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Stable continuous-state branching processes with immigration and Beta-Fleming-Viot processes with immigration*

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Abstract

Branching processes and Fleming-Viot processes are two main models in stochastic population theory. Incorporating an immigration in both models, we generalize the results of Shiga (1990) and Birkner *et al.* (2005) which respectively connect the Feller diffusion with the classical Fleming-Viot process and the α-stable continuous state branching process with the $Beta(2-\alpha,\alpha)$-generalized Fleming-Viot process. In a recent work, a new class of probability-measure valued processes, called M-generalized Fleming-Viot processes with immigration, has been set up in duality with the so-called M-coalescents. The purpose of this article is to investigate the links between this new class of processes and the continuous-state branching processes with immigration. In the specific case of the α-stable branching process conditioned to be never extinct, we get that its genealogy is given, up to a random time change, by a $Beta(2-\alpha,\alpha-1)$-coalescent.

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1 Introduction

The connections between the Fleming-Viot processes and the continuous-state branching processes have been intensively studied. Shiga established in 1990 that a Fleming-Viot process may be recovered from the *ratio process* associated with a Feller diffusion up to a random time change, see [25]. This result has been generalized in 2005 by Birkner *et al* in [6] in the setting of Λ-generalized Fleming-Viot processes and continuous-state branching processes (CBs for short). In that paper they proved that the ratio process associated with an α-stable branching process is a time-changed $Beta(2-\alpha,\alpha)$-Fleming-Viot process for $\alpha \in (0,2)$. The main goal of this article is to study such connections when immigration is incorporated in the underlying population. The continuous-state branching processes with immigration (CBIs for short) are a class

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of time-homogeneous Markov processes with values in $\mathbb{R}_+$. They have been introduced by Kawazu and Watanabe in 1971, see [18], as limits of rescaled Galton-Watson processes with immigration. These processes are characterized by two functions Φ and Ψ respectively called the immigration mechanism and the branching mechanism. A new class of measure-valued processes with immigration has been recently set up in Foucart [15]. These processes, called M-generalized Fleming-Viot processes with immigration (M-GFVIs for short) are valued in the space of probability measures on $[0,1]$. The notation M stands for a couple of finite measures (Λ_0, Λ_1) encoding respectively the rates of immigration and of reproduction. The genealogies of the M-GFVIs are given by the so-called M-coalescents. These processes are valued in the space of the partitions of $\mathbb{Z}_+$, denoted by $\mathcal{P}^0_\infty$.

In the same manner as Birkner *et al.* in [6], Perkins in [23] and Shiga in [25], we shall establish some relations between continuous-state branching processes with immigration and M-GFVIs. In order to compare the two notions of continuous populations provided respectively by the CBIs and by the M-GFVIs, we shall use the notion of measure-valued branching process. Using calculations of generators, we show in Theorem 3.3 that the following self-similar CBIs admit time-changed M-GFVIs for ratio processes:

- the Feller branching diffusion with branching rate σ^2 and immigration rate β (namely the CBI with $\Phi(q) = \beta q$ and $\Psi(q) = \frac{1}{2}\sigma^2 q^2$) which has for ratio process a time-changed M-Fleming-Viot process with immigration where $M = (\beta\delta_0, \sigma^2\delta_0)$,
- the CBI process with $\Phi(q) = d'\alpha q^{\alpha-1}$ and $\Psi(q) = dq^\alpha$ for some $d, d' \geq 0$, $\alpha \in (1,2)$ which has for ratio process a time-changed M-generalized Fleming-Viot process with immigration where $M = (c'Beta(2-\alpha, \alpha-1), cBeta(2-\alpha, \alpha))$, $c' = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}d'$ and $c = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}d$.

We stress that the CBIs may reach 0, see Proposition 3.1, in which case the M-GFVIs involved describe the ratio process up to this hitting time only. When $d = d'$ or $\beta = \sigma^2$, the corresponding CBIs are respectively the α-stable branching process and the Feller branching diffusion *conditioned to be never extinct*. In that case, the M-coalescents are genuine Λ-coalescents viewed on $\mathcal{P}^0_\infty$. We obtain a $Beta(2-\alpha, \alpha-1)$-coalescent when $\alpha \in (1,2)$ and a Kingman's coalescent for $\alpha = 2$, see Theorem 4.5. This differs from the α-stable branching process *without immigration* studied in [6] for which the coalescent involved is a $Beta(2-\alpha, \alpha)$-coalescent.
Last, ideas provided to establish our main theorem have been used by Handa [17] to study stationary distributions for another class of generalized Fleming-Viot processes.

Outline. The paper is organized as follows. In Section 2, we recall the definition of a continuous-state branching process with immigration (CBI) and of an M-generalized Fleming-Viot process with immigration. Using the framework of measure-valued Markov processes, we define a continuous population with immigration associated with a CBI. We state in Section 3 the connections between the CBIs and M-GFVIs, mentioned in the Introduction, and study the random time change. After recalling the definition of an M-coalescent, we focus in Section 4 on the genealogy of the M-GFVIs involved. We establish that, when the CBIs correspond to CB-processes conditioned to be never extinct, the M-coalescents involved are actually classical Λ-coalescents, and identify them as $Beta(2-\alpha, \alpha-1)$-coalescent. In Section 5, we compare the generators of the M-GFVI and CBI processes and prove the main result.

2 A measure-valued branching process with immigration and the M-generalized Fleming-Viot process with immigration

2.1 Background on continuous state branching processes with immigration

We will focus on critical continuous-state branching processes with immigration characterized by two functions of the variable $q \geq 0$:

$$\Psi(q) = \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\hat{\nu_1}(du)$$
$$\Phi(q) = \beta q + \int_0^\infty (1 - e^{-qu})\hat{\nu_0}(du)$$

where $\sigma^2, \beta \geq 0$ and $\hat{\nu_0}$, $\hat{\nu_1}$ are two Lévy measures such that $\int_0^\infty (1 \wedge u)\hat{\nu_0}(du) < \infty$ and $\int_0^\infty (u \wedge u^2)\hat{\nu_1}(du) < \infty$. The measure $\hat{\nu_1}$ is the Lévy measure of a spectrally positive Lévy process which characterizes the reproduction. We point out that the critical mechanism Ψ is not the most general branching mechanism since we assume that $\hat{\nu}_1$ integrates the identity function near ∞. However, we shall see that there is no loss of generality in assuming this integrability condition on $\hat{\nu}_1$ for our study. The measure $\hat{\nu_0}$ characterizes the jumps of the subordinator that describes the arrival of immigrants in the population. The non-negative constants σ^2 and β correspond respectively to the continuous reproduction and the continuous immigration. Let $\mathbb{P}_x$ be the law of a CBI $(Y_t, t \geq 0)$ started at x, and denote by $\mathbb{E}_x$ the associated expectation. The law of the Markov process $(Y_t, t \geq 0)$ can then be characterized by the Laplace transform of its marginal as follows: for every $q > 0$ and $x \in \mathbb{R}_+$,

$$\mathbb{E}_x[e^{-qY_t}] = \exp\left(-xv_t(q) - \int_0^t \Phi(v_s(q))ds\right)$$

where v is the unique non-negative solution of $\frac{\partial}{\partial t} v_t(q) = -\Psi(v_t(q))$, $v_0(q) = q$.

The pair (Ψ, Φ) is known as the branching-immigration mechanism. The critical CBI process $(Y_t, t \geq 0)$ is conservative in the sense that for every $t > 0$ and $x \in [0, \infty[, \mathbb{P}_x[Y_t < \infty] = 1$. To define a genuine continuous population model with immigration on $[0, 1]$ associated with a CBI, we shall work in the framework of measure-valued Markov processes. Emphasizing the rôle of the initial value, we denote by $(Y_t(x), t \geq 0)$ a CBI started at $x \in \mathbb{R}_+$. The branching property ensures that $(Y_t(x + y), t \geq 0) \overset{law}{=} (Y_t(x) + X_t(y), t \geq 0)$ where $(X_t(y), t \geq 0)$ is a CBI$(\Psi, 0)$ starting from y (that is a CB-process without immigration and with branching mechanism Ψ) independent of $(Y_t(x), t \geq 0)$. The Kolmogorov's extension theorem allows one to construct a flow $(Y_t(x), t \geq 0, x \geq 0)$ such that for every $y \geq 0$, $(Y_t(x + y) - Y_t(x), t \geq 0)$ has the same law as $(X_t(y), t \geq 0)$ a CB-process started from y.
We denote by $(M_t, t \geq 0)$ the Stieltjes-measure associated with the increasing process $x \in [0, 1] \mapsto Y_t(x)$. Namely, define

$$M_t(]x, y]) := Y_t(y) - Y_t(x), \quad 0 \leq x \leq y \leq 1.$$
$$M_t(\{0\}) := Y_t(0).$$

