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Central limit theorem for $\mathbb{Z}_+^d$-actions by toral endomorphisms

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Abstract

In this paper we prove the central limit theorem for the following multisequence

$$\sum_{n_1=1}^{N_1} ... \sum_{n_d=1}^{N_d} f(A_1^{n_1}...A_d^{n_d}\mathbf{x})$$

where f is a Hölder's continue function, $A_1, ..., A_d$ are $s \times s$ partially hyperbolic commuting integer matrices, and $\mathbf{x}$ is a uniformly distributed random variable in $[0,1]^s$. Then we prove the functional central limit theorem, and the almost sure central limit theorem. The main tool is the S-unit theorem.

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1 Introduction.

In [F], [K], Fortet and Kac proved the central limit theorem (abbreviated CLT) for the sum $\sum_{n=0}^{N-1} f(q^n x)$ where $q \geq 2$ is an integer, $x \in [0,1)$ and f is 1-periodic function. Let $(\omega_{q_1,...,q_d}(n))_{n\geq 1}$ be a so-called Hardy-Littlewood-Pólya sequence, i.e. let $(\omega_{q_1,...,q_d}(n))_{n\geq 1}$ consist of the elements of the multiplicative semigroup generated by a finite set $(q_1, ..., q_d)$ of coprime integers, arranged in increasing order. In [P], [FP], Philipp, Fukuyama and Petit obtained limit theorems for the sum $\sum_{n=0}^{N-1} f(\omega_{q_1,...,q_d}(n)x)$. In this paper, we prove some limit theorems for the sum $\sum_{n_1=0}^{N_1-1} ... \sum_{n_d=0}^{N_d-1} f(q_1^{n_1}...q_d^{n_d}x)$ as $N_1, ..., N_d \to \infty$, where $q_1, ..., q_d$ may be not coprime integers (see Theorem 5).

In [L1], [L2], Leonov proved CLT for endomorphisms of s-torus and Hölder's continuous functions (see also [LB]). In this paper, we extend Leonov's result to the case of $\mathbb{Z}_+^d$-actions by endomorphisms of s-torus (this result were announced in [Le1], [Le2]). Note that mixing properties of $\mathbb{Z}^d$-actions by commuting automorphisms of s-torus was investigated earlier by Schmidt and Ward [ScWa].

Let us describe the structure of the paper. In §2 we fix some definitions and present our results. In §3 we examine questions of normalizations (determination of the variance

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limit). In §4 we obtain growth estimates from above and from below for the multisequence $(|A_1^{n_1}...A_d^{n_d}\mathbf{m}|)_{n_i\in\mathbb{Z}, i=1,...,d}$. In §5 we prove a multidimensional CLT, a functional CLT and an almost sure CLT.

2 Notations and results.

Let A be an invertible $s\times s$ matrix with integer entries. It generates a surjective endomorphism on the s-dimensional torus $[0,1)^s$ which we will denote by the same letter A. The dual endomorphism $A^*: \mathbb{Z}^s \to \mathbb{Z}^s$ is given by the transpose matrix $A^{(t)}$. It induces a dual map on the characters:

$$e(\langle \mathbf{m}, \mathbf{x}\rangle) \quad \text{to} \quad e(\langle A\mathbf{m}, \mathbf{x}\rangle),$$

where $e(x) = \exp(2\pi\sqrt{-1}x)$, and $\langle \mathbf{m}, \mathbf{x}\rangle = m_1x_1 + ... + m_sx_s$. Let f be a $\mathbb{Z}^s$-periodic local integrable real function. In terms of Fourier coefficients, A sends

$$f \sim \sum_{\mathbf{m}\in\mathbb{Z}^s} \widehat{f}(\mathbf{m})e(\langle \mathbf{m}, \mathbf{x}\rangle) \quad \text{to} \quad f\circ A \sim \sum_{\mathbf{m}\in\mathbb{Z}^s} \widehat{f\circ A}(\mathbf{m})e(\langle \mathbf{m}, \mathbf{x}\rangle), \tag{2.1}$$

where

$$\widehat{f\circ A}(\mathbf{m}) = \begin{cases} \widehat{f}(\widetilde{\mathbf{m}}), & \text{if } \mathbf{m} = A^{(t)}\widetilde{\mathbf{m}} \text{ for some } \widetilde{\mathbf{m}} \in \mathbb{Z}^s, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Throughout this paper $\widehat{f}(\mathbf{y}) = 0$ for $\mathbf{y} \notin \mathbb{Z}^s$. To simplify the notation in the rest of the paper, whenever there is no confusion as to which map we refer to, we will denote the dual map by the same symbol A. Also we will denote the transposed matrices $A^{(t)}$, $\mathbf{m}^{(t)}$ by the symbols A and $\mathbf{m}$.

Definition 1. *An action $\mathcal{A}$ by surjectives endomorphisms $A_1, ..., A_d$ of $[0,1)^s$ is called* partially hyperbolic *if for all $(n_1, ..., n_d) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ none of the eigenvalues of the matrix $A_1^{n_1}...A_d^{n_d}$ are roots of unity.*

Examples of partially hyperbolic actions :

1. Let $\mathbb{I}$ be the $s\times s$ identity matrix, $q_1, ..., q_d \geq 2$ pairwise coprime integers, $A_i = q_i\mathbb{I}$, $i = 1, ..., d$.

2. Let K be an algebraic number field of degree s, $\eta_1, ..., \eta_d$ $(d \leq s-1)$ a set of fundamental units of K, $\phi_i(x)$ the minimal polynomial of η_i, and A_i the companion matrix of $\phi_i(x)$ $(1 \leq i \leq d)$.

Denote

$$\mathbf{m} \lessdot \mathbf{m}' \quad \text{if} \quad |\mathbf{m}| < |\mathbf{m}'|, \quad \text{or if} \quad |\mathbf{m}| = |\mathbf{m}'| \tag{2.3}$$

and there exists $k \in [0, s)$ with $m_1 = m_1', ..., m_k = m_k'$ and $m_{k+1} < m_{k+1}'$, where $|\mathbf{m}| = (m_1^2 + ... + m_s^2)^{1/2}$.

Let

$$B(\mathbf{m}) = \{\widetilde{\mathbf{m}} \in \mathbb{Z}^s \setminus \mathbf{0} \mid \exists \mathbf{n} = (n_1, ..., n_d) \in \mathbb{Z}^d \quad \text{with} \quad \widetilde{\mathbf{m}} = A_1^{n_1}...A_d^{n_d}\mathbf{m}\}, \tag{2.4}$$

$$W = \{\mathbf{m} \in \mathbb{Z}^s \setminus \mathbf{0} \mid \nexists \mathbf{m}_1 \in \mathbb{Z}^s \setminus \mathbf{0} \text{ with } B(\mathbf{m}) = B(\mathbf{m}_1) \text{ and } \mathbf{m}_1 \lessdot \mathbf{m}\}. \tag{2.5}$$

It is easy to see that

$$\bigcup_{\mathbf{m}\in W} B(\mathbf{m}) = \mathbb{Z}^s \setminus \mathbf{0}, \quad \text{and} \quad B(\mathbf{m}_1) \bigcap B(\mathbf{m}_2) = \emptyset \quad \text{for} \quad \mathbf{m}_1, \mathbf{m}_2 \in W, \ \mathbf{m}_1 \neq \mathbf{m}_2. \tag{2.6}$$

Let $\mathbb{Z}_+^d = \{\mathbf{n} \in \mathbb{Z}^d \mid n_i \geq 0, \ i = 1, ..., d\}$, $\mathbf{A}^{\mathbf{n}} = A_1^{n_1}...A_d^{n_d}$, $\|f\|_p^p = \int_{[0,1)^s} |f(\mathbf{x})|^p d\mathbf{x}$, $\mathbf{N} = (N_1, ..., N_d)$, $N_i \in \mathbb{N}$ $(i = 1, ..., d)$, $\check{\mathbf{N}} = N_1N_2\cdots N_d$, and

$$S_{\mathbf{N}}(f)) := \sum_{0\leq n_i<N_i, \ i=1,...,d} f(\mathbf{A}^{\mathbf{n}}\mathbf{x}). \tag{2.7}$$

Theorem 1. *Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1, ..., A_d$ of $[0,1)^s$, f a real $\mathbb{Z}^s$-periodic locally integrable function with mean zero with*

$$S(f) := \sum_{\mathbf{m}\in W} \Big(\sum_{\mathbf{n}\in\mathbb{Z}^d} |\widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m})| \Big)^2 < +\infty. \tag{2.8}$$

Then

$$\sigma^2(f) := \lim_{\min_i N_i\to\infty} \frac{1}{\check{\mathbf{N}}} \Big\| S_{\mathbf{N}}(f(\mathbf{x})) \Big\|_2^2 = \sum_{\mathbf{m}\in W} \Big| \sum_{\mathbf{n}\in\mathbb{Z}^d} \widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m}) \Big|^2 \tag{2.9}$$

$$= \sum_{\mathbf{n},\mathbf{n}'\in\mathbb{Z}^d_+, \mathbf{n}\cdot\mathbf{n}'=\mathbf{0}} \int_{[0,1)^s} f(\mathbf{A}^{\mathbf{n}}\mathbf{x}) f(\mathbf{A}^{\mathbf{n}'}\mathbf{x}) d\mathbf{x} < +\infty, \tag{2.10}$$

where $\mathbf{n}\cdot\mathbf{n}' = (n_1 n_1', ..., n_d n_d')$.

Let $\mathbf{u} = (u_1, ..., u_s), \mathbf{v} = (v_1, ..., v_s) \in [0,1)^s$, $u_i < v_i$, $i = 1, ..., s$, and $[\mathbf{u},\mathbf{v}) = [u_1, v_1) \times \cdots \times [u_s, v_s)$. We denote by $\mathbf{1}_{[\mathbf{u},\mathbf{v})}$ the indicator function of the box $[\mathbf{u},\mathbf{v})$. Let $f_{[\mathbf{u},\mathbf{v})}(\mathbf{x}) = \mathbf{1}_{[\mathbf{u},\mathbf{v})}(\mathbf{x}) - (v_1 - u_1)\dots(v_s - u_s)$. In the next theorem we show two examples of $f_{[\mathbf{u},\mathbf{v})}$ with $\sigma(f_{[\mathbf{u},\mathbf{v})}) > 0$:

Theorem 2. *Let $\sigma(f_{[\mathbf{u},\mathbf{v})})$ be the variance limit of $f_{[\mathbf{u},\mathbf{v})}$. Then $f_{[\mathbf{u},\mathbf{v})}$ satisfies the condition (2.8) and $\sigma(f_{[\mathbf{u},\mathbf{v})}) > 0$ for each of the following cases :*
(i) $\mathbf{u} = \mathbf{0}$ and $1, v_1, ..., v_s$ are rational independents numbers;
(ii) $1, u_1, ..., u_s, v_1, ..., v_s$ are rational independents numbers.

The third result permits to give a functional characterization of functions with variance limit zero (see also [FP, Theorem 3], and [KaNi, Theorem 6.2.2, Corollary 6.2.7]) :

Theorem 3. *Let $d \geq 2$, f be a real $\mathbb{Z}^s$-periodic function locally integrable with mean zero and*

$$\sum_{n\geq 1} n^{d-1} \|f - f_{2^n}\|_2 < +\infty, \tag{2.11}$$

where

$$f_L(\mathbf{x}) := \sum_{|m_i|<L,\ i=1,...,s} \widehat{f}(\mathbf{m}) e(\langle \mathbf{m}, \mathbf{x}\rangle). \tag{2.12}$$

Then (2.8) is true, and $\sigma(f) = 0$ if and only if there exist $f^{(1)}, ..., f^{(d)} \in L^2([0,1)^s)$ such that (2.8) is true for all g_i with $g_i(\mathbf{x}) = f^{(i)}(\mathbf{x}) - f^{(i)}(A_i\mathbf{x}),\ i = 1, ..., d$ and

$$f(\mathbf{x}) = \sum_{1\leq i\leq d} (f^{(i)}(\mathbf{x}) - f^{(i)}(A_i\mathbf{x})) \tag{2.13}$$

for almost all $\mathbf{x} \in [0,1)^s$.

It is easy to verify that the condition (2.11) of the theorem is satisfied under the following decreasing property of Fourier coefficients of f:

$$|\widehat{f}(\mathbf{m})| \leq c_0 \prod_{i=1}^{s} \frac{1}{(1+|m_i|)^{1/2}(\ln(2+|m_i|))^{\beta}} \tag{2.14}$$

with $c_0 > 0$ and $\beta > d + 0.5$.

Using the approach of ([Ah], p. 222, Theorem 1, see also [Z], p. 241, (3.3) and [Ba], p. 160, (2.6)), we get that all Hölder's continuous functions satisfy the condition (2.11).

In [Ka], A.Katok and S.Katok proved the following theorem:

Theorem A. ([Ka], Theorem 2.1, [KaNi], Theorem 6.2.12) *Let $\mathcal{A}$ be an action by commuting partially hyperbolic automorphisms of $[0,1)^s$. Then there exist constants $a_1, a_2, c_1, c_2 > 0$ depending on the action only such that for any initial point $\mathbf{m} \in \mathbb{Z}^s \setminus 0$*

$$c_1|\mathbf{m}|^{-s}\exp(a_1|\mathbf{n}|) \leq |\mathbf{A}^{\mathbf{n}}\mathbf{m}| \leq c_2|\mathbf{m}|\exp(a_2|\mathbf{n}|).$$

In this paper we extend this result to the case of endomorphisms:

Theorem 4. *Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1, ..., A_d$ of $[0,1)^s$. Then there exist constants $a_1, a_2, b_1, c_1, c_2 > 0$ depending on the action only such that for any $\mathbf{n} \in \mathbb{Z}^d$, and any initial point $\mathbf{m} \in \mathbb{Z}^s \setminus 0$ with $\mathbf{A}^{\mathbf{n}}\mathbf{m} \in \mathbb{Z}^s$*

$$c_1|\mathbf{m}|^{-b_1}\exp(a_1|\mathbf{n}|) \leq |\mathbf{A}^{\mathbf{n}}\mathbf{m}| \leq c_2|\mathbf{m}|\exp(a_2|\mathbf{n}|). \tag{2.15}$$

Let $q \geq 1, d \geq 2, N_{i,j} \geq 1, R_{i,j}$ be integers, $\mathbf{N}_i = (N_{i,1}, ..., N_{i,d})$ $(i = 1, ..., q, j = 1, ..., d)$, $\breve{\mathbf{N}}_i = N_{i,1} \cdots N_{i,d}$,

$$\mathfrak{R}_i = \mathfrak{R}_i(\mathbf{N}_i) = [R_{i,1}, R_{i,1} + N_{i,1}) \times \cdots \times [R_{i,d}, R_{i,d} + N_{i,d}). \tag{2.16}$$

Theorem 5. *Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1, ..., A_d$ of $[0,1)^s$, f a real $\mathbb{Z}^s$-periodic locally integrable function with mean zero satisfy the condition (2.8) and $\sigma(f) > 0$, $\mathbf{x}$ a uniformly distributed random variable in $[0,1]^s$, $\mathfrak{R}_i(\mathbf{N}_i) \cap \mathfrak{R}_j(\mathbf{N}_j) = \emptyset$ for $i \neq j \in [1,q]$. Then*

$$\Big(\frac{1}{\sigma(f)\sqrt{\breve{\mathbf{N}}_1}} \sum_{\mathbf{n_1} \in \mathfrak{R}_1(\mathbf{N}_1)} f(\mathbf{A}^{\mathbf{n}_1}\mathbf{x}), ..., \frac{1}{\sigma(f)\sqrt{\breve{\mathbf{N}}_q}} \sum_{\mathbf{n_q} \in \mathfrak{R}_q(\mathbf{N}_q)} f(\mathbf{A}^{\mathbf{n}_q}\mathbf{x})\Big)$$

converges in distribution to a Gaussian $\mathcal{N}(0, \mathbb{I})$-distribution, where $\mathbb{I}$ is the $q \times q$ identity matrix, as $\min_{i,j} N_{i,j} \longrightarrow \infty$.

Related questions

1. *Hardy-Littlewood-Pólya (HLP) sequence.* In [Fu], Furstenberg studied denseness properties of HLP sequence $(\omega_{2,3}(n))_{n\geq 1}$ (see Introduction) from an ergodic point of view. He also asked in [Fu] the celebrated question on ergodic properties of this sequence (see e.g. [EiWa, p.7]). In [P], Philipp proved the almost sure invariance principle (ASIP) for the sequence $(\cos(\omega_{q_1,...,q_d}(n)x))_{n\geq 1}$ and the law of the iterated logarithm (LIL) for the discrepancy of the sequence $(\{\omega_{q_1,...,q_d}(n)x\})_{n\geq 1}$ (see also [BPT]). We consider the following s-dimensional variant of HLP sequence:

Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1, ..., A_d$ of $[0,1)^s$. Denote $A_1^{n_1}...A_d^{n_d} \lessdot A_1^{\dot{n}_1}...A_d^{\dot{n}_d}$ if $(n_1, ..., n_d) \lessdot (\dot{n}_1, ..., \dot{n}_d)$ (see (2.3)). Let $(\Omega_n)_{n\geq 1}$ consist of the elements of the multiplicative semigroup generated by a finite set $(A_1, ..., A_d)$ arranged in increasing order. In a forthcoming paper, we will show that the approach of [P] and [BPT] can be applied to the proof of ASIP for the sequence $(\cos(\Omega_n\mathbf{x}))_{n\geq 1}$ (the result announced in [Le1]) and to the proof of LIL for the discrepancy of the sequence $(\{\Omega_n\mathbf{x}\})_{n\geq 1}$.

2. *Salem-Zygmund CLT on lacunary trigonometric series.* In 1948, Salem and Zygmund proved the following theorem: *Let $\lambda_n \geq 1$ be integers, $\lambda_{n+1}/\lambda_n \geq c > 1$ for $n = 1, 2, ...$, and let a_n, ϕ_n be reals, $\mathcal{A}_N = (1/2(a_1^2 + ... + a_N^2))^{1/2} \to \infty$, $\max_{1\leq n\leq N}|a_n|/\mathcal{A}_N \to$*

$0 \quad \text{as} \quad N \to \infty$ *and let* $S_N = \frac{1}{\mathcal{A}_N} \sum_{n=1}^N a_n \cos(2\pi\lambda_n x + \phi_n)$*. Then* S_N *over any set* D*,* $\mathrm{mes} D > 0$*, tends to the Gaussian distribution with mean value* 0 *and dispersion* 1 *as* $N \to \infty$ (see [Z, p. 233]).

In [PhSt], Philipp and Stout proved that if for the coefficient a_N we assume the stronger condition $a_N = O(A_N^{1-\delta})$ for some $\delta > 0$, then S_N obeys ASIP. In [Le4], we proved the following multiparameter variant of the Salem-Zygmund theorem: Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1, ..., A_d$ of $[0,1)^s$, $\mathbf{x}$ a uniformly distributed random variable in $[0,1)^s$. Let $\mathbf{m} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$, $R(\mathbf{N}) = [1, N_1] \times ... \times [1, N_d],\ N_0 = \min(N_1, ..., N_d)$, $a_{\mathbf{n}} \geq 0$, $\phi_{\mathbf{n}}$ be reals,

$$\mathcal{A}(\mathbf{N}) = (1/2 \sum_{\mathbf{n} \in R(\mathbf{N})} a_{\mathbf{n}}^2)^{1/2} \to \infty, \quad \text{and} \quad \rho(\mathbf{N}) = \max_{\mathbf{n} \in R(\mathbf{N})} a_{\mathbf{n}} / \mathcal{A}(\mathbf{N}) \stackrel{N_0 \to \infty}{\to} 0,$$

$$S_{\mathbf{N}} = \frac{1}{\mathcal{A}(\mathbf{N})} \sum_{\mathbf{n} \in R(\mathbf{N})} a_{\mathbf{n}} \cos(2\pi \langle \mathbf{m}, A_1^{n_1} ... A_d^{n_d} \mathbf{x} \rangle + \phi_{\mathbf{n}}).$$

Then $S_{\mathbf{N}}$ over any set $D \subset [0,1]^s$, $\mathrm{mes} D > 0$, tends to the Gaussian distribution with mean value 0 and dispersion 1 for $N_0 \to \infty$.

We consider the order (2.3). Let $(g_n)_{n \geq 1}$ consist of the elements of $\mathbb{Z}_+^d$ arranged in increasing order. Let

$$\dot{\mathcal{A}}(L) = (1/2 \sum_{1 \leq n \leq L} a_{g_n}^2)^{1/2}, \quad \text{and} \quad \dot{S}_L = \frac{1}{\dot{\mathcal{A}}(L)} \sum_{1 \leq n \leq L} a_{g_n} \cos(2\pi \langle \mathbf{m}, \Omega_{g_n} \mathbf{x} \rangle + \phi_{g_n}).$$

In a forthcoming paper, we will show that the approach of [PhSt] can be applied to the proof of ASIP for the sequence $(\dot{S}_L)_{L \geq 1}$ for the case $a_{\mathbf{N}} = O(\mathcal{A}(\mathbf{N}^{1-\delta})$ for some $\delta > 0$.

3. *Randomness in lattice point problems.* In 1992, Beck (see [Be]) discovered a very surprising phenomenon of randomness of the number of the lattice points $\{(n, n\sqrt{2} + m) | (n,m) \in \mathbb{Z}^2\}$ in a rectangular domain and in a hyperbolic domain. According to [Be, p.41], the generalizations of his results to the multidimensional case for a Kronecker's lattice $\{(n, n\alpha_1 + m_1, ..., n\alpha_{s-1} + m_{s-1}) \mid (n, m_1, ..., m_{s-1}) \in \mathbb{Z}^s\}$ is very difficult because of the problems connected to the Littlewood's conjecture: $\underline{\lim}_{n \to \infty}\ n \ll n\alpha \gg \ll n\beta \gg = 0$ for all reals α, β, where $\ll x \gg = \min(\{x\}, 1 - \{x\})$.

In [Le5], we consider a lattice obtained from a module in a totally real algebraic number field to avoid the mentioned problem. Let $K(r_1, r_2)$ be an algebraic number field with signature (r_1, r_2), $r_1 + 2r_2 = s$, $\Gamma = \Gamma(M, r_1, r_2) \subset \mathbb{R}^s$ a lattices obtained from a module M in $K(r_1, r_2)$, $\mathbf{N} = (N_1^{'}, ..., N_{r_1}^{'}, N_1, ..., N_{r_2}) \in \mathbb{Z}_+^{r_1 + r_2}$, $\boldsymbol{\gamma} = (\gamma_1^{'}, ..., \gamma_{r_1}^{'}, \gamma_1, ..., \gamma_{r_2}) \in \mathbb{R}^s$ $(\gamma_i^{'} \in \mathbb{R},\ \gamma_j \in \mathbb{R}^2, i = 1, ..., r_1, j = 1, ..., r_2)$, $\mathbf{y} = (y_1^{'}, ..., y_{r_1}^{'}, y_1, ..., y_{r_2})$, $V = \mathbb{R}^s / \Gamma$, $(\mathbf{y}, \mathbf{x})$ uniformly distributed random variable in $[0,1]^{r_1 + r_2} \times V$, $\mathbf{1}_G$ the indicator function of the domain G,

$$G(\mathbf{N}) = \prod_{i=1}^{r_1} [-N_i y_i, N_i y_i] \prod_{j=1}^{r_2} \{z \in \mathbb{R}^2 \mid |z| \leq N_j y_j\},$$

and let

$$\xi_1(\mathbf{N}) = \xi_{1, r_1, r_2}(\mathbf{N}) = \sum_{\boldsymbol{\gamma} \in \Gamma + \mathbf{x}} \mathbf{1}_{G(\mathbf{N})}(\boldsymbol{\gamma}), \quad \xi_2(\mathbf{N}) = \sum_{\boldsymbol{\gamma} \in \Gamma + \mathbf{x}} \mathbf{1}_{G(\mathbf{N})}(\boldsymbol{\gamma}) \prod_{j=1}^{r_2} \sqrt{N_j^2 y_j^2 - \gamma_j^2}.$$

We consider the group of units of $K(s, 0)$ and the corresponding group $(\mathbf{A}^{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^{s-1}}$ of hyperbolic automorphisms of $[0,1)^s$. In [Le5], using the Poisson summation formula, we have shown that $\xi_{1,s,0}(\mathbf{N}) = S_{\dot{\mathbf{N}}}(f))$ (see (2.7)) for some f and $\dot{\mathbf{N}}$. Applying the S-unit theorem and the approach of this paper, we have proved in [Le5] that $\xi_{1,s,0}(\mathbf{N})$ (the number of lattice points in a shifted and dilated rectangular domain) obeys CLT.

In a forthcoming paper, we will prove CLT for the multisequence $\xi_i(\mathbf{N})$, where $i = 1$ if $r_2 \geq 2$ and $i = 2$ if $r_2 = 1, r_1 \geq 1$. The case $r_2 = 1, r_1 = 0$ was investigated earlier by Hughes and Rudnick [HuRu]. Using the approach of this paper, in a forthcoming paper we will prove CLT for the number of lattice in a hyperbolic domain.

4. *Randomness of low discrepancy sequences.* Let $((\beta_n)_{n=0}^{N-1})$ be a sequence in the unit cube $[0,1)^s$. We define the *local discrepancy* of an N-point set $(\beta_n)_{n=0}^{N-1}$ as $\Delta(\mathbf{y}, (\beta_n)_{n=0}^{N-1}) = \#\{0 \leq n < N \mid \beta_n \in [0, y_1) \times \cdots \times [0, y_s)\} - N y_1 \ldots y_s$. We define the *discrepancy* of a N-point set $(\beta_n)_{n=0}^{N-1}$ as $D((\beta_n)_{n=0}^{N-1}) = \sup_{0<y_1,\ldots,y_s\leq 1} \ |\Delta(\mathbf{y}, (\beta_n)_{n=0}^{N-1})|/N$, A sequence $(\beta_n)_{n\geq 0}$ is of *low discrepancy* (abbreviated l.d.s.) if $D((\beta_n)_{n=0}^{N-1}) = O(N^{-1}(\ln N)^s)$ for $N \to \infty$.

Let $(z_n)_{n\geq 1}$ be a l.d.s. obtained from a lattice $\Gamma(M, s, 0)$ [Le2], and let $(v_n)_{n\geq 1}$ be a l.d.s. described in [Le3]. We consider the following classes of s-dimensional l.d.s.: $(z_n)_{n\geq 1}$, $(v_n)_{n\geq 1}$, Halton's sequence (see [DrTi]) and digital (t,s) sequence (see [DiPi]).

In [Le5], we proved that the local discrepancy of the sequence $(z_n)_{n\geq 1}$ obeys CLT. In a forthcoming paper, we will prove a similar result for the sequence $(v_n)_{n\geq 1}$ and for the s-dimensional Halton's sequence. Note that CLT for the 1-dimensional Halton's sequence is proved in [LeMe].

Let $(w_n)_{n\geq 1}$ be a digital (t,s) sequence in base b, and let $\mathbf{x} \oplus \mathbf{y}$ be a digital summation (see def. in [DiPi]). In a forthcoming paper, we will prove that the local discrepancy $\Delta(\mathbf{y}, (w_n \oplus \mathbf{x})_{n=0}^{N-1})$ obeys CLT, where $(\mathbf{y}, \mathbf{x})$ is uniformly distributed random variable in $[0,1)^{2s}$.

The proofs of the CLT for the mentioned sequences, similar to the proof of the CLT for the sequence $\xi_{1,s,0}(\mathbf{N})$.

5. In this paper, we use Theorem 4 to prove CLT and to give a functional characterization of functions with variance limit zero. Similarly to the proof of Lemma 2.3, we can apply Theorem 4 to obtain the rate of mixing of the action $\mathcal{A}$. Analogously to [Ka, Proposition 3.1], we can use Theorem 4 to analyze periodic orbits of the action $\mathcal{A}$. We note that in [MiWa] was described a much more general method of analyze rates of mixing and periodic points distribution of actions generated by commuting automorphisms of a compact abelian group.

