing and $a_{n-k_n} = a_n(1+o(1))$ for any sequence $0 \leq k_n \ll n$. Assume that for all $\gamma \in (0,1)$ we have*

$$\mathbb{E}[(W_n^{\cdot,\delta})^\gamma] = e^{\gamma a_n(1+o(1))}.$$

Then almost surely

$$W_n^{\cdot,\delta} \geq e^{a_n(1+o(1))}.$$

The lower bounds of Theorem 1.1 are therefore derived from this lemma and Theorem 1.4 using the same sequence a_n as in the last remark. Note, however, that in this case the lower bounds are almost sure rather than in probability.

Proof. Let $\varepsilon > 0$. By assumption, for any $\gamma \in (0, \frac{1}{2})$ we have that $\mathbb{E}[(W_n^{\cdot,\delta})^\gamma] \geq e^{\gamma a_n - \frac{\varepsilon}{4}\gamma|a_n|}$ and $\mathbb{E}[(W_n^{\cdot,\delta})^{2\gamma}] \leq e^{2\gamma a_n + \frac{\varepsilon}{4}|a_n|}$, for all n sufficiently large. By the Paley-Zygmund inequality, we have that

$$\begin{aligned} \mathbb{P}\left(W_n^{\cdot,\delta} > e^{a_n - \varepsilon|a_n|}\right) &\geq \left(1 - \frac{e^{\gamma(a_n - \varepsilon|a_n|)}}{\mathbb{E}[(W_n^{\cdot,\delta})^\gamma]}\right)^2 \frac{\mathbb{E}[(W_n^{\cdot,\delta})^\gamma]^2}{\mathbb{E}[(W_n^{\cdot,\delta})^{2\gamma}]} \\ &\geq \left(1 - e^{-3\varepsilon\gamma|a_n|/4}\right)^2 e^{-\frac{3\varepsilon}{4}|a_n|} \geq e^{-\varepsilon|a_n|}, \end{aligned} \tag{A.1}$$

for all n sufficiently large. Now define $\tau_n = \lceil \frac{2\varepsilon|a_n|}{\log 2} \rceil$ so that $\tau_n < n$ for all n sufficiently large. Then

$$\begin{aligned} W_n^{\cdot,\delta} &= \sum_{|w|=\tau_n} e^{-(1\pm n^{-\delta})V(w)} \sum_{\substack{v\in T(w) \\ |v|=n-\tau_n}} e^{-(1\pm n^{-\delta})(V(v)-V(w))} \\ &\geq \exp\left\{ -(1\pm n^{-\delta}) \max_{|w|=\tau_n} V(w)\right\} \sum_{|w|=\tau_n} \sum_{\substack{v\in T(w) \\ |v|=n-\tau_n}} e^{-(1\pm n^{-\delta})(V(v)-V(w))}. \end{aligned}$$

Call the rightmost sum $Y_{n-\tau_n}(w)$. Then the above implies the estimate

$$\begin{aligned}\mathbb{P}\Big(W_n^{\cdot,\delta} \le e^{a_{n-\tau_n}-\varepsilon|a_{n-\tau_n}|}\exp\{-(1\pm n^{-\delta})\max_{|w|=\tau_n} V(w)\}\Big) \\ \le \mathbb{P}\Big(\sum_{|w|=\tau_n} Y_{n-\tau_n}(w) \le e^{a_{n-\tau_n}-\varepsilon|a_{n-\tau_n}|}\Big) \\ \le \mathbb{P}\left(W_{n-\tau_n}^{\cdot,\delta} \le e^{a_{n-\tau_n}-\varepsilon|a_{n-\tau_n}|}\right)^{2^{\tau_n}} \\ \le \left(1-e^{-\varepsilon|a_{n-\tau_n}|}\right)^{2^{\tau_n}} \le \exp\{-e^{\varepsilon|a_n|}\},\end{aligned}$$

where we used first Equation (A.1) and finally that $n \mapsto |a_n|$ is increasing. Therefore, by the assumption that $|a_n| \gg (\log n)^{1/2}$ the probabilities are summable and so by Borel-Cantelli we have that with probability one

$$W_n^{\cdot,\delta} \ge e^{a_{n-\tau_n}-\varepsilon|a_{n-\tau_n}|}\exp\big\{-(1\pm n^{-\delta})\max_{|w|=\tau_n} V(w)\big\}\,, \tag{A.2}$$

for n sufficiently large. However it is well known that there is an explicit constant $C > 0$ such that $\frac{1}{\tau_n}\max_{|v|=\tau_n} V(v) \to C$ with probability one (the max is the position of the rightmost particle in the system of branching random walks), so that

$$\exp\big\{-(1\pm n^{-\delta})\max_{|w|=\tau_n} V(w)\big\} = e^{C^*\varepsilon|a_n|(1+o(1))}$$

for some $C^* > 0$ (not depending on ε). Hence, also using that $a_{n-\tau_n} = a_n(1+o(1))$ by assumption, we have with probability one $W_n^{\cdot,\delta} \ge e^{a_n(1+o(1))}$. □

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