The measure-valued process $(M_t, t \geq 0)$ over the space $[0, 1]$ admits a càdlàg version, see for instance Proposition 2 in [20], and we shall work with such a version in the rest of this paper. Let $x \in [0, 1]$, the process $(Y_t(x), t \geq 0) := (M_t([0, x]), t \geq 0)$ is a CBI(Ψ, Φ) started at x. By a slight abuse of notation, we denote by $(Y_t, t \geq 0)$ the process $(Y_t(1), t \geq 0)$. The process $(M_t, t \geq 0)$ is valued in the space $\mathcal{M}_f$ of finite measures on $[0, 1]$. The framework of measure-valued processes allows us to consider

an infinitely many types model. Namely each individual has initially its own type (which lies in $[0,1]$) and transmits it to its progeny. People issued from the immigration have a *distinguished* type fixed at 0. Since the types do not evolve in time, they allow us to track the ancestors at time 0. This model can be viewed as a superprocess without spatial motion (or without mutation in population genetics vocable).
Let $\mathcal{C}$ be the class of functions on $\mathcal{M}_f$ of the form

$$F(\eta) := G\left(\langle f_1, \eta\rangle, ..., \langle f_n, \eta\rangle\right),$$

where $\langle f, \eta\rangle := \int_{[0,1]} f(x)\eta(dx)$, $G \in C^2(\mathbb{R}^n)$ and $f_1, ..., f_n$ are bounded measurable functions on $[0,1]$. The following operator acting on the space $\mathcal{M}_f$ is an extended generator of $(M_t, t \geq 0)$. For any $\eta \in \mathcal{M}_f$,

$$\mathcal{L}F(\eta) := \sigma^2/2 \int_0^1 \int_0^1 \eta(da)\delta_a(db)F''(\eta; a, b) \tag{2.1}$$

$$+ \beta F'(\eta; 0) \tag{2.2}$$

$$+ \int_0^1 \eta(da) \int_0^\infty \hat{\nu_1}(dh)[F(\eta + h\delta_a) - F(\eta) - hF'(\eta, a)] \tag{2.3}$$

$$+ \int_0^\infty \hat{\nu_0}(dh)[F(\eta + h\delta_0) - F(\eta)] \tag{2.4}$$

where $F'(\eta; a) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}[F(\eta + \epsilon\delta_a) - F(\eta)]$ is the Gateaux derivative of F at η in direction δ_a, and $F''(\eta; a, b) := G'(\eta; b)$ with $G(\eta) = F'(\eta; a)$. The terms (2.1) and (2.3) correspond to the reproduction, see for instance Section 6.1 p. 106 of Dawson [8]. The terms (2.2) and (2.4) correspond to the immigration. In our model the immigration is concentrated on 0, contrary to other works which consider infinitely many types for the immigrants.

Remark 2.1. *We stress that the definition of $(M_t, t \geq 0)$ does not yield plainly the form of the generator $\mathcal{L}$. However one can easily prove that the process $(M_t, t \geq 0)$ has the same law as the càdlàg process*

$$(Y_t(0)\delta_0 + Z_t, t \geq 0),$$

where $(Z_t, t \geq 0)$ is a measure-valued branching process with branching mechanism Ψ started at the Lebesgue measure (see Example 2.45 of Li [22] with $\phi(x, z) = \Psi(z)$ in the notation of this book), and $(Y_t(0), t \geq 0)$ is an independent CBI(Ψ, Φ) started at 0. Corollary 9.3 and Theorem 9.18 p. 218 of [22] then ensure that $\mathcal{L}$ is the generator of $(M_t, t \geq 0)$. Notice that the operator $\mathcal{L}$ corresponds to the one given in equation (9.25) of Section 9 of [22] with $H(d\mu) = \int_0^\infty \hat{\nu_0}(dh)\delta_{h\delta_0}(d\mu)$ and $\eta = \beta\delta_0$.

For $\eta \in \mathcal{M}_f$, we denote by $|\eta|$ the total mass $|\eta| := \eta([0,1])$. If $(M_t, t \geq 0)$ is a Markov process with the above operator for generator, the process $(|M_t|, t \geq 0)$ is by construction a CBI. This is also plain from the form of the generator $\mathcal{L}$: let ψ be a twice differentiable function on $\mathbb{R}_+$ and define $F : \eta \mapsto \psi(|\eta|)$, we find $\mathcal{L}F(\eta) = zG_B\psi(z) + G_I\psi(z)$ for $z = |\eta|$, where

$$G_B\psi(z) = \frac{\sigma^2}{2}\psi''(z) + \int_0^\infty [\psi(z + h) - \psi(z) - h\psi'(z)]\hat{\nu_1}(dh) \tag{2.5}$$

$$G_I\psi(z) = \beta\psi'(z) + \int_0^\infty [\psi(z + h) - \psi(z)]\hat{\nu_0}(dh). \tag{2.6}$$

2.2 Background on M-generalized Fleming-Viot processes with immigration

We denote by $\mathcal{M}_1$ the space of probability measures on $[0,1]$. Let c_0, c_1 be two non-negative real numbers and ν_0, ν_1 be two measures on $[0,1]$ such that $\int_0^1 x\nu_0(dx) < \infty$ and $\int_0^1 x^2\nu_1(dx) < \infty$. Following the notation of [15], we define the couple of finite measures $M = (\Lambda_0, \Lambda_1)$ such that

$$\Lambda_0(dx) = c_0\delta_0(dx) + x\nu_0(dx),\ \Lambda_1(dx) = c_1\delta_0(dx) + x^2\nu_1(dx).$$

The M-generalized Fleming-Viot process with immigration describes a population with *constant size* which evolves by resampling. Let $(\rho_t, t \geq 0)$ be an M-generalized Fleming-Viot process with immigration. The evolution of this process is a superposition of a continuous evolution, and a discontinuous one. The continuous evolution can be described as follows: every couple of individuals is sampled at constant rate c_1, in which case one of the two individuals gives its type to the other: this is a reproduction event. Furthermore, any individual is picked at constant rate c_0, and its type replaced by the distinguished type 0 (the immigrant type): this is an immigration event. The discontinuous evolution is prescribed by two independent Poisson point measures N_0 and N_1 on $\mathbb{R}_+ \times [0,1]$ with respective intensity $dt \otimes \nu_0(dx)$ and $dt \otimes \nu_1(dx)$. More precisely, if (t,x) is an atom of $N_0 + N_1$ then t is a jump time of the process $(\rho_t, t \geq 0)$ and the conditional law of ρ_t given ρ_{t-} is:

- $(1-x)\rho_{t-} + x\delta_U$, if (t,x) is an atom of N_1, where U is distributed according to ρ_{t-}
- $(1-x)\rho_{t-} + x\delta_0$, if (t,x) is an atom of N_0.

If (t,x) is an atom of N_1, an individual is picked at random in the population at generation $t-$ and generates a proportion x of the population at time t: this is a reproduction event, as for the genuine generalized Fleming-Viot process (see [4] p. 278). If (t,x) is an atom of N_0, the individual 0 at time $t-$ generates a proportion x of the population at time t: this is an immigration event. In both cases, the population at time $t-$ is reduced by a factor $1-x$ so that, at time t, the total size is still 1. The genealogy of this population (which is identified as a probability measure on $[0,1]$) is given by an M-coalescent (see Section 4 below). This description is purely heuristic (we stress for instance that the atoms of $N_0 + N_1$ may form an infinite dense set), to make a rigorous construction of such processes, we refer to the Section 5.2 of [15] (or alternatively Section 3.2 of [16]). For any $p \in \mathbb{N}$ and any continuous function f on $[0,1]^p$, we denote by G_f the map

$$\rho \in \mathcal{M}_1 \mapsto \langle f, \rho^{\otimes p}\rangle := \int_{[0,1]^p} f(x)\rho^{\otimes p}(dx) = \int_{[0,1]^p} f(x_1, ..., x_p)\rho(dx_1)...\rho(dx_p).$$

Let $(\mathcal{F}, \mathcal{D})$ denote the generator of $(\rho_t, t \geq 0)$ and its domain. The vector space generated by the functionals of the type G_f forms a core of $(\mathcal{F}, \mathcal{D})$ and we have (see Lemma 5.2 in [15]):

$$\mathcal{F}G_f(\rho) = c_1 \sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f(\mathbf{x}^{i,j}) - f(\mathbf{x})]\rho^{\otimes p}(d\mathbf{x}) \tag{2.1'}$$

$$+ c_0 \sum_{1\leq j\leq p} \int_{[0,1]^p} [f(\mathbf{x}^{0,j}) - f(\mathbf{x})]\rho^{\otimes p}(d\mathbf{x}) \tag{2.2'}$$

$$+ \int_0^1 \nu_1(dr) \int \rho(da)[G_f((1-r)\rho + r\delta_a) - G_f(\rho)] \tag{2.3'}$$

$$+ \int_0^1 \nu_0(dr)[G_f((1-r)\rho + r\delta_0) - G_f(\rho)]. \tag{2.4'}$$

where $\mathbf{x}$ denotes the vector $(x_1, ..., x_p)$ and

- the vector $\mathbf{x}^{0,j}$ is defined by $\mathbf{x}^{0,j}_k = x_k$, for all $k \neq j$ and $\mathbf{x}^{0,j}_j = 0$,
- the vector $\mathbf{x}^{i,j}$ is defined by $\mathbf{x}^{i,j}_k = x_k$, for all $k \neq j$ and $\mathbf{x}^{i,j}_j = x_i$.

We mention that when $\nu_0 = 0$, the generator $\mathcal{F}$ is a special case of the generator L defined at (4.12) of Dawson and Li [9], with $\gamma = \delta_0$.

3 Relations between CBIs and M-GFVIs

3.1 Forward results

The expressions of the generators of $(M_t, t \geq 0)$ and $(\rho_t, t \geq 0)$ lead us to specify the connections between CBIs and GFVIs. We add a cemetery point Δ to the space $\mathcal{M}_1$ and define $(R_t, t \geq 0) := (\frac{M_t}{|M_t|}, t \geq 0)$, the ratio process with lifetime $\tau := \inf\{t \geq 0; |M_t| = 0\}$. By convention, for all $t \geq \tau$, we set $R_t = \Delta$. As mentioned in the Introduction, we shall focus our study on the two following critical CBIs:

(i) $(Y_t, t \geq 0)$ is a CBI with parameters $\sigma^2, \beta \geq 0$ and $\hat{\nu}_0 = \hat{\nu}_1 = 0$, so that $\Psi(q) = \frac{\sigma^2}{2} q^2$ and $\Phi(q) = \beta q$.

(ii) $(Y_t, t \geq 0)$ is a CBI with $\sigma^2 = \beta = 0$, $\hat{\nu}_0(dh) = c' h^{-\alpha} 1_{h>0} dh$ and $\hat{\nu}_1(dh) = c h^{-1-\alpha} 1_{h>0} dh$ for $1 < \alpha < 2$, so that $\Psi(q) = dq^\alpha$ and $\Phi(q) = d' \alpha q^{\alpha-1}$ with $d' = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} c'$ and $d = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} c$

Notice that the CBI in (i) may be seen as a limit case of the CBIs in (ii) for $\alpha = 2$. We first establish in the following proposition a dichotomy for the finiteness of the lifetime, depending on the ratio immigration over reproduction.