3 Proofs of Theorems 1 - 3.

Lemma 2.1. *Let (2.8) be true. Then*

$$\sum_{\mathbf{n},\mathbf{n}' \in \mathbb{Z}_+^d, \ \mathbf{n}\cdot\mathbf{n}' = \mathbf{0}} \left| \int_{[0,1)^s} f(\mathbf{A}^{\mathbf{n}} \mathbf{x}) f(\mathbf{x} \mathbf{A}^{\mathbf{n}'}) d\mathbf{x} \right| < +\infty. \tag{3.1}$$

Proof. Bearing in mind that for all $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{Z}^d$, there exists the unique $(\mathbf{n}, \mathbf{n}') \in \mathbb{Z}_+^{2d}$ with $\mathbf{n} \cdot \mathbf{n}' = \mathbf{0}$ and $\mathbf{n}_2 = \mathbf{n}_1 + \mathbf{n} - \mathbf{n}'$, we have from (2.8)

$$S(f) = \sum_{\mathbf{m}\in W} \sum_{\mathbf{n}_1,\mathbf{n}_2 \in \mathbb{Z}^d} \left| \widehat{f}(\mathbf{A}^{\mathbf{n}_1} \mathbf{m}) \overline{\widehat{f}(\mathbf{A}^{\mathbf{n}_1+\mathbf{n}_2} \mathbf{m})} \right|$$

$$= \sum_{\mathbf{m}\in W} \sum_{\mathbf{n}_1 \in \mathbb{Z}^d} \sum_{\mathbf{n},\mathbf{n}' \in \mathbb{Z}_+^d, \ \mathbf{n}\cdot\mathbf{n}' = \mathbf{0}} \left| \widehat{f}(\mathbf{A}^{\mathbf{n}_1} \mathbf{m}) \overline{\widehat{f}(\mathbf{A}^{\mathbf{n}_1+\mathbf{n}-\mathbf{n}'} \mathbf{m})} \right|$$

$$= \sum_{\substack{\mathbf{n},\mathbf{n}'\in\mathbb{Z}^d_+\\ \mathbf{n}\cdot\mathbf{n}'=\mathbf{0}}} \sum_{\mathbf{m}\in\mathbb{Z}^s} \Big|\widehat{f}(\mathbf{m})\overline{\widehat{f}(\mathbf{A}^{\mathbf{n}-\mathbf{n}'}\mathbf{m})}\Big| = \sum_{\substack{\mathbf{n},\mathbf{n}'\in\mathbb{Z}^d_+\\ \mathbf{n}\cdot\mathbf{n}'=\mathbf{0}}} \sum_{\substack{\mathbf{m},\mathbf{m}'\in\mathbb{Z}^s\\ \mathbf{A}^{\mathbf{n}}\mathbf{m}=\mathbf{A}^{\mathbf{n}'}\mathbf{m}'}} \Big|\widehat{f}(\mathbf{m})\overline{\widehat{f}(\mathbf{m}')}\Big|.$$

Taking into account that f is a real function, we get that

$$\overline{\widehat{f}}(\mathbf{m}) = \widehat{f}(-\mathbf{m}). \tag{3.2}$$

Hence

$$S(f) \geq \sum_{\substack{\mathbf{n},\mathbf{n}'\in\mathbb{Z}^d_+\\ \mathbf{n}\cdot\mathbf{n}'=\mathbf{0}}} \Big| \sum_{\substack{\mathbf{m},\mathbf{m}'\in\mathbb{Z}^s\\ \mathbf{A}^{\mathbf{n}}\mathbf{m}=-\mathbf{A}^{\mathbf{n}'}\mathbf{m}'}} \widehat{f}(\mathbf{m})\widehat{f}(\mathbf{m}')\Big| = \sum_{\substack{\mathbf{n},\mathbf{n}'\in\mathbb{Z}^d_+\\ \mathbf{n}\cdot\mathbf{n}'=\mathbf{0}}} \Big|\int_{[0,1)^s} f(\mathbf{A}^{\mathbf{n}}\mathbf{x})f(\mathbf{A}^{\mathbf{n}'}\mathbf{x})d\mathbf{x}\Big|. \tag{3.3}$$

Therefore Lemma 2.1 is proved. ∎

Lemma 2.2. *Let (2.8) be true, $\widehat{f}(\mathbf{0}) = 0$, $\mathbb{E} \subset \mathbb{Z}^d$ and $\#\mathbb{E} < \infty$. Then*

$$\varphi(\mathbb{E}) := \int_{[0,1)^s} \Big(\sum_{\mathbf{n}\in\mathbb{E}} f(\mathbf{A}^{\mathbf{n}}\mathbf{x})\Big)^2 d\mathbf{x} \leq S(f)\#\mathbb{E}.$$

Proof. We have

$$\varphi(\mathbb{E}) = \sum_{\mathbf{n},\mathbf{n}'\in\mathbb{E}} \tilde{\varphi}(\mathbf{n},\mathbf{n}') \quad \text{with} \quad \tilde{\varphi}(\mathbf{n},\mathbf{n}') = \int_{[0,1)^s} f(\mathbf{A}^{\mathbf{n}}\mathbf{x})f(\mathbf{A}^{\mathbf{n}'}\mathbf{x})d\mathbf{x}.$$

It is easy to see

$$\int_{[0,1)^s} f(\mathbf{A}^{\mathbf{n}}\mathbf{x})f(\mathbf{A}^{\mathbf{n}'}\mathbf{x})d\mathbf{x} = \sum_{\substack{\mathbf{m},\mathbf{m}'\in\mathbb{Z}^s\\ \mathbf{A}^{\mathbf{n}}\mathbf{m}=\mathbf{A}^{\mathbf{n}'}\mathbf{m}'}} \widehat{f}(\mathbf{m})\widehat{f}(-\mathbf{m}').$$

Let $\mathbf{m}_0 = B(\mathbf{m}) \cap W = B(\mathbf{m}') \cap W$. Then there exist $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{Z}^d$ with $\mathbf{m} = \mathbf{A}^{\mathbf{n}_1}\mathbf{m}_0$ and $\mathbf{m}' = \mathbf{A}^{\mathbf{n}_2}\mathbf{m}_0$. Hence

$$\tilde{\varphi}(\mathbf{n},\mathbf{n}') = \sum_{\mathbf{m}_0\in W} \sum_{\substack{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^d\\ \mathbf{n}_1+\mathbf{n}=\mathbf{n}_2+\mathbf{n}'}} \widehat{f}(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_0)\widehat{f}(-\mathbf{A}^{\mathbf{n}_2}\mathbf{m}_0).$$

Therefore

$$\begin{aligned}\varphi(\mathbb{E}) &\leq \sum_{\mathbf{m}_0\in W} \sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^d} \Big|\widehat{f}(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_0)\widehat{f}(\mathbf{A}^{-\mathbf{n}_2}\mathbf{m}_0)\Big| \sum_{\substack{\mathbf{n},\mathbf{n}'\in\mathbb{E}\\ \mathbf{n}_1+\mathbf{n}=\mathbf{n}_2+\mathbf{n}'}} 1\\ &\leq \#\mathbb{E} \sum_{\mathbf{m}_0\in W} \sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^d} \Big|\widehat{f}(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_0)\overline{\widehat{f}(\mathbf{A}^{\mathbf{n}_2}\mathbf{m}_0)}\Big| = S(f)\#\mathbb{E}.\end{aligned}$$

Thus Lemma 2.2 is proved. ∎

Let

$$\delta(\mathfrak{T}) = \begin{cases} 1, & \text{if } \mathfrak{T} \text{ is true,}\\ 0, & \text{otherwise.}\end{cases} \tag{3.4}$$

Proof of Theorem 1. Let

$$\Xi(f) = \sum_{\mathbf{m}\in W} \Big| \sum_{\mathbf{n}\in\mathbb{Z}^d} \widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m})\Big|^2. \tag{3.5}$$

First we consider the case when f is a polynomial trigonometric (see (2.12)) :

Repeating the proof of Lemma 2.2, we obtain

$$\frac{1}{\breve{\mathbf{N}}}\int_{[0,1)^s}\left|S_{\mathbf{N}}(f_L(\mathbf{x}))\right|^2 d\mathbf{x} = \sum_{\mathbf{m}_0\in W}\sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^d}\widehat{f_L}(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_0)\overline{\widehat{f_L}(\mathbf{A}^{\mathbf{n}_2}\mathbf{m}_0)}\Psi_{\mathbf{N}}(\mathbf{m}_0,\mathbf{n}_1,\mathbf{n}_2),$$

where

$$\Psi_{\mathbf{N}}(\mathbf{m}_0,\mathbf{n}_1,\mathbf{n}_2) = \frac{1}{\breve{\mathbf{N}}}\sum_{\substack{\mathbf{n},\mathbf{n}'\in\Re(\mathbf{N})\\ \mathbf{n}_1+\mathbf{n}=\mathbf{n}_2+\mathbf{n}'}} 1, \qquad \text{with} \qquad \Re(\mathbf{N}) = \prod_{i=1}^d [0,N_i-1], \quad \breve{\mathbf{N}} = N_1\cdots N_d.$$

It is easy to see that

$$\frac{1}{\breve{\mathbf{N}}}\prod_{i=1}^d (N_i - 2|n_{1,i}| - 2|n_{2,i}|) \le \Psi_{\mathbf{N}}(\mathbf{m}_0,\mathbf{n}_1,\mathbf{n}_2) \le 1.$$

Hence

$$\lim_{\min_i N_i\to\infty}\Psi_{\mathbf{N}}(\mathbf{m}_0,\mathbf{n}_1,\mathbf{n}_2) = 1. \tag{3.6}$$

By (2.4), (2.5) and (2.12), we have that $\widehat{f_L}(\mathbf{m}) = 0$ for $|\mathbf{m}| \ge L$, and $\widehat{f_L}(\mathbf{A}^{\mathbf{n}}\mathbf{m}_0) \ne 0$ only for $|\mathbf{m}_0| < L$. Using Theorem 4, we have that the set $\{\mathbf{m}_0 \in W, \mathbf{n} \in \mathbb{Z}^d \mid \widehat{f_L}(\mathbf{A}^{\mathbf{n}}\mathbf{m}_0) \ne 0\}$ is finite. So, from (2.9) and (3.5)-(3.6), we get

$$\sigma^2(f_L) = \lim_{\min_i N_i\to\infty}\sum_{\mathbf{m}_0\in W}\sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^d}\widehat{f_L}(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_0)\overline{\widehat{f_L}(\mathbf{A}^{\mathbf{n}_2}\mathbf{m}_0)}\Psi_{\mathbf{N}}(\mathbf{m}_0,\mathbf{n}_1,\mathbf{n}_2) = \Xi(f_L). \tag{3.7}$$

We will need the following equality (obtained from (3.5), (3.2) and (2.6)) :

$$\sigma^2(f_L) = \sum_{\mathbf{m}_0\in W}\sum_{\mathbf{n}_1\in\mathbb{Z}^d}\sum_{\mathbf{m}_2\in\mathbb{Z}^s}\widehat{f_L}(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_0)\widehat{f_L}(-\mathbf{m}_2)\delta(\mathbf{m}_2\in B(\mathbf{m}_0)) \tag{3.8}$$

$$= \sum_{\mathbf{m}_1,\mathbf{m}_2\in\mathbb{Z}^s}\widehat{f_L}(\mathbf{m}_1)\widehat{f_L}(\mathbf{m}_2)\delta(-\mathbf{m}_2\in B(\mathbf{m}_1)) = \sum_{|\mathbf{m}_i|<L,i=1,2}\widehat{f}(\mathbf{m}_1)\widehat{f}(\mathbf{m}_2)\delta(\mathbf{m}_1\in B(-\mathbf{m}_2)).$$

Now we consider the general case. It follows from (2.8) and (3.5) that $\Xi(f) < \infty$. Using Lemma 2.2 and the Cauchy–Schwartz inequality, we have

$$\frac{1}{\sqrt{\breve{\mathbf{N}}}}|\|S_{\mathbf{N}}(f)\|_2 - \|S_{\mathbf{N}}(f_L)\|_2| \le \frac{1}{\sqrt{\breve{\mathbf{N}}}}\|S_{\mathbf{N}}(f-f_L)\|_2 \le (S(f-f_L))^{1/2}. \tag{3.9}$$

By (2.8) , we get

$$S(f-f_L) \le \sum_{\mathbf{m}\in W,|\mathbf{m}|\ge L}\left(\sum_{\mathbf{n}\in\mathbb{Z}^d}|\widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m})|\right)^2.$$

Hence

$$S(f-f_L)\to 0 \qquad \text{as} \quad L\to\infty. \tag{3.10}$$

Therefore, for all $\epsilon > 0$, there exist L_0 such that $S(f-f_L) < \epsilon$ for $L > L_0$. Using (3.9), we obtain

$$\frac{1}{\sqrt{\breve{\mathbf{N}}}}\|S_{\mathbf{N}}(f_L)\|_2 - \epsilon \le \frac{1}{\sqrt{\breve{\mathbf{N}}}}\|S_{\mathbf{N}}(f)\|_2 \le \frac{1}{\sqrt{\breve{\mathbf{N}}}}\|S_{\mathbf{N}}(f_L)\|_2 + \epsilon.$$

From (2.9) and (3.7), we have

$$(\Xi(f_L))^{1/2} - \epsilon \le \liminf_{\mathbf{N}\to\infty} \frac{1}{\sqrt{\breve{\mathbf{N}}}} \|S_{\mathbf{N}}(f)\|_2 \le \limsup_{\mathbf{N}\to\infty} \frac{1}{\sqrt{\breve{\mathbf{N}}}} \|S_{\mathbf{N}}(f)\|_2 \le (\Xi(f_L))^{1/2} + \epsilon. \quad (3.11)$$

Using (3.5), we get

$$\Xi(f) - \Xi(f_L) = \sum_{\mathbf{m}\in W} \sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^d} \Big(\widehat{f}(\mathbf{A}^{\mathbf{n}_1}\mathbf{m})\widehat{f}(\mathbf{A}^{\mathbf{n}_2}\mathbf{m}) - (\widehat{f} - \widehat{f - f_L})(\mathbf{A}^{\mathbf{n}_1}\mathbf{m})(\widehat{f} - \widehat{f - f_L})(\mathbf{A}^{\mathbf{n}_2}\mathbf{m}) \Big).$$

Hence

$$|\Xi(f) - \Xi(f_L)| \le \Xi(f - f_L)) + 2 \sum_{\mathbf{m}\in W} \sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^d} |\widehat{f}(\mathbf{A}^{\mathbf{n}_1}\mathbf{m})\widehat{f - f_L}(\mathbf{A}^{\mathbf{n}_2}\mathbf{m})|.$$

Applying the Cauchy–Schwartz inequality, we obtain from (2.8), (3.5) and (3.10):

$$|\Xi(f) - \Xi(f_L)| \le \Xi(f - f_L)) + 2\Xi^{1/2}(f - f_L)\Xi^{1/2}(f) \le (S(f - f_L) + 2(S(f - f_L)\Xi(f))^{1/2}) \to 0$$

as $L \to \infty$. By (3.7)

$$\Xi(f) = \lim_{L\to\infty} \Xi(f_L) = \lim_{L\to\infty} \sigma^2(f_L). \quad (3.12)$$

From (3.11), we have $\sigma^2(f) = \Xi(f)$ and (2.9) follows. To obtain (2.10), we repeat the proof of Lemma 2.1. This is possible because the series (3.1) and (3.3) converges absolutely. Hence Theorem 1 is proved. ■

Proof of Theorem 2. We will prove the case (i). The proof of the case (ii) is similar. From Theorem 3, (2.14) and (3.15) we get that $f_{[\mathbf{u},\mathbf{v})}$ satisfy the condition (2.8). By (2.9), it is enough to prove that there exists $\mathbf{m} \in \mathbb{Z}^s$ with

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \widehat{f}_{[\mathbf{u},\mathbf{v})}(\mathbf{A}^{\mathbf{n}}\mathbf{m}) \ne \mathbf{0}. \quad (3.13)$$

It is easy to verify that

$$\widehat{f}_{[\mathbf{u},\mathbf{v})}(\mathbf{0}) = 0, \quad \widehat{f}_{[\mathbf{u},\mathbf{v})}(\mathbf{m}) = \widehat{1}_{[\mathbf{u},\mathbf{v})}(\mathbf{m}) \qquad \text{for} \quad \mathbf{m} \ne \mathbf{0}, \quad (3.14)$$

and

$$\widehat{1}_{[\mathbf{u},\mathbf{v})}(\mathbf{m}) = \prod_{i=1}^{s} \widehat{1}_{[u_i,v_i)}(\mathbf{m}_i), \quad \text{where} \quad \widehat{1}_{[a,b)}(m) = \begin{cases} \frac{e(bm)-e(am)}{2\pi\sqrt{-1}m}, & \text{if } m \ne 0, \\ b - a, & \text{otherwise.} \end{cases} \quad (3.15)$$

Suppose that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \widehat{f}_{[\mathbf{u},\mathbf{v})}(\mathbf{A}^{\mathbf{n}}\mathbf{m}) = \mathbf{0} \qquad \forall \mathbf{m} \in \mathbb{Z}^s. \quad (3.16)$$

Let

$$\dot{\Xi}(\mathbf{n},\mathbf{m}) = \{j \in [1,s] \mid (\mathbf{A}^{\mathbf{n}}\mathbf{m})_j = 0\},$$

$$\Psi(i,\mathbf{m}) = \{\mathbf{n} \in \mathbb{Z}^d \mid \mathbf{A}^{\mathbf{n}}\mathbf{m} \in \mathbb{Z}^s, \text{ and } \#\dot{\Xi}(\mathbf{n},\mathbf{m}) = i\}.$$

We fix $\mathbf{m} \in \mathbb{Z}^s$ with $m_i \ne 0$ for all $i = 1, ..., s$. It is easy to see that

$$\Psi(0,\mathbf{m}) \ne \emptyset. \quad (3.17)$$

Let

$$\psi_i(\mathbf{v},k) = \sum_{\mathbf{n}\in\Psi(i,\mathbf{m})} \prod_{\mu\in\dot{\Xi}(\mathbf{n},\mathbf{m})} v_\mu \prod_{\mu\in[1,s]\setminus\dot{\Xi}(\mathbf{n},\mathbf{m})} \frac{e((\mathbf{A}^{\mathbf{n}}\mathbf{m}k)_\mu v_\mu) - 1}{2\pi\sqrt{-1}(\mathbf{A}^{\mathbf{n}}\mathbf{m})_\mu}$$

and

$$\tilde{\psi}_i(\mathbf{x}) = \sum_{\mathbf{n}\in\Psi(i,\mathbf{m})} \prod_{\mu\in\dot{\Xi}(\mathbf{n},\mathbf{m})} v_\mu \prod_{\mu\in[1,s]\setminus\dot{\Xi}(\mathbf{n},\mathbf{m})} \frac{e((\mathbf{A^n m})_\mu x_\mu) - 1}{2\pi\sqrt{-1}(\mathbf{A^n m})_\mu}. \tag{3.18}$$

From (3.14) and (3.16), we have

$$k^s \sum_{\mathbf{n}\in\mathbb{Z}^d} \widehat{f_{[\mathbf{0},\mathbf{v})}}(\mathbf{A^n m}k) = \sum_{i=0}^{s-1} k^i \psi_i(\mathbf{v}, k) = \mathbf{0} \quad \text{for} \quad k = 1, 2, ... \tag{3.19}$$

Applying Theorem 4, we get that the series (3.18) converges absolutely and uniformly continuously and there exists $c_0(\mathbf{m}) > 0$ with

$$\sup_{\mathbf{v},\mathbf{x},i,k} (|\tilde{\psi}_i(\mathbf{x})|, |\psi_i(\mathbf{v}, k)|) \le c_0(\mathbf{m}). \tag{3.20}$$

Thus $\tilde{\psi}_i(\mathbf{x})$ are continuous functions. We will prove that

$$\sup_{\mathbf{x}\in[0,1]^s} (|\tilde{\psi}_i(\mathbf{x})|) = 0. \tag{3.21}$$

Let $i_0 \in [1, s-1]$, (3.21) be true for $i_0 < i \le s-1$ and

$$\sup_{\mathbf{x}\in[0,1]^s} (|\tilde{\psi}_{i_0}(\mathbf{x})|) = \epsilon > 0. \tag{3.22}$$

Let $|\tilde{\psi}_{i_0}(\mathbf{x}_0)|) = \epsilon$. There exists $\epsilon_0 > 0$ such that if $|\mathbf{x}-\mathbf{x}_0| < \epsilon_0$, then $|\tilde{\psi}_{i_0}(\mathbf{x})|) \ge \epsilon/2$. From the condition (i) and the Kronnecker-Weil's theorem, the sequence $(\{kv_1\}, ..., \{kv_s\})_{k\ge1}$ is uniformly distributed in $[0, 1)^s$ (see, e.g., [DrTi], p. 66). Hence, there exists a subsequence $(k_n)_{n\ge1}$ such that $|\{k_n\mathbf{v}\} - \mathbf{x}_0| < \epsilon_0$ and $|\psi_{i_0}(\mathbf{v}, k_n)|) \ge \epsilon/2 > 0$. From (3.19) and (3.20), we get that

$$\psi_{i_0}(\mathbf{v}, k) = -\sum_{i=0}^{i_0-1} k^{i-i_0}\psi_i(\mathbf{v}, k) \qquad \text{and} \qquad \epsilon/2 \le |\psi_{i_0}(\mathbf{v}, k)| \le c_0(\mathbf{m})s/k, \quad k = 1, 2, ...$$

We have a contradiction ($\epsilon = O(1/k)$). Thus (3.21) is true for $i \in [1, s-1]$. By (3.19), we have that (3.21) is true also for $i = 0$.

Using Definition 1 we get: if $\mathbf{A^n m} = \mathbf{m}$, then 1 is the eigenvalue of $\mathbf{A^n}$ and $\mathbf{n} = 0$. Therefore, if $\mathbf{A^{n_1} m} = \mathbf{A^{n_2} m}$, then $\mathbf{n}_1 = \mathbf{n}_2$. So

$$\int_{[0,1)^s} e(\langle \mathbf{A^n m}, \mathbf{x}\rangle)d\mathbf{x} = 0 = \int_{[0,1)^s} e(\langle(\mathbf{A^{n_1}} - \mathbf{A^{n_2}})\mathbf{m}, \mathbf{x}\rangle) \quad \text{for} \quad \mathbf{n}_1 \ne \mathbf{n}_2. \tag{3.23}$$

Let $\mathbf{n}_0 \in \Psi(0, \mathbf{m}) \ne \emptyset$ (see (3.17)). We have $(\mathbf{A^{n_0} m})_i \ne 0$ for $i = 1, ..., s$. Consider $\tilde{\psi}_0(\mathbf{x}) = 0$ for $\mathbf{x} \in [0, 1)^s$ (see (3.21)). Applying (3.23), we obtain from (3.18)

$$0 = \int_{[0,1)^s} \tilde{\psi}_0(\mathbf{x})e(< -\mathbf{A^{n_0} m}, \mathbf{x} >)d\mathbf{x} = \prod_{\mu\in[1,s]} \frac{-1}{2\pi\sqrt{-1}(\mathbf{A^{n_0} m})_\mu} \ne 0.$$

We have a contradiction. Thus (3.13) is true. Hence Theorem 2 is proved. ■

Proof of Theorem 3.
Lemma 2.3. *Let (2.11) be true. Then*

$$S(f) < +\infty. \tag{3.24}$$

Proof. Let

$$S_i(f) = \sum_{\mathbf{n}\in\mathbb{Z}^d} g_i(\mathbf{n}), \quad \text{with} \quad g_1(\mathbf{n}) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^s \\ |\mathbf{m}|\leq\exp(a_0|\mathbf{n}|)}} \left|\widehat{f}(\mathbf{m})\widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m})\right|,$$

and

$$g_2(\mathbf{n}) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^s \\ |\mathbf{m}|>\exp(a_0|\mathbf{n}|)}} \left|\widehat{f}(\mathbf{m})\widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m})\right|,$$

where $a_0 = a_1/(1+2b_1)$ and $i = 1,2$ (see (2.15)). We have

$$g_1(\mathbf{n}) \leq \Big(\sum_{\mathbf{m}\in\mathbb{Z}^s, |\mathbf{m}|\leq\exp(a_0|\mathbf{n}|)} \left|\widehat{f}(\mathbf{m})\right|^2\Big)^{1/2} \Big(\sum_{\mathbf{m}\in\mathbb{Z}^s, |\mathbf{m}|\leq\exp(a_0|\mathbf{n}|)} \left|\widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m})\right|^2\Big)^{1/2}.$$

Applying Theorem 4 with $|\mathbf{m}| \leq \exp(a_0|\mathbf{n}|)$, we get

$$|\mathbf{A}^{\mathbf{n}}\mathbf{m}| \geq c_1|\mathbf{m}|^{-b_1}\exp(a_1|\mathbf{n}|) \geq c_1\exp((a_1 - a_0 b_1)|\mathbf{n}|) \geq c_1\exp(a_1|\mathbf{n}|/2).$$

Hence

$$g_1(\mathbf{n}) \leq \|f\|_2 \|f - f_{c_1\exp(a_1|\mathbf{n}|/2)}\|_2$$

and

$$S_1(f) \leq \sum_{\mathbf{n}\in\mathbb{Z}^d} \|f\|_2 \|f - f_{c_1\exp(a_1|\mathbf{n}|/2)}\|_2 \;=\; O\Big(\sum_{k=1}^{\infty} \|f\|_2$$

$$\times \sum_{\mathbf{n}\in\mathbb{Z}^d, c_1\exp(a_1|\mathbf{n}|/2\in[2^k,2^{k+1})} \|f - f_{2^k}\|_2\Big) = O\Big(\sum_{k=1}^{\infty} k^{d-1}\|f - f_{2^k}\|_2\Big) < +\infty.$$

Similarly, we have

$$g_2(\mathbf{n}) \leq \|f\|_2 \|f - f_{\exp(a_0|\mathbf{n}|)}\|_2 \quad \text{and} \quad S_2(f) = O(1).$$

From (2.8), we get

$$S(f) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \sum_{\mathbf{n}\in\mathbb{Z}^d} \left|\widehat{f}(\mathbf{m})\widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m})\right| = S_1(f) + S_2(f).$$

Therefore Lemma 2.3 is proved. ■

Let $\widehat{h}^{(0)} = \widehat{f}$, and

$$\widehat{h^{(i)}}(\mathbf{m}) = \begin{cases} \sum_{n_1,\dots,n_i\in\mathbb{Z}} \widehat{f}(A_1^{n_1}\cdots A_i^{n_i}\widetilde{\mathbf{m}}), & \text{if } \mathbf{m} = A_{i+1}^{n_{i+1}}\cdots A_d^{n_d}\widetilde{\mathbf{m}} \\ 0, & \text{otherwise} \end{cases} \tag{3.25}$$

for some $\widetilde{\mathbf{m}} \in W$, and $n_{i+1},...,n_d \in \mathbb{Z}$.
Let $\widehat{g}^{(i)} = \widehat{h}^{(i-1)} - \widehat{h}^{(i)}$ and

$$\widehat{f^{(i)}}(\mathbf{m}) = \sum_{k\leq 0} \widehat{g^{(i)}}(A_i^k\mathbf{m}), \qquad 1 \leq i \leq d. \tag{3.26}$$

Using (2.8) and Lemma 2.3, we get that the series (3.25) and (3.26) converges. By (3.25) we get

$$\sum_{k\in\mathbb{Z}} \widehat{g^{(i)}}(A_i^k\mathbf{m}) = 0, \qquad \forall\mathbf{m} \in \mathbb{Z}^s \setminus \mathbf{0}, \qquad i = 1,...,d. \tag{3.27}$$

Let $\mathbf{n}^{(1)} = (n_1, ..., n_{i-1})$, $\mathbf{A}_1^{\mathbf{n}^{(1)}} = A_1^{n_1} \cdots A_{i-1}^{n_{i-1}}$ for $i \geq 2$, and $\mathbf{n}^{(1)} = \mathbf{0}$, $\mathbf{A}_1^{\mathbf{n}^{(1)}} = 1$ for $i = 1$. Let $\mathbf{n}^{(2)} = (n_{i+1}, ..., n_d)$, $\mathbf{A}_2^{\mathbf{n}^{(2)}} = A_{i+1}^{n_{i+1}} \cdots A_d^{n_d}$ for $i < d$, and $\mathbf{n}^{(2)} = \mathbf{0}$, $\mathbf{A}_2^{\mathbf{n}^{(2)}} = 1$ for $i = d$.

By (2.4) and (2.6), we get that for all $m \in \mathbb{Z}^d \setminus \mathbf{0}$ there exists unique $\mathbf{n}^{(1)} \in \mathbb{Z}^{i-1}$, $n_i \in \mathbb{Z}$, $\mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}$ and $\widetilde{\mathbf{m}} \in W$ such that, $\mathbf{m} = \mathbf{A}_1^{\mathbf{n}^{(1)}} A_i^{n_i} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}$.

Let $n_i < 0$. Using (3.25) and (3.26), we derive the following expression for $\widehat{f^{(i)}}(\mathbf{m})$. Next by (3.27), we obtain a similar expression for the case $n_i \geq 0$:

$$\widehat{f^{(i)}}(\mathbf{m}) = \begin{cases} \sum_{\mathbf{k}^{(1)} \in \mathbb{Z}^{i-1}} \sum_{k \leq 0} \widehat{f}(\mathbf{A}_1^{\mathbf{k}^{(1)}} A_i^{n_i+k} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}), & \text{if } \mathbf{n}^{(1)} = \mathbf{0}, \text{ and } n_i < 0, \\ -\sum_{\mathbf{k}^{(1)} \in \mathbb{Z}^{i-1}} \sum_{k > 0} \widehat{f}(\mathbf{A}_1^{\mathbf{k}^{(1)}} A_i^{n_i+k} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}), & \text{if } \mathbf{n}^{(1)} = \mathbf{0}, \text{ and } n_i \geq 0. \\ 0. & \text{otherwise} \end{cases} \quad (3.28)$$

Lemma 2.4. *Let (2.11) be true, $i \in [1, d]$. Then*

$$\varkappa^{(i)} := \sum_{\mathbf{m} \in \mathbb{Z}^s} |\widehat{f^{(i)}}(\mathbf{m})|^2 = \sum_{\widetilde{\mathbf{m}} \in W} \sum_{\mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}} \sum_{n_i \in \mathbb{Z}} \left| \widehat{f^{(i)}}(A_i^{n_i} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}) \right|^2 < +\infty. \quad (3.29)$$

Proof. By (3.28) we have $\varkappa^{(i)} = \varkappa_1^{(i)} + \varkappa_2^{(i)}$, where

$$\varkappa_1 = \varkappa_1^{(i)} = \sum_{\widetilde{\mathbf{m}} \in W} \sum_{\mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}} \sum_{\mathbf{n}_i \geq 0} \Big| \sum_{\mathbf{k}^{(1)} \in \mathbb{Z}^{i-1}} \sum_{k > 0} \widehat{f}(\mathbf{A}_1^{\mathbf{k}^{(1)}} A_i^{n_i+k} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}) \Big|^2, \quad (3.30)$$

and

$$\varkappa_2 = \varkappa_2^{(i)} = \sum_{\widetilde{\mathbf{m}} \in W} \sum_{\mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}} \sum_{\mathbf{n}_i < 0} \big| \sum_{\mathbf{k}^{(1)} \in \mathbb{Z}^{i-1}} \sum_{k \leq 0} \widehat{f}(\mathbf{A}_1^{\mathbf{k}^{(1)}} A_i^{n_i+k} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}) \big|^2.$$

We will prove that $\varkappa_1 < +\infty$. Analogously, we obtain that $\varkappa_2 < +\infty$. We see that

$$\varkappa_1 \leq 2 \sum_{\substack{\widetilde{\mathbf{m}} \in W \\ \mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}}} \sum_{\mathbf{n}_i, k_1, k_2 \geq 0} \sum_{\mathbf{k}_1^{(1)}, \mathbf{k}_2^{(1)} \in \mathbb{Z}^{i-1}} \big| \widehat{f}(\mathbf{A}_1^{\mathbf{k}_1^{(1)}} A_i^{n_i+k_1} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}) \widehat{f}(\mathbf{A}_1^{\mathbf{k}_1^{(1)}+\mathbf{k}_2^{(1)}} A_i^{n_i+k_1+k_2} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}) \big|$$

$$\leq 4 \sum_{\substack{\widetilde{\mathbf{m}} \in W \\ \mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}}} \sum_{\mathbf{n}_i, k_1, k_2 \geq 0, \mathbf{n}_i \geq k_2} \sum_{\mathbf{k}_1^{(1)}, \mathbf{k}_2^{(1)} \in \mathbb{Z}^{i-1}} \big| \widehat{f}(\mathbf{A}_1^{\mathbf{k}_1^{(1)}} A_i^{n_i+k_1} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}) \widehat{f}(\mathbf{A}_1^{\mathbf{k}_1^{(1)}+\mathbf{k}_2^{(1)}} A_i^{n_i+k_1+k_2} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}) \big|.$$

We have that $\varkappa_1 \leq 4(\varkappa_{1,1} + \varkappa_{1,2})$, where

$$\varkappa_{1,1} = \sum_{\widetilde{\mathbf{m}} \in W,\ \mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}} \sum_{\mathbf{n}_i, k_1, k_2 \geq 0, \mathbf{n}_i \geq k_2} \sum_{\mathbf{k}_1^{(1)}, \mathbf{k}_2^{(1)} \in \mathbb{Z}^{i-1},\ |\dot{\mathbf{m}}| \geq \exp(a_0(|\mathbf{k}_2^{(1)}| + k_2)/2)} \big| \widehat{f}(\dot{\mathbf{m}}) \widehat{f}(\mathbf{A}_1^{\mathbf{k}_2^{(1)}} A_i^{k_2} \dot{\mathbf{m}}) \big|,$$

and

$$\varkappa_{1,2} = \sum_{\widetilde{\mathbf{m}} \in W,\ \mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}} \sum_{\mathbf{n}_i, k_1, k_2 \geq 0, \mathbf{n}_i \geq k_2} \sum_{\mathbf{k}_1^{(1)}, \mathbf{k}_2^{(1)} \in \mathbb{Z}^{i-1},\ |\dot{\mathbf{m}}| < \exp(a_0(|\mathbf{k}_2^{(1)}| + k_2)/2)} \big| \widehat{f}(\dot{\mathbf{m}}) \widehat{f}(\mathbf{A}_1^{\mathbf{k}_2^{(1)}} A_i^{k_2} \dot{\mathbf{m}}) \big|,$$

with $\dot{\mathbf{m}} = \mathbf{A}_1^{\mathbf{k}_1^{(1)}} A_i^{n_i+k_1} \mathbf{A}_2^{\mathbf{n}^{(2)}} \widetilde{\mathbf{m}}$, and $a_0 = a_1/(1 + b_1)$.

Consider $\varkappa_{1,1}$. Applying the Cauchy–Schwartz inequality, we get:

$$\varkappa_{1,1} \leq \sum_{\mathbf{n}_i \geq k_2 \geq 0} \sum_{\mathbf{k}_2^{(1)} \in \mathbb{Z}^{i-1}} Q_1(\mathbf{0}, 0)^{1/2} Q_1(\mathbf{k}_2^{(1)}, k_2)^{1/2}, \quad (3.31)$$

where

$$Q_1(\mathbf{k}^{(1)},k)=\sum_{\widetilde{\mathbf{m}}\in W}\sum_{\mathbf{n}^{(2)}\in\mathbb{Z}^{d-i}}\sum_{k_1\geq 0,|\dot{\mathbf{m}}|\geq\exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)}\sum_{\mathbf{k}_1^{(1)}\in\mathbb{Z}^{i-1}}\left|\widehat{f}(\mathbf{A}_1^{\mathbf{k}^{(1)}}A_i^k\dot{\mathbf{m}})\right|^2. \quad (3.32)$$

It is east to see that

$$Q_1(\mathbf{0},0)\leq\|f-f_{\exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)}\|_2^2. \quad (3.33)$$

We have $Q_1(\mathbf{k}^{(1)},k)=\dot{Q}_1(\mathbf{k}^{(1)},k)+\ddot{Q}_1(\mathbf{k}^{(1)},k)$, where

$$\dot{Q}_1(\mathbf{k}^{(1)},k)=\sum_{\widetilde{\mathbf{m}}\in W,|\widetilde{\mathbf{m}}|\geq\exp(a_0n_i)}\sum_{\mathbf{n}^{(2)}\in\mathbb{Z}^{d-i}}\sum_{k_1\geq 0}\sum_{\mathbf{k}_1^{(1)}\in\mathbb{Z}^{i-1},|\dot{\mathbf{m}}|\geq\exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)}\left|\widehat{f}(\mathbf{A}_1^{\mathbf{k}^{(1)}}A_i^k\dot{\mathbf{m}})\right|^2,$$

and

$$\ddot{Q}_1(\mathbf{k}^{(1)},k)=\sum_{\widetilde{\mathbf{m}}\in W,|\widetilde{\mathbf{m}}|<\exp(a_0n_i)}\sum_{\mathbf{n}^{(2)}\in\mathbb{Z}^{d-i}}\sum_{k_1\geq 0}\sum_{\mathbf{k}_1^{(1)}\in\mathbb{Z}^{i-1},|\dot{\mathbf{m}}|\geq\exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)}\left|\widehat{f}(\mathbf{A}_1^{\mathbf{k}^{(1)}}A_i^k\dot{\mathbf{m}})\right|^2.$$

From definition of the set W (see (2.5)), we get

$$\dot{Q}_1(\mathbf{k}_2^{(1)},k_2)\leq\|f-f_{\exp(a_0n_i)}\|_2^2. \quad (3.34)$$

Consider the case $|\widetilde{\mathbf{m}}|<\exp(a_0n_i)$. Using Theorem 4 and that $\dot{\mathbf{m}}=\mathbf{A}_1^{\mathbf{k}_1^{(1)}}A_i^{n_i+k_1}\mathbf{A}_2^{\mathbf{n}^{(2)}}\widetilde{\mathbf{m}}$, we obtain

$$|\mathbf{A}_1^{\mathbf{k}_2^{(1)}}A_i^{k_2}\dot{\mathbf{m}}|=|\mathbf{A}_1^{\mathbf{k}_1^{(1)}+\mathbf{k}_2^{(1)}}A_i^{n_i+k_1+k_2}\mathbf{A}_2^{\mathbf{n}^{(2)}}\widetilde{\mathbf{m}}|\geq c_1\exp\left(a_1(n_i+k_1+k_2)-b_1a_0n_i\right)\geq c_1\exp(a_0n_i).$$

Hence

$$\ddot{Q}_1(\mathbf{k}_2^{(1)},k_2)\leq\|f-f_{c_1\exp(a_0n_i)}\|_2^2.$$

By (3.34), we have

$$Q_1(\mathbf{k}_2^{(1)},k_2)\leq 2\|f-f_{\exp(\dot{c_1}a_0n_i)}\|_2^2,\quad\text{with}\quad\dot{c_1}=\min(1,c_1). \quad (3.35)$$

From (3.31), (3.33) and (3.35), we derive

$$\varkappa_{1,1}\leq 2\sum_{\mathbf{n}_i,k_2\geq 0}\sum_{\mathbf{k}_2^{(1)}\in\mathbb{Z}^{i-1}}\|f-f_{\exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)}\|_2\;\|f-f_{\dot{c_1}\exp(a_0n_i)}\|_2. \quad (3.36)$$

Thus

$$\varkappa_{1,1}\leq 2\sum_{j_1\geq 0}\sum_{\mathbf{k}_2^{(1)}\in\mathbb{Z}^{i-1}}\sum_{k_2\geq 0,\exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)\in[2^{j_1},2^{j_1}+1)}\|f-f_{2^{j_1}}\|_2$$
$$\times\sum_{j_2\geq 0}\sum_{n_i\geq 0,\ \dot{c_1}\exp(a_0n_i)\in[2^{j_2},2^{j_2}+1)}\|f-f_{2^{j_2}}\|_2.$$

By (2.11), we have

$$\varkappa_{1,1}=O\Big(\sum_{j_1\geq 1}j_1^{i-1}\|f-f_{2^{j_1}}\|_2\Big)=O(1). \quad (3.37)$$

Now we consider $\varkappa_{1,2}$. Applying the Cauchy–Schwartz inequality, we get:

$$\varkappa_{1,2}\leq\sum_{n_i\geq k_2\geq 0}\sum_{\mathbf{k}_2^{(1)}\in\mathbb{Z}^{i-1}}Q_2(\mathbf{0},0)^{1/2}Q_2(\mathbf{k}_2^{(1)},k_2)^{1/2}, \quad (3.38)$$

where

$$Q_2(\mathbf{k}^{(1)},k)=\sum_{\widetilde{\mathbf{m}}\in W}\sum_{\mathbf{n}^{(2)}\in\mathbb{Z}^{d-i}}\sum_{k_1\geq 0,|\dot{\mathbf{m}}|<\exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)}\sum_{\mathbf{k}_1^{(1)}\in\mathbb{Z}^{i-1}}\left|\widehat{f}(\mathbf{A}_1^{\mathbf{k}^{(1)}}A_i^k\dot{\mathbf{m}})\right|^2.$$

Using Theorem 4 with $|\dot{\mathbf{m}}| < \exp(a_0(|\mathbf{k}_2^{(1)}| + k_2)/2)$ and bearing in mind that $|(\mathbf{k}_2^{(1)}, k_2)| \geq (|\mathbf{k}_2^{(1)}| + k_2)/2$, we obtain

$$|\mathbf{A}_1^{\mathbf{k}_2^{(1)}} A_i^{k_2}\dot{\mathbf{m}}| \geq c_1 \exp\Big(a_1|(\mathbf{k}_2^{(1)}, k_2)| - b_1 a_0(|\mathbf{k}_2^{(1)}| + k_2)/2\Big) \geq c_1 \exp(a_0(|\mathbf{k}_2^{(1)}| + k_2)/2).$$

Hence

$$Q_2(\mathbf{k}_2^{(1)}, k_2) \leq \|f - f_{c_1 \exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)}\|_2^2. \tag{3.39}$$

We have $Q_2(\mathbf{0}, 0) = \dot{Q}_2(\mathbf{0}, 0) + \ddot{Q}_2(\mathbf{0}, 0)$, where

$$\dot{Q}_2(\mathbf{0}, 0) = \sum_{\widetilde{\mathbf{m}} \in W, |\widetilde{\mathbf{m}}| \geq \exp(a_0 n_i)} \sum_{\mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}} \sum_{k_1 \geq 0} \sum_{\mathbf{k}_1^{(1)} \in \mathbb{Z}^{i-1}, |\dot{\mathbf{m}}| < \exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)} \big|\widehat{f}(\dot{\mathbf{m}})\big|^2,$$

and

$$\ddot{Q}_2(\mathbf{0}, 0) = \sum_{\widetilde{\mathbf{m}} \in W, |\widetilde{\mathbf{m}}| < \exp(a_0 n_i)} \sum_{\mathbf{n}^{(2)} \in \mathbb{Z}^{d-i}} \sum_{k_1 \geq 0} \sum_{\mathbf{k}_1^{(1)} \in \mathbb{Z}^{i-1}, |\dot{\mathbf{m}}| < \exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)} \big|\widehat{f}(\dot{\mathbf{m}})\big|^2.$$

Similarly to (3.34) - (3.35) , we obtain

$$\dot{Q}_2(\mathbf{0}, 0) \leq \|f - f_{\exp(a_0 n_i)}\|_2^2, \qquad \ddot{Q}_2(\mathbf{0}, 0) \leq \|f - f_{\dot{c_1} \exp(a_0 n_i)}\|_2^2,$$

and

$$Q_2(\mathbf{0}, 0) \leq 2\|f - f_{\dot{c_1} \exp(a_0 n_i)}\|_2^2. \tag{3.40}$$

By (3.38), (3.39) and (3.40), we have

$$\varkappa_{1,2} \leq \sum_{n_i \geq k_2 \geq 0} \sum_{\mathbf{k}_2^{(1)} \in \mathbb{Z}^{i-1}} 2\|f - f_{c_1 \exp(a_0(|\mathbf{k}_2^{(1)}|+k_2)/2)}\|_2 \; \|f - f_{\dot{c_1} \exp(a_0 n_i)}\|_2.$$

Similarly to (3.36) and (3.37), we obtain

$$\varkappa_{1,2} = O(1) \quad \text{and} \quad \varkappa_1 \leq 4(\varkappa_{1,1} + \varkappa_{1,2}) = O(1). \tag{3.41}$$

Hence Lemma 2.4 is proved. ■

End of the proof of Theorem 3. Consider the case $\sigma(f) = 0$. By (3.25), (2.9) and Lemma 2.3, we get that $\widehat{h^{(d)}}(\mathbf{m}) = 0$ for all $\mathbf{m} \in \mathbb{Z}^s$. Hence $\widehat{f}(\mathbf{m}) = \widehat{h^{(0)}}(\mathbf{m}) = \sum_{1 \leq i \leq d} \widehat{g^{(i)}}(\mathbf{m})$. Using Lemma 2.4, we obtain that $\widehat{f^{(i)}} \in l^2$. Bearing in mind that $\widehat{g^{(i)}}(\mathbf{m}) = \widehat{f^{(i)}}(\mathbf{m}) - \widehat{f^{(i)}}(A_i^{-1}\mathbf{m}) \in l^2$ (see (3.26)), we get that $\widehat{g^{(i)}} \in l^2$ and $\widehat{h^{(i)}} \in l^2$ $(i = 1, ..., d)$. Let $f^{(i)}, g^{(i)}$ and $h^{(i)}$ be the correspondent functions of L^2. We have that $g^{(i)}(\mathbf{x}) = f^{(i)}(\mathbf{x}) - f^{(i)}(A_i\mathbf{x})$ (see (2.2)) and $f(\mathbf{x}) = g^{(1)}(\mathbf{x}) + \cdots + g^{(d)}(\mathbf{x})$ for almost all $\mathbf{x} \in [0, 1)^s$. The assertion (2.13) is proved. Next we have that $h^{(i)}$ (and hence $g^{(i)}$) verify (2.8) :

$$\sum_{\mathbf{m} \in W} \Bigg(\sum_{\mathbf{n} \in \mathbb{Z}^d} \Big|\widehat{h^{(i)}}(\mathbf{A^n m})\Big|\Bigg)^2$$

$$\leq \sum_{\mathbf{m} \in W} \Bigg(\sum_{n_{i+1}, ..., n_d \in \mathbb{Z}} \sum_{n_1, ..., n_i \in \mathbb{Z}} \Big|\widehat{f}(\mathbf{A^n m})\Big|\Bigg)^2 = S(f) < +\infty, \quad i = 1, ..., d.$$

Now let f satisfy (2.13), $f = \sum_{1 \leq i \leq d} g^{(i)}$, $g^{(i)}(\mathbf{x}) = f^{(i)}(\mathbf{x}) - f^{(i)}(A_i\mathbf{x})$), $f^{(i)} \in L^2$, and $g_i^{(i)}$ satisfy (2.8) $(i = 1, ..., d)$. By (2.8), the series

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} \widehat{f}(\mathbf{A^n m}) = \sum_{1 \leq i \leq d} \sum_{\mathbf{n} \in \mathbb{Z}^d} \widehat{g}_i(\mathbf{A^n m}) \quad \text{with} \quad \mathbf{m} \in W$$

converges absolutely. From (2.1) and (2.2), we get

$$\sum_{n_i \in \mathbb{Z}} \widehat{g}_i(A_i^{n_i}\mathbf{m}) = \sum_{n_i \in \mathbb{Z}} (\widehat{f^{(i)}}(A_i^{n_i}\mathbf{m}) - \widehat{f^{(i)}}(A_i^{n_i-1}\mathbf{m})) = 0, \qquad \text{with} \quad \mathbf{m} \in \mathbb{Z}^s.$$

Hence

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} \widehat{g}_i(\mathbf{A}^{\mathbf{n}}\mathbf{m}) = 0, \quad i = 1, ..., d, \qquad \sum_{\mathbf{n} \in \mathbb{Z}^d} \widehat{f}(\mathbf{A}^{\mathbf{n}}\mathbf{m}) = 0 \quad \text{and} \quad \sigma = 0.$$

Thus Theorem 3 is proved. ■

4 Proof of Theorem 4.

The upper bound in (2.15) follows from the formula for a degree of Jordan matrix (see, e.g., [Ga, pp.157,158]). We can take for example $a_2 = d \max_{i,j} |\ln|\lambda_{i,j}|| + 1$, where $\lambda_{i,j}$ are eigenvalues of A_i $(i = 1, ..., d)$. Let us consider the lower bound :

4.1. Preliminary lemmas.

Let K_1 be an algebraic number field of degree s_1 over $\mathbb{Q}$. Then there are s_1 distinct monomorphisms $\sigma_i : K_1 \to \mathbb{C}$, $i = 1, ..., s_1$ [see, e.g., Al, p.112]. By [BS, p.401], [Al, p.222], we get

$$N_{K_1/\mathbb{Q}}(\xi) = \sigma_1(\xi) \cdots \sigma_{s_1}(\xi). \tag{4.1}$$

If $\xi \in K_1 \setminus 0$ is an algebraic integer, then

$$|N_{K_1/\mathbb{Q}}(\xi)| \geq 1. \tag{4.2}$$

Let $\eta_1, ..., \eta_d$ be units of K_1 with $\eta_1^{n_1} \cdots \eta_d^{n_d} = 1 \Longleftrightarrow n_1 = ... = n_d = 0$. Let

$$\chi_i(\mathbf{n}) = \sum_{j=1}^{d} n_j \ln|\sigma_i(\eta_j)| \quad i = 1, ..., s_1.$$

Repeating the proof of ([KaNi], Lemma 6.2.14), we obtain :

Lemma 4.1. *There exists a constant* $a_3 = a_3(\eta_1, ..., \eta_d, K_1) > 0$ *such that*

$$\max_{i \in [1,s_1]} \chi_i(\mathbf{n}) \geq a_3|\mathbf{n}|.$$

We need the following lemma on abelian groups (see [Ln], Lemma 7.2, p. 40) :

Lemma 4.2 *Let* $\mathbb{V} \xrightarrow{\vartheta} \mathbb{V}'$ *be a surjective homomorphism of abelian groups, and assume that* $\mathbb{V}'$ *is free. Let* $\mathbb{W}_1$ *be the kernel of* ϑ*. Then there exists a subgroup* $\mathbb{W}_2$ *of* $\mathbb{V}$ *such that the restriction of* ϑ *to* $\mathbb{W}_2$ *induces an isomorphism of* $\mathbb{W}_2$ *with* $\mathbb{V}'$*, and such that* $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$.

We recall some lemmas from linear algebra :

Lemma 4.3 ([Ho], p.267, Theorem 13) *Let* $\mathbb{C}_1$ *be a subfield of the field of complex numbers* $\mathbb{C}$*, let* $\mathbb{V}$ *a finite-dimensional vector space over* $\mathbb{C}_1$*, and let* $\mathbb{T}$ *be a linear operator on* $\mathbb{V}$*. There is a semi-simple operator* $\mathbb{S}$ *on* $\mathbb{V}$ *and a nilpotent operator* $\mathbb{H}$ *on* $\mathbb{V}$ *such that*

(i) $\mathbb{T} = \mathbb{S} + \mathbb{H}$*;*

(ii) $\mathbb{S}\mathbb{H} = \mathbb{H}\mathbb{S}$*.*

Furthermore, the semi-simple $\mathbb{S}$ *and nilpotent* $\mathbb{H}$ *satisfying (i) and (ii) are unique, and* *each is a polynomial in* $\mathbb{T}$.

Lemma 4.4 ([Ma], p.77, ref. 4.21.1) *Let* $M_s(\mathbb{C})$ *be the set of* s*-square matrices with entries in* $\mathbb{C}$*. If* $B_i \in M_s(\mathbb{C})$ $(i = 1, ..., d)$ *pairwise commute [i.e.* $B_iB_j = B_jB_i,\ (i, j = 1, ..., d)$*], then there exists a unitary matrix* U *(i.e.* $U^* = U^{-1}$*) such that* U^*B_iU *is an upper triangular matrix for* $i = 1, ..., d$*, where* U^* *- conjugate transpose of* $U \in M_s(\mathbb{C})$*.*

Lemma 4.5 ([Ga], p.224, Corollary 2) *If the linear operators* $A, B, ..., L$ *pairwise commute and all the eigenvalues of these operators belong to the ground field K, then the whole space R can be split into subspaces* $I_1, ..., I_w$ *invariant with respect to all the operators such that each operator* $A, B, ..., L$ *has equal eigenvalues in each of them.*

4.2. Invariant subspaces.

We consider matrices $A_1, ..., A_d$, the space $\mathbb{C}^s$ and we apply Lemma 4.5:

Let $I_1, ..., I_w$ be corresponding invariant subspaces of $\mathbb{C}^s$ with $\dim I_j = r_j,\ j = 1, ..., w$, $r_1 + \cdots + r_w = s$. There exists a matrix $U_1 \in M_s(\mathbb{C})$ such that $T_i = U_1A_iU_1^{-1}$ have the following block diagonal structure: $T_i = T_{1,i} \oplus \cdots \oplus T_{w,i}$ with $r_j \times r_j$ commuting matrices $T_{j,i}$ with equal eigenvalues $(j = 1, ..., w,\ i = 1, ..., d)$. We denote by $\lambda_{j,i}$ the unique eigenvalue of $T_{j,i}$ in the subspace I_i. It is easy to see that $\lambda_{1,i}, ..., \lambda_{w,i}$ are all eigenvalues of A_i $(i = 1, ..., d)$.

Now we consider matrices $T_{j,1}, ..., T_{j,d}$ and we use Lemma 4.4. We have that there exists a matrix $U_2 \in M_s(\mathbb{C})$ such that

$$\Lambda_i = U_2A_iU_2^{-1}, \qquad i = 1, ..., d, \tag{4.3}$$

have the following block diagonal structure:

$$\Lambda_1 = \begin{pmatrix} \Lambda_{1,1} & & 0 \\ & \ddots & \\ 0 & & \Lambda_{w,1} \end{pmatrix}, \cdots, \Lambda_d = \begin{pmatrix} \Lambda_{1,d} & & 0 \\ & \ddots & \\ 0 & & \Lambda_{w,d} \end{pmatrix},$$

with $r_j \times r_j$ commuting upper triangular matrices $\Lambda_{j,i}$ $(j = 1, ..., w,\ i = 1, ..., d)$. Hence

$$A_1^{n_1} \cdots A_d^{n_d} = U_2^{-1}\Lambda_1^{n_1} \cdots \Lambda_d^{n_d}U_2, \quad \text{and} \quad \Lambda_1^{n_1} \cdots \Lambda_d^{n_d} = \begin{pmatrix} \widetilde{\Lambda}_1(\mathbf{n}) & & 0 \\ & \ddots & \\ 0 & & \widetilde{\Lambda}_w(\mathbf{n}) \end{pmatrix}, \tag{4.4}$$

where $\mathbf{n} = (n_1, ..., n_d)$, and $\widetilde{\Lambda}_j(\mathbf{n})$ is an upper-triangular matrix with $\lambda_{j,1}^{n_1} \cdots \lambda_{j,d}^{n_d}$ on the diagonal $(1 \leq j \leq w)$. Let $\widetilde{\Lambda}_j(\mathbf{n}) = (\widetilde{\lambda}^{(j)}_{\nu_1,\nu_2}(\mathbf{n}))_{1 \leq \nu_1, \nu_2 \leq r_j}$. Using the formula for the degree of Jordan's normal form of matrices $\Lambda_{j,i}$ (see, e.g., [Ga, pp. 157,158]), we get that

$$\widetilde{\lambda}^{(j)}_{\nu_1,\nu_2}(\mathbf{n}) = \lambda_{j,1}^{n_1} \cdots \lambda_{j,d}^{n_d} P^{(j)}_{\nu_1,\nu_2}(\mathbf{n}) \tag{4.5}$$

for some polynomial $P^{(j)}_{\nu_1,\nu_2}$. It is easy to see that

$$P^{(j)}_{\nu_1,\nu_1}(\mathbf{n}) = 1 \quad \text{and} \quad P^{(j)}_{\nu_1,\nu_2}(\mathbf{n}) = 0 \quad \text{for} \quad \nu_1 > \nu_2. \tag{4.6}$$

Taking into account that $\lambda_{j,1}^{n_1} \cdots \lambda_{j,d}^{n_d}$ is an eigenvalue of $A_1^{n_1} ... A_d^{n_d}$, we obtain from Definition 1 that

$$\lambda_{j,1}^{n_1} \cdots \lambda_{j,d}^{n_d} = 1 \quad \Longleftrightarrow \quad (n_1, ..., n_d) = \mathbf{0}, \quad \text{with} \quad j \in [1, r]. \tag{4.7}$$

Now we decompose $\Lambda_{j,i}$ to semisimple (i.e. diagonalizable) and nilpotent components. Let $\mathbb{I}_r$ be an $r \times r$ identity matrix, $\Lambda_{j,i,1} = \lambda_{j,i}\mathbb{I}_{r_j}$, $\Lambda_{j,i,2} = \Lambda_{j,i} - \Lambda_{j,i,1}$, $\Lambda_{j,i,3} = \mathbb{I}_{r_j} - \lambda_{j,i}^{-1}\Lambda_{j,i,2}$

$$\Lambda_{i,l} = \Lambda_{1,i,l} \oplus \cdots \oplus \Lambda_{w,i,l}, \qquad A_{i,l} = U_2^{-1}\Lambda_{i,l}U_2, \quad l = 1,2,3.$$

We see that $\Lambda_{j,i} = \Lambda_{j,i,1}\Lambda_{j,i,3}$ and $A_{i,1}$ are the semisimple matrices, $A_{i,2}$ is the nilpotent matrix, $A_{i,3}$ is the unipotent matrix,

$$A_i = A_{i,1} + A_{i,2}, \qquad \text{and} \quad A_i = A_{i,1}A_{i,3}, \quad i = 1,...,d.$$

By Lemma 4.3 there exists only one decomposition of a matrix to semisimple and nilpotent components. Applying Lemma 4.3 we obtain that $A_{i,l}$ is a polynomial of A_i $(i = 1,...,d,$
$l = 1,2,3)$. Hence, they are commuting matrices, and

$$A_1^{n_1} \cdots A_d^{n_d} = A_{1,1}^{n_1} \cdots A_{d,1}^{n_d}(A_{1,3}^{n_1} \cdots A_{d,3}^{n_d}). \tag{4.8}$$

Applying (4.5), we get

$$|A_{1,3}^{n_1} \cdots A_{d,3}^{n_d}\mathbf{m}| = O(|\mathbf{n}|^{sd}|\mathbf{m}|).$$

Therefore there exists a constant $\dot{c}_0 > 0$, such that

$$|A_{1,3}^{n_1} \cdots A_{d,3}^{n_d}\mathbf{m}| \leq \dot{c}_0|\mathbf{n}|^{sd}|\mathbf{m}| \qquad \text{and} \qquad 1 \leq |\mathbf{m}| \leq \dot{c}_0|\mathbf{n}|^{sd}|A_{1,3}^{n_1} \cdots A_{d,3}^{n_d}\mathbf{m}|. \tag{4.9}$$

From (4.8) and (4.9), we get

$$|A_1^{n_1} \cdots A_d^{n_d}\mathbf{m}| \geq \dot{c}_0^{-1}|\mathbf{n}|^{-sd}|A_{1,1}^{n_1} \cdots A_{d,1}^{n_d}\mathbf{m}|. \tag{4.10}$$

Thus, to prove Theorem 4, it is enough to verify (2.15) for the semisimple case, i.e. when $A_i = A_{i,1}$ $(i = 1,...,d)$. In this case,

$$\Lambda_i = \mathrm{diag}[\theta_{1,i},...,\theta_{s,i}], \qquad \text{with} \qquad \theta_{l,i} = \lambda_{j,i}, \quad \text{for} \quad l \in (r'_{j-1}, r'_j], \tag{4.11}$$

where $r'_j = r_1 + \cdots + r_j$, $r'_0 = 0$ $(l \in [1,s], j \in [1,w], i \in [1,d])$.