Proposition 3.1. *Recall the notation* $\tau = \inf\{t \geq 0, Y_t = 0\}$.

- *If* $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$ *in case (i) or* $\frac{c'}{c} \geq \frac{\alpha-1}{\alpha}$ *in case (ii), then* $\mathbb{P}[\tau = \infty] = 1$.
- *If* $\frac{\beta}{\sigma^2} < \frac{1}{2}$ *in case (i) or* $\frac{c'}{c} < \frac{\alpha-1}{\alpha}$ *in case (ii), then* $\mathbb{P}[\tau < \infty] = 1$.

We then deal with the random change of time. In the case of a CB-process (that is a CBI process without immigration), Birkner *et al.* used the Lamperti representation and worked on the embedded stable spectrally positive Lévy process. We shall work directly on the CBI process instead. For $0 \leq t \leq \tau$, we define:

$$C(t) = \int_0^t Y_s^{1-\alpha} ds,$$

in case (ii) and set $\alpha = 2$ in case (i).

Proposition 3.2. *In both cases (i) and (ii), we have:*

$$\mathbb{P}\left(C(\tau) = \infty\right) = 1.$$

In other words, the additive functional C *maps* $[0, \tau[$ *to* $[0, \infty[$.

By convention, if τ is almost surely finite we set $C(t) = C(\tau) = \infty$ for all $t \geq \tau$. Denote by C^{-1} the right continuous inverse of the functional C. This maps $[0, \infty[$ to $[0, \tau[$, a.s. We stress that in most cases, $(R_t, t \geq 0)$ is not a Markov process. Nevertheless, in some cases, through a change of time, the process $(R_t, t \geq 0)$ may be changed into a Markov process. This shall be stated in the following Theorem where the functional C is central.

For every $x, y > 0$, we denote by $Beta(x, y)(dr)$ the unnormalized finite measure with density

$$r^{x-1}(1-r)^{y-1} 1_{(0,1)}(r) dr.$$

Theorem 3.3. *Let $(M_t, t \geq 0)$ be the measure-valued branching process with immigration as defined in Section 2.1.*

- *In case (i), the process $(R_{C^{-1}(t)})_{t\geq 0}$ is a M-Fleming-Viot process with immigration with*

$$\Lambda_0(dr) = \beta\delta_0(dr) \text{ and } \Lambda_1(dr) = \sigma^2\delta_0(dr).$$

- *In case (ii), the process $(R_{C^{-1}(t)})_{t\geq 0}$ is a M-generalized Fleming-Viot process with immigration with*

$$\Lambda_0(dr) = c' Beta(2-\alpha, \alpha-1)(dr) \text{ and } \Lambda_1(dr) = cBeta(2-\alpha, \alpha)(dr).$$

The proof requires rather technical arguments on the generators and is given in Section 5.

Remark 3.4.
- *The CBIs in the statement of Theorem 3.3 with $\sigma^2 = \beta$ in case (i) or $c = c'$ in case (ii), are also CBs conditioned on non extinction and are studied further in Section 4.*
- *Contrary to the case without immigration, see Theorem 1.1 in [6], we have to restrict ourselves to $\alpha \in (1, 2]$.*
- *It is worth to notice that the relation obtained in Theorem 3.3 holds only in the stable case (see Lemma 5.4 below).*

So far, we state that the ratio process $(R_t, t \geq 0)$ associated to $(M_t, t \geq 0)$, once time changed by C^{-1}, is a M-GFVI process. Conversely, starting from a M-GFVI process, we could wonder how to recover the measure-valued CBI process $(M_t, t \geq 0)$. This lead us to investigate the relation between the time changed ratio process $(R_{C^{-1}(t)}, t \geq 0)$ and the process $(Y_t, t \geq 0)$.

Proposition 3.5. *In case (i) of Theorem 3.3, the additive functional $(C(t), t \geq 0)$ and $(R_{C^{-1}(t)}, 0 \leq t < \tau)$ are independent.*

This proves that in case (i) we need additional randomness to reconstruct M from the M-GFVI process. On the contrary, in case (ii), the process $(Y_t, t \geq 0)$ is clearly not independent of the ratio process $(R_t, t \geq 0)$, since both processes jump at the same time.
The proof of Propositions 3.1, 3.2 are given in the next Subsection. Some rather technical arguments are needed to prove Proposition 3.5. We postpone its proof to the end of Section 5.

3.2 Proofs of Propositions 3.1, 3.2

Proof of Proposition 3.1. Let $(X_t(x), t \geq 0)$ denote an α-stable branching process started at x (with $\alpha \in (1, 2]$). Denote ζ its absorption time, $\zeta := \inf\{t \geq 0; X_t(x) = 0\}$. The following construction of the process $(Y_t(0), t \geq 0)$ may be deduced from the expression of the Laplace transform of the CBI process. We shall need the canonical measure $\mathbb{N}$ which is a sigma-finite measure on càdlàg paths and represents informally the "law" of the population generated by one single individual in a CB(Ψ), see Chu and Ren [7] or Li [22] in the general framework of measure-valued Markov processes. We write:

$$(Y_t(0), t \geq 0) = \left(\sum_{i \in \mathcal{I}} X^i_{(t-t_i)_+}, t \geq 0 \right) \tag{3.1}$$

with $\sum_i \delta_{(t_i, X^i)}$ a Poisson random measure on $\mathbb{R}_+ \times \mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ with intensity $dt \otimes \mu$, where $\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$ denotes the space of càdlàg functions, and μ is defined as follows:

- in case (ii), $\mu(dX) = \int \hat{\nu_0}(dx)\mathbb{P}_x(dX)$, where $\mathbb{P}_x$ is the law of a CB(Ψ) with $\Psi(q) = dq^\alpha$ starting from initial mass x. Formula (3.1) may be understood as follows: at the jump times t_i of a pure jump stable subordinator with Lévy measure $\hat{\nu_0}$, a new arrival of immigrants, of size X_0^i, occurs in the population. Each of these "packs", labelled by $i \in \mathcal{I}$, generates its own descendance $(X_t^i, t \geq 0)$, which is a CB(Ψ) process.
- in case (i), $\mu(dX) = \beta\ \mathbf{N}(dX)$, where $\mathbf{N}$ is the canonical measure associated to the CB(Ψ) with $\Psi(q) = \frac{\sigma^2}{2}q^2$. The canonical measure may be thought of as the "law" of the population generated by one single individual. The link with case (ii) is the following: the pure jump subordinator degenerates into a continuous subordinator equal to $(t \mapsto \beta t)$. The immigrants no more arrive by packs, but appear continuously.

Actually, the canonical measure $\mathbf{N}$ is defined in both cases (i) and (ii), and we may always write $\mu(dX) = \Phi(\mathbf{N}(dX))$. The process $(Y_t(0), t \geq 0)$ is a CBI(Ψ, Φ) started at 0. We call $\mathcal{R}$ the set of zeros of $(Y_t(0), t > 0)$:

$$\mathcal{R} := \{t > 0; Y_t(0) = 0\}.$$

Denote $\zeta_i = \inf\{t > 0, X_t^i = 0\}$ the lifetime of the branching process X^i. The intervals $]t_i, t_i + \zeta_i[$ and $[t_i, t_i + \zeta_i[$ represent respectively the time where X^i is alive in case (i) and in case (ii) (in this case, we have $X_{t_i}^i > 0$.) Therefore, if we define $\tilde{\mathcal{R}}$ as the set of the positive real numbers left uncovered by the random intervals $]t_i, t_i + \zeta_i[$, that is:

$$\tilde{\mathcal{R}} := \mathbb{R}_+^\star \setminus \bigcup_{i \in \mathcal{I}}]t_i, t_i + \zeta_i[.$$

we have $\mathcal{R} \subset \tilde{\mathcal{R}}$.

The lengths ζ_i have law $\mu(\zeta \in dt)$ thanks to the Poisson construction of $Y(0)$. We now distinguish the two cases:

- Feller case: this corresponds to $\alpha = 2$. We have $\Psi(q) := \frac{\sigma^2}{2}q$ and $\Phi(q) := \beta q$, and thus
$$\mu[\zeta > t] = \beta\ \mathbf{N}[\zeta > t] = \frac{2\beta}{\sigma^2}\frac{1}{t},$$
see Li [22] p. 62. Using Example 1 p. 180 of Fitzsimmons et al. [14], we deduce that
$$\tilde{\mathcal{R}} = \emptyset \text{ a.s. if and only if } \frac{2\beta}{\sigma^2} \geq 1. \tag{3.2}$$
- Stable case: this corresponds to $\alpha \in (1, 2)$. Recall $\Psi(q) := dq^\alpha, \Phi(q) := d'\alpha q^{\alpha-1}$. In that case, we have,
$$\mathbf{N}(\zeta > t) = d^{-\frac{1}{\alpha-1}}[(\alpha-1)t]^{-\frac{1}{\alpha-1}}.$$
Thus, $\mu[\zeta > t] = \Phi(\mathbf{N}(\zeta > t)) = \frac{\alpha}{\alpha-1}\frac{d'}{d}\frac{1}{t}$. Recall that $\frac{d'}{d} = \frac{c'}{c}$. Therefore, using reference [14], we deduce that
$$\tilde{\mathcal{R}} = \emptyset \text{ a.s. if and only if } \frac{c'}{c} \geq \frac{\alpha-1}{\alpha}. \tag{3.3}$$

This allows us to establish the first point of Proposition 3.1: we get $\mathcal{R} \subset \tilde{\mathcal{R}} = \emptyset$, and the inequality $Y_t(1) \geq Y_t(0)$ for all t ensures that $\tau = \infty$.