Let $K_2 = \mathbb{Q}(\lambda_{1,1},...,\lambda_{w,1},...,\lambda_{1,d},...,\lambda_{w,d})$, be the algebraic number field of degree s_2, and let $\sigma_1,...,\sigma_{s_2}$ be distinct monomorphisms $\sigma_i : K_2 \to \mathbb{C}$, $i = 1,...,s_2$. The first part of the following result is mentioned without the complete proof found in [Ga, p. 220]:

Lemma 4.6. *There exist an invertible matrix* $T = (t_{i,j})_{1\leq i,j\leq s}$ *with* $t_{i,j} \in K_2$, $(1 \leq i,j \leq s)$ *and constant* $c_3 > 0$ *such that*

$$\Lambda_i = TA_iT^{-1} \qquad (i = 1,...,d) \qquad \text{and} \qquad |\widetilde{m}_j| \geq c_3|\mathbf{m}|^{-s_2+1}, \quad \text{for} \quad \widetilde{m}_j \neq 0, \tag{4.12}$$

where $\widetilde{\mathbf{m}} = (\widetilde{m}_1,...,\widetilde{m}_s)^{(t)} = T\mathbf{m}$.

Proof. We consider the following system of linear equations:

$$XA_i = \Lambda_iX, \quad i = 1,...,d \quad \text{with} \quad X = (x_{j,\nu})_{1\leq j,\nu\leq s}. \tag{4.13}$$

By (4.3) there exists the nontrivial solution $U_2 \in M_s(\mathbb{C})$ of this system. Hence there exists a partition G_1, G_2 of $[1,s]^2$ with $G_1 \cup G_2 = [1,s]^2$, $G_1 \cap G_2 = \emptyset$, $\min(\#G_1, \#G_2) \geq 1$, and

$$x_\kappa = \mathfrak{g}_\kappa(\tilde{X}), \quad \text{with} \quad \tilde{X} = \{x_\omega \mid \omega \in G_2\}, \tag{4.14}$$

where $\mathfrak{g}_\kappa$ is a linear form with coefficients in K_2, $\kappa \in G_1$. We see that

$$\det X = \mathfrak{g}(\tilde{X}),$$

where $\mathfrak{g}$ is some polynomial with coefficients in K_2.

Bearing in mind that $\det U_2 \neq 0$, we get that $\mathfrak{g}(\tilde{X}) \not\equiv 0$ for $\tilde{X} \in \mathbb{C}^{\#G_2}$. Taking into account that K_2 contains infinitely many elements, we obtain (by induction on $\#G_2$) that $\mathfrak{g}(\tilde{X}) \not\equiv 0$ for $\tilde{X} \in K_2^{\#G_2}$. Let $\mathfrak{g}(\tilde{T}) \neq 0$ with $\tilde{T} \in K_2^{\#G_2}$. From (4.14), we get that there exists a solution $T = (t_{j,\nu})_{1 \leq j,\nu \leq s}$ of the system (4.13) with $t_{j,\nu} \in K_2,\ (1 \leq j, \nu \leq s)$ and $\det T \neq 0$. Let $\mathcal{D}(K_2)$ be the ring of algebraic integers of the field K_2. We take an integer $q_0 \geq 1$ such that

$$q_0 t_{j,\nu} \in \mathcal{D}(K_2), \qquad j, \nu = 1, ..., s. \tag{4.15}$$

Let

$$\widetilde{m}_i = \sum_{j=1}^{s} t_{i,j} m_j, \quad \text{and} \quad \widetilde{\mathbf{m}} = (\widetilde{m}_1, ..., \widetilde{m}_s)^{(t)} = T\mathbf{m}. \tag{4.16}$$

By (4.1) and (4.2), we have

$$|N_{K_2/\mathbb{Q}}(q_0 \widetilde{m}_i)| = q_0^{s_2} |\sigma_1(\widetilde{m}_i) \cdots \sigma_{s_2}(\widetilde{m}_i)| \geq 1 \quad \text{for} \quad \widetilde{m}_i \neq 0. \tag{4.17}$$

Using (4.16) and (4.17), we get

$$|\widetilde{m}_i| \geq c_3 |\mathbf{m}|^{-s_2+1}, \quad \text{for} \quad \widetilde{m}_i \neq 0, \quad \text{where} \quad c_3 = q_0^{-s_2} (s \max_{i,j,k} \sigma_k(t_{i,j}))^{-s_2+1}.$$

Hence Lemma 4.6 is proved. ■

Bearing in mind that

$$A_1^{n_1} \cdots A_s^{n_s} \mathbf{m} = T^{-1} \Lambda_1^{n_1} \cdots \Lambda_s^{n_s} \widetilde{\mathbf{m}}, \tag{4.18}$$

we obtain that (2.15) is a result the following inequality

$$|\Lambda_1^{n_1} \cdots \Lambda_d^{n_d} \widetilde{\mathbf{m}}| \geq c_4 |\mathbf{m}|^{-b_1} \exp(a_1 |\mathbf{n}|) \quad \text{for} \quad \mathbf{m} \neq \mathbf{0}$$

with some $c_4 > 0$. Let

$$\mathbb{G} = \{i \in [1, s] \mid \widetilde{m}_i \neq 0\}. \tag{4.19}$$

By (4.11) and (4.12), to obtain (2.15), it is enough to prove that

$$\max_{j \in \mathbb{G}} |\theta_{j,1}^{n_1} \cdots \theta_{j,d}^{n_d}| \geq c_5 |\mathbf{m}|^{-b_2} \exp(a_1 |\mathbf{n}|), \qquad \forall \mathbf{n} \in \mathbb{Z}^d \ \text{with} \ A_1^{n_1} \cdots A_s^{n_s} \mathbf{m} \in \mathbb{Z}^s \setminus \mathbf{0} \tag{4.20}$$

for some $a_1, b_2, c_5 > 0$.

Let $\mathbf{e}_1, ..., \mathbf{e}_s$ be a standard basis of $\mathbb{Z}^s$, $T^{-1} = (\tilde{t}_{i,j})_{1 \leq i,j \leq s}$,

$$\widetilde{\mathbf{e}}_i = \sum_{j=1}^{s} \tilde{t}_{j,i} \mathbf{e}_j, \quad \text{and} \quad \widetilde{\mathbf{m}} = (\widetilde{m}_1, ..., \widetilde{m}_s)^{(t)} = T\mathbf{m}.$$

By ([Ga], pp. 59, 60 and 73), $\widetilde{m}_1, ..., \widetilde{m}_s$ are coordinates of vector $\mathbf{m}$ in the basis $\widetilde{\mathbf{e}}_1, ..., \widetilde{\mathbf{e}}_s$, Λ_i is the matrix of the operator A_i in the basis $\widetilde{\mathbf{e}}_1, ..., \widetilde{\mathbf{e}}_s$ $(i = 1, ..., d)$, and $\widetilde{\mathbf{e}}_1, ..., \widetilde{\mathbf{e}}_s$ are eigenvectors of $A_1, ..., A_d$ in $\mathbb{C}^s$. Hence

$$\mathbf{m} = \widetilde{m}_1 \widetilde{\mathbf{e}}_1 + \cdots + \widetilde{m}_s \widetilde{\mathbf{e}}_s = \sum_{i \in \mathbb{G}} \widetilde{m}_i \widetilde{\mathbf{e}}_i.$$

Let $\mathbb{V}$ be a subspace of $\mathbb{C}^s$ with basis $\{\widetilde{\mathbf{e}}_i \mid i \in \mathbb{G}\}$, $\Gamma_0 = \mathbb{V} \cap \mathbb{Z}^s$, and let $\mathfrak{G}$ be the set of all of distinct lattices Γ_0 . Note that $\#\mathfrak{G} \leq 2^s$ (the number of subsets $\mathbb{G}$ of $[1, s]$, see (4.19)). We denote by $\mathbb{V}_0$ the $\mathbb{C}$-linear span of Γ_0. We see that $\mathbb{V}, \Gamma_0$ and $\mathbb{V}_0$ are $A_1, ..., A_d$ invariant subsets in $\mathbb{C}^s$. Let $d_0 = \dim \Gamma_0$. Taking into account that $\mathbf{m} \in \mathbb{V}$ and $\mathbf{m} \in \Gamma_0$, we get that $d_0 \geq 1$. Let $\check{\mathbf{e}}_1, ..., \check{\mathbf{e}}_{d_0}$ be a basis of Γ_0, and let $\check{A}_1, ..., \check{A}_d$ be matrices of operators $A_i: \ \mathbb{C}^s \to \mathbb{C}^s$ $(i = 1, ..., d)$ restricted in $\mathbb{V}_0$ in the basis $\check{\mathbf{e}}_1, ..., \check{\mathbf{e}}_{d_0}$.

It is easy to see that $\check{A}_1, ..., \check{A}_d$ are integer matrices, and $\check{\mathbf{A}}^{\mathbf{n}} := \check{A}_1^{n_1}, ..., \check{A}_d^{n_d}$ is a matrix with rational coefficients. Hence the characteristic polynomial $\phi_{\mathbf{n}}$ of $\check{\mathbf{A}}^{\mathbf{n}}$ has rational coefficients. Let $\mathbf{h} \in \mathbb{V}_0$ be an eigenvector of $\check{\mathbf{A}}^{\mathbf{n}}$, and β a corresponding eigenvalue. We see that $\mathbf{h} \in \mathbb{V}$ is an eigenvector of $A_1^{n_1} \cdots A_d^{n_d}$ restricted on $\mathbb{V}$. Therefore β is an eigenvalue of $A_1^{n_1} \cdots A_d^{n_d}|_{\mathbb{V}}$. Taking into account that all eigenvalues of $A_1^{n_1} \cdots A_d^{n_d}|_{\mathbb{V}}$ are $\theta_{l,1}^{n_1} \cdots \theta_{l,d}^{n_d}$ with $l \in \mathbb{G}$, we get that there exists $l_0 \in \mathbb{G}$ such that $\beta = \theta_{l_0,1}^{n_1} \cdots \theta_{l_0,d}^{n_d}$. By (4.11) there exists $j_0 \in [1, w]$, such that

$$\beta = \theta_{l_0,1}^{n_1} \cdots \theta_{l_0,d}^{n_d} = \lambda_{j_0,1}^{n_1} \cdots \lambda_{j_0,d}^{n_d}. \tag{4.21}$$

In §4.4 we will prove that there exists $a_1, b_2, c_5 > 0$ such that

$$|\sigma_\nu(\beta)| = |\sigma_\nu(\theta_{l_0,1}^{n_1}) \cdots \sigma_\nu(\theta_{l_0,d}^{n_d})| \geq c_5 \exp(a_1 |\mathbf{n}|) |\mathbf{m}|^{-b_2} \qquad (\mathbf{m} \neq \mathbf{0}) \tag{4.22}$$

for some $\nu \in [1, s_2]$. Bearing in mind that for all $\nu \in [1, s_2]$: $\sigma_\nu(\beta)$ is a root of $\phi_{\mathbf{n}}$, we get that there exists an eigenvector $\mathbf{h}_\nu \in \mathbb{V}_0$ be of $\check{\mathbf{A}}^{\mathbf{n}}$. We have that $\mathbf{h}_\nu \in \mathbb{V}$ is the eigenvector of $A_1^{n_1} \cdots A_d^{n_d}|_{\mathbb{V}}$, and $\sigma_\nu(\beta)$ is an eigenvalue of $A_1^{n_1} \cdots A_d^{n_d}|_{\mathbb{V}}$. Similarly to (4.21), we obtain that there exists $l_1 \in \mathbb{G}$ with

$$\sigma_\nu(\beta) = \theta_{l_1,1}^{n_1} \cdots \theta_{l_1,d}^{n_d}.$$

Now Theorem 4 follows from (4.22) and (4.20).

4.3. Some notations and inequalities from divisor theory.

Let $\mathfrak{D}$ be the group of divisors of the field K_2, $K_2^* = K_2 \setminus 0$. Consider the homomorphism from K_2^* to $\mathfrak{D}$. We denote the image of the element $\xi \in K_2^*$ by $\mathrm{div}(\xi)$. By [BS, p.217],

$$N_{K_2/\mathbb{Q}}(\mathrm{div}(\xi)) = |\mathrm{N}_{\mathrm{K_2/\mathbb{Q}}}(\xi)|. \tag{4.23}$$

If $\mathfrak{d}$ divides the rational prime p and if $\mathfrak{d}$ has degree $\dot{\mathfrak{f}}$, then ([BS, p.217])

$$N_{K_2/\mathbb{Q}}(\mathfrak{d}) = p^{\dot{\mathfrak{f}}}.$$

Let $\mathfrak{d}_1, ..., \mathfrak{d}_\mu$ be the set of all prime divisors of $\mathfrak{D}$ such that for all $\nu \in [1, \mu]$ there exists $(i, j) \in [1, d] \times [1, w]$ with $\lambda_{j,i} \equiv 0 \mod \mathfrak{d}_\nu$. Thus

$$\mathrm{div}(\lambda_{\mathrm{j,i}}) = \prod_{\nu=1}^{\mu} \mathfrak{d}_\nu^{\mathrm{b_{i,j,\nu}}}$$

for some nonnegative integers $b_{i,j,\nu}$, $(i, j, \nu) \in [1, d] \times [1, w] \times [1, \mu]$. Let

$$N_{K_2/\mathbb{Q}}(\mathfrak{d}_\nu) = p_\nu^{\dot{\mathfrak{f}}_\nu}. \tag{4.24}$$

Fixing $j_0 \in [1, w]$, we obtain

$$\mathrm{div}(\lambda_{\mathrm{j_0,1}}^{\mathrm{n_1}} \cdots \lambda_{\mathrm{j_0,d}}^{\mathrm{n_d}}) = \prod_{\nu=1}^{\mu} \mathfrak{d}_\nu^{\mathrm{l}_\nu(\mathbf{n})} \tag{4.25}$$

where

$$l_\nu(\mathbf{n}) = \sum_{i=1}^{d} n_i b_{i,j_0,\nu}. \tag{4.26}$$

Let

$$\mathbf{l}(\mathbf{n}) = (l_1(\mathbf{n}), ..., l_\mu(\mathbf{n})),$$

and let

$$l^+(\mathbf{n}) = \max_{i\in[1,\mu]}(0, l_i(\mathbf{n})), \qquad\qquad l^-(\mathbf{n}) = \max_{i\in[1,\mu]}(0, -l_i(\mathbf{n})).$$

We see that

$$\max(l^+(\mathbf{n}), l^-(\mathbf{n})) \leq |\mathbf{l}(\mathbf{n})| \leq \mu \max(l^+(\mathbf{n}), l^-(\mathbf{n})). \tag{4.27}$$

Let $\mathbf{m}^{'} = A_1^{n_1} \cdots A_d^{n_d}\mathbf{m} \in \mathbb{Z}^s \setminus \mathbf{0}$, and $\widetilde{\mathbf{m}}^{'} = (\widetilde{m}_1^{'}, ..., \widetilde{m}_s^{'})^{(t)} = T\mathbf{m}^{'}$.

By (4.15), (4.16) and (4.18), we have that $\widetilde{\mathbf{m}}^{'} = \Lambda_1^{n_1} \cdots \Lambda_d^{n_d}\widetilde{\mathbf{m}}$ and $q_0\widetilde{m}_l^{'} \in K_2$ $(l = 1, ..., s)$ are algebraic integers. From (4.19) and (4.21), we obtain

$$\widetilde{m}_{l_0}^{'} = \theta_{l_0,1}^{n_1} \cdots \theta_{l_0,d}^{n_d}\widetilde{m}_{l_0} = \lambda_{j_0,1}^{n_1} \cdots \lambda_{j_0,d}^{n_d}\widetilde{m}_{l_0} \neq 0 \qquad \text{for} \quad l_0 \in \mathbb{G},$$

with some $j_0 \in [1, w]$. Hence

$$\mathrm{div}(\mathrm{q_0\widetilde{m}_{l_0}}) = \mathrm{div}(\mathrm{q_0\widetilde{m}_{l_0}^{'}})\mathrm{div}(\lambda_{\mathrm{j_0,1}}^{-\mathrm{n_1}} \cdots \lambda_{\mathrm{j_0,d}}^{-\mathrm{n_d}}). \tag{4.28}$$

Let $l^-(\mathbf{n}) > 0$. Then there exists $i_0 \in [1, \mu]$ with $-l_{i_0}(\mathbf{n}) = l^-(\mathbf{n})$. We have $\mathbf{m}, \mathbf{m}^{'} \in \mathbb{Z}^s \setminus \mathbf{0}$ and $q_0\widetilde{m}_{l_0}, q_0\widetilde{m}_{l_0}^{'}$ are algebraic integers. Bearing in mind (4.28) and (4.25), we get that

$$\mathrm{div}(\mathrm{q_0\widetilde{m}_{l_0}}) \equiv 0 \mod \mathfrak{d}_{\mathrm{i_0}}^{-\mathrm{l_{i_0}}(\mathbf{n})}.$$

By (4.23), (4.24) and (4.17), we obtain

$$1 \leq |N_{K_2/\mathbb{Q}}(q_0\widetilde{m}_{l_0})| = N_{K_2/\mathbb{Q}}(\mathrm{div}(\mathrm{q_0\widetilde{m}_{l_0}})) \equiv 0 \mod \mathrm{p}_{\mathrm{i_0}}^{-\mathrm{l_{i_0}}(\mathbf{n})},$$

and

$$2^{-l_{i_0}(\mathbf{n})} \leq |N_{K_2/\mathbb{Q}}(q_0\widetilde{m}_{l_0})| \leq (q_0 s \max_{i,j,\nu}|\sigma_\nu(t_{i,j})||\mathbf{m}|)^{s_2}.$$

Hence

$$l^-(\mathbf{n}) \leq c_6 + s_2\log_2|\mathbf{m}| \quad \text{with} \quad c_6 = s_2|\log_2(q_0 s \max_{i,j,\nu}|\sigma_\nu(t_{i,j})|)|. \tag{4.29}$$

We see that (4.29) is also true for $l^-(\mathbf{n}) = 0$. By (4.23), (4.24), and (4.25), we have that

$$|N_{K_2/\mathbb{Q}}(\lambda_{j_0,1}^{n_1} \cdots \lambda_{j_0,d}^{n_d})| = N_{K_2/\mathbb{Q}}\big(\mathrm{div}(\lambda_{\mathrm{j_0,1}}^{\mathrm{n_1}} \cdots \lambda_{\mathrm{j_0,d}}^{\mathrm{n_d}})\big) = \prod_{\nu=1}^{\mu} \mathrm{p}_\nu^{\dot{\mathfrak{f}}_\nu \mathrm{l}_\nu(\mathbf{n})} \geq 2^{\mathrm{l}^+(\mathbf{n}) - \mathrm{c_7}\mathrm{l}^-(\mathbf{n})},$$

where $c_7 = \mu\max_{\nu\in[1,\mu]}\dot{\mathfrak{f}}_\nu\log_2(p_\nu)$. Using (4.29), we obtain

$$|N_{K_2/\mathbb{Q}}(\lambda_{j_0,1}^{n_1} \cdots \lambda_{j_0,d}^{n_d})| \geq 2^{l^+(\mathbf{n}) - c_6c_7}|\mathbf{m}|^{-c_7s_2}. \tag{4.30}$$

4.4. End of the proof of Theorem 4. Let

$$\Gamma^{'} = \{\mathbf{l}(\mathbf{n}) \mid \mathbf{n} \in \mathbb{Z}^d\} \subseteq \mathbb{Z}^\mu, \quad \Gamma_1 = \{\mathbf{n} \in \mathbb{Z}^d \mid \mathbf{l}(\mathbf{n}) = \mathbf{0}\}. \tag{4.31}$$

Applying Lemma 4.2 with $\mathbb{V} = \mathbb{Z}^d$, $\mathbb{V}^{'} = \Gamma^{'}$ and $\mathbb{W}_1 = \Gamma_1$, we get that there exists a subgroup Γ_2 of $\mathbb{Z}^d$ isomorphic with $\Gamma^{'}$, and such that $\mathbb{Z}^d = \Gamma_1 \oplus \Gamma_2$.

Let $\kappa_1 = \dim \Gamma_1$, and $\kappa_2 = d - \kappa_1$. Consider the case of $\min(\kappa_1, \kappa_2) \geq 1$. Let $\mathbf{f}_1, ..., \mathbf{f}_d$ ($\mathbf{f}_i = (\tilde{f}_{1,i}, ..., \tilde{f}_{d,i})$) be a basis of $\mathbb{Z}^d$ such that $\mathbf{f}_1, ..., \mathbf{f}_{\kappa_1}$ is the basis of Γ_1 and $\mathbf{f}_{\kappa_1+1}, ..., \mathbf{f}_d$ is the basis of Γ_2.

For all $\mathbf{n} \in \mathbb{Z}^d$ there exist $\mathbf{n}_1 = (n_1^{(1)}, ..., n_d^{(1)}) \in \Gamma_1$, $\mathbf{n}_2 = (n_1^{(2)}, ..., n_d^{(2)}) \in \Gamma_2$, $\mathbf{k}_1 = (k_1, ..., k_{\kappa_1}) \in \mathbb{Z}^{\kappa_1}$ and $\mathbf{k}_2 = (k_{\kappa_1+1}, ..., k_d) \in \mathbb{Z}^{\kappa_2}$ such that

$$\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2, \quad \mathbf{n}_1 = k_1\mathbf{f}_1 + \cdots + k_{\kappa_1}\mathbf{f}_{\kappa_1}, \quad \text{and} \quad \mathbf{n}_2 = k_{\kappa_1+1}\mathbf{f}_{\kappa_1+1} + \cdots + k_d\mathbf{f}_d. \tag{4.32}$$

By (4.26), (4.31), (4.32) and Lemma 4.2, we have that there exists $c_0 > 1$ such that

$$c_0^{-1}|\mathbf{n}_i| \leq |\mathbf{k}_i| \leq c_0|\mathbf{n}_i|, \quad i = 1,2 \quad \text{and} \quad c_0^{-1}|\mathbf{k}_2| \leq |\mathbf{l}(\mathbf{n})| \leq c_0|\mathbf{k}_2|. \tag{4.33}$$

If $\kappa_i = 0$, then we will use (4.33) with $\mathbf{n}_i = \mathbf{0}$ and $\mathbf{k}_i = \mathbf{0}$ ($i = 1, 2$). By (4.32), we have

$$\dot{\theta}_0 := \lambda_{j_0,1}^{n_1} \cdots \lambda_{j_0,d}^{n_d} = \dot{\theta}_1\dot{\theta}_2, \quad \text{where} \quad \dot{\theta}_i := \lambda_{j_0,1}^{n_1^{(i)}} \cdots \lambda_{j_0,d}^{n_d^{(i)}} \quad i = 1, 2,$$

and

$$\dot{\theta}_1 = \eta_1^{k_1} \cdots \eta_{\kappa_1}^{k_{\kappa_1}}, \quad \text{where} \quad \eta_i := \lambda_{j_0,1}^{\tilde{f}_{1,i}} \cdots \lambda_{j_0,d}^{\tilde{f}_{d,i}} \quad i = 1, ..., \kappa_1.$$

From (4.25), (4.31) and (4.32), we obtain that $\eta_1, ..., \eta_{\kappa_1}, \dot{\theta}_1$ are units in K_2. Let $\mathbf{n}_2 = \mathbf{0}$. Using (4.7), we get that $\dot{\theta}_0 = \dot{\theta}_1 = 1$ if and only if $\mathbf{n}_1 = \mathbf{0}$, and

$$\eta_1^{k_1} \cdots \eta_{\kappa_1}^{k_{\kappa_1}} = 1, \quad \Longleftrightarrow \quad k_1 = \cdots = k_{\kappa_1} = 0.$$

Applying Lemma 4.1 and (4.11), we get that there exists a constant $a_4(l_0) > 0$, such that

$$|\max_{\nu \in [1,s_2]} \sigma_\nu(\dot{\theta}_1)| \geq \exp(a_4(l_0)|\mathbf{k}_1|) \geq \exp(a_4(l_0)|\mathbf{n}_1|/c_0).$$

Let $a_5 = c_0^{-1} \min_{l_0 \in \mathbb{G}} a_4(l_0)$. Hence, there exists $\nu_0 \in [1, s_2]$ such that

$$|\sigma_{\nu_0}(\lambda_{j_0,1}^{n_1^{(1)}} \cdots \lambda_{j_0,d}^{n_d^{(1)}})| \geq \exp(a_5|\mathbf{n}_1|). \tag{4.34}$$

We will need the following notations :

$$b_0 = 0.25a_5(1 + a_5)^{-1}d^{-1}(1 + \max_{i,j,\nu} |\ln|\sigma_\nu(\lambda_{j,i})||)^{-1}, \quad a_6 = d \max_{i,j,\nu} |\ln|\sigma_\nu(\lambda_{j,i})||,$$

$$b_4 = 2b_0^{-1}c_0^2\mu s_2 d \max_{i,j,\nu} |\ln|\sigma_\nu(\lambda_{j,i})||/\ln 2, \qquad a_1 = \min(a_5/4, a_6, b_0c_0^{-2}\mu^{-1}s_2^{-1}\ln 2),$$

$$\varkappa(\mathbf{m}) = b_0^{-1}c_0^2\mu(c_6 + s_2\log_2(|\mathbf{m}|)), \qquad b_2 = \max(b_4, c_7), \tag{4.35}$$

$$c_8 = \exp(-2b_0^{-1}c_0^2\mu c_6 d \max_{i,j,\nu} |\ln|\sigma_\nu(\lambda_{j,i})||), \qquad c_5 = \min(c_8, 2^{-c_6c_7/s_2}).$$

Case 1. Let $\kappa_2 = 0$. Then $\mathbf{n}_1 = \mathbf{n}$ and (4.22) follows from (4.34) and (4.35).

Case 2. Let $|\mathbf{n}| \leq \varkappa(\mathbf{m})$. Then $-|\mathbf{n}| \geq |\mathbf{n}| - 2\varkappa(\mathbf{m})$, and

$$\min_{j,\nu} |\sigma_\nu(\lambda_{j,1}^{n_1} \cdots \lambda_{j,d}^{n_d}| \geq \exp(-|\mathbf{n}|d \max_{i,j,\nu} |\ln|\sigma_\nu(\lambda_{j,i})||) \tag{4.36}$$

$$\geq \exp((|\mathbf{n}| - 2\varkappa(\mathbf{m}))d \max_{i,j,\nu} |\ln|\sigma_\nu(\lambda_{j,i})||) \geq c_8 \exp(a_6|\mathbf{n}|)|\mathbf{m}|^{-b_4} \geq c_5 \exp(a_1|\mathbf{n}|)|\mathbf{m}|^{-b_2}.$$

Case 3. Let $l^+(\mathbf{n}) \geq b_0c_0^{-2}\mu^{-1}|\mathbf{n}|$. By (4.30) and (4.1), we have that there exists $\nu_0 \in [1, s_2]$ such that

$$|\sigma_{\nu_0}(\lambda_{j_0,1}^{n_1} \cdots \lambda_{j_0,d}^{n_d}| \geq 2^{l^+(\mathbf{n})/s_2 - c_6c_7/s_2}|\mathbf{m}|^{-c_7}$$

$$\geq 2^{b_0c_0^{-2}\mu^{-1}|\mathbf{n}|/s_2 - c_6c_7/s_2}|\mathbf{m}|^{-c_7} \geq c_5 \exp(a_1|\mathbf{n}|)|\mathbf{m}|^{-b_2}. \tag{4.37}$$

Case 4. Let $|\mathbf{n}| \geq \varkappa(\mathbf{m})$, and $\kappa_2 = d$. We see that $\kappa_1 = 0$, $\mathbf{n}_1 = \mathbf{0}$, and $\mathbf{n}_2 = \mathbf{n}$. By (4.27), (4.33) and (4.35), we have $b_0^{-1} > 4$ and

$$\max(l^+(\mathbf{n}), l^-(\mathbf{n})) \geq |\mathbf{l}(\mathbf{n})|/\mu \geq c_0^{-2}\mu^{-1}|\mathbf{n}_2| = c_0^{-2}\mu^{-1}|\mathbf{n}| \geq b_0^{-1}(c_6 + s_2\log_2(|\mathbf{m}|).$$

Bearing in mind (4.29), we obtain that $l^-(\mathbf{n}) \leq c_6 + s_2\log_2(|\mathbf{m}|)$, and $l^+(\mathbf{n}) > l^-(\mathbf{n})$. Thus

$$l^+(\mathbf{n}) \geq c_0^{-2}\mu^{-1}|\mathbf{n}| > b_0 c_0^{-2}\mu^{-1}|\mathbf{n}|.$$

Hence we can use the inequality (4.37).

Case 5. Let $|\mathbf{n}| \geq \varkappa(\mathbf{m})$, $d > \kappa_2 \geq 1$ and $l^+(\mathbf{n}) \leq b_0 c_0^{-2}\mu^{-1}|\mathbf{n}|$. By (4.29), (4.27), (4.35) and (4.33), we have that $l^-(\mathbf{n}) \leq b_0 c_0^{-2}\mu^{-1}\varkappa(\mathbf{m}) \leq b_0 c_0^{-2}\mu^{-1}|\mathbf{n}|$ and

$$|\mathbf{n}_2| \leq c_0|\mathbf{k}_2| \leq c_0^2|\mathbf{l}(\mathbf{n})| \leq c_0^2\mu\max(l^+(\mathbf{n}), l^-(\mathbf{n})) \leq b_0|\mathbf{n}| \leq |\mathbf{n}|/2.$$

Thus

$$|\mathbf{n}_1| \geq |\mathbf{n}| - |\mathbf{n}_2| \geq |\mathbf{n}|/2. \tag{4.38}$$

Using the definition of b_0 (see (4.35)), we obtain

$$\min_{j,\nu}|\sigma_\nu(\lambda_{j,1}^{n_1^{(2)}}\cdots\lambda_{j,d}^{n_d^{(2)}})| \geq \exp(-d|\mathbf{n}_2|\max_{i,j,\nu}|\ln|\sigma_\nu(\lambda_{j,i})||)$$

$$\geq \exp(-db_0|\mathbf{n}|\max_{i,j,\nu}|\ln|\sigma_\nu(\lambda_{j,i})||) \geq \exp(-a_5|\mathbf{n}|/4).$$

Applying (4.34) and (4.38), we have

$$|\sigma_{\nu_0}(\lambda_{j_0,1}^{n_1}\cdots\lambda_{j_0,d}^{n_d}| = |\sigma_{\nu_0}(\lambda_{j_0,1}^{n_1^{(1)}}\cdots\lambda_{j_0,d}^{n_d^{(1)}})||\sigma_{\nu_0}(\lambda_{j_0,1}^{n_1^{(2)}}\cdots\lambda_{j_0,d}^{n_d^{(2)}})|$$

$$\geq \exp(a_5|\mathbf{n}_1| - a_5|\mathbf{n}|/4) \geq \exp(a_5|\mathbf{n}|/4) \geq c_5\exp(a_1|\mathbf{n}|)|\mathbf{m}|^{-b_2}. \tag{4.39}$$

Now from (4.36) - (4.39), we get (4.22) and Theorem 4 for the semisimple case. Bearing in mind (4.10), we obtain that Theorem 4 is true for the general case. ■

5 Proof of Limit Theorems.

5.1 Proof of Theorem 5.

By the Cramér-Wold device, it is enough to prove that for arbitrary reals $\alpha_1, ..., \alpha_q$

$$\upsilon(\mathbf{N}, f, \mathbf{x}) = \frac{1}{\sigma(f)\sqrt{\alpha_1^2 + \cdots + \alpha_q^2}}\sum_{i=1}^{q}\frac{\alpha_i}{\sqrt{\breve{\mathbf{N}}_i}}\sum_{\mathbf{n_i}\in\mathfrak{R}_i(\mathbf{N}_i)} f(\mathbf{A}^{\mathbf{n}_i}\mathbf{x}) \xrightarrow{d} \mathcal{N}(0,1). \tag{5.1}$$

We consider first the case that f has a finite Fourier expansion :

Lemma 5.1. *Let $\sigma(f_L) > 0$. With notations as above :*

$$\iota(\hbar) = \lim_{\min_{i,j} N_{i,j}\to\infty}\int_{[0,1)^s}|\upsilon(\mathbf{N}, f_L, \mathbf{x})|^\hbar d\mathbf{x} = \begin{cases}\frac{\hbar!}{2^{\hbar/2}(\hbar/2)!}, & \text{if } \hbar \text{ is even},\\ 0, & \text{if } \hbar \text{ is odd}.\end{cases} \tag{5.2}$$

By the moment method, (5.1) follows from (5.2) for $f = f_L$ (see (2.12)). The proofs of the general case and of Lemma 5.1 are given below. We consider the following variant of the *S-unit* theorem (see, [SS], Theorem 1):

Let K be an algebraic number field of degree $s_1 \geq 1$. Write K^* for its multiplicative group of nonzero elements. We consider the equation

$$\sum_{i=1}^{h_1} P_i(\mathbf{n})\boldsymbol{\vartheta}_i^{\mathbf{n}} = 0 \tag{5.3}$$

in variables $\mathbf{n} = (n_1, ..., n_{d_1}) \in \mathbb{Z}^{d_1}$, where the P_i are polynomials with coefficients in K, $\boldsymbol{\vartheta}_i^{\mathbf{n}} = \vartheta_{i,1}^{n_1} \cdots \vartheta_{i,d_1}^{n_{d_1}}$, and $\vartheta_{i,j} \in K^*$ $(1 \leq i \leq h_1,\ 1 \leq j \leq d_1)$. Let U_1 be the potential number of nonzero coefficients of the polynomials $P_1, ..., P_{h_1}$, and $U = \max(d_1, U_1)$. A solution $\mathbf{n}$ of (5.3) is called *non-degenerate* if $\sum_{i \in I} P_i(\mathbf{n})\boldsymbol{\vartheta}_i^{\mathbf{n}} \neq 0$ for every nonempty subset I of $\{1, ..., h_1\}$. Let G be the subgroup of $\mathbb{Z}^{d_1}$ consisting of vectors $\mathbf{n}$ with $\boldsymbol{\vartheta}_1^{\mathbf{n}} = \cdots = \boldsymbol{\vartheta}_{h_1}^{\mathbf{n}}$.

Theorem B. ([SS]) *Suppose $G = \{\mathbf{0}\}$. Then the number $\mathfrak{U}(P_1, ..., P_{h_1})$ of non-degenerate solutions $\mathbf{n} \in \mathbb{Z}^{d_1}$ of equation (5.3) satisfies the estimate*

$$\mathfrak{U}(P_1, ..., P_{h_1}) \leq \mathbb{U}(d_1, \mathbb{P}) = 2^{35U^3} s_1^{6U^2}.$$

It is easy to get the following

Corollary 5.1. *Let $d_1 = d(h_1 - 1)$, $\vartheta_{h_1,j} = 1$ $(j = 1, ..., d)$, $\vartheta_{i,j+(i-1)d} = \vartheta_j \in K^*$ and $\vartheta_{i,j+\mu d} = 1$ $(\mu \in [0, h_1 - 2],\ \mu \neq i - 1,\ i = 1, ..., h_1 - 1,\ j = 1, ..., d)$, $\overline{\mathbf{n}} = (\mathbf{n}_1, ..., \mathbf{n}_{h_1-1})$, $\mathbf{n}_i = (n_{i,1}, ..., n_{i,d})$ with $i = 1, ..., h_1 - 1$, $P_{h_1}(\overline{\mathbf{n}}) \equiv -1$. Suppose*

$$\vartheta_1^{n_1} \cdots \vartheta_d^{n_d} = 1 \quad \Longleftrightarrow \quad (n_1, ..., n_d) = \mathbf{0}. \tag{5.4}$$

Then the number $\mathfrak{U}'(P_1, ..., P_{h_1-1})$ of non-degenerate solutions $\overline{\mathbf{n}} \in \mathbb{Z}^{d_1}$ of the equation

$$\sum_{i=1}^{h_1-1} P_i(\overline{\mathbf{n}})\boldsymbol{\vartheta}_i^{\overline{\mathbf{n}}} \;=\; \sum_{i=1}^{h_1-1} P_i(\overline{\mathbf{n}})\vartheta_1^{n_{i,1}} \cdots \vartheta_d^{n_{i,d}} = 1$$

satisfies the estimate

$$\mathfrak{U}'(P_1, ..., P_{h_1-1}) \leq \mathbb{U}(d_1, \mathbb{P}).$$

Remark 1. In this paper we need only the estimate $\mathfrak{U}'(P_1, ..., P_{h_1-1}) \leq \mathbb{U}$, where a constant $\mathbb{U}$ depends only on s, d and h_1.

Remark 2. The condition defining the group G is equivalent to the condition (5.4) in terms of Corollary 5.1. In this paper, the validity of (5.4) follows from the partially hyperbolic property of the action $\mathcal{A}$ (see (4.7) and Definition 1). It is known that if $\mathcal{A}$ has the partially hyperbolic property, then $\mathbf{A}^{\mathbf{n}}$ is ergodic with respect to the Lebesgue measure for all $\mathbf{n} \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$. According to [ScWa] the partially hyperbolic action $\mathcal{A}$ is mixing of all orders.

Definition 5.1. *Let $F^{(\hbar)} = \{1, ..., \hbar\}$, $F \subseteq F^{(\hbar)}$, $\beta_F = \#F$, $F = (F(1), ..., F(\beta_F))$, $\overline{\mathbf{n}} = (\mathbf{n}_1, ..., \mathbf{n}_\hbar)$, $\overline{\mathbf{n}}^{(F^{(\hbar)})} = \overline{\mathbf{n}}$, $\overline{\mathbf{n}}^{(F)} = (\mathbf{n}_{F(1)}, ..., \mathbf{n}_{F(\beta_F)})$, with $\mathbf{n}_i = (n_{i,1}, ..., n_{i,d})$, $\mathcal{P} = \{\mathbf{p} = (p_1, p_2) \mid 1 \leq p_1 \leq w,\ 1 \leq p_2 \leq r_{p_1}\}$ (see (4.3), (4.4)), $\mathbf{p}^{(1)} \prec \mathbf{p}^{(2)}$ if $p_1^{(1)} < p_1^{(2)}$ or if $p_1^{(1)} = p_1^{(2)}$ and $p_2^{(1)} < p_2^{(2)}$. Let*

$$\mathbf{C}(\overline{\mathbf{n}}^{(F)}) = \sum_{\mu \in F} T A_1^{n_{\mu,1}} \cdots A_d^{n_{\mu,d}} \mathbf{m}^{(\mu)} = \sum_{\mu \in F} \Lambda_1^{n_{\mu,1}} \cdots \Lambda_d^{n_{\mu,d}} \widetilde{\mathbf{m}}^{(\mu)}, \qquad \mathbf{C}(\overline{\mathbf{n}}^{(\emptyset)}) = \mathbf{0}, \tag{5.5}$$

where $\widetilde{\mathbf{m}} = (\widetilde{m}_{1,1}, ..., \widetilde{m}_{1,r_1}, ..., \widetilde{m}_{w,1}, ..., \widetilde{m}_{w,r_w})^{(t)} = T\mathbf{m}$ *(see (4.12)).*

We have that coordinates of a vector $\mathbf{x} \in \mathbb{R}^s$ can be enumerated by the set $\mathcal{P}$:

$$\mathbf{x} = (x_{(1,1)}, ..., x_{(1,r_1)}, ..., x_{(w,1)}, ..., x_{(w,r_w)}), \quad \text{with} \quad x_{(p_1,p_2)} = x_{\mathbf{p}} = x_{p_1,p_2}.$$

Hence $\mathbf{C}(\overline{\mathbf{n}}^{(F)}) = (\mathbf{C}(\overline{\mathbf{n}}^{(F)})_{\mathbf{p}})_{\mathbf{p}\in\mathcal{P}}$, with $\mathbf{C}(\overline{\mathbf{n}}^{(F)})_{\mathbf{p}} := (\mathbf{C}(\overline{\mathbf{n}}^{(F)}))_{\mathbf{p}}$. By (5.5) and (4.4)-(4.6), we get

$$\mathbf{C}(\overline{\mathbf{n}}^{(F)})_{p_1,p_2} = \sum_{\mu\in F} \sum_{1\leq\nu\leq r_{p_1}} \widetilde{\lambda}^{(p_1)}_{p_2,\nu}(\mathbf{n}_\mu)\widetilde{m}^{(\mu)}_{p_1,\nu} = \sum_{\mu\in F} \lambda^{n_{\mu,1}}_{p_1,1} \cdots \lambda^{n_{\mu,d}}_{p_1,d} \sum_{p_2\leq\nu\leq r_{p_1}} P^{(p_1)}_{p_2,\nu}(\mathbf{n}_\mu)\widetilde{m}^{(\mu)}_{p_1,\nu}. \tag{5.6}$$

Definition 5.2. *Let* $\mathbb{F}_0 = \widetilde{\mathbb{F}}_1 = \emptyset$, $\widetilde{\mathbf{m}}^{(j)} \neq \mathbf{0}$ $(j = 1, ..., \hbar)$, $\mathbf{p}_0 = (w, r_w)$,

$$\mathcal{P}_1 = \Big\{\mathbf{p} \in \mathcal{P} \mid \exists j \in [1,\hbar] \text{ with } \widetilde{m}^{(j)}_{\mathbf{p}} \neq 0\Big\}, \ \ \mathbf{p}_1 = \max_{\mathbf{p}\in\mathcal{P}_1} \mathbf{p}, \ \ \mathbb{F}_1 = \{j \in [1,\hbar] \mid \widetilde{m}^{(j)}_{\mathbf{p}_1} \neq 0\}. \tag{5.7}$$

For $i \geq 2$ *we denote* $\mathcal{P}_i, \mathbf{p}_i, \mathbb{F}_i$ *and* $\widetilde{\mathbb{F}}_i$ *recursively :*

$$\mathbf{p}_i = \max_{\mathbf{p}\in\mathcal{P}_i} \mathbf{p}, \qquad \mathfrak{f}_i = \#\mathbb{F}_i, \tag{5.8}$$

$$\mathbb{F}_i = \{j \in [1,\hbar] \setminus \widetilde{\mathbb{F}}_i \mid \widetilde{m}^{(j)}_{\mathbf{p}_i} \neq 0\}, \qquad \mathbb{F}_i = \{\mathbb{F}_i(1), ..., \mathbb{F}_i(\mathfrak{f}_i)\}, \qquad \widetilde{\mathbb{F}}_i = \cup_{l=1}^{i-1}\mathbb{F}_l, \tag{5.9}$$

where

$$\mathcal{P}_i = \Big\{\mathbf{p} \in \mathcal{P} \mid \mathbf{p} \prec \mathbf{p}_{i-1} \quad \text{and} \quad \exists j \in [1,\hbar] \setminus \widetilde{\mathbb{F}}_i \quad \text{with} \quad \widetilde{m}^{(j)}_{\mathbf{p}} \neq 0\Big\}. \tag{5.10}$$

Let $\mathfrak{k} = \max\{i \in [1, s] \mid \mathcal{P}_i \neq \emptyset\}$.

We have

$$\bigcup_{i=1}^{\mathfrak{k}} \mathbb{F}_i = [1,\hbar]. \tag{5.11}$$

Lemma 5.2. *Let* $\mathbf{C}(\overline{\mathbf{n}}^{(F^{(\hbar)})}) = \mathbf{0}$, *and* $i \in [1,\mathfrak{k}]$. *Then*

$$\mathbf{C}(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i}, \tag{5.12}$$

$$\mathbf{C}(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = L(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i}, \quad \text{where} \quad L(\overline{\mathbf{n}}^{(F)})_{\mathbf{p}} = \sum_{\mu\in F} \lambda^{n_{\mu,1}}_{p_1,1} \cdots \lambda^{n_{\mu,d}}_{p_1,d} \widetilde{m}^{(\mu)}_{p_1,p_2}, \tag{5.13}$$

and

$$L(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i}. \tag{5.14}$$

Proof. We need the following equality

$$\widetilde{m}^{(j)}_{\mathbf{p}} = 0 \quad \text{for} \quad \mathbf{p}_k \prec \mathbf{p} \quad \text{and} \qquad j \in \mathbb{F}_k, \qquad k = 1, ..., \mathfrak{k}. \tag{5.15}$$

Let $k = 1$. We see that (5.15) follows from (5.7). Consider the case $k \geq 2$. We have that $\mathbf{p}_l \prec \mathbf{p} \preceq \mathbf{p}_{l-1}$ for some $l \in [2, k]$. Let $\mathbf{p}_l \prec \mathbf{p} \prec \mathbf{p}_{l-1}$. We derive from (5.8) that $\mathbf{p} \notin \mathcal{P}_l$. By (5.10) and (5.11), we obtain that $\widetilde{m}^{(j)}_{\mathbf{p}} = 0$ for all $j \in [1,\hbar] \setminus \widetilde{\mathbb{F}}_l = \cup_{\nu\geq l}\mathbb{F}_\nu$. Bearing in mind that $l \leq k$, we get that $\mathbb{F}_k \subseteq \cup_{\nu\geq l}\mathbb{F}_\nu$ and (5.15) follows. Let $\mathbf{p} = \mathbf{p}_{l-1}$. We get from

(5.9) that if $\widetilde{m}^{(j)}_{\mathbf{p}_{l-1}} \neq 0$ for some $j \in [1,\hbar] \setminus \widetilde{\mathbb{F}}_{l-1}$, then $j \in \mathbb{F}_{l-1}$ and $j \notin \mathbb{F}_i, \ i \geq l$. Hence, for all $j \in \mathbb{F}_k$ we have $\widetilde{m}^{(j)}_{\mathbf{p}_{l-1}} = 0$. Thus (5.15) is true.

Let $k > i$, then $\mathbf{p}_k \prec \mathbf{p}_i$. From (5.15) we obtain that $\widetilde{m}^{(\mu)}_{\mathbf{p}} = 0$ for $\mu \in \mathbb{F}_k$, $\mathbf{p}_i \preceq \mathbf{p}$. Let $\mathbf{p}_i = (p_{i,1}, p_{i,2})$, then $\widetilde{m}^{(\mu)}_{p_{i,1},\nu} = 0$ for $\mu \in \mathbb{F}_k$, $p_{i,2} \leq \nu$. Using (5.6) and (5.9), we get (5.12).

By (5.15), we have that $\widetilde{m}^{(\mu)}_{p_{i,1},\nu} = 0$ for $\mu \in \mathbb{F}_i$, $p_{i,2} < \nu$. Applying (4.6) and (5.6), we obtain (5.13). Now from (5.12) and (5.13), we obtain (5.14). Hence Lemma 5.2 is proved. ■

Let $\eth_i \in [1,q], \ i = 1,...,h$ and

$$R(\mathbf{N},F,\mathbf{p}) = \{(\mathbf{n}_{F(1)},...,\mathbf{n}_{F(\beta)}) \mid \mathbf{n}_i \in \mathfrak{R}_{\eth_{F(i)}}, \ i = 1,...,\beta, \ \beta = \#F \quad \text{and} \quad \nexists F^{'} \subsetneq F \quad \text{with} \quad L(\overline{\mathbf{n}}^{(F^{'})})_{\mathbf{p}} = 0\}. \tag{5.16}$$

We do not suppose that $\mathfrak{R}_i(\mathbf{N}_i) \cap \mathfrak{R}_j(\mathbf{N}_i) = \emptyset$ for $i \neq j \in [1,q]$ in the following Lemma 5.3-Lemma 5.8 (see (2.16)).

Lemma 5.3. *Let $F \subseteq F^{(\hbar)}$, $\beta = \#F$, $\breve{\mathbf{N}}_F = \prod_{i \in F} \breve{\mathbf{N}}_i$, with $\breve{\mathbf{N}}_i = \prod_{j \in [1,d]} N_{i,j}$, and*

$$\varpi := \frac{1}{\sqrt{\breve{\mathbf{N}}_F}} \sum_{\overline{\mathbf{n}}^{(F)} \in R(\mathbf{N},F,\mathbf{p})} \delta(L(\overline{\mathbf{n}}^{(F)})_{\mathbf{p}} = \gamma).$$

Then

$$\varpi \leq \begin{cases} 1, & if \ \gamma = 0, \ \beta = 2, \\ c\rho(\mathbf{N}), & otherwise, \end{cases}$$

where a constant c depend only on $\hbar$, and $\rho(\mathbf{N}) = \max_i (\breve{\mathbf{N}}_i)^{-1/2}$.

Proof. Let $\gamma \neq 0$. Applying Corollary 5.1 with $h_1 = \hbar + 1$, $d_1 = d\hbar$, $s_1 = s_2 \in [1, s^s]$, $U = sd\hbar$ and $\mathbb{U}(d_1, \mathbb{P}) = 2^{35U^3} s^{6sU^2}$, from (4.7), (5.4), (5.13) and (5.16), we get that

$$\varpi \leq \mathbb{U}(d_1, \mathbb{P}) \frac{1}{\sqrt{\breve{\mathbf{N}}_F}} \leq \mathbb{U}(d_1, \mathbb{P}) \rho(\mathbf{N}).$$

Let $\gamma = 0$ and $\beta = 1$. We see that there are no solutions of the equation $L(\overline{\mathbf{n}}^{(F)})_{\mathbf{p}} = 0$.

Let $\gamma = 0$ and $\beta \geq 3$. By (5.16) there are no non-degenerate solutions of the equation $L(\overline{\mathbf{n}}^{(F)})_{\mathbf{p}} = 0$. Hence $\widetilde{m}^{(i)}_{\mathbf{p}} \neq 0$ for all $i \in F$. Let $\min_{i \in F} \breve{\mathbf{N}}_{\eth_i} = \breve{\mathbf{N}}_{\eth_{\mu_0}}$. We fix $\mathbf{n}_{\mu_0}$. Let $n^{'}_{\mu,j} = n_{\mu,j} - n_{\mu_0,j}$ $(\mu \in F)$. We see that

$$-\sum_{\mu \in F, \mu \neq \mu_0} \lambda_{1,p_1}^{n^{'}_{\mu,1}} \cdots \lambda_{d,p_1}^{n^{'}_{\mu,d}} \widetilde{m}^{(\mu)}_{p_1,p_2} / \widetilde{m}^{(\mu_0)}_{p_1,p_2} = 1. \tag{5.17}$$

Bearing in mind that $\lambda_{i,j}$ are algebraic integers, we can apply Corollary 5.1. We get that the number of solutions of (5.17) is equal to $O(\hbar)$. Taking into account that $\beta \geq 3$ and $\breve{\mathbf{N}}_F \geq (\breve{\mathbf{N}}_{\mu_0})^3$, we obtain

$$\varpi = O\big(\breve{\mathbf{N}}_{\mu_0} / \sqrt{\breve{\mathbf{N}}_F}\big) = O((\breve{\mathbf{N}}_{\mu_0})^{-1/2}) = O(\rho(\mathbf{N})).$$

Let $\gamma = 0$, $\beta = 2$. Using Definition 1, we get that

$$\#\{\mathbf{n}^{'} \in \mathbb{Z}^d \mid \lambda_{1,p_1}^{n^{'}_{\mu,1}} \cdots \lambda_{d,p_1}^{n^{'}_{\mu,d}} = -\widetilde{m}^{(\mu)}_{p_1,p_2} / \widetilde{m}^{(\mu_0)}_{p_1,p_2}\} \leq 1. \tag{5.18}$$

Therefore

$$\varpi \leq (\breve{\mathbf{N}}_F)^{-1/2} \prod_{j \in [1,d]} \min(N_{F(1),j}, N_{F(2),j}) \leq 1. \tag{5.19}$$

Thus Lemma 5.3 is proved. ■

Let $\mathcal{F}_r^{(i)} = (F_1, ..., F_r)$ be a partition of $\mathbb{F}_i$, i.e.

$$F_1 \cup \cdots \cup F_r = \mathbb{F}_i, \quad F_j \cap F_k = \emptyset, \quad j \neq k \quad \text{and} \quad F_i(j) < F_i(k), \quad \text{for} \quad j < k.$$

Let $(F_1, ..., F_{r_1}) \equiv (F_1^{'}, ..., F_{r_2}^{'})$ if $r_1 = r_2$, and for all $i \in [1, r_1]$ $\exists k \in [1, r_1]$ such that $F_i = F_k^{'}$. We denote by $\mathfrak{F}_i$ the set of all nonequivalent partition of $\mathbb{F}_i$, and by $\mathfrak{F}_0$ the set of all nonequivalent partition of $F^{(\hbar)}$.

Definition 5.3. *Let $\dot{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 0$, if $\mathfrak{f}_i = \#\mathbb{F}_i$ is odd, or $\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i} \neq 0$, and let $\dot{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 1$ otherwise. Let $\mathcal{F}_r^{(i)} = (F_1, ..., F_r) \in \mathfrak{F}_i$. Let $\ddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)}) = 0$, if $\beta_{F_k} = \#F_k \neq 2$ for some $k \in [1, r]$, and let $\ddot{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 1$ otherwise. Let*

$$\gamma_j = L(\overline{\mathbf{n}}^{(F_j)})_{\mathbf{p}_i}, \quad \overline{\mathbf{n}}^{(F_j)} \in R(\mathbf{N}, F_j, \mathbf{p}_i), \quad \text{where} \quad j = 1, ..., r,$$
$$\text{and} \quad \gamma_1 = ... = \gamma_{r-1} = 0, \gamma_r = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i}. \tag{5.20}$$

Let $\dddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)}) = 0$, if (5.20) is true, and let $\dddot{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 1$ otherwise. Let $\mathfrak{g}_i(\overline{\mathbf{n}}) = 1$, if there exists a partition $\mathcal{F}_r^{(i)} \in \mathfrak{F}_i$ with $\dot{\mathfrak{g}}_i(\overline{\mathbf{n}})\ddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)})\dddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)}) = 1$. Let $\mathfrak{g}_i(\overline{\mathbf{n}}) = 0$ otherwise $(i = 1, ..., \mathfrak{k})$, and let $\mathfrak{g}(\overline{\mathbf{n}}) = \mathfrak{g}_1(\overline{\mathbf{n}}) \cdots \mathfrak{g}_{\mathfrak{k}}(\overline{\mathbf{n}})$.