We deal now with the second point of Proposition 3.1. Assume that $\frac{c'}{c} < \frac{\alpha-1}{\alpha}$ or $\frac{\beta}{\sigma^2} < \frac{1}{2}$. By assertions (3.2) and (3.3), we already know that $\tilde{\mathcal{R}} \neq \emptyset$. However, what we really need is that $\tilde{\mathcal{R}}$ is a.s. not bounded. To that aim, observe that, in both cases (i) and (ii),

$$\mu[\zeta > s] = \Phi(\mathbf{N}(\zeta > s)) = \frac{\kappa}{s}$$

with $\kappa = \frac{\alpha}{\alpha-1}\frac{d'}{d} = \frac{\alpha}{\alpha-1}\frac{c'}{c} < 1$ if $1 < \alpha < 2$ and $\kappa = \frac{2\beta}{\sigma^2} < 1$ if $\alpha = 2$. Thus $\int_1^u \mu[\zeta > s]ds = \kappa \ln(u)$ and we obtain

$$\exp\left(-\int_1^u \mu[\zeta > s]ds\right) = \left(\frac{1}{u}\right)^{\kappa}.$$

Therefore, since $\kappa < 1$,

$$\int_1^\infty \exp\left(-\int_1^u \mu[\zeta > s]ds\right) du = \infty,$$

which implies thanks to Corollary 4 (Equation 17 p. 183) of [14] that $\tilde{\mathcal{R}}$ is a.s. not bounded.

Since $\mathcal{R} = \tilde{\mathcal{R}}$ in case (i), the set $\mathcal{R}$ is a.s. not bounded in that case. Now, we prove that $\mathcal{R}$ is a.s. not bounded in case (ii). The set $\tilde{\mathcal{R}}$ is almost surely not empty and not bounded. Moreover this is a perfect set (Corollary 1 of [14]). Since there are only countable points $(t_i, i \in \mathcal{I})$, the set $\tilde{\mathcal{R}} = \mathcal{R} \setminus \bigcup_{i\in I}\{t_i\}$ is also uncountable and not bounded.

Last, recall from Subsection 2.1 that we may write $Y_t(1) = Y_t(0) + X_t(1)$ for all $t \geq 0$ with $(X_t(1), t \geq 0)$ a CB-process independent of $(Y_t(0), t \geq 0)$. Let $\xi := \inf\{t \geq 0, X_t(1) = 0\}$ be the extinction time of $(X_t(1), t \geq 0)$. Since $\mathcal{R}$ is a.s. not bounded in both cases (i) and (ii), $\mathcal{R} \cap (\xi, \infty) \neq \emptyset$, and $\tau < \infty$ almost surely. □

Proof of Proposition 3.2. Recall that $Y_t(x)$ is the value of the CBI started at x at time t. We will denote by $\tau^x(0) := \inf\{t > 0, Y_t(x) = 0\}$. With this notation, $\tau^1(0) = \tau$ introduced in Section 3.1. In both cases (i) and (ii), the processes are self-similar, see Kyprianou and Pardo [19]. Namely, we have

$$(xY_{x^{1-\alpha}t}(1), t \geq 0) \overset{law}{=} (Y_t(x), t \geq 0),$$

where we take $\alpha = 2$ in case (i). Performing the change of variable $s = x^{1-\alpha}t$, we obtain

$$\int_0^{\tau^x(0)} dt\, Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\tau^1(0)} ds\, Y_s(1)^{1-\alpha}. \tag{3.4}$$

According to Proposition 3.1, depending on the values of the parameters:

- Either $\mathbb{P}(\tau^x(0) < \infty) = 1$ for every x. Let $x > 1$. Denote $\tau^x(1) = \inf\{t > 0, Y_t(x) \leq 1\}$. We have $\mathbb{P}(\tau^x(1) < \infty) = 1$. We have:

  $$\int_0^{\tau^x(0)} dt\, Y_t(x)^{1-\alpha} = \int_0^{\tau^x(1)} dt\, Y_t(x)^{1-\alpha} + \int_{\tau^x(1)}^{\tau^x(0)} dt\, Y_t(x)^{1-\alpha}$$

  By the strong Markov property applied at the stopping time $\tau^x(1)$, since Y has no negative jumps:

  $$\int_{\tau^x(1)}^{\tau^x(0)} dt\, Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\tau^1(0)} dt\, \tilde{Y}_t(1)^{1-\alpha},$$

with $(\tilde{Y}_t(1), t \geq 0)$ an independent copy started from 1. Since

$$\int_0^{\tau^x(1)} dt\ Y_t(x)^{1-\alpha} > 0,\ a.s.,$$

the equality (3.4) is impossible unless both sides of the equality are infinite almost surely. We thus get that $C(\tau) = \infty$ almost surely in that case.

- Either $\mathbb{P}(\tau^x(0) = \infty) = 1$ for every x, on which case we may rewrite (3.4) as follows:

$$\int_0^{\infty} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\infty} ds\ Y_s(1)^{1-\alpha}.$$

Since, for $x > 1$, the difference $(Y_t(x) - Y_t(1), t \geq 0)$ is an α-stable CB-process started at $x - 1 > 0$, we deduce that $C(\tau) = \infty$ almost surely again.

This proves the statement. □

Remark 3.6. *The situation is quite different when the CBI process starts at 0, in which case the time change also diverges in the neighbourhood of 0. The same change of variables as in (3.4) yields, for all $0 < x < k$,*

$$\int_0^{\iota^x(k)} dt\ Y_t(x)^{1-\alpha} \overset{law}{=} \int_0^{\iota^1(k/x)} dt\ Y_t(1)^{1-\alpha},$$

with $\iota^x(k) = \inf\{t > 0, Y_t(x) \geq k\} \in [0, \infty]$. Letting x tend to 0, we get $\iota^1(k/x) \longrightarrow \infty$ and the right hand side diverges to infinity. Thus, the left hand side also diverges, which implies that:

$$\mathbb{P}\left(\int_0^{\iota^0(k)} dt\ Y_t(0)^{1-\alpha} = \infty\right) = 1.$$

4 Genealogy of the Beta-Fleming-Viot processes with immigration

To describe the genealogy associated with stable CBs, Bertoin and Le Gall [5] and Birkner et al. [6] used partition-valued processes called Beta-coalescents. These processes form a subclass of Λ-coalescents, introduced independently by Pitman and Sagitov in 1999. Let us also mention that Donnelly and Kurtz [10] found at about the same time an embedding of the Λ-coalescents in their look-down particle system. A Λ-coalescent is an exchangeable process in the sense that its law is invariant under the action of any permutation. In words, there is no distinction between the individuals. Although these processes arise as models of genealogy for a wide range of stochastic populations, they are not in general adapted to describe the genealogy of a population with immigration. Recently, a larger class of processes called M-coalescents has been defined in [15] (see Section 5). These processes are precisely those describing the genealogy of M-GFVIs.

Remark 4.1. *We mention that the use of the lookdown construction in Birkner et al. [6] may be easily adapted to our framework and yields a genealogy for any conservative CBI. Moreover, other genealogies, based on continuous trees, have been investigated by Lambert [20] and Duquesne [11].*

4.1 Background on M-coalescents

Before focusing on the M-coalescents involved in the context of Theorem 3.3, we recall their general definition and the duality with the M-GFVIs. Contrary to the Λ-coalescents, the M-coalescents are only invariant by permutations letting 0 fixed. The individual 0 represents the immigrant lineage and is distinguished from the others. We denote by $\mathcal{P}_\infty^0$ the space of partitions of $\mathbb{Z}_+ := \{0\} \bigcup \mathbb{N}$. Let $\pi \in \mathcal{P}_\infty^0$. By convention, we identify π with the sequence $(\pi_0, \pi_1, ...)$ of the blocks of π enumerated in increasing order of their smallest element: for every $i \leq j$, $\min \pi_i \leq \min \pi_j$. Let $[\![n]\!]$ denote the set $\{0, ..., n\}$ and $\mathcal{P}_n^0$ the space of partitions of $[\![n]\!]$. The partition of $[\![n]\!]$ into singletons is denoted by $0_{[\![n]\!]}$. As in Section 2.2, the notation M stands for a pair of finite measures (Λ_0, Λ_1) such that:

$$\Lambda_0(dx) = c_0\delta_0(dx) + x\nu_0(dx),\ \Lambda_1(dx) = c_1\delta_0(dx) + x^2\nu_1(dx),$$

where c_0, c_1 are two non-negative real numbers and ν_0, ν_1 are two measures on $[0, 1]$ subject to the same conditions as in Section 2.2. Let N_0 and N_1 be two Poisson point measures with respective intensity $dt \otimes \nu_0(dx)$ and $dt \otimes \nu_1(dx)$. An M-coalescent is a Feller process $(\Pi(t), t \geq 0)$ valued in $\mathcal{P}_\infty^0$ with the following dynamics.

- At an atom (t, x) of N_1, flip a coin with probability of "heads" x for each block not containing 0. All blocks flipping "heads" are merged immediately in one block. At time t, a proportion x share a common parent in the population.
- At an atom (t, x) of N_0, flip a coin with probability of "heads" x for each block not containing 0. All blocks flipping "heads" coagulate immediately with the distinguished block. At time t, a proportion x of the population is children of immigrant.

It must be highlighted that this definition is informal since points (t, x) may be dense in time, so one would have to consider the restriction to $[\![n]\!]$ first. In order to take into account the parameters c_0 and c_1, imagine that at constant rate c_1, two blocks (not containing 0) merge *continuously* in time, and at constant rate c_0, one block (not containing 0) merged with the distinguished one. We refer to Section 4.2 of [15] for a rigorous definition. Let $\pi \in \mathcal{P}_n^0$. The jump rate of an M-coalescent from $0_{[\![n]\!]}$ to π, denoted by q_π, is given as follows:

- If π has one block not containing 0 with k elements and $2 \leq k \leq n$, then

$$q_\pi = \lambda_{n,k} := \int_0^1 x^{k-2}(1-x)^{n-k}\Lambda_1(dx).$$

- If the distinguished block of π has $k+1$ elements (counting 0) and $1 \leq k \leq n$ then

$$q_\pi = r_{n,k} := \int_0^1 x^{k-1}(1-x)^{n-k}\Lambda_0(dx).$$

The next duality property is a key result and links the M-GFVIs to the M-coalescents. For any π in $\mathcal{P}_\infty^0$, define

$$\alpha_\pi : k \mapsto \text{the index of the block of } \pi \text{ containing } k.$$

We have the duality relation (see Lemma 4 in [16]): for any $p \geq 1$ and $f \in C([0, 1]^p)$,

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(t)}(1)}, ..., x_{\alpha_{\Pi(t)}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)\rho_t(dx_1)...\rho_t(dx_p)\right],$$

where $(\rho_t, t \geq 0)$ is a M-GFVI started from the Lebesgue measure on $[0, 1]$. We establish a useful lemma relating genuine Λ-coalescents and M-coalescents. Consider a

Λ-coalescent taking values in the set $\mathcal{P}_\infty^0$; this differs from the usual convention, according to which they are valued in the set $\mathcal{P}_\infty$ of the partitions of $\mathbb{N}$ (see Chapters 1 and 3 of [2] for a complete introduction to these processes). In that framework, Λ-coalescents appear as a subclass of M-coalescents and the integer 0 may be viewed as a typical individual. The proof is postponed in Section 4.3.

Lemma 4.2. *A M-coalescent, with $M = (\Lambda_0, \Lambda_1)$ is also a Λ-coalescent on $\mathcal{P}_\infty^0$ if and only if*

$$(1-x)\Lambda_0(dx) = \Lambda_1(dx).$$

In that case $\Lambda = \Lambda_0$.

4.2 The $Beta(2-\alpha, \alpha-1)$-coalescent

The aim of this Section is to show how a $Beta(2-\alpha, \alpha-1)$-coalescent is embedded in the genealogy of an α-stable CB-process conditioned to be never extinct. Along the way, we also derive the fixed time genealogy of the Feller CBI.

We first state the following straightforward Corollary of Theorem 3.3, which gives the genealogy of the ratio process at the random time $C^{-1}(t)$:

Corollary 4.3. *Let $(R_t, t \geq 0)$ be the ratio process of a CBI in case (i) or (ii). We have for all $t \geq 0$:*

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(t)}(1)}, ..., x_{\alpha_{\Pi(t)}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_{C^{-1}(t)}(dx_1)...R_{C^{-1}(t)}(dx_p)\right],$$

where:

- *In case (i), $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (\beta\delta_0, \sigma^2\delta_0)$,*
- *In case (ii), $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (c'Beta(2-\alpha, \alpha-1), cBeta(2-\alpha, \alpha))$.*

In general, we cannot set the random quantity $C(t)$ instead of t in the equation of Corollary 4.3. Nevertheless, using the independence property proved in Proposition 3.5, we get the following Corollary, whose proof may be found in Section 4.3.

Corollary 4.4. *In case (i), assume $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$, then for all $t \geq 0$,*

$$\mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(C(t))}(1)}, ..., x_{\alpha_{\Pi(C(t))}(p)})\delta_0(dx_0)dx_1...dx_p\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_t(dx_1)...R_t(dx_p)\right],$$

where $(\Pi(t), t \geq 0)$ is a M-coalescent with $M = (\beta\delta_0, \sigma^2\delta_0)$, $(Y_t, t \geq 0)$ is a CBI in case (i) independent of $(\Pi(t), t \geq 0)$ and $(C(t), t \geq 0) = \left(\int_0^t \frac{1}{Y_s}ds, t \geq 0\right)$.

We stress on a fundamental difference between Corollaries 4.3 and 4.4. Whereas the first gives the genealogy of the ratio process R at the random time $C^{-1}(t)$, the second gives the genealogy of the ratio process R at a fixed time t. Notice that we impose the additional assumption that $\frac{\beta}{\sigma^2} \geq \frac{1}{2}$ in Corollary 4.4 for ensuring that the lifetime is infinite. Therefore, $R_t \neq \Delta$ for all $t \geq 0$, and we may consider its genealogy.

We easily check that the M-coalescents for which $M = (\sigma^2\delta_0, \sigma^2\delta_0)$ and $M = (cBeta(2-\alpha, \alpha-1), cBeta(2-\alpha, \alpha))$ fulfill the conditions of Lemma 4.2. Recall from Section 3.1 the definitions of the CBIs in case (i) and (ii) .

Theorem 4.5. *(i) If the process $(Y_t, t \geq 0)$ is a CBI such that $\sigma^2 = \beta > 0$, $\hat{\nu}_1 = \hat{\nu}_0 = 0$, then the process $(\Pi(t/\sigma^2), t \geq 0)$ defined in Corollary 4.3 is a Kingman's coalescent valued in $\mathcal{P}_\infty^0$.*

(ii) If the process $(Y_t, t \geq 0)$ is a CBI such that $\sigma^2 = \beta = 0$ and $\hat{\nu}_0(dh) = ch^{-\alpha}dh$, $\hat{\nu}_1(dh) = ch^{-\alpha-1}dh$ for some constant $c > 0$ then the process $(\Pi(t/c), t \geq 0)$ defined in Corollary 4.3 is a $Beta(2-\alpha, \alpha-1)$-coalescent valued in $\mathcal{P}^0_\infty$.

In both cases, the process $(Y_t, t \geq 0)$ involved in that Theorem may be interpreted as a CB-process $(X_t, t \geq 0)$ without immigration ($\beta = 0$ or $c' = 0$) conditioned on non-extinction, see e.g. the discussion and references in Lambert [21]. We then notice that both the genealogies of the time changed Feller diffusion and of the time changed Feller diffusion conditioned on non extinction are given by the same Kingman's coalescent. On the contrary, the genealogy of the time changed α-stable CB-process is a $Beta(2-\alpha, \alpha)$-coalescent, whereas the genealogy of the time changed α-stable CB-process conditioned on non-extinction is a $Beta(2-\alpha, \alpha-1)$-coalescent. We stress that for any $\alpha \in (1, 2)$ and any borelian B of $[0, 1]$, we have $Beta(2-\alpha, \alpha-1)(B) \geq Beta(2-\alpha, \alpha)(B)$. This may be interpreted as the additional reproduction events needed for the process to be never extinct.

4.3 Proofs.

Proof of Lemma 4.2. Let $(\Pi'(t), t \geq 0)$ be a Λ-coalescent on $\mathcal{P}^0_\infty$. Let $n \geq 1$, we may express the jump rate of $(\Pi'_{|[\![n]\!]}(t), t \geq 0)$ from $0_{[\![n]\!]}$ to π by

$$q'_\pi = \begin{cases} 0 \text{ if } \pi \text{ has more than one non-trivial block} \\ \\ \int_{[0,1]} x^k(1-x)^{n+1-k}x^{-2}\Lambda(dx) \text{ if the non trivial block has } k \text{ elements.} \end{cases}$$

Consider now a M-coalescent, denoting by q_π the jump rate from $0_{[\![n]\!]}$ to π, we have

$$q_\pi = \begin{cases} 0 \text{ if } \pi \text{ has more than one non-trivial block} \\ \\ \int_{[0,1]} x^k(1-x)^{n-k}x^{-2}\Lambda_1(dx) \text{ if } \pi_0 = \{0\} \text{ and the non trivial block has } k \text{ elements} \\ \\ \int_{[0,1]} x^{k-1}(1-x)^{n+1-k}x^{-1}\Lambda_0(dx) \text{ if } \#\pi_0 = k. \end{cases}$$

Since the law of a Λ-coalescent is entirely described by the family of the jump rates of its restriction on $[\![n]\!]$ from $0_{[\![n]\!]}$ to π for π belonging to $\mathcal{P}^0_n$ (see Section 4.2 of [3]), the processes Π and Π' have the same law if and only if for all $n \geq 0$ and $\pi \in \mathcal{P}^0_n$, we have $q_\pi = q'_\pi$, that is if and only if $(1-x)\Lambda_0(dx) = \Lambda_1(dx)$. □

Proof of Corollary 4.4. Since $C^{-1}(C(t)) = t$,

$$\mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_t(dx_1)...R_t(dx_p)\right] = \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_{C^{-1}(C(t))}(dx_1)...R_{C^{-1}(C(t))}(dx_p)\right].$$

Then, using the independence between $R_{C^{-1}}$ and C, the right hand side above is also equal to:

$$\int \mathbb{P}(C(t) \in ds)\, \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p)R_{C^{-1}(s)}(dx_1)...R_{C^{-1}(s)}(dx_p)\right].$$

Using Corollary 4.3 and choosing $(\Pi(t), t \geq 0)$ independent of $(C(t), t \geq 0)$, we find:

$$\int \mathbb{P}(C(t) \in ds)\, \mathbb{E}\left[\int_{[0,1]^p} f(x_1, ..., x_p) R_{C^{-1}(s)}(dx_1)...R_{C^{-1}(s)}(dx_p)\right]$$
$$= \int \mathbb{P}(C(t) \in ds)\, \mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(s)}(1)}, ..., x_{\alpha_{\Pi(s)}(p)})\delta_0(dx_0)dx_1...dx_p\right]$$
$$= \mathbb{E}\left[\int_{[0,1]^{p+1}} f(x_{\alpha_{\Pi(C(t))}(1)}, ..., x_{\alpha_{\Pi(C(t))}(p)})\delta_0(dx_0)dx_1...dx_p\right].$$

□

Remark 4.6. *Notice the crucial rôle of the independence in order to establish Corollary 4.4. When this property fails, as in the case (ii), the question of describing the fixed time genealogy of the α-stable CB or CBI remains open. We refer to the discussion in Section 2.2 of Berestycki et. al [1].*