Lemma 5.4. *Let $i \in [1, \mathfrak{k}]$, $l \in \{0, 1\}$, $\breve{\mathbf{N}}_F = \prod_{i \in F} \breve{\mathbf{N}}_i$ and*

$$\dot{\varpi}_i(l) := \frac{1}{\sqrt{\breve{\mathbf{N}}_{\mathbb{F}_i}}} \sum_{\substack{\mathbf{n}_{\mathbb{F}_i(j)} \in \mathfrak{R}_{\mathfrak{d}_{\mathbb{F}_i(j)}} \\ j=1,...,\mathfrak{f}_i}} \delta(L(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i})\delta(\mathfrak{g}_i(\overline{\mathbf{n}}) = l). \tag{5.21}$$

Then

$$\dot{\varpi}_i(1) = O(1) \quad \text{and} \quad \dot{\varpi}_i(0) = O(\rho(\mathbf{N})), \tag{5.22}$$

where O-constants depend only on $\hbar$.

Proof. Let $L(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i}$. Using (5.16), we see that there exists a partition $\mathcal{F}_r^{(i)} = (F_1, ..., F_r) \in \mathfrak{F}_i$ satisfying (5.20). By Definition 5.3, we get

$$\delta(L(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i})\delta(\mathfrak{g}_i(\overline{\mathbf{n}}) = l) \leq \sum_{r=1}^{\mathfrak{f}_i} \sum_{(F_1,...,F_r) \in \mathfrak{F}_i} \prod_{j=1}^{r} \delta(L(\overline{\mathbf{n}}^{(F_j)})_{\mathbf{p}_i} = \gamma_j)$$

$$\times \delta(\overline{\mathbf{n}}^{(F_j)} \in R(\mathbf{N}, F_j, \mathbf{p}_i))\delta(\dot{\mathfrak{g}}_i(\overline{\mathbf{n}})\ddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)}) = l)\dddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)}).$$

Let $\beta_j = \#F_j$, and let

$$\mathfrak{e}_j = \begin{cases} 1, & \text{if } \gamma_j \neq 0, \\ 2, & \text{if } \beta_j = 1, \text{ and } \gamma_j = 0, \\ 3, & \text{if } \beta_j \geq 3, \text{ and } \gamma_j = 0, \\ 4, & \text{if } \beta_j = 2 \text{ and } \gamma_j = 0. \end{cases}$$

Changing the order of the summation, we obtain

$$\dot{\varpi}_i(l) \leq \sum_{r=1}^{\mathfrak{f}_i} \sum_{(F_1,...,F_r) \in \mathfrak{F}_i} \prod_{j=1}^{r} \sum_{k=1}^{4} \varkappa_{j,l,k}, \tag{5.23}$$

where

$$\varkappa_{j,l,k} = \frac{1}{\sqrt{\breve{\mathbf{N}}_{F_j}}} \sum_{\overline{\mathbf{n}}^{(F_j)} \in R(\mathbf{N}, F_j, \mathbf{p}_i)} \delta(L(\overline{\mathbf{n}}^{(F_j)})_{\mathbf{p}_i} = \gamma_j)\delta(\dot{\mathfrak{g}}_i(\overline{\mathbf{n}})\ddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)}) = l)\dddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)})\delta(\mathfrak{e}_j = k).$$

Using Lemma 5.3 we get that $\varkappa_{j,l,k} \le 1$ for $k=4$, and $\varkappa_{j,l,k} = O(\rho(\mathbf{N}))$ for $k \in [1,3]$. Hence (5.22) is true for $l=1$. Consider the case $l=0$. From Definition 5.3 we get that if $\dot{\mathfrak{g}}_i(\overline{\mathbf{n}})\ddot{\mathfrak{g}}_i(\overline{\mathbf{n}}, \mathcal{F}_r^{(i)}) = 0$, then $\mathfrak{e}_{j_0} \in [1,3]$ for some $j_0 \in [1,r]$. Hence

$$\sum_{k=1}^{4} \varkappa_{j_0,0,k} = O(\rho(\mathbf{N})) \quad \text{and} \quad \prod_{j=1}^{r}\sum_{k=1}^{4} \varkappa_{j,0,k} = O(\rho(\mathbf{N})).$$

By (5.23) Lemma 5.4 is proved. ∎

Lemma 5.5. *Let* $\breve{\mathbf{N}} = \breve{\mathbf{N}}_1 \cdots \breve{\mathbf{N}}_d = \breve{\mathbf{N}}_{\mathbb{F}_1} \cdots \breve{\mathbf{N}}_{\mathbb{F}_{\mathfrak{k}}}$ *and*

$$\varpi_1 := \frac{1}{\sqrt{\breve{\mathbf{N}}}} \sum_{\substack{\mathbf{n}_i \in \mathfrak{R}_{\eth_i} \\ i=1,\dots,\hbar}} \delta(\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0})\delta(\mathfrak{g}(\overline{\mathbf{n}}) = 0).$$

Then

$$\varpi_1 = O(\rho(\mathbf{N})),$$

where O-constant depends only on $\hbar$.

Proof. Using (5.14) we get

$$\delta(\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0}) \le \prod_{i=1}^{\mathfrak{k}} \delta(L(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i}).$$

Hence

$$\varpi_1 \le \prod_{i=1}^{\mathfrak{k}} \frac{1}{\sqrt{\breve{\mathbf{N}}_{\mathbb{F}_i}}} \sum_{\substack{\mathbf{n}_{\mathbb{F}_i(j)} \in \mathfrak{R}_{\eth_{\mathbb{F}_i(j)}} \\ j=1,\dots,\mathfrak{f}_i}} \delta(L(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i})\delta(\mathfrak{g}(\overline{\mathbf{n}}) = 0).$$

It is easy to see that if $\mathfrak{g}(\overline{\mathbf{n}}) = 0$, then there exists $\mu \in [1,\mathfrak{k}]$ with $\mathfrak{g}_\mu(\overline{\mathbf{n}}) = 0$. By (5.21), we obtain

$$\varpi_1 \le \sum_{\mu \in [1,\mathfrak{k}]} \dot{\varpi}_\mu(0) \prod_{i \in [1,\mathfrak{k}], i \ne \mu} (\dot{\varpi}_i(0) + \dot{\varpi}_i(1)).$$

Applying Lemma 5.4, we get the assertion of Lemma 5.5. ∎

Definition 5.4. *Let* $\breve{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 0$*, if there exists a partition* $(F_1, ..., F_r) \in \mathfrak{F}_i$ *and* $j \in [1,r]$ *such that*

$$L(\overline{\mathbf{n}}^{(F_k)})_{\mathbf{p}_i} = 0,\ \beta_{F_k} = 2,\ \overline{\mathbf{n}}^{(F_k)} \in R(\mathbf{N}, F_k, \mathbf{p}_i), \qquad \forall k \in [1,r], \tag{5.24}$$

and $\mathbf{C}(\overline{\mathbf{n}}^{(F_j)}) \ne \mathbf{0}$*. Let* $\breve{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 1$ *otherwise* $(i = 1, ..., \mathfrak{k})$*, and let* $\breve{\mathfrak{g}}(\overline{\mathbf{n}}) = \breve{\mathfrak{g}}_1(\overline{\mathbf{n}}) \cdots \breve{\mathfrak{g}}_{\mathfrak{k}}(\overline{\mathbf{n}})$*.*

Lemma 5.6. *Let*

$$\varpi_2 := \frac{1}{\sqrt{\breve{\mathbf{N}}}} \sum_{\substack{\mathbf{n}_i \in \mathfrak{R}_{\eth_i} \\ i=1,\dots,\hbar}} \delta(\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0})\delta(\mathfrak{g}(\overline{\mathbf{n}}) = 1)\delta(\breve{\mathfrak{g}}(\overline{\mathbf{n}}) = 0).$$

Then

$$\varpi_2 = O(\rho(\mathbf{N})),$$

where O-constant depends only on $\hbar$.

Proof. Let $\breve{\mathfrak{g}}(\overline{\mathbf{n}}) = 0$. By Definition 5.4, we have that there exist $i_1 \in [1,\mathfrak{k}]$ and a partition $(F_1^{(i_1)}, ..., F_r^{(i_1)}) \in \mathfrak{F}_{i_1}$ satisfying (5.24). We consider the conditions $\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0}$

and $\mathfrak{g}_i(\overline{\mathbf{n}}) = 1$, $i \in [1, \mathfrak{k}] \setminus \{i_1\}$. From Definition 5.3 and (5.14), we obtain that there exists a partition $(F_1^{(i)}, ..., F_r^{(i)}) \in \mathfrak{F}_i$ satisfying (5.20) with $r = \mathfrak{f}_i/2$, $\beta_{F_j^{(i)}} = 2$, $j = 1, ..., \mathfrak{f}_i/2$, $i \in [1, \mathfrak{k}] \setminus \{i_1\}$. Hence we get the following inequality

$$\delta(\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0})\delta(\mathfrak{g}(\overline{\mathbf{n}}) = 1)\delta(\breve{\mathfrak{g}}(\overline{\mathbf{n}}) = 0) \leq \sum_{i_1=1}^{\mathfrak{k}} \sum_{j_1=1}^{\mathfrak{f}_{i_1}/2} \prod_{i=1}^{\mathfrak{k}} \sum_{\substack{(F_1^{(i)}, ..., F_{\mathfrak{f}_i/2}^{(i)}) \in \mathfrak{F}_i \\ \#F_j^{(i)} = 2,\ j \in [1, \mathfrak{f}_i/2]}} 1$$

$$\times \prod_{j=1}^{\mathfrak{f}_i/2} \delta(L(\overline{\mathbf{n}}^{(F_j^{(i)})})_{\mathbf{p}_i} = 0)\delta(\mathbf{C}(\overline{\mathbf{n}}^{(F_{j_1}^{(i_1)})}) \neq \mathbf{0})\delta(\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0}),$$

with $\overline{\mathbf{n}}^{(F_j^{(i)})} \in R(\mathbf{N}, F_j^{(i)}, \mathbf{p}_i)$. Let

$$R^{'}(\mathbf{N}, F, \mathbf{p}) = \{(\mathbf{n}_{F(1)}, ..., \mathbf{n}_{F(\beta)}) \mid \mathbf{n}_i \in \mathfrak{R}_{\eth_{F(i)}},\ i = 1, ..., \beta, \quad \beta = \#F, \qquad (5.25)$$
$$\text{and} \quad \nexists F^* \subsetneq F \quad \text{with} \quad C(\overline{\mathbf{n}}^{(F^*)})_{\mathbf{p}} = 0\}.$$

Consider the conditions $\mathbf{C}(\overline{\mathbf{n}}^{(F_{j_1}^{(i_1)})}) \neq \mathbf{0}$ and $\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0}$. We see that there exists $\mathbf{p} \in \mathcal{P}$ with $\mathbf{C}(\overline{\mathbf{n}}^{(F_{j_1}^{(i_1)})})_{\mathbf{p}} \neq 0$. Therefore, there exists a partition $(F_1^{'}, ..., F_r^{'})$ of $\mathbb{F}^{(\hbar)} \setminus F_{j_1}^{(i_1)}$ such that $\mathbf{C}(\overline{\mathbf{n}}^{(F_j^{'})})_{\mathbf{p}} = 0$ $(j = 1, ..., r-1)$, $\mathbf{C}(\overline{\mathbf{n}}^{(F_r^{'})})_{\mathbf{p}} = -\mathbf{C}(\overline{\mathbf{n}}^{(F_{j_1}^{(i_1)})})_{\mathbf{p}} \neq 0$, and $\overline{\mathbf{n}}^{(F_j^{'})} \in R^{'}(\mathbf{N}, F_j, \mathbf{p})$, $j = 1, ..., r$. Thus

$$\varpi_2 \leq \sum_{i_1=1}^{\mathfrak{k}} \sum_{j_1=1}^{\mathfrak{f}_{i_1}/2} \prod_{i=1}^{\mathfrak{k}} \sum_{\substack{(F_1^{(i)}, ..., F_{\mathfrak{f}_i/2}^{(i)}) \in \mathfrak{F}_i \\ \#F_j^{(i)} = 2,\ j \in [1, \mathfrak{f}_i/2]}} \sum_{\mathbf{p} \in \mathcal{P}} \sum_{r=1}^{\hbar-1} \sum_{(F_1^{'}, ..., F_r^{'}, F_{j_1}^{(i_1)}) \in \mathfrak{F}_0} \prod_{j=1}^{\mathfrak{f}_i/2} \varkappa_{i, i_1, j, j_1} \qquad (5.26)$$

where

$$\varkappa_{i, i_1, j, j_1} = \frac{1}{\sqrt{\breve{\mathbf{N}}_{F_j^{(i)}}}} \sum_{\overline{\mathbf{n}}^{(F_j^{(i)})} \in R(\mathbf{N}, F_j^{(i)}, \mathbf{p}_i)} \delta\big(L(\overline{\mathbf{n}}^{(F_j^{(i)})})_{\mathbf{p}_i} = 0\big) \qquad (5.27)$$

$$\times \delta\big(\overline{\mathbf{n}}^{(F_r^{'})} \in R^{'}(\mathbf{N}, F_r^{'}, \mathbf{p})\big)\delta\big(\mathbf{C}(\overline{\mathbf{n}}^{(F_r^{'})})_{\mathbf{p}} = -\mathbf{C}(\overline{\mathbf{n}}^{(F_{j_1}^{(i_1)})})_{\mathbf{p}} \neq 0\big).$$

By Lemma 5.3, we have

$$\varkappa_{i, i_1, j, j_1} = O(1), \qquad \text{with} \qquad j, j_1 \in [1, \mathfrak{f}_i/2],\ i, i_1 \in \mathfrak{k}. \qquad (5.28)$$

For $\varsigma \in \{1, 2\}$, we denote

$$\varsigma^{'} \equiv \varsigma + 1 \mod 2, \qquad \varsigma^{'} \in \{1, 2\}.$$

Let $F_r^{'}(1) = F_{j_2}^{(i_2)}(\varsigma)$ for some $j_2 \in [1, \mathfrak{f}_{i_2}/2]$, $i_2 \in \mathfrak{k}$ and $\varsigma \in \{1, 2\}$. Bearing in mind that $F_{j_1}^{(i_1)} \cap F_r^{'} = \emptyset$, we get $(i_1, j_1) \neq (i_2, j_2)$. We fix $i_1, j_1, F_{j_1}^{(i_1)}, F_r^{'}$ and $\mathbf{p}$. Using (5.11), (5.13) and (5.16), we obtain from the condition $\overline{\mathbf{n}}^{(F_{j_1}^{(i_1)})} \in R(\mathbf{N}, F_{j_1}^{(i_1)}, \mathbf{p}_i)$ that $\widetilde{m}_{\mathbf{p}_{i_1}}^{(\mu)} \neq 0$ for all $\mu \in F_{j_1}^{(i_1)}$, $i_1 = 1, ..., \mathfrak{k}, j_1 = 1, ..., \mathfrak{f}_{i_1}/2$. For given $\mathbf{n}_{F_{j_1}^{(i_1)}(\varsigma_1)}$, we derive from (5.18)

$$\#\{\mathbf{n}^{(F_{j_1}^{(i_1)}(\varsigma_1^{'}))} \in \mathfrak{R}_{\eth_{F_{j_1}^{(i_1)}(\varsigma_1^{'})}} \mid L(\overline{\mathbf{n}}^{(F_{j_1}^{(i_1)})})_{\mathbf{p}_{i_1}} = 0\} \leq 1, \qquad \varsigma_1 = 1, 2.$$

Similarly to (5.19), we have

$$\#\{\mathbf{n}^{(F_{j_1}^{(i_1)})} \in \mathfrak{R}_{\eth_{F_{j_1}^{(i_1)}(1)}} \times \mathfrak{R}_{\eth_{F_{j_1}^{(i_1)}(2)}} \mid L(\overline{\mathbf{n}}^{(F_{j_1}^{(i_1)})})_{\mathbf{p}_{i_1}} = 0\} \leq (\breve{\mathbf{N}}_{F_{j_1}^{(i_1)}})^{1/2}. \qquad (5.29)$$

We fix $\overline{\mathbf{n}}^{(F^{(i_1)}_{j_1})}$. Let

$$\mathfrak{B} = \{\overline{\mathbf{n}}^{(F^{'}_r)} \in R^{'}(\mathbf{N}, F^{'}_r, \mathbf{p}) \mid \mathbf{C}(\overline{\mathbf{n}}^{(F^{'}_r)})_{\mathbf{p}} = -\mathbf{C}(\overline{\mathbf{n}}^{(F^{(i_1)}_{j_1})})_{\mathbf{p}} \neq 0\}.$$

Applying (5.25), (5.6) and Corollary 5.1 with $h_1 = \hbar + 1$, $d_1 = d\hbar$, $s_1 = s_2 \in [1, s^s]$, $U = sd\hbar$ and $\mathbb{U}(d_1, \mathbb{P}) = 2^{35U^3} s^{6sU^2}$, we get

$$\#\mathfrak{B} \leq \mathbb{U}(d_1, \mathbb{P}).$$

Taking into account that $F^{'}_r(1) = F^{(i_2)}_{j_2}(\varsigma)$, we obtain from (5.13) and (5.18) that

$$\#\{\overline{\mathbf{n}}^{(F^{(i_2)}_{j_2})} \in R(\mathbf{N}, F^{(i_2)}_{j_2}, \mathbf{p}_{i_2}) \mid L(\overline{\mathbf{n}}^{(F^{(i_2)}_{j_2})})_{\mathbf{p}_{i_2}} = 0, \qquad \mathbf{n}_{F^{(i_2)}_{j_2}(\varsigma)} = \mathbf{n}_{F^{'}_r(1)} \quad \text{and} \quad \overline{\mathbf{n}}^{(F^{'}_r)} \in \mathfrak{B}\} \leq \mathbb{U}(d_1, \mathbb{P}). \tag{5.30}$$

From (5.29) and (5.30), we derive

$$\#\{\overline{\mathbf{n}}^{(F^{(i_k)}_{j_k})} \in R(\mathbf{N}, F^{(i_k)}_{j_k}, \mathbf{p}_{i_k}),\ k = 1,2 \ \mid \ L(\overline{\mathbf{n}}^{(F^{(i_k)}_{j_k})})_{\mathbf{p}_{i_k}} = 0, \qquad k = 1,2,$$
$$\mathbf{n}_{F^{(i_2)}_{j_2}(\varsigma)} = \mathbf{n}_{F^{'}_r(1)} \quad \text{and} \quad \overline{\mathbf{n}}^{(F^{'}_r)} \in \mathfrak{B}\} \quad \leq \mathbb{U}(d_1, \mathbb{P})(\breve{\mathbf{N}}_{F^{(i_1)}_{j_1}})^{1/2}.$$

Using (5.27), we get

$$\sqrt{\breve{\mathbf{N}}_{F^{(i_1)}_{j_1}}}\varkappa_{i_1,i_1,j_1,j_1}\sqrt{\breve{\mathbf{N}}_{F^{(i_2)}_{j_2}}}\varkappa_{i_2,i_1,j_2,j_1} \leq \mathbb{U}(d_1, \mathbb{P})(\breve{\mathbf{N}}_{F^{(i_1)}_{j_1}})^{1/2}$$

and

$$\varkappa_{i_1,i_1,j_1,j_1}\varkappa_{i_2,i_1,j_2,j_1} = O(\rho(\mathbf{N})). \tag{5.31}$$

Consider (5.26). Applying (5.28) for $(i,j) \notin \{(i_1,j_1),(i_2,j_2)\}$ and (5.31) for $(i,j) \in \{(i_1,j_1),(i_2,j_2)\}$, we obtain the assertion of Lemma 5.6. ■

Definition 5.5. *Let* $\widetilde{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 0$. *If there exists two partitions* $(F_1, ..., F_{\mathfrak{f}_i/2}), (F^{'}_1, ..., F^{'}_{\mathfrak{f}_i/2}) \subset \mathfrak{F}_i$ *such that* $\beta_{F_j} = \beta_{F^{'}_j} = 2$, $L(\overline{\mathbf{n}}^{(F_j)})_{\mathbf{p}_i} = L(\overline{\mathbf{n}}^{(F^{'}_j)})_{\mathbf{p}_i} = 0$ *for* $j = 1, ..., \mathfrak{f}_i/2$, $F_{j_1}(\varsigma_1) = F^{'}_{j_2}(\varsigma_2)$ *and* $F_{j_1}(\varsigma^{'}_1) \neq F^{'}_{j_2}(\varsigma^{'}_2)$ *for some* $j_1, j_2 \in [1, \mathfrak{f}_i/2]$, $\varsigma_1, \varsigma_2 \in \{1,2\}$. *Let* $\widetilde{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 1$, *otherwise* $(i = 1, ..., \mathfrak{k})$, *and let* $\widetilde{\mathfrak{g}}(\overline{\mathbf{n}}) = \widetilde{\mathfrak{g}}_1(\overline{\mathbf{n}}) \cdots \widetilde{\mathfrak{g}}_{\mathfrak{k}}(\overline{\mathbf{n}})$.

Lemma 5.7. *Let*

$$\varpi_3 := \frac{1}{\sqrt{\breve{\mathbf{N}}}} \sum_{\substack{\mathbf{n}_i \in \mathfrak{R}_{\mathfrak{d}_i} \\ i=1,\dots,\hbar}} \delta(\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0})\delta(\mathfrak{g}(\overline{\mathbf{n}}) = 1)\delta(\widetilde{\mathfrak{g}}(\overline{\mathbf{n}}) = 0).$$

Then

$$\varpi_3 = O(\rho(\mathbf{N})),$$

where O*-constant depends only on* $\hbar$ *and* $\rho(\mathbf{N}) = \max_i (N_{i,1} \cdots N_{i,d})^{-1/2}$.

Proof. Using (5.14), we have

$$\varpi_3 \leq \sum_{i \in [1,\mathfrak{k}]} \ddot{\varpi}_3(i) \prod_{\substack{i_1 \in [1,\mathfrak{k}] \\ i_1 \neq i}} \breve{\mathbf{N}}^{-1/2}_{\mathbb{F}_{i_1}} \sum_{\substack{\mathbf{n}_{\mathbb{F}_{i_1}(j)} \in \mathfrak{R}_{\mathfrak{d}_{\mathbb{F}_{i_1}(j)}} \\ j=1,\dots,\mathfrak{f}_{i_1}}} \delta(L(\overline{\mathbf{n}}^{(\mathbb{F}_{i_1})})_{\mathbf{p}_{i_1}} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_{i_1})})_{\mathbf{p}_{i_1}})\delta(\mathfrak{g}_{i_1}(\overline{\mathbf{n}}) = 1),$$

where

$$\ddot{\varpi}_3(i) = \frac{1}{\sqrt{\breve{\mathbf{N}}_{\mathbb{F}_i}}} \sum_{\substack{\mathbf{n}_{\mathbb{F}_i(j)} \in \mathfrak{R}_{\mathfrak{d}_{\mathbb{F}_i(j)}} \\ j=1,\dots,\mathfrak{f}_i}} \delta(L(\overline{\mathbf{n}}^{(\mathbb{F}_i)})_{\mathbf{p}_i} = -\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i})\delta(\mathfrak{g}_i(\overline{\mathbf{n}}) = 1)\delta(\widetilde{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 0).$$

Applying Lemma 5.4, we obtain that the assertion of Lemma 5.7 is obtained using the following estimate:

$$\ddot{\varpi}_3(i) = O(\rho(\mathbf{N})), \qquad i = 1, ..., \mathfrak{k}. \tag{5.32}$$

From Definition 5.5, we derive

$$\delta(\widetilde{\mathfrak{g}}_i(\overline{\mathbf{n}}) = 0) \leq \sum_{j_1,j_2=1}^{\mathfrak{f}_i/2} \sum_{\varsigma_1,\varsigma_2=1}^{2} \sum_{\substack{(F_1^{(i)},...,F_{\mathfrak{f}_i/2}^{(i)}) \in \mathfrak{F}_i \\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \sum_{\substack{(F_1^{'(i)},...,F_{\mathfrak{f}_i/2}^{'(i)}) \in \mathfrak{F}_i \\ \#F_j^{'(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \prod_{j=1}^{\mathfrak{f}_i/2} 1$$

$$\times\, \delta(L(\overline{\mathbf{n}}^{(F_j^{(i)})})_{\mathbf{p}_i} = 0)\delta(L(\overline{\mathbf{n}}^{(F_j^{'(i)})})_{\mathbf{p}_i} = 0)\delta(F_{j_1}^{(i)}(\varsigma_1) = F_{j_2}^{'(i)}(\varsigma_2))\delta(F_{j_1}^{(i)}(\varsigma_1^{'}) \neq F_{j_2}^{'(i)}(\varsigma_2^{'})).$$

By Definition 5.3, we get

$$\ddot{\varpi}_3(i) \leq \sum_{j_2=1}^{\mathfrak{f}_i/2} \sum_{\varsigma_2=1}^{2} \widetilde{\varpi}_3(i, j_2, \varsigma_2), \tag{5.33}$$

with

$$\widetilde{\varpi}_3(i, j_2, \varsigma_2) \leq \sum_{j_1=1}^{\mathfrak{f}_i/2} \sum_{\varsigma_1=1}^{2} \sum_{\substack{(F_1^{(i)},...,F_{\mathfrak{f}_i/2}^{(i)}) \in \mathfrak{F}_i \\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \sum_{\substack{(F_1^{'(i)},...,F_{\mathfrak{f}_i/2}^{'(i)}) \in \mathfrak{F}_i \\ \#F_j^{'(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \prod_{j=1}^{\mathfrak{f}_i/2} \dot{\varkappa}_{j_2,\varsigma_1,\varsigma_2}^{(i,j,j_1)}, \tag{5.34}$$

where

$$\dot{\varkappa}_{j_2,\varsigma_1,\varsigma_2}^{(i,j,j_1)} = (\breve{\mathbf{N}}_{F_j^{(i)}})^{-1/2} \sum_{\overline{\mathbf{n}}^{(F_j^{(i)})} \in R(\mathbf{N}, F_j^{(i)}, \mathbf{p}_i)} \delta(L(\overline{\mathbf{n}}^{(F_j^{(i)})})_{\mathbf{p}_i} = 0)\delta(L(\overline{\mathbf{n}}^{(F_j^{'(i)})})_{\mathbf{p}_i} = 0)$$

$$\times\, \delta(F_{j_1}^{(i)}(\varsigma_1) = F_{j_2}^{'(i)}(\varsigma_2))\delta(F_{j_1}^{(i)}(\varsigma_1^{'}) \neq F_{j_2}^{'(i)}(\varsigma_2^{'})).$$

By Lemma 5.3, we have

$$\dot{\varkappa}_{j_2,\varsigma_1,\varsigma_2}^{(i,j,j_1)} = O(1), \quad \text{with} \quad j, j_1, j_2 \in [1, \mathfrak{f}_i/2], \quad \varsigma_1, \varsigma_2 \in [1, 2],\ i \in \mathfrak{k}. \tag{5.35}$$

Consider the conditions $F_{j_1}^{(i)}(\varsigma_1) = F_{j_2}^{'(i)}(\varsigma_2))$ and $F_{j_1}^{(i)}(\varsigma_1^{'}) \neq F_{j_2}^{'(i)}(\varsigma_2^{'})$. It is easy to see that for given (j_2, ς_2) there exists at most one such $(j_1, \varsigma_1) \in [1, \mathfrak{f}_i/2] \times [1, 2]$. Using (5.13) and (5.16), we get from the conditions $\overline{\mathbf{n}}^{(F_j^{(i)})} \in R(\mathbf{N}, F_j^{(i)}, \mathbf{p}_i)$ $(j = 1, ..., \mathfrak{f}_i/2)$ that $\widetilde{m}_{\mathbf{p}_i}^{(\mu)} \neq 0$ for all $\mu \in F_j^{(i)},\ j = 1, ..., \mathfrak{f}_i/2$. Hence for given $\mathbf{n}_{F_{j_2}^{'(i)}(\varsigma_2)}$ there exists only one $\mathbf{n}_{F_{j_2}^{'(i)}(\varsigma_2^{'})}$ and only one $\mathbf{n}_{F_{j_1}^{(i)}(\varsigma_1^{'})}$ satisfying the following equations

$$L(\overline{\mathbf{n}}^{F_{j_2}^{'(i)}})_{\mathbf{p}_i} = 0 \qquad \text{and} \qquad L(\overline{\mathbf{n}}^{(F_{j_1}^{(i)})})_{\mathbf{p}_i} = 0.$$

It is easy to see that there exists only one $(j_3, \varsigma_3) \in [1, \mathfrak{f}_i/2] \times \{1, 2\}$ with $F_{j_3}^{(i)}(\varsigma_3) = F_{j_2}^{'(i)}(\varsigma_2^{'})$. Therefore for given $\overline{\mathbf{n}}^{F_{j_2}^{'(i)}}$ there exists only one $\mathbf{n}_{F_{j_3}^{(i)}(\varsigma_3^{'})}$ satisfying to $L(\overline{\mathbf{n}}^{(F_{j_3}^{(i)})})_{\mathbf{p}_i} = 0$. Similarly to (5.29) - (5.31), we get

$$\dot{\varkappa}_{j_2,\varsigma_1,\varsigma_2}^{(i,j_1,j_1)} \dot{\varkappa}_{j_2,\varsigma_1,\varsigma_2}^{(i,j_3,j_1)} \leq (\breve{\mathbf{N}}_{F_{j_1}^{(i)}} \breve{\mathbf{N}}_{F_{j_3}^{(i)}})^{-1/2} \sum_{\substack{\overline{\mathbf{n}}^{(F_{j_1}^{(i)})} \in R(\mathbf{N}, F_{j_1}^{(i)}, \mathbf{p}_i) \\ \overline{\mathbf{n}}^{(F_{j_3}^{(i)})} \in R(\mathbf{N}, F_{j_3}^{(i)}, \mathbf{p}_i)}} \delta(L(\overline{\mathbf{n}}^{(F_{j_1}^{(i)})})_{\mathbf{p}_i} = 0)$$

$$\times\, \delta(L(\overline{\mathbf{n}}^{(F_{j_3}^{(i)})})_{\mathbf{p}_i}=0)\delta(L(\overline{\mathbf{n}}^{(F_{j_2}^{'(i)})})_{\mathbf{p}_i}=0)\delta(F_{j_1}^{(i)}(\varsigma_1)=F_{j_2}^{'(i)}(\varsigma_2))\delta(F_{j_1}^{(i)}(\varsigma_1^{'})\neq F_{j_2}^{'(i)}(\varsigma_2^{'}))$$
$$= O((\breve{\mathbf{N}}_{F_{j_3}^{(i)}})^{-1/2}) = O(\rho(\mathbf{N})). \tag{5.36}$$

Consider (5.34). Applying (5.35) for $j\notin\{j_1,j_3\}$ and (5.36) for $j\in\{j_1,j_3\}$, we obtain $\widetilde{\varpi}_3(i,j_2,\varsigma_2)=O(\rho(\mathbf{N}))$. Now by (5.32) and (5.33), we get the assertion of Lemma 5.7. ∎

Lemma 5.8. *Let*

$$\varpi_4 := \frac{1}{\sqrt{\breve{\mathbf{N}}}}\sum_{\mathbf{n}_i\in\mathfrak{R}_{\mathfrak{V}_i},\, i=1,\dots,\hbar}\delta(\mathbf{C}(\overline{\mathbf{n}})=\mathbf{0}).$$

Then

$$\varpi_4 = O(\rho(\mathbf{N})) \qquad \text{if } \hbar \text{ is odd,} \tag{5.37}$$

and

$$\varpi_4 = \varpi_4^{'} + O(\rho(\mathbf{N})) \qquad \text{if } \hbar \text{ is even,}$$

with

$$\varpi_4^{'} = \prod_{i=1}^{\mathfrak{k}}\sum_{\substack{(F_1^{(i)},\dots,F_{\mathfrak{f}_i/2}^{(i)})\in\mathfrak{F}_i\\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}}\prod_{j=1}^{\mathfrak{f}_i/2}\frac{1}{\sqrt{\breve{\mathbf{N}}_{F_j^{(i)}}}}\sum_{\substack{\mathbf{n}_{\mu_{i,j,k}}\in\mathfrak{R}_{\mathfrak{V}_{\mu_{i,j,k}}}\\ k=1,2}} 1 \tag{5.38}$$
$$\times\, \delta(\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})} = -\mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})})$$

where $\mu_{i,j,k}=F_j^{(i)}(k)$, $\rho(\mathbf{N})=\max_i(\breve{\mathbf{N}}_i)^{-1/2}$ and O-constants depend only on $\hbar$.

Proof. Let

$$\varpi_5(\nu) := \frac{1}{\sqrt{\breve{\mathbf{N}}}}\sum_{\substack{\mathbf{n}_i\in\mathfrak{R}_{\mathfrak{V}_i}\\ i=1,\dots,\hbar}}\delta(\mathbf{C}(\overline{\mathbf{n}})=\mathbf{0})\delta\big(\mathfrak{g}(\overline{\mathbf{n}})\breve{\mathfrak{g}}(\overline{\mathbf{n}})\widetilde{\mathfrak{g}}(\overline{\mathbf{n}})=\nu\big) \quad \text{with} \quad \nu=0,1.$$

By Lemma 5.5, Lemma 5.6 and Lemma 5.7, we get

$$\varpi_5(0) = O(\rho(\mathbf{N})).$$

By Definition 5.3, we get that if $\hbar$ is odd, then $\mathfrak{g}(\overline{\mathbf{n}})=0$. The assertion (5.37) is proved.

It is easy to see that

$$\varpi_4 = \varpi_5(0)+\varpi_5(1) = \varpi_5(1)+O(\rho(\mathbf{N})).$$

Consider $\varpi_5(1)$. Let $\mathbf{C}(\overline{\mathbf{n}})=\mathbf{0}$ and $\mathfrak{g}(\overline{\mathbf{n}})=1$. Applying (5.14) and Definition 5.3, we get that for all $i=1,\dots,\mathfrak{k}$ there exists a partition $(F_1^{(i)},\dots,F_r^{(i)})\in\mathfrak{F}_i$ with $\mathfrak{f}_i=\#\mathbb{F}_i$ is even, $\mathbf{C}(\overline{\mathbf{n}}^{(\widetilde{\mathbb{F}}_i)})_{\mathbf{p}_i}=0$, $L(\overline{\mathbf{n}}^{(F_j^{(i)})})_{\mathbf{p}_i}=0$, and $\beta_{F_j^{(i)}}=2$, for all $j\in[1,r]$, $r=\mathfrak{f}_i/2$. By Definition 5.5 this partition is unique for $\widetilde{\mathfrak{g}}(\overline{\mathbf{n}})=1$. Using Definition 5.4 for $\breve{\mathfrak{g}}_i(\overline{\mathbf{n}})=1$, we have that $\mathbf{C}(\overline{\mathbf{n}}^{(F_j^{(i)})})=\mathbf{0}$. Hence $\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})}=-\mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})}$ with $\mu_{i,j,k}=F_j^{(i)}(k),\ k=1,2$ (see (5.5)). Therefore

$$\delta(\mathbf{C}(\overline{\mathbf{n}})=\mathbf{0})\delta\big(\mathfrak{g}(\overline{\mathbf{n}})\breve{\mathfrak{g}}(\overline{\mathbf{n}})\widetilde{\mathfrak{g}}(\overline{\mathbf{n}})=1\big) = \prod_{i=1}^{\mathfrak{k}}\sum_{\substack{(F_1^{(i)},\dots,F_{\mathfrak{f}_i/2}^{(i)})\in\mathfrak{F}_i\\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}}\prod_{j=1}^{\mathfrak{f}_i/2}\delta\big(\mathfrak{g}(\overline{\mathbf{n}})\breve{\mathfrak{g}}(\overline{\mathbf{n}})\widetilde{\mathfrak{g}}(\overline{\mathbf{n}})=1\big)$$

$$\times\, \delta(\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})} = -\mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})})\delta(\overline{\mathbf{n}}^{(F_j^{(i)})}\in R(\mathbf{N},F_j^{(i)},\mathbf{p}_i)).$$

Bearing in mind that $\mathbf{m}^{(\mu_{i,j,k})} \neq \mathbf{0} \, \forall \, i,j,k$, we get that if $\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})} = -\mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})}$, then $\overline{\mathbf{n}}^{(F_j^{(i)})} \in R(\mathbf{N}, F_j^{(i)}, \mathbf{p}_i)$. Hence

$$\delta(\mathbf{C}(\overline{\mathbf{n}}) = \mathbf{0})\delta\big(\mathfrak{g}(\overline{\mathbf{n}})\breve{\mathfrak{g}}(\overline{\mathbf{n}})\widetilde{\mathfrak{g}}(\overline{\mathbf{n}}) = 1\big) = \prod_{i=1}^{\mathfrak{k}} \sum_{\substack{(F_1^{(i)},\dots,F_{\mathfrak{f}_i/2}^{(i)}) \in \mathfrak{F}_i \\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \prod_{j=1}^{\mathfrak{f}_i/2} 1$$

$$\times\, \delta\big(\mathfrak{g}(\overline{\mathbf{n}})\breve{\mathfrak{g}}(\overline{\mathbf{n}})\widetilde{\mathfrak{g}}(\overline{\mathbf{n}}) = 1\big)\delta(\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})} = -\mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})}).$$

Changing the order of summations, we obtain

$$\varpi_5(1) = \varpi_6(1), \tag{5.39}$$

where

$$\varpi_6(\nu) = \prod_{i=1}^{\mathfrak{k}} \sum_{\substack{(F_1^{(i)},\dots,F_{\mathfrak{f}_i/2}^{(i)}) \in \mathfrak{F}_i \\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \prod_{j=1}^{\mathfrak{f}_i/2} \frac{1}{\sqrt{\breve{\mathbf{N}}_{F_j^{(i)}}}} \delta\big(\mathfrak{g}(\overline{\mathbf{n}})\breve{\mathfrak{g}}(\overline{\mathbf{n}})\widetilde{\mathfrak{g}}(\overline{\mathbf{n}}) = \nu\big)$$

$$\times \sum_{\mathbf{n}_{\mu_{i,j,k}} \in \mathfrak{R}_{\eth_{\mu_{i,j,k}}},k=1,2} \delta(\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})} = -\mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})}).$$

It is easy to see that

$$\varpi_6(0) \leq 2^{\hbar}\varpi_5(0) = O(\rho(\mathbf{N})). \tag{5.40}$$

Now from (5.38)-(5.40), we get

$$\varpi_4 = \varpi_6(1) + O(\rho(\mathbf{N})) = \varpi_6(0) + \varpi_6(1) + O(\rho(\mathbf{N})) = \varpi_4^{'} + O(\rho(\mathbf{N})).$$

Thus Lemma 5.8 is proved. ■

We assume in the following that $\mathfrak{R}_i(\mathbf{N}_i) \cap \mathfrak{R}_j(\mathbf{N}_i) = \emptyset$ for $i \neq j \in [1,q]$ (see (2.16)).

Lemma 5.9. *Let $0 < |\mathbf{m}^{(i)}| < L$ $(1 \leq i \leq \hbar)$, $\hbar$ be an even. Then*

$$\varpi_4 = \sum_{\substack{(F_1,\dots,F_{\hbar/2}) \in \mathfrak{F}_0 \\ \#F_i=2,\ i\in[1,\hbar/2]}} \prod_{i=1}^{\hbar/2} \delta(\eth_{F_i(1)} = \eth_{F_i(2)})\delta(\mathbf{m}^{(F_i(1))} \in B(-\mathbf{m}^{(F_i(2))})) + O(\rho_1(\mathbf{N})), \tag{5.41}$$

where O-constant depends only on $\hbar$ and L, and $\rho_1(\mathbf{N}) = \max_{i,j}(N_{i,j})^{-1}$.

Proof. Consider the equation (5.38). Let $\mu_{i,j,k} = F_j^{(i)}(k),\ k = 1,2$. Bearing in mind that $|\mathbf{m}^{(\mu_{i,j,k})}| < L$, we get from Theorem 4 that there exists $L^{'} > 0$ such that $|\mathbf{n}_0| < L^{'}$ if $\mathbf{A}^{\mathbf{n}_0}\mathbf{m}^{(\mu_{i,j,2})} = -\mathbf{m}^{(\mu_{i,j,1})}$. From Definition 1, we obtain that there are no two solutions of this equation. Let $\eth_{\mu_{i,j,1}} = \eth_{\mu_{i,j,2}}$, $\mathbf{m}_{\mu_{i,j,2}} \in B(-\mathbf{m}_{\mu_{i,j,1}})$, and let

$$\beta = \#\big\{\mathbf{n}_{\mu_{i,j,k}} \in \mathfrak{R}_{\eth_{\mu_{i,j,k}}}, k=1,2 \mid \mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})} = -\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})}\big\}. \tag{5.42}$$

We see that $\mathfrak{R}_{\eth_{\mu_{i,j,1}}} = \mathfrak{R}_{\eth_{\mu_{i,j,2}}}$, $\breve{\mathbf{N}}_{F_j^{(i)}} = (N_{\eth_{\mu_{i,j,1}},1}\cdots N_{\eth_{\mu_{i,j,1}},d})^2$ and

$$(N_{\eth_{\mu_{i,j,1}},1} - L^{'})\cdots(N_{\eth_{\mu_{i,j,1}},d} - L^{'}) \leq \beta \leq N_{\eth_{\mu_{i,j,1}},1}\cdots N_{\eth_{\mu_{i,j,1}},d} = (\breve{\mathbf{N}}_{F_j^{(i)}})^{1/2}.$$

Hence

$$(1 - L^{'}\rho_1(\mathbf{N}))^d \leq \beta(\breve{\mathbf{N}}_{F_j^{(i)}})^{-1/2} \leq 1 \qquad \text{and} \qquad \beta(\breve{\mathbf{N}}_{F_j^{(i)}})^{-1/2} = 1 + O(\rho_1(\mathbf{N})). \tag{5.43}$$

Let $\eth_{\mu_{i,j,1}} \neq \eth_{\mu_{i,j,2}}$, $\breve{\mathbf{N}}_{F_j^{(i)}(\nu)} = \min(\breve{\mathbf{N}}_{F_j^{(i)}(1)}, \breve{\mathbf{N}}_{F_j^{(i)}(2)})$ for some $\nu \in [1,2]$ and

$$\beta = \#\{\mathbf{n}_{\mu_{i,j,k}} \in \mathfrak{R}_{\eth_{\mu_{i,j,k}}}, k=1,2 \mid \mathbf{n}_{\mu_{i,j,2}} = \mathbf{n}_{\mu_{i,j,1}} + \mathbf{n}_0\}.$$

Taking into account that $|\mathbf{n}_0| < L^{'}$, (2.16) and that $\mathfrak{R}_{i_1} \cap \mathfrak{R}_{i_2} = \emptyset$ for $i_1 \neq i_2 \in [1,q]$, we get $\exists l \in [1,d]$ with $[R_{\eth_{\mu_{i,j,1}},l}, R_{\eth_{\mu_{i,j,1}},l} + N_{\eth_{\mu_{i,j,1}},l}) \cap [R_{\eth_{\mu_{i,j,2}},l}, R_{\eth_{\mu_{i,j,2}},l} + N_{\eth_{\mu_{i,j,2}},l}) = \emptyset$,

$$\#\{n_{\mu_{i,j,k},l} \in [R_{\eth_{\mu_{i,j,k}},l}, R_{\eth_{\mu_{i,j,k}},l} + N_{\eth_{\mu_{i,j,k}},l}), k=1,2 \mid n_{\mu_{i,j,1},l} = n_{\mu_{i,j,2},l} + n_{0,l}\} \leq L^{'},$$

and

$$\beta \leq L^{'} \prod_{k\in[1,d], k\neq l} N_{\eth_{\mu_{i,j,\nu}},k} \leq L^{'} \breve{N}_{F_j^{(i)}(\nu)} / \min_{i,j} N_{i,j} = O((\breve{N}_{F_j^{(i)}})^{1/2} \rho_1(\mathbf{N})). \qquad (5.44)$$

Note that $\rho_1(\mathbf{N}) \geq \rho(\mathbf{N}) = \max_i (N_{i,1} \cdots N_{i,d})^{-1/2} \ \ (d \geq 2)$. By (5.38), (5.43) and (5.44), we have

$$\varpi_4 = \prod_{i=1}^{\mathfrak{k}} \sum_{\substack{(F_1^{(i)},\dots,F_{\mathfrak{f}_i/2}^{(i)}) \in \mathfrak{F}_i \\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \prod_{j=1}^{\mathfrak{f}_i/2} \delta(\eth_{F_j^{(i)}(1)} = \eth_{F_j^{(i)}(2)}) \delta(\mathbf{m}^{(\mu_{i,j,1})} \in B(-\mathbf{m}^{(\mu_{i,j,2})})) + O(\rho_1(\mathbf{N})).$$

Thus

$$\varpi_4 = \sum_{\substack{(F_1,\dots,F_{\hbar/2}) \in \mathfrak{F}_0 \\ \#F_i=2,\ i\in[1,\hbar/2]}} \prod_{i=1}^{\hbar/2} \delta(\eth_{F_i(1)} = \eth_{F_i(2)}) \delta(\mathbf{m}^{(F_i(1))} \in B(-\mathbf{m}^{(F_i(2))}))$$

$$\times \sum_{j\in[1,\hbar/2]} \delta(F_i \subseteq \mathbb{F}_j) + O(\rho_1(\mathbf{N})).$$

Now to obtain (5.41) it is enough to prove that if $F_i(1) \in \mathbb{F}_j$ for some $j \in [1,\mathfrak{k}]$ and $\mathbf{m}^{(F_i(1))} \in B(-\mathbf{m}^{(F_i(2))})$, then $F_i(2) \in \mathbb{F}_j$ $(i=1,...,\hbar/2)$. Let $j_1 = F_i(1)$ and $j_2 = F_i(2)$. Suppose that there exists $1 \leq i_1 < i_2 \leq \mathfrak{k}$ with $j_1 \in \mathbb{F}_{i_1}$ and $j_2 \in \mathbb{F}_{i_2}$. Using (5.9), (5.10) and (5.15), we get $\mathbf{p}_{i_2} \prec \mathbf{p}_{i_1}$,

$$\widetilde{m}^{(j_1)}_{\mathbf{p}_{i_1}} \neq 0,\ \widetilde{m}^{(j_1)}_{\mathbf{p}} = 0 \ \text{ for } \ \mathbf{p}_{i_1} \prec \mathbf{p} \quad \text{and} \quad \widetilde{m}^{(j_2)}_{\mathbf{p}_{i_2}} \neq 0,\ \widetilde{m}^{(j_2)}_{\mathbf{p}} = 0 \ \text{ for } \ \mathbf{p}_{i_2} \prec \mathbf{p}. \qquad (5.45)$$

Let $\mathbf{m}^{(F_i(1))} \in B(-\mathbf{m}^{(F_i(2))})$. Hence $\mathbf{m}^{(F_i(1))} = -\mathbf{A}^{\mathbf{n}} \mathbf{m}^{(F_i(2))}$ for some $\mathbf{n}$. By (4.12) we have $-\widetilde{\mathbf{m}}^{(F_i(1))} = \mathbf{\Lambda}^{\mathbf{n}} \widetilde{\mathbf{m}}^{(F_i(2))}$. Bearing in mind that $(\dot{\lambda}_{\mathbf{p}_1,\mathbf{p}_2})_{\mathbf{p}_1,\mathbf{p}_2\in\mathcal{P}} := \mathbf{\Lambda}^{\mathbf{n}}$ is an upper triangular matrix, we get from (5.45)

$$\dot{\lambda}_{\mathbf{p}_{i_1},\mathbf{p}} = 0 \quad \text{for} \quad \mathbf{p} \prec \mathbf{p}_{i_1}, \qquad \text{and} \qquad \widetilde{m}^{(j_2)}_{\mathbf{p}} = 0 \quad \text{for} \quad \mathbf{p}_{i_1} \preceq \mathbf{p}.$$

Thus $\widetilde{m}^{(j_1)}_{\mathbf{p}_{i_1}} = \sum_{\mathbf{p}_{i_1} \preceq \mathbf{p}} \dot{\lambda}_{\mathbf{p}_{i_1},\mathbf{p}} \widetilde{m}^{(j_2)}_{\mathbf{p}} = 0$. By (5.45), we have a contradiction. Therefore Lemma 5.9 is proved. ∎

Proof of Lemma 5.1. Using (5.1) we get

$$(\upsilon(\mathbf{N}, f_L, \mathbf{x}))^{\hbar} = \Big(\frac{\beta_1}{\sigma(f_L)}\Big)^{\hbar} \sum_{\eth_1,\dots,\eth_\hbar=1}^{q} \frac{\alpha_{\eth_1}\cdots\alpha_{\eth_\hbar}}{\sqrt{\breve{\mathbf{N}}_{\eth_1}\cdots\breve{\mathbf{N}}_{\eth_\hbar}}} \qquad (5.46)$$

$$\times \sum_{|\mathbf{m}^{(i)}|<L,\ i=1,\dots,\hbar} \widehat{f}(\mathbf{m}^{(1)}) \cdots \widehat{f}(\mathbf{m}^{(\hbar)}) \sum_{\mathbf{n_i}\in\mathfrak{R}_{\eth_i}(\mathbf{N}_{\eth_i}),i=1,\dots,\hbar} e\Big(\Big\langle \mathbf{x}, \sum_{i=1}^{\hbar} \mathbf{A}^{\mathbf{n}_i} \mathbf{m}^{(i)} \Big\rangle\Big),$$

where $\beta_1 = (\alpha_1^2 + \cdots + \alpha_q^2)^{-1/2}$. Hence

$$\kappa := \int_{[0,1)^s} (\upsilon(\mathbf{N}, f_L, \mathbf{x}))^{\hbar} d\mathbf{x}$$

$$= \Big(\frac{\beta_1}{\sigma(f_L)}\Big)^{\hbar} \sum_{\eth_1,\dots,\eth_\hbar=1}^{q} \alpha_{\eth_1}\cdots\alpha_{\eth_\hbar} \sum_{|\mathbf{m}^{(i)}|<L,\ i=1,\dots,\hbar} \widehat{f}(\mathbf{m}^{(1)})\cdots\widehat{f}(\mathbf{m}^{(\hbar)})\kappa_1$$

where

$$\kappa_1 = (\breve{\mathbf{N}}_{\eth_1}\cdots\breve{\mathbf{N}}_{\eth_\hbar})^{-1/2} \sum_{\mathbf{n_i}\in\mathfrak{R}_{\eth_i}(\mathbf{N}_{\eth_i}),i=1,\dots,\hbar} \delta\Big(\sum_{i=1}^{\hbar} \mathbf{A}^{\mathbf{n}_i}\mathbf{m}^{(i)}\Big).$$

Applying (5.5) and Lemma 5.8 for odd $\hbar$, we obtain

$$\kappa = O(\rho(\mathbf{N})),$$

where O-constants depend only on $\hbar$, f, and L. Hence (5.2) is true for odd $\hbar$.

Let $\hbar$ be even. Using (5.5) and Lemma 5.9, we get

$$\kappa = \Big(\frac{\beta_1}{\sigma(f_L)}\Big)^{\hbar} \sum_{\eth_1,\dots,\eth_\hbar=1}^{q} \alpha_{\eth_1}\cdots\alpha_{\eth_\hbar} \sum_{|\mathbf{m}^{(i)}|<L,\ i=1,\dots,\hbar} \widehat{f}(\mathbf{m}^{(1)})\cdots\widehat{f}(\mathbf{m}^{(\hbar)})$$

$$\times \sum_{\substack{(F_1,\dots,F_{\hbar/2})\in\mathfrak{F}_0\\ \#F_i=2,\ i\in[1,\hbar/2]}} \prod_{i=1}^{\hbar/2} \delta(\eth_{F_i(1)} = \eth_{F_i(2)})\delta\Big(\mathbf{m}^{(F_i(1))} \in B(-\mathbf{m}^{(F_i(2))})\Big) + O(\rho_1(\mathbf{N})),$$

where O-constant depends only on $\hbar$, f, and L. Changing the order of the summation, we obtain

$$\kappa = \Big(\frac{\beta_1}{\sigma(f_L)}\Big)^{\hbar} \sum_{\substack{(F_1,\dots,F_{\hbar/2})\in\mathfrak{F}_0\\ \#F_i=2,\ i\in[1,\hbar/2]}} \sum_{\eth_1,\dots,\eth_\hbar=1}^{q} \alpha_{\eth_1}\cdots\alpha_{\eth_\hbar} \prod_{i=1}^{\hbar/2} \delta(\eth_{F_i(1)} = \eth_{F_i(2)})$$

$$\times \sum_{|\mathbf{m}^{(F_i(j))}|<L,\ j=1,2} \widehat{f}(\mathbf{m}^{(F_i(1))})\widehat{f}(\mathbf{m}^{(F_i(1))})\delta\Big(\mathbf{m}^{(F_i(1))} \in B(-\mathbf{m}^{(F_i(2))})\Big) + O(\rho_1(\mathbf{N})).$$

By (3.8) and (5.46), we have that $\beta_1 = (\alpha_1^2 + \cdots + \alpha_q^2)^{-1/2}$ and

$$\kappa = \Big(\frac{\beta_1}{\sigma(f_L)}\Big)^{\hbar} \sum_{\substack{(F_1,\dots,F_{\hbar/2})\in\mathfrak{F}_0\\ \#F_i=2,\ i\in[1,\hbar/2]}} \sum_{\eth_1,\dots,\eth_{\hbar/2}=1}^{q} \alpha_{\eth_1}^2\cdots\alpha_{\eth_{\hbar/2}}^2$$

$$\times \Big(\sum_{|\mathbf{m}^{(i)}|<L,\ i=1,2} \widehat{f}(\mathbf{m}^{(1)})\widehat{f}(\mathbf{m}^{(2)})\delta(\mathbf{m}^{(1)} \in B(-\mathbf{m}^{(2)}))\Big)^{\hbar/2} + O(\rho_1(\mathbf{N}))$$

$$= \Big(\frac{\beta_1}{\sigma(f_L)}\Big)^{\hbar} \sum_{\substack{(F_1,\dots,F_{\hbar/2})\in\mathfrak{F}_0\\ \#F_i=2,\ i\in[1,\hbar/2]}} \Big(\sum_{i=1}^{q}\alpha_{\eth_1}^2\Big)^{\hbar/2}(\sigma(f_L))^{\hbar} + O(\rho_1(\mathbf{N})) = \sum_{\substack{(F_1,\dots,F_{\hbar/2})\in\mathfrak{F}_0\\ \#F_i=2,\ i\in[1,\hbar/2]}} 1$$

$$+ O(\rho_1(\mathbf{N})) = \frac{1}{(\hbar/2)!}\binom{\hbar}{2}\binom{\hbar-2}{2}\cdots\binom{2}{2} + O(\rho_1(\mathbf{N})) = \frac{\hbar!}{(\hbar/2)!2^{\hbar/2}} + O(\rho_1(\mathbf{N})).$$

Therefore Lemma 5.1. is proved . ∎

Lemma 5.10. [Bi, Theorem 3.2, p. 28] *Suppose that* $X_{L,n}, X_n$ *are random variables. If* $X_{L,n} \xrightarrow{d} Z_L$ *as* $n \to \infty$, $Z_L \xrightarrow{d} X$ *as* $L \to \infty$, *and*

$$\lim_{L\to\infty} \limsup_{n\to\infty} P(|X_{L,n} - X_n| > \epsilon) = 0 \tag{5.47}$$

for each $\epsilon > 0$, *then* $X_n \xrightarrow{d} X$ *as* $\to \infty$.

End of the proof of Theorem 5. We have $\sigma(f) > 0$. To prove (5.1), we will use Lemma 5.10 with $X = \mathcal{N}(0,1)$, $Z_L = X\sigma(f_L)/\sigma(f)$, $X_{L,n} = \upsilon(\overline{\mathbf{N}}_n, f_L, \mathbf{x})\sigma(f_L)/\sigma(f)$, and $X_n = \upsilon(\overline{\mathbf{N}}_n, f, \mathbf{x})$, where $\overline{\mathbf{N}}_n = (\mathbf{N}_1^{(n)}, ..., \mathbf{N}_q^{(n)})$, $\mathbf{N}_i^{(n)} = (N_{i,1}^{(n)}, ..., N_{i,d}^{(n)})$, with $\lim_{n\to\infty} \min_{i,j} N_{i,j}^{(n)} \to \infty$.