5 Proof of Theorem 3.3 and Proposition 3.5

We first deal with Theorem 3.3. The proof of Proposition 3.5 is rather technical and is postponed at the end of this Section. In order to get the connection between the two measure-valued processes $(R_t, t \geq 0)$ and $(M_t, t \geq 0)$, we may follow the ideas of Birkner *et al.* [6] and rewrite the generator of the process $(M_t, t \geq 0)$ using the "polar coordinates": for any $\eta \in \mathcal{M}_f$, we define

$$z := |\eta| \text{ and } \rho := \frac{\eta}{|\eta|}.$$

This convention will be used throughout the rest of this section. The proof relies on five lemmas. Lemma 5.1 establishes that the law of a generalized Fleming-Viot process with immigration is entirely determined by the generator $\mathcal{F}$ on the test functions of the form $\rho \mapsto \langle \phi, \rho \rangle^m$ with ϕ a measurable non-negative bounded map and $m \in \mathbb{N}$. Lemmas 5.2, 5.3 and 5.5 allow us to study the generator $\mathcal{L}$ on the class of functions of the type $F : \eta \mapsto \frac{1}{|\eta|^m}\langle \phi, \eta \rangle^m$. Lemma 5.4 (lifted from Lemma 3.5 of [6]) relates stable Lévy-measures and Beta-measures. We end the proof using results on time change by the inverse of an additive functional. We conclude thanks to a result due to Volkonskiĭ in [26] about the generator of a time-changed process. Recall the notation G_f and $\mathcal{F}G_f$ from Section 2.2.

Lemma 5.1. *The following martingale problem is well-posed: for any function f of the form:*

$$(x_1, ..., x_p) \mapsto \prod_{i=1}^{p} \phi(x_i)$$

with ϕ a non-negative measurable bounded map and $p \geq 1$, the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale.

Proof. Only the uniqueness has to be checked. We shall establish that the martingale problem of the statement is equivalent to the following martingale problem: for any continuous function f on $[0,1]^p$, the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale. This martingale problem is well posed, see Proposition 5.2 of [15]. Notice that we can focus on continuous and symmetric functions since for any continuous f, $G_f = G_{\tilde{f}}$ with $\tilde{f}$ the symmetrized version of f. Moreover, by the Stone-Weierstrass theorem, any symmetric continuous function f from $[0,1]^p$ to $\mathbb{R}$ can be uniformly approximated by linear combination of functions of the form $(x_1, ..., x_p) \mapsto \prod_{i=1}^p \phi(x_i)$ for some function ϕ continuous on $[0,1]$. We now take f symmetric and continuous, and let f_k be an approximating sequence. Plainly, we have

$$|G_{f_k}(\rho) - G_f(\rho)| \leq ||f_k - f||_\infty$$

Assume that $(\rho_t, t \geq 0)$ is a solution of the martingale problem stated in the lemma. Since the map $h \mapsto G_h$ is linear, the process

$$G_{f_k}(\rho_t) - \int_0^t \mathcal{F}G_{f_k}(\rho_s)ds$$

is a martingale for each $k \geq 1$. We want to prove that the process

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds$$

is a martingale, knowing it holds for each f_k. We will show the following convergence

$$\mathcal{F}G_{f_k}(\rho) \underset{k\to\infty}{\longrightarrow} \mathcal{F}G_f(\rho) \text{ uniformly in } \rho.$$

Recall expressions (2.1') and (2.2') in Subsection 2.2, one can check that the following limits are uniform in the variable ρ

$$\sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f_k(\mathbf{x}^{i,j}) - f_k(\mathbf{x})]\rho^{\otimes p}(d\mathbf{x}) \underset{k\to\infty}{\longrightarrow} \sum_{1\leq i<j\leq p} \int_{[0,1]^p} [f(\mathbf{x}^{i,j}) - f(\mathbf{x})]\rho^{\otimes p}(d\mathbf{x})$$

and

$$\sum_{1\leq i\leq m} \int_{[0,1]^p} [f_k(\mathbf{x}^{0,i}) - f_k(\mathbf{x})]\rho^{\otimes p}(d\mathbf{x}) \underset{k\to\infty}{\longrightarrow} \sum_{1\leq i\leq p} \int_{[0,1]^p} [f(\mathbf{x}^{0,i}) - f(\mathbf{x})]\rho^{\otimes p}(d\mathbf{x}).$$

We have now to deal with the terms (2.3') and (2.4'). In order to get that the quantity

$$\int_0^1 \nu(dr) \int_0^1 [G_{f_k}((1-r)\rho + r\delta_a) - G_{f_k}(\rho)]\rho(da)$$

converges toward

$$\int_0^1 \nu(dr) \int_0^1 [G_f((1-r)\rho + r\delta_a) - G_f(\rho)]\rho(da),$$

we compute

$$\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle.$$

Since the function $f_k - f$ is symmetric, we may expand the p-fold product $\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle$, this yields

$$\begin{aligned}
&\langle f_k - f, ((1-r)\rho + r\delta_a)^{\otimes p}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle \\
&\qquad = \sum_{i=0}^p \binom{p}{i} r^i (1-r)^{p-i} \left(\langle f_k - f, \rho^{\otimes p-i} \otimes \delta_a^{\otimes i}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle\right) \\
&\qquad = pr(1-r)^{p-1} \left(\langle f_k - f, \rho^{\otimes p-1} \otimes \delta_a\rangle - \langle f_k - f, \rho^{\otimes p}\rangle\right) \\
&\qquad\quad + \sum_{i=2}^p \binom{p}{i} r^i (1-r)^{p-i} \left(\langle f_k - f, \rho^{\otimes p-i} \otimes \delta_a^{\otimes i}\rangle - \langle f_k - f, \rho^{\otimes p}\rangle\right).
\end{aligned}$$

We use here the notation

$$\langle g, \mu^{\otimes m-i} \otimes \delta_a^{\otimes i} \rangle := \int g(x_1, ..., x_{m-i}, \underbrace{a, ..., a}_{i \text{ terms}}) \mu(dx_1) ... \mu(dx_{m-i}).$$

Therefore, integrating with respect to ρ, the first term in the last equality vanishes and we get

$$\left| \int_0^1 \rho(da) \left(G_{f-f_k}((1-r)\rho + r\delta_a) - G_{f-f_k}(\rho) \right) \right| \leq 2^{p+1} ||f - f_k||_\infty r^2$$

where $||f_k - f||_\infty$ denotes the supremum of the function $|f_k - f|$. Recall that the measure ν_1 verifies $\int_0^1 r^2 \nu_1(dr) < \infty$, moreover the quantity $||f_k - f||_\infty$ is bounded. Thus appealing to the Lebesgue Theorem, we get the sought-after convergence. Same arguments hold for the immigration part (2.4') of the operator $\mathcal{F}$. Namely we have

$$|G_{f-f_k}((1-r)\rho + r\delta_0) - G_{f-f_k}(\rho)| \leq 2^{p+1} r ||f_k - f||_\infty$$

and the measure ν_0 satisfies $\int_0^1 r\nu_0(dr) < \infty$. Combining our results, we obtain

$$|\mathcal{F}G_{f_k}(\rho) - \mathcal{F}G_f(\rho)| \leq C||f - f_k||_\infty$$

for a positive constant C independent of ρ. Therefore the sequence of martingales $G_{f_k}(\rho_t) - \int_0^t \mathcal{F}G_{f_k}(\rho_s)ds$ converges toward

$$G_f(\rho_t) - \int_0^t \mathcal{F}G_f(\rho_s)ds,$$

which is then a martingale. □

Lemma 5.2. *Assume that $\hat{\nu_0} = \hat{\nu_1} = 0$ the generator $\mathcal{L}$ of $(M_t, t \geq 0)$ is reduced to the expressions (2.1) and (2.2):*

$$\mathcal{L}F(\eta) = \sigma^2/2 \int_0^1 \int_0^1 \eta(da)\delta_a(db)F''(\eta; a, b) + \beta F'(\eta; 0)$$

Let ϕ be a measurable bounded function on $[0, 1]$ and F be the map $\eta \mapsto G_f(\rho) := \langle f, \rho^{\otimes m} \rangle$ with $f(x_1, ..., x_p) = \prod_{i=1}^p \phi(x_i)$. We have the following identity

$$|\eta|\mathcal{L}F(\eta) = \mathcal{F}G_f(\rho),$$

for $\eta \neq 0$, where $\mathcal{F}$ is the generator of a Fleming-Viot process with immigration with reproduction rate $c_1 = \sigma^2$ and immigration rate $c_0 = \beta$, see expressions (2.1') and (2.2').

Proof. By the calculations in Section 4.3 of Etheridge [12] (but in a non-spatial setting, see also the proof of Theorem 2.1 p. 249 of Shiga [25]), we get:

$$\begin{aligned} \frac{\sigma^2}{2} \int_0^1 \int_0^1 \eta(da)\delta_a(db)F''(\eta; a, b) &= |\eta|^{-1} \frac{\sigma^2}{2} \int_0^1 \int_0^1 \frac{\partial^2 G_f}{\partial\rho(a)\partial\rho(b)}(\rho)[\delta_a(db) - \rho(db)]\rho(da) \\ &= |\eta|^{-1}\sigma^2 \sum_{1 \leq i < j \leq m} \int_{[0,1]^p} [f(\mathbf{x}^{i,j}) - f(\mathbf{x})]\rho^{\otimes m}(d\mathbf{x}). \end{aligned}$$

We focus now on the immigration part. We take f a function of the form $f:(x_1,...,x_m)\mapsto \prod_{i=1}^m \phi(x_i)$ for some function ϕ, and consider $F(\eta):=G_f(\rho)=\langle f,\rho^{\otimes m}\rangle$. We may compute:

$$\begin{aligned} F(\eta+h\delta_a)-F(\eta) &= \left\langle \phi, \frac{\eta+h\delta_a}{z+h}\right\rangle^m - \langle\phi,\rho\rangle^m \\ &= \sum_{j=2}^m \binom{m}{j}\left(\frac{z}{z+h}\right)^{m-j}\left(\frac{h}{z+h}\right)^j [\langle\phi,\rho\rangle^{m-j}\phi(a)^j-\langle\phi,\rho\rangle^m] \quad (5.1)\\ &\quad + m\left(\frac{z}{z+h}\right)^{m-1}\left(\frac{h}{z+h}\right)[\langle\phi,\rho\rangle^{m-1}\phi(a)-\langle\phi,\rho\rangle^m]\,. \quad (5.2)\end{aligned}$$

We get that:

$$F'(\eta;a)=\frac{m}{z}\left[\phi(a)\langle\phi,\rho\rangle^{m-1}-\langle\phi,\rho\rangle^m\right].$$

Thus,

$$F'(\eta;0)=|\eta|^{-1}\sum_{1\le i\le m}\int_{[0,1]^p}[f(\mathbf{x}^{0,i})-f(\mathbf{x})]\rho^{\otimes m}(d\mathbf{x})$$

and

$$\int F'(\eta;a)\eta(da)=0 \quad (5.3)$$

for such function f. This proves the Lemma. □

This first lemma will allow us to prove the case (i) of Theorem 3.3. We now focus on the case (ii). Assuming that $\sigma^2=\beta=0$, the generator of $(M_t,t\ge 0)$ reduces to

$$\mathcal{L}F(\eta)=\mathcal{L}_0F(\eta)+\mathcal{L}_1F(\eta) \quad (5.4)$$

where, as in equations (2.3) and (2.4) of Subsection 2.1,

$$\begin{aligned}\mathcal{L}_0F(\eta)&=\int_0^\infty \hat{\nu_0}(dh)[F(\eta+h\delta_0)-F(\eta)]\\ \mathcal{L}_1F(\eta)&=\int_0^1\eta(da)\int_0^\infty\hat{\nu_1}(dh)[F(\eta+h\delta_a)-F(\eta)-hF'(\eta,a)].\end{aligned}$$

The following lemma is a first step to understand the infinitesimal evolution of the non-markovian process $(R_t,t\ge 0)$ in the purely discontinuous case.

Lemma 5.3. *Let f be a continuous function on $[0,1]^p$ of the form $f(x_1,...,x_p)=\prod_{i=1}^p\phi(x_i)$ and F be the map $\eta\mapsto G_f(\rho)=\langle\phi,\rho\rangle^p$. Recall the notation $\rho:=\eta/|\eta|$ and $z=|\eta|$. We have the identities:*

$$\begin{aligned}\mathcal{L}_0F(\eta)&=\int_0^\infty\hat{\nu_0}(dh)\left[G_f\left([1-\frac{h}{z+h}]\rho+\frac{h}{z+h}\delta_0\right)-G_f(\rho)\right]\\ \mathcal{L}_1F(\eta)&=z\int_0^\infty\hat{\nu_1}(dh)\int_0^1\rho(da)\left[G_f\left([1-\frac{h}{z+h}]\rho+\frac{h}{z+h}\delta_a\right)-G_f(\rho)\right].\end{aligned}$$

Proof. The identity for $\mathcal{L}_0$ is plain, we thus focus on $\mathcal{L}_1$. Combining Equation (13) and the term (5.2) we get

$$\int_0^1\rho(da)\left[m\left(\frac{z}{z+h}\right)^{m-1}\left(\frac{h}{z+h}\right)[\langle\phi,\rho\rangle^{m-1}\phi(a)-\langle\phi,\rho\rangle^m]-hF'(\eta;a)\right]=0.$$

We easily check from the terms of (5.1) that the map $h \mapsto \int_0^1 \rho(da)[F(\eta + h\delta_a) - F(\eta) - hF'(\eta, a)]$ is integrable with respect to the measure $\hat{\nu}_1$. This allows us to interchange the integrals and yields:

$$\mathcal{L}_1 F(\eta) = z \int_0^\infty \hat{\nu_1}(dh) \int_0^1 \rho(da) \left[G_f \left(\frac{\eta + h\delta_a}{z+h} \right) - G_f(\rho) \right]. \tag{5.5}$$

□

The previous lemma leads us to study the images of the measures $\hat{\nu}_0$ and $\hat{\nu}_1$ by the map $\phi_z : h \mapsto r := \frac{h}{h+z}$, for every $z > 0$. Denote $\lambda_z^0(dr) = \hat{\nu}_0 \circ \phi_z^{-1}$ and $\lambda_z^1(dr) = \hat{\nu}_1 \circ \phi_z^{-1}$. The following lemma is lifted from Lemma 3.5 of [6].

Lemma 5.4. *There exist two measures ν_0, ν_1 such that $\lambda_z^0(dr) = s_0(z)\nu_0(dr)$ and $\lambda_z^1(dr) = s_1(z)\nu_1(dr)$ for some maps s_0, s_1 from $\mathbb{R}_+$ to $\mathbb{R}$ if and only if for some $\alpha \in (0,2), \alpha' \in (0,1)$ and $c, c' > 0$:*

$$\hat{\nu_1}(dx) = cx^{-1-\alpha}dx, \ \hat{\nu_0}(dx) = c'x^{-1-\alpha'}dx.$$

In this case, assuming $s_0(1) = s_1(1) = 1$, we have:

$$s_1(z) = z^{-\alpha}, \ \nu_1(dr) = r^{-2}cBeta(2-\alpha, \alpha)(dr)$$

and

$$s_0(z) = z^{-\alpha'}, \ \nu_0(dr) = r^{-1}c'Beta(1-\alpha', \alpha')(dr).$$

Proof. The necessary part is given by the same arguments as in Lemma 3.5 of [6]. We focus on the sufficient part. Assuming that $\hat{\nu}_0, \hat{\nu}_1$ are as above, we have

- $\lambda_z^1(dr) = cz^{-\alpha}r^{-1-\alpha}(1-r)^{-1+\alpha}dr = z^{-\alpha}r^{-2}cBeta(2-\alpha, \alpha)(dr)$, and thus $s_1(z) = z^{-\alpha}$.
- $\lambda_z^0(dr) = c'z^{-\alpha'}r^{-1-\alpha'}(1-r)^{-1+\alpha'}dr = z^{-\alpha'}r^{-1}c'Beta(1-\alpha', \alpha')(dr)$ and thus $s_0(z) = z^{-\alpha'}$.

□

The next lemma allows us to deal with the second statement of Theorem 3.3.

Lemma 5.5. *Assume that $\sigma^2 = \beta = 0$, $\hat{\nu}_0(dh) = c'h^{-\alpha}1_{h>0}dh$ and $\hat{\nu}_1(dh) = ch^{-1-\alpha}1_{h>0}dh$. Let f be a function on $[0,1]^p$ of the form $f(x_1, ..., x_p) = \prod_{i=1}^p \phi(x_i)$, and F be the map $\eta \mapsto G_f(\rho)$. We have*

$$|\eta|^{\alpha-1}\mathcal{L}F(\eta) = \mathcal{F}G_f(\rho),$$

for $\eta \neq 0$, where $\mathcal{F}$ is the generator of a M-Fleming-Viot process with immigration, with $M = (c'Beta(2-\alpha, \alpha-1), cBeta(2-\alpha, \alpha))$, see expressions $(2.3')$, $(2.4')$.

Proof. Recall Equation (5.4):

$$\mathcal{L}F(\eta) = \mathcal{L}_0 F(\eta) + \mathcal{L}_1 F(\eta)$$

Recall from Equation (13) that we have $\int_0^1 F'(\eta; a)\eta(da) = 0$ for $F(\eta) = G_f(\rho)$. Applying Lemma 5.3 and Lemma 5.4, we get that in the case $\sigma^2 = \beta = 0$ and $\hat{\nu_1}(dx) = cx^{-1-\alpha}dx, \hat{\nu_0}(dx) = c'x^{-1-\alpha'}dx$:

$$\begin{aligned}\mathcal{L}F(\eta) = \mathcal{L}G_f(\rho) = \ & s_0(z) \int_0^1 r^{-1}c'Beta(1-\alpha', \alpha')(dr)[G_f((1-r)\rho + r\delta_0) - G_f(\rho)] \\ & + zs_1(z) \int_0^1 r^{-2}cBeta(2-\alpha, \alpha)(dr) \int_0^1 \rho(da)[G_f((1-r)\rho + r\delta_a) - G_f(\rho)].\end{aligned}$$

Recalling the expressions (2.3'), (2.4'), the factorization $h(z)\mathcal{L}F(\eta) = \mathcal{F}G(\rho)$ holds for some function h if

$$s_0(z) = zs_1(z),$$

if $\alpha' = \alpha - 1$. In that case, $h(z) = z^{\alpha-1}$. □

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. To treat the case (i), replace α by 2 in the sequel. The process $(Y_t, R_t)_{t\geq 0}$ with lifetime τ has the Markov property. The additive functional $C(t) = \int_0^t \frac{1}{Y_s^{\alpha-1}} ds$ maps $[0,\tau)$ to $[0,\infty)$. From Theorem 65.9 of [24] and Proposition 3.2, the process $(Y_{C^{-1}(t)}, R_{C^{-1}(t)})_{t\geq 0}$ is a strong Markov process with infinite lifetime. Denote by $\mathcal{U}$ the generator of $(Y_t, R_t)_{t\geq 0}$. As explained in Birkner et al. [6] (Equation (2.6) p. 314), the law of $(Y_t, R_t)_{t\geq 0}$ is characterized by $\mathcal{U}$ acting on the following class of test functions:

$$(z,\rho) \in \mathbb{R}_+ \times \mathcal{M}_1 \mapsto F(z,\rho) := \psi(z)\langle \phi, \rho\rangle^m$$

for ϕ a non-negative measurable bounded function on $[0,1]$, $m \geq 1$ and ψ a twice differentiable non-negative map. Theorem 3 of Volkonskiĭ, see [26] (or Theorem 1.4 Chapter 6 of [13]) states that the Markov process with generator

$$\tilde{\mathcal{U}}F(z,\rho) := z^{\alpha-1}\mathcal{U}F(z,\rho)$$

coincides with $(Y_{C^{-1}(t)}, R_{C^{-1}(t)})_{t\geq 0}$. We establish now that $(R_{C^{-1}(t)}, t \geq 0)$ is a Markov process with the same generator as the Fleming-Viot processes involved in Theorem 3.3. Let $G(z,\rho) = G_f(\rho) = \langle \phi, \rho\rangle^m$ (taking $f : (x_1, ..., x_m) \mapsto \prod_{i=1}^m \phi(x_i)$). In both cases (i) and (ii) of Theorem 3.3, we have:

$$\begin{aligned} z^{\alpha-1}\mathcal{U}G(z,\rho) &= z^{\alpha-1}\mathcal{L}F(\eta) \text{ with } F : \eta \mapsto G_f(\rho) \\ &= \mathcal{F}G_f(\rho). \end{aligned}$$