From (3.12) we have $\sigma(f_L) \to \sigma(f)$ and $Z_L \xrightarrow{d} X$ as $L \to \infty$. Using Lemma 5.1, we get that $X_{L,n} \xrightarrow{d} X$. Let

$$\upsilon'(\overline{\mathbf{N}}, f, f_L, \mathbf{x}) = \upsilon(\overline{\mathbf{N}}, f, \mathbf{x}) - \frac{\sigma(f_L)}{\sigma(f)}\upsilon(\overline{\mathbf{N}}, f_L, \mathbf{x}). \tag{5.48}$$

Applying Chebyshev's inequality, we get that to obtain (5.47) it is enough to verify that

$$\lim_{L\to\infty} \limsup_{n\to\infty} \left\|\upsilon'(\overline{\mathbf{N}}_n, f, f_l, \mathbf{x})\right\|_2 = 0. \tag{5.49}$$

By (5.1) and (2.12) we have

$$\upsilon'(\overline{\mathbf{N}}, f, f_L, \mathbf{x}) = \frac{1}{\sigma(f)} \sum_{\eth=1}^{q} \frac{\alpha_\eth}{\sqrt{\alpha_1^2 + \cdots + \alpha_q^2}} \dot{S}_\eth, \tag{5.50}$$

where

$$\dot{S}_\eth = \breve{\mathbf{N}}_\eth^{-1/2} \sum_{|\mathbf{m}|\geq L} \widehat{f}(\mathbf{m}) \sum_{\mathbf{n}\in\mathfrak{R}_\eth(\mathbf{N}_\eth)} e(\langle \mathbf{x}, \mathbf{A}^{\mathbf{n}}\mathbf{m}\rangle). \tag{5.51}$$

Bearing in mind that

$$\sum_{\mathbf{n_1},\mathbf{n_2}\in\mathfrak{R}_\eth(\mathbf{N}_\eth)} \delta(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_1 = \mathbf{A}^{\mathbf{n}_2}\mathbf{m}_2) = \sum_{0\leq n_{i,j}<N_{\eth,i}, i=1,\dots,d, j=1,2} \delta(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_1 = \mathbf{A}^{\mathbf{n}_2}\mathbf{m}_2),$$

from (2.7) we obtain

$$\breve{\mathbf{N}}_\eth \|\dot{S}_\eth\|_2^2 = \sum_{|\mathbf{m}_1|,|\mathbf{m}_2|\geq L} \widehat{f}(\mathbf{m}_1)\widehat{f}(-\mathbf{m}_2) \sum_{\mathbf{n_1},\mathbf{n_2}\in\mathfrak{R}_\eth(\mathbf{N}_\eth)} \delta(\mathbf{A}^{\mathbf{n}_1}\mathbf{m}_1 = \mathbf{A}^{\mathbf{n}_2}\mathbf{m}_2) = \|S_{\mathbf{N}_\eth}(f - f_L)\|_2^2.$$

Now by the triangle inequality

$$\sigma(f)\left\|\upsilon'(\overline{\mathbf{N}}, f, f_L, \mathbf{x})\right\|_2 \leq \sum_{\eth=1}^{q} \frac{1}{\sqrt{\breve{\mathbf{N}}_\eth}} \|S_{\mathbf{N}_\eth}(f - f_L)\|_2.$$

Using (3.9), we get

$$\frac{1}{\sqrt{\breve{\mathbf{N}}_\eth}} \|S_{\mathbf{N}_\eth}(f - f_L)\|_2 \leq (S(f - f_L))^{1/2}.$$

By (3.10), $S(f - f_L) \to 0$ and (5.49) follows. Hence Theorem 5 is proved. ■

5.2 Functional CLT.

Let $D([0,1]^d)$ be the Skorokhod space of functions (see def., e.g.,[BuSh, p.252]), $(\zeta_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}_+^d}$ a random multisequence. We introduce the *partial sums process* by the following formula

$$W_{\mathbf{N}}(\mathbf{t}) = \frac{1}{\sqrt{\breve{\mathbf{N}}}} \sum_{0\le n_i<t_iN_i,\ i=1,\dots,d} \zeta_{\mathbf{n}} \qquad \text{where} \qquad \mathbf{t}\in[0,1]^d \quad \text{and} \quad \breve{\mathbf{N}} = N_1\cdots N_d.$$

Definition 5.6. (see, e.g., [BuSh], p.255) *One says that the multisequence $(\zeta_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}_+^d}$ satisfies the weak invariance principle or a functional CLT (abbreviated FCLT) if there exist $\sigma^2>0$ and a multiparameter Brownian motion W defined on $[0,1]^d$ such that the law of $W_{\mathbf{N}}$ weakly converges to the law of σW in the space $D([0,1]^d)$ as $\min_i N_i\to\infty$.*

Theorem 6. *Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1,\dots,A_d$ of $[0,1)^s$, f a real $\mathbb{Z}^s$-periodic local integrable function with absolutely convergent Fourier series, with mean zero and $\sigma(f)>0$. Then $(f(\mathbf{A}^{\mathbf{n}})\mathbf{x})_{\mathbf{n}\in\mathbb{Z}_+^d}$ satisfies the FCLT .*

Proof. By Prohorov's theorem (see, e.g., [Bi], p.66, Th. 6.1, 6.2) the necessary and sufficient condition for the weak convergence of a sequence of processes $(W_{\mathbf{n}}(\mathbf{t}))_{\mathbf{n}\in\mathbb{Z}_+^d}$ where $\mathbf{t}\in[0,1]^d$ is the tightness (see def., e.g., [BuSh] p.253) of the sequence of their distributions in the Skorokhod space $D([0,1]^d)$ and weak convergence of the finite-dimensional distributions. The weak convergence of the finite-dimensional distributions follows from Theorem 5. Let

$$S_{\mathbf{N}}(f,\mathfrak{R}) = \sum_{\mathbf{n}\in\mathfrak{R}} f(\mathbf{A}^{\mathbf{n}}\mathbf{x}), \quad \text{with} \quad \mathfrak{R}=\mathfrak{R}(\mathbf{N}) = [R_1,R_1+N_1)\times\cdots\times[R_d,R_d+N_d).$$

By [BW, Theorem 3, p.1665], to prove the tightness condition it is enough to verify that $S_{\mathbf{N}}(f,\mathfrak{R})$ belong to the class $\mathfrak{T}(2,4)$ defined in [BW] (see inequalities 2,3 p.1658), i.e.

$$E\Big(\Big(\min\big(|S_{\mathbf{N}_1}(f,\mathfrak{R}_1)|,|S_{\mathbf{N}_2}(f,\mathfrak{R}_2)|\big)\Big)^4\Big) \le c_0(\breve{N}_1+\breve{N}_2)^2 \quad \text{for} \quad \mathfrak{R}_1\cap\mathfrak{R}_2=\emptyset,$$

with some constant $c_0>0$. It is easy to see that this inequality follows from the estimate

$$E(|S_{\mathbf{N}}(f,\mathfrak{R})|^4) = O(\breve{\mathbf{N}}^2). \tag{5.52}$$

Applying (5.1) and Lemma 6.1 with $q=1,\ \hbar=4$, we get (5.52). Hence Theorem 6 is proved. ■

Lemma 6.1. *With notations as above*

$$\lim_{\min_{i,j} N_{i,j}\to\infty} \big\|\upsilon(\mathbf{N},f,\mathbf{x})\big\|_{\hbar}^{\hbar} = \begin{cases} \frac{\hbar!}{2^{\hbar/2}(\hbar/2)!}, & \text{if } \hbar \text{ is even,} \\ 0, & \text{if } \hbar \text{ is odd.} \end{cases} \tag{5.53}$$

Proof. Using (5.1), (5.48), (5.50), (5.51) and the Minkowski's inequality, we get

$$\frac{\sigma(f_L)}{\sigma(f)}\big\|\upsilon(\mathbf{N},f_L,\mathbf{x})\big\|_{\hbar} - \big\|\upsilon^{'}(\mathbf{N},f,f_L,\mathbf{x})\big\|_{\hbar} \le \big\|\upsilon(\mathbf{N},f,\mathbf{x})\big\|_{\hbar} \tag{5.54}$$

$$\le \frac{\sigma(f_L)}{\sigma(f)}\big\|\upsilon(\mathbf{N},f_L,\mathbf{x})\big\|_{\hbar} + \big\|\upsilon^{'}(\mathbf{N},f,f_L,\mathbf{x})\big\|_{\hbar}$$

and

$$\sigma(f)\big\|\upsilon^{'}(\mathbf{N},f,f_L,\mathbf{x})\big\|_{\hbar} \leq \sum_{\eth=1}^{q} \big\|\dot{S}_{\eth}\big\|_{\hbar}. \tag{5.55}$$

We have for $\eth \in [1,q]$

$$\big\|\dot{S}_{\eth}\big\|_{\hbar}^{\hbar} = \sum_{|\mathbf{m}^{(i)}|\geq L,\ i=1,\dots,\hbar} \widehat{f}(\mathbf{m}^{(1)})\cdots\widehat{f}(\mathbf{m}^{(\hbar)})(\breve{\mathbf{N}}_{\eth})^{-\hbar/2} \sum_{\mathbf{n_i}\in\mathfrak{R}_{\eth}(\mathbf{N}_{\eth}),i=1,\dots,\hbar} \delta(\sum_{i=1}^{\hbar}\mathbf{A}^{\mathbf{n}_i}\mathbf{m}^{(i)}).$$

Let $\hbar$ is even. By (5.5) and Lemma 5.8, we obtain

$$\big\|\dot{S}_{\eth}\big\|_{\hbar}^{\hbar} = \sum_{|\mathbf{m}^{(i)}|\geq L,\ i=1,\dots,\hbar} \widehat{f}(\mathbf{m}^{(1)})\cdots\widehat{f}(\mathbf{m}^{(\hbar)})$$

$$\times\Big(O(\rho(\mathbf{N})) + \prod_{i=1}^{\mathfrak{k}} \sum_{\substack{(F_1^{(i)},\dots,F_{\mathfrak{f}_i/2}^{(i)})\in\mathfrak{F}_i \\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \prod_{j=1}^{\mathfrak{f}_i/2} \varkappa_{i,j}\Big),$$

with

$$\varkappa_{i,j} = \frac{1}{\sqrt{\breve{\mathbf{N}}_{F_j^{(i)}}}} \sum_{\mathbf{n}_{\mu_{i,j,k}}\in\mathfrak{R}_{\eth_{\mu_{i,j,k}}},k=1,2} \delta(\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})} = -\mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})}),$$

where O-constant depends only on $\hbar$. It is easy to verify that $\varkappa_{i,j} \leq 1$ (see Definition 1 and (5.19)). Therefore

$$\big\|\dot{S}_{\eth}\big\|_{\hbar}^{\hbar} = O\Big((1+\rho(\mathbf{N})) \sum_{|\mathbf{m}^{(i)}|\geq L,\ i=1,\dots,\hbar} \widehat{f}(\mathbf{m}^{(1)})\cdots\widehat{f}(\mathbf{m}^{(\hbar)})\Big),$$

where O-constant depends only on $\hbar$, and $\rho(\mathbf{N}) = (\min_i \breve{\mathbf{N}}_i)^{-1/2}$.

Bearing in mind that Fourier series of the function f converge absolutely, we get that for all $\epsilon > 0$ $\exists$ $L(\epsilon) > 0$ with $\big\|\dot{S}_{\eth}\big\|_{\hbar} \leq \epsilon\sigma(f)/q$ for all $\mathbf{N}$ and $L \geq L(\epsilon)$. From (5.54) and (5.55), we get for $L \geq L(\epsilon)$

$$\frac{\sigma(f_L)}{\sigma(f)}\big\|\upsilon(\mathbf{N},f_L,\mathbf{x})\big\|_{\hbar} - \epsilon \leq \big\|\upsilon(\mathbf{N},f,\mathbf{x})\big\|_{\hbar} \leq \frac{\sigma(f_L)}{\sigma(f)}\big\|\upsilon(\mathbf{N},f_L,\mathbf{x})\big\|_{\hbar} + \epsilon.$$

Applying Lemma 5.1, we get

$$\frac{\sigma(f_L)}{\sigma(f)}\frac{\hbar!}{2^{\hbar/2}(\hbar/2)!} - \epsilon \leq \liminf_{\min_{i,j} N_{i,j}\to\infty} \big\|\upsilon(\mathbf{N},f,\mathbf{x})\big\|_{\hbar}$$

$$\leq \limsup_{\min_{i,j} N_{i,j}\to\infty} \big\|\upsilon(\mathbf{N},f,\mathbf{x})\big\|_{\hbar} \leq \frac{\sigma(f_L)}{\sigma(f)}\frac{\hbar!}{2^{\hbar/2}(\hbar/2)!} + \epsilon. \tag{5.56}$$

By (3.12), we obtain that $\sigma(f_L) \to \sigma(f) > 0$ as $L \to \infty$. Now from (5.56), we get (5.53) for $\hbar$ is even. Using Lemma 5.8 we obtain (5.53) for $\hbar$ is odd similarly. Hence Lemma 6.1 is proved. ■

5.3 Almost sure CLT.

Let $\zeta_{\mathbf{n}}$ be a random multisequence with $Var(\zeta_{\mathbf{n}}) = 1$ ($\mathbf{n} \in \mathbb{Z}_+^d$), $\tilde{\delta}(\mathbf{x})$ denotes the point mass at $\mathbf{x} \in \mathbb{R}^s$. We say that $\zeta_{\mathbf{N}}$ satisfies the *almost sure central limit theorem* (abbreviated ASCLT) (see, e.g., [FR]) if with probability one

$$\frac{1}{\ln N_1 \cdots \ln N_d} \sum_{n_i \in [1,N_i], i=1,\dots d} \frac{\tilde{\delta}(\zeta_{\mathbf{n}})}{n_1 \cdots n_d} \xrightarrow{w} \mathcal{N}(0,1) \quad \text{as} \quad \min_i N_i \to \infty. \tag{5.57}$$

Similarly to [Li, Lemma 6.1], we have that it is enough to verify the almost sure convergence

$$\frac{1}{\ln N_1 \cdots \ln N_d} \sum_{n_i \in [1,N_i],\ i=1,\dots,d} \frac{g(\zeta_{\mathbf{n}})}{n_1 \cdots n_d} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) \exp(-y^2/2) dy \tag{5.58}$$

for each fixed bounded Lipschitz function g on $\mathbb{R}^s$ to obtain (5.57).

We say that the multisequence $\zeta_{\mathbf{n}}$ satisfies the polynomial ASCLT if (5.58) is true for arbitrary polynomial $g(x)$. One can observe that the polynomial ASCLT implies a standard ASCLT.

Theorem 7. *Let $\mathcal{A}$ be an action by commuting partially hyperbolic endomorphisms $A_1, ..., A_d$ of $[0,1)^s$, f a real $\mathbb{Z}^s$-periodic local integrable function with absolutely convergent Fourier series, with mean zero and $\sigma(f) > 0$,*

$$S_{\mathbf{N}}(f) = \left(\sigma_f \breve{\mathbf{N}}^{1/2}\right)^{-1} \sum_{\mathbf{n} \in \mathfrak{R}(\mathbf{N})} f(\mathbf{A}^{\mathbf{n}} \mathbf{x}), \qquad \text{with} \qquad \mathfrak{R}(\mathbf{N}) = [0, N_1) \times \cdots \times [0, N_d). \tag{5.59}$$

Then $S_{\mathbf{N}}(f)$ satisfies the polynomial ASCLT .

Proof. Clearly, that is enough to prove (5.58) for $g(x) = x^{\hbar_1}$ ($\hbar_1 = 1, 2, ...$). Applying Theorem 6, we get

$$\gamma := \lim_{\min_i N_i \to \infty} E((S_{\mathbf{N}}(f))^{\hbar_1}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^{\hbar_1} \exp(-y^2/2) dy.$$

Hence

$$\lim_{\min_i N_i \to \infty} \frac{1}{\ln N_1 \cdots \ln N_d} \sum_{n_i \in [1,N_i], i=1,\dots d} \frac{E((S_{\mathbf{n}}(f))^{\hbar_1})}{n_1 \cdots n_d} = \gamma.$$

Let

$$\xi_{\mathbf{n}} = \left((S_{\mathbf{n}}(f))^{\hbar_1} - E((S_{\mathbf{n}}(f))^{\hbar_1})\right) / (n_1 \cdots n_d). \tag{5.60}$$

To prove Theorem 7, it is enough to verify that

$$\frac{1}{\ln N_1 \cdots \ln N_d} \sum_{n_i \in [1,N_i], i=1,\dots d} \xi_{\mathbf{n}} \to 0 \quad \text{a.s.} \tag{5.61}$$

Lemma 7.1. *Let $\mathbf{N}_i = (N_{i,1}, ..., N_{i,d}) \in \mathbb{N}^d$ $(i = 1, 2)$, $\dot{N}_i = \min(N_{1,i}, N_{2,i})$ and $\ddot{N}_i = \max(N_{1,i}, N_{2,i})$ $(i = 1, ..., d)$. Then there exists a constant $C > 0$ with*

$$|E(\xi_{\mathbf{N}_1} \xi_{\mathbf{N}_2})| \le C (\prod_{i=1}^d (\dot{N}_i)^{-3/2} (\ddot{N}_i)^{-1} + \prod_{i=1}^d (\dot{N}_i)^{-1/2} (\ddot{N}_i)^{-3/2}). \tag{5.62}$$

The proof of Lemma 7.1 is given after Lemma 7.3. But first we give some definitions. From (5.62), we get

$$Q := E(|\sum_{I_i \le n_i \le J_i, i=1,\dots,d} \xi_{\mathbf{n}}|^2) \le \sum_{N_{j,i} \in [I_i, J_i], i=1,\dots d, j=1,2} |E(\xi_{\mathbf{N}_1}\xi_{\mathbf{N}_2})|$$

$$\le C2^d \sum_{I_i \le \dot{N}_i \le \ddot{N}_i \le J_i, i=1,\dots d} (\prod_{i=1}^{d} (\dot{N}_i)^{-3/2}(\ddot{N}_i)^{-1} + \prod_{i=1}^{d} (\dot{N}_i)^{-1/2}(\ddot{N}_i)^{-3/2}).$$

Hence

$$Q \le C2^{4d} \sum_{I_i \le \ddot{N}_i \le J_i, i=1,\dots d} \prod_{i \in [1,d]} \frac{1}{\ddot{N}_i}$$

By Jensen's inequality and Lemma 7.3, we obtain

$$E\Big(\max_{1 \le I_i \le J_i \le N_i, i=1,\dots,d} \Big| \sum_{I_i \le n_i \le J_i, i=1,\dots,d} \xi_{\mathbf{n}} \Big|^{\sqrt{2}}\Big)$$

$$\le \Big(E\Big(\max_{1 \le I_i \le J_i \le N_i, i=1,\dots,d} \Big| \sum_{I_i \le n_i \le J_i, i=1,\dots,d} \xi_{\mathbf{n}} \Big|^2\Big)\Big)^{1/\sqrt{2}} \le C_2^{1/\sqrt{2}} C2^{4d} \sum_{1 \le n_i \le N_i, i=1,\dots d} \frac{1}{n_1 \cdots n_d}.$$

Applying Lemma 7.2, we get (5.61) and the assertion of Theorem 7. ∎

Using [NT, Theorem 3] with $a_{\mathbf{N}} = (N_1 \cdots N_d)^{-1}$, $b_{\mathbf{N}} = \ln(N_1) \cdots \ln(N_d)$, $\mathbf{N} = (N_1, \dots, N_d)$ and $r = \sqrt{2}$, we obtain

Lemma 7.2. *Let* $\zeta_{\mathbf{n}}$ *be the random multisequence,* $C_1 > 0$ *and*

$$E\Big(\max_{1 \le n_i \le N_i, i=1,\dots d} \Big| \sum_{1 \le k_i \le n_i, i=1,\dots,d} \zeta_{\mathbf{k}} \Big|^{\sqrt{2}}\Big) \le \sum_{n_i \in [1,N_i], i=1,\dots d} \frac{C_1}{n_1 \cdots n_d} \qquad \forall \mathbf{N} \in \mathbb{N}^d. \quad (5.63)$$

Then

$$\lim_{\min_i N_i \to \infty} \frac{1}{\ln N_1 \cdots \ln N_d} \sum_{1 \le n_i \le N_i, i=1,\dots,d} \zeta_{\mathbf{n}} = 0 \qquad \text{a.s.} \qquad (5.64)$$

Applying Móricz's maximal inequality [Mo, Corollary 1, p. 340] with $\gamma = 2$ and $\alpha = \sqrt{2}$, we get

Lemma 7.3. *Let* $\zeta_{\mathbf{n}}$ *be the random multisequence,* $C_2 = (5/2)^d(1 - 2^{(1-\sqrt{2})/2})^{-2d}$ *and*

$$E(|\sum_{I_i \le n_i \le J_i, i=1,\dots,d} \zeta_{\mathbf{n}}|^2) \le \Big(\sum_{I_i \le n_i \le J_i, i=1,\dots d} \frac{1}{n_1 \cdots n_d}\Big)^{\sqrt{2}} \quad \forall\, I_i \le J_i \in \mathbb{N},\ i = 1, \dots, d.$$

Then

$$E\Big(\max_{1 \le I_i \le J_i \le N_i, i=1,\dots,d} | \sum_{I_i \le n_i \le J_i, i=1,\dots,d} \zeta_{\mathbf{n}}|^2\Big) \le C_2\Big(\sum_{1 \le n_i \le N_i, i=1,\dots d} \frac{1}{n_1 \cdots n_d}\Big)^{\sqrt{2}}.$$

Proof of Lemma 7.1. From (5.59) and (5.60), we have

$$\breve{\mathbf{N}}\xi_{\mathbf{N}} = (\sigma_f \breve{\mathbf{N}}^{1/2})^{-\hbar_1} \sum_{\mathbf{m}^{(i)} \in \mathbb{Z}^s,\ i=1,\dots,\hbar_1} \widehat{f}(\mathbf{m}^{(1)}) \cdots \widehat{f}(\mathbf{m}^{(\hbar_1)})$$

$$\sum_{\mathbf{n_i} \in \mathfrak{R}_{\mathbf{N}}, i=1,\dots,\hbar_1} e\Big(\langle \mathbf{x}, \sum_{i=1}^{\hbar_1} \mathbf{A}^{\mathbf{n}_i} \mathbf{m}^{(i)} \rangle\Big) \delta\Big(\sum_{i=1}^{\hbar_1} \mathbf{A}^{\mathbf{n}_i} \mathbf{m}^{(i)} \ne \mathbf{0}\Big).$$

Let $\hbar = 2\hbar_1$, $\mathfrak{R}_i = \mathfrak{R}(\mathbf{N}_i)$ $(i \in [1,2])$, $\eth_i = 1$ for $l \in [1,\hbar_1])$ and $\eth_i = 2$ for $l \in [\hbar_1+1,\hbar])$. We see

$$\breve{\mathbf{N}}_1\breve{\mathbf{N}}_2 E(\xi_{\mathbf{N}_1}\xi_{\mathbf{N}_2}) = \sigma_f^{-\hbar} \sum_{\mathbf{m}^{(i)}\in\mathbb{Z}^s,\ i=1,\dots,\hbar} \widehat{f}(\mathbf{m}^{(1)})\cdots\widehat{f}(\mathbf{m}^{(\hbar)})\varphi, \tag{5.65}$$

where

$$\varphi = (\breve{\mathbf{N}}_1\breve{\mathbf{N}}_2)^{-\hbar_1/2} \sum_{\mathbf{n}_i\in\mathfrak{R}_{\eth_i},i=1,\dots,\hbar} \delta\Big(\sum_{i\in[1,\hbar]} \mathbf{A}^{\mathbf{n}_i}\mathbf{m}^{(i)} = \mathbf{0}\Big)\psi(\overline{\mathbf{n}})$$

with

$$\psi(\overline{\mathbf{n}}) = \delta\Big(\sum_{i\in[1,\hbar_1]} \mathbf{A}^{\mathbf{n}_i}\mathbf{m}^{(i)} \neq \mathbf{0}\Big)\delta\Big(\sum_{i\in[\hbar_1+1,\hbar]} \mathbf{A}^{\mathbf{n}_i}\mathbf{m}^{(i)} \neq \mathbf{0}\Big). \tag{5.66}$$

Applying (5.5) and Lemma 5.8 with $q = 2$, we get

$$\varphi = O(\rho(\mathbf{N})) + \sigma_f^{-\hbar} \prod_{i=1}^{\mathfrak{k}} \sum_{\substack{(F_1^{(i)},\dots,F_{\mathfrak{f}_i/2}^{(i)})\in\mathfrak{F}_i \\ \#F_j^{(i)}=2,\ j\in[1,\mathfrak{f}_i/2]}} \prod_{j=1}^{\mathfrak{f}_i/2} \varkappa_{i,j} \tag{5.67}$$

with

$$\varkappa_{i,j} = (\breve{\mathbf{N}}_{F_j^{(i)}})^{-1/2} \sum_{\mathbf{n}_{\mu_{i,j,k}}\in\mathfrak{R}_{\eth_{\mu_{i,j,k}}},k=1,2} \delta(\mathbf{A}^{\mathbf{n}_{\mu_{i,j,1}}}\mathbf{m}^{(\mu_{i,j,1})} = -\mathbf{A}^{\mathbf{n}_{\mu_{i,j,2}}}\mathbf{m}^{(\mu_{i,j,2})})\psi(\overline{\mathbf{n}}), \tag{5.68}$$

where $\mu_{i,j,k} = F_j^{(i)}(k) \in [1,\hbar],\ k = 1,2$, $\mu_{i,j,1} < \mu_{i,j,2}$, O-constant depends only on $\hbar$ and

$$\rho(\mathbf{N}) = \max_{i=1,2}(N_{i,1}\cdots N_{i,d})^{-1/2} \leq (\dot{N}_1\cdots\dot{N}_d)^{-1/2}. \tag{5.69}$$

For given partition $(F_j^{(i)})_{i,j}$, consider the case $(\mu_{i,j,1}-\hbar_1-1/2)(\mu_{i,j,2}-\hbar_1-1/2) > 0$ $\forall(i,j)$. From (5.66) and (5.68), we get $\varkappa_{i,j} = 0$ $\forall(i,j)$. Now consider the case that there exists i_0, j_0 such that $\mu_{i_0,j_0,1} \leq \hbar_1$ and $\mu_{i_0,j_0,2} > \hbar_1$. We see $\breve{\mathbf{N}}_{F_{j_0}^{(i_0)}} = \breve{\mathbf{N}}_1\breve{\mathbf{N}}_2$. Let $\mathbf{A}^{\mathbf{n}_0}\mathbf{m}^{(\mu_{i_0,j_0,1})} = -\mathbf{m}^{(\mu_{i_0,j_0,2})}$. Similarly to (5.42)-(5.44), we obtain from (5.68)

$$\varkappa_{i_0,j_0}(\breve{\mathbf{N}}_{F_{j_0}^{(i_0)}})^{1/2} \leq \#\{\mathbf{n}_{\mu_{i_0,j_0,k}} \in \mathfrak{R}_{\eth_{\mu_{i_0,j_0,k}}}, k=1,2 \mid \mathbf{n}_{\mu_{i_0,j_0,1}} = \mathbf{n}_{\mu_{i_0,j_0,2}} + \mathbf{n}_0\} \leq \dot{N}_1\cdots\dot{N}_d.$$

Taking into account that $\breve{\mathbf{N}}_{F_{j_0}^{(i_0)}} = \breve{\mathbf{N}}_1\breve{\mathbf{N}}_2 = \dot{N}_1\cdots\dot{N}_d\ddot{N}_1\cdots\ddot{N}_d$, we have

$$\varkappa_{i_0,j_0} \leq \prod_{i=1}^{d}(\dot{N}_i/\ddot{N}_i)^{1/2}. \tag{5.70}$$

Using Definition 1 and (5.19), we obtain from (5.68)

$$\varkappa_{i,j} = O(1), \qquad \text{for} \quad i \in [1,\mathfrak{k}], j \in [1,\mathfrak{f}_i/2],$$

with O-constant depending only on $\hbar$.

By (5.67), (5.70) and (5.69), we get

$$\varphi = O\Big(\prod_{i=1}^{d}(\dot{N}_i)^{-1/2} + \prod_{i=1}^{d}(\dot{N}_i/\ddot{N}_i)^{1/2}\Big),$$

with O-constant depending only on $\hbar$.

Bearing in mind that Fourier series of the function f converge absolutely, we get from (5.65)

$$E(\xi_{\mathbf{N}_1}\xi_{\mathbf{N}_2}) = O\Big(\prod_{i=1}^{d}(\dot{N}_i)^{-3/2}(\ddot{N}_i)^{-1} + \prod_{i=1}^{d}(\dot{N}_i)^{-1/2}(\ddot{N}_i)^{-3/2}\Big),$$

with O-constant depending only on $\hbar$. Therefore Lemma 7.1 is proved. ■

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