First equality holds since we took $\psi \equiv 1$ and the second uses Lemma 5.2 and Lemma 5.5. Since it does not depend on z, the process $(R_{C^{-1}(t)}, t \geq 0)$ is a Markov process, moreover it is a generalized Fleming-Viot process with immigration with parameters as stated. □

Proof of Proposition 3.5. Let $(Y_t)_{t\geq 0}$ be a Feller branching diffusion with continuous immigration with parameters (σ^2, β). Consider an independent M-Fleming-Viot $(\rho_t, t \geq 0)$ with $M = (\beta\delta_0, \sigma^2\delta_0)$. We first establish that $(Y_t\rho_{C(t)}, 0 \leq t < \tau)$ has the same law as the measure-valued branching process $(M_t, 0 \leq t < \tau)$. Recall that $\mathcal{L}$ denotes the generator of $(M_t, t \geq 0)$ (here only the terms (2.1) and (2.2) are considered). Consider $F(\eta) := \psi(z)\langle \phi, \rho\rangle^m$ with $z = |\eta|$, ψ a twice differentiable map valued in $\mathbb{R}_+$ and ϕ a non-negative bounded measurable function. Note that the generator acting on such functions F characterizes the law of $(M_{t\wedge\tau}, t \geq 0)$. First we easily obtain that

$$\begin{aligned} F'(\eta; 0) &= \psi'(z)\langle \phi, \rho\rangle^m + m\frac{\psi(z)}{z}[\phi(0)\langle \phi, \rho\rangle^{m-1} - \langle \phi, \rho\rangle^m], \\ F''(\eta; a, b) &= \psi''(z)\langle \phi, \rho\rangle^m + m\frac{\psi'(z)}{z}\left[(\phi(b) + \phi(a))\langle \phi, \rho\rangle^{m-1} - 2\langle \phi, \rho\rangle^m\right] \\ &+ m\frac{\psi(z)}{z^2}\left[(m-1)\phi(a)\phi(b)\langle \phi, \rho\rangle^{m-2} - m(\phi(a) + \phi(b))\langle \phi, \rho\rangle^{m-1} + (m+1)\langle \phi, \rho\rangle^m\right]. \end{aligned}$$

Simple calculations yield,

$$\mathcal{L}F(\eta) = \left[z\left(\frac{\sigma^2}{2}\psi''(z)\right) + \beta\psi'(z)\right]\langle\phi,\rho\rangle^m$$
$$+ \frac{\psi(z)}{z}\left[\sigma^2\frac{m(m-1)}{2}\left(\langle\phi^2,\rho\rangle\langle\phi,\rho\rangle^{m-2} - \langle\phi,\rho\rangle^m\right) + \beta m\left(\phi(0)\langle\phi,\rho\rangle^{m-1} - \langle\phi,\rho\rangle^m\right)\right].$$

We recognize in the first line the generator of $(Y_t, t \geq 0)$ and in the second, $\frac{1}{z}\mathcal{F}G_f(\rho)$ with $f(x_1, ..., x_m) = \prod_{i=1}^m \phi(x_i)$ and $c_0 = \beta, c_1 = \sigma^2$. We easily get that this is the generator of the Markov process $(Y_t\rho_{C(t)}, t \geq 0)$ with lifetime τ. We conclude that it has the same law as $(M_{t\wedge\tau}, t \geq 0)$. We rewrite this equality in law as follows:

$$(Y_t\ \rho_{C(t)}, 0 \leq t < \tau) \overset{law}{=} (|M_t|\ R_{C^{-1}(C(t))}, 0 \leq t < \tau), \tag{5.6}$$

with C defined by $C(t) = \int_0^t |M_s|^{-1}ds$ for $0 \leq t < \tau$ on the right hand side. Since $(C(t), t \geq 0)$ and $(\rho_t, t \geq 0)$ are independent on the left hand side and the decomposition in (5.6) is unique, we have also $(C(t), 0 \leq t < \tau)$ and $(R_{C^{-1}(t)}, 0 \leq t < \tau)$ independent on the right hand side.

Concerning the case (ii) of Theorem 3.3, we easily observe that the presence of jumps implies that such a decomposition of the generator cannot hold. See for instance Equation (2.7) of [6] p344. The processes $(R_{C^{-1}(t)}, t \geq 0)$ and $(Y_t, \geq 0)$ are not independent. □

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References

[1] J. Berestycki and N. Berestycki. Kingman's coalescent and Brownian motion. *ALEA Lat. Am. J. Probab. Math. Stat.*, 6:239–259, 2009. MR-2534485

[2] N. Berestycki. *Recent progress in coalescent theory*, volume 16 of *Ensaios Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2009. MR-2574323

[3] J. Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006. MR-2253162

[4] J. Bertoin and J.-F. Le Gall. Stochastic flows associated to coalescent processes. *Probab. Theory Related Fields*, 126(2):261–288, 2003. MR-1990057

[5] J. Bertoin and J.-F. Le Gall. Stochastic flows associated to coalescent processes. III. Limit theorems. *Illinois J. Math.*, 50(1-4):147–181 (electronic), 2006. MR-2247827

[6] M. Birkner, J. Blath, M. Capaldo, A. Etheridge, M. Möhle, J. Schweinsberg, and A. Wakolbinger. Alpha-stable branching and beta-coalescents. *Electron. J. Probab.*, 10:no. 9, 303–325 (electronic), 2005. MR-2120246

[7] W. Chu and Y.-X. Ren. N-measure for continuous state branching processes and its application. *Frontiers of Mathematics in China*, 6:1045–1058, 2011. 10.1007/s11464-011-0122-0. MR-2862645

[8] D. A. Dawson. Measure-valued Markov processes. In *École d'Été de Probabilités de Saint-Flour XXI—1991*, volume 1541 of *Lecture Notes in Math.*, pages 1–260. Springer, Berlin, 1993. MR-1242575

[9] D. A. Dawson and Z. Li. Stochastic equations, flows and measure-valued processes. *Ann. Probab.*, 40(2):813–857, 2012. MR-2952093

[10] P. Donnelly and T. G. Kurtz. Particle representations for measure-valued population models. *Ann. Probab.*, 27(1):166–205, 1999. MR-1681126

[11] T. Duquesne. Continuum random trees and branching processes with immigration. *Stochastic Process. Appl.*, 119(1):99–129, 2009. MR-2485021

[12] A. M. Etheridge. *An introduction to superprocesses*, volume 20 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2000. MR-1779100

[13] S. N. Ethier and T. G. Kurtz. *Markov processes: Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. MR-0838085

[14] P. J. Fitzsimmons, B. Fristedt, and L. A. Shepp. The set of real numbers left uncovered by random covering intervals. *Z. Wahrsch. Verw. Gebiete*, 70(2):175–189, 1985. MR-0799145

[15] C. Foucart. Distinguished exchangeable coalescents and generalized Fleming-Viot processes with immigration. *Adv. in Appl. Probab.*, 43(2):348–374, 2011. MR-2848380

[16] C. Foucart. Generalized Fleming-Viot processes with immigration via stochastic flows of partitions. *ALEA Lat. Am. J. Probab. Math. Stat.*, 9:451–472, 2012.

[17] K. Handa. Stationary distributions for a class of generalized Fleming-Viot processes. To appear in *Ann. Probab.*, preprint May 2012.

[18] K. Kawazu and S. Watanabe. Branching processes with immigration and related limit theorems. *Teor. Verojatnost. i Primenen.*, 16:34–51, 1971. MR-0290475

[19] A. Kyprianou and J. Pardo. Continuous-state branching processes and self-similarity. *Journal of Applied Probability*, 45(4):1140–1160, 2008. MR-2484167

[20] A. Lambert. The genealogy of continuous-state branching processes with immigration. *Probab. Theory Related Fields*, 122(1):42–70, 2002. MR-1883717

[21] A. Lambert. Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. *Electron. J. Probab.*, 12:no. 14, 420–446, 2007. MR-2299923

[22] Z. H. Li. *Measure-Valued Branching Markov Processes*. Springer, 2011. MR-2760602

[23] E. A. Perkins. Conditional Dawson-Watanabe processes and Fleming-Viot processes. In *Seminar on Stochastic Processes, 1991 (Los Angeles, CA, 1991)*, volume 29 of *Progr. Probab.*, pages 143–156. Birkhäuser Boston, Boston, MA, 1992. MR-1172149

[24] M. Sharpe. *General theory of Markov processes*, volume 133 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988. MR-0958914

[25] T. Shiga. A stochastic equation based on a Poisson system for a class of measure-valued diffusion processes. *J. Math. Kyoto Univ.*, 30(2):245–279, 1990. MR-1068791

[26] V. A. Volkonskiĭ. Random substitution of time in strong Markov processes. *Teor. Veroyatnost. i Primenen*, 3:332–350, 1958. MR-0100919