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Chaotic extensions and the lent particle method for Brownian motion*

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Abstract

In previous works, we have developed a new Malliavin calculus on the Poisson space based on *the lent particle formula*. The aim of this work is to prove that, on the Wiener space for the standard Ornstein-Uhlenbeck structure, we also have such a formula which permits to calculate easily and intuitively the Malliavin derivative of a functional. Our approach uses chaos extensions associated to stationary processes of rotations of normal martingales.

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1 Introduction

When a measurable space with a σ-finite measure ν is equipped on $L^2(\nu)$ with a local Dirichlet form with carré du champ γ, the associated Poisson space, i.e. the probability space of a random Poisson measure with intensity ν, may itself be endowed with a local Dirichlet structure with carré du champ Γ (cf. [16], [17]). If a gradient $\flat$ has been chosen associated with the operator γ, a gradient $\sharp$ associated with Γ may be constructed on the Poisson space (cf [1],[10],[14],[2]) and we have shown [2],[3], that such a gradient is provided by the lent particle formula which amounts to add a point to the configuration, to derive with respect to this point, and then to take back the point before integrating with respect to a random Poisson measure variant of the initial one.

On the example of a Lévy process, in order to find the gradient of the functional $V = \int_0^t \varphi(Y_{u-})dY_u$, this method consists in adding a jump to the process Y at time s and then deriving with respect to the size of this jump.

If we think the Brownian motion as a Lévy process, this addresses naturally the question of knowing whether to obtain the Malliavin derivative of a Wiener functional we could add a

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jump to the Brownian path and derive with respect to the size of the jump, in other words whether we have, denoting D_sF the Malliavin derivative of F

$$D_sF = \lim_{a\to 0} \frac{1}{a}(F(\omega + a1_{\{.\,\geqslant\,s\}}) - F(\omega)). \tag{1.1}$$

Formula (1.1) is satisfied in the case $F = \Phi(\int_0^1 h_1 dB, \ldots, \int_0^1 h_n dB)$ with Φ regular and h_i continuous. But this formula has no sense in general, since the mapping $t \mapsto 1_{\{t\,\geqslant\,s\}}$ does not belong to the Cameron-Martin space.

We tackle this question by means of the *chaotic extension* of a Wiener functional to a normal martingale weighted combination of a Brownian motion and a Poisson process, and we show that the gradient and its domain are characterized in terms of derivative of a second order stationary process.

We show that a formula similar to (1.1) is valid and yields the gradient if F belongs to the domain of the Ornstein-Uhlenbeck Dirichlet form, but whose meaning and justification involve chaotic decompositions. This gives rise to a concrete calculus allowing $\mathcal{C}^1$ changes of variables.

Let us also mention the works of B. Dupire ([8]), R. Cont and D.A. Fournié ([5]) which use an idea somewhat similar but in a completely different mathematical approach and context.

2 Second order stationary process of rotations of normal martingales

Let B be a standard one-dimensional Brownian motion defined on the Wiener space Ω_1 under the Wiener measure $\mathbb{P}_1$.

In this section, we consider $\tilde{N}$ a standard compensated Poisson process independent of B. We denote by $\mathbb{P}_2$ the law of the Poisson process N and $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$.

Let us point out that in the next sections, we shall replace $\tilde{N}$ by any normal martingale.

2.1 The chaotic extension

For real θ, let us consider the normal martingale

$$X_t^\theta = B_t \cos\theta + \tilde{N}_t sin\theta.$$

If f_n is a symmetric function of $L^2(\mathbb{R}^n, \lambda_n)$, we denote $I_n(f_n)$ the Brownian multiple stochastic integral and $I_n^\theta(f_n)$ the multiple stochastic integral with respect to X^θ. We have classically cf [6]

$$\|I_n(f_n)\|^2_{L^2(\mathbb{P}_1)} = \|I_n^\theta(f_n)\|^2_{L^2(\mathbb{P})} = n!\|f_n\|^2_{L^2(\lambda_n)}.$$

It follows that if $F \in L^2(\mathbb{P}_1)$ has the expansion on the Wiener chaos

$$F = \sum_{n=0}^\infty I_n(f_n)$$

the same sequence f_n defines *a chaotic extension* of F : $F^\theta = \sum_{n=0}^\infty I_n^\theta(f_n)$.

Remark 2.1. *Let us emphasize that the chaotic extension $F \mapsto F^\theta$ is not compatible with composition of applications : $\Phi \circ \Psi^\theta \neq (\Phi \circ \Psi)^\theta$ except obvious cases as seen by taking $\Phi(x) = x^2$, $\Psi = I_1(f)$ and $\theta = \pi/2$. Thus it is important that the sequence $(f_n)_n$ appears in the notation: we will use the "short notation" of [6].*

We denote $\mathcal{P}$ (resp. $\mathcal{P}(t)$) the set of finite subsets of $]0, \infty[$ (resp. $]0, t]$). We write $A = \{s_1 < \cdots < s_n\}$ for the generic element of $\mathcal{P}$ and dA for the measure whose restriction to each simplex is the Lebesgue measure, cf [6] p201 et seq.

If $F \in L^2(\mathbb{P}_1)$ expans in $F = \sum_{n=0}^{\infty} I_n(f_n)$ we denote $f \in L^2(\mathcal{P})$ the sequence $f = (n! f_n)_{n\in\mathbb{N}}$ and

$$\begin{aligned} I^\theta(f) &= \int_{\mathcal{P}} f(A) dX_A^\theta \\ &= f(\emptyset) + \sum_{n>0} \int_{s_1<\cdots<s_n} f(s_1,\ldots,s_n) dX_{s_1}^\theta \cdots dX_{s_n}^\theta. \end{aligned}$$

Thus we have $f(\emptyset) = \mathbb{E}[F]$, $F = I^0(f)$ and $I^{\pi/2}(f) = \int_{\mathcal{P}} f(A) d\tilde{N}_A = \sum_n I_n^{\pi/2}(f_n)$.

In all the paper we confond the stochastic integrals $\int H_{s-} dX_s^\theta$ and $\int H_s dX_s^\theta$ thanks to the fact that X^θ is normal.

Proposition 2.2. *Let be f and $g \in L^2(\mathcal{P})$, and $h = f + ig \in L^2_{\mathbb{C}}(\mathcal{P})$. The random variable $H^\theta = \int_{\mathcal{P}} h(A) dX_A^\theta$ defines a second order stationary process continuous in $L^2_{\mathbb{C}}(\mathbb{P})$.*

Proof. Let us denote similarly $F^\theta = \int_{\mathcal{P}} f(A) dX_A^\theta$ and $G^\theta = \int_{\mathcal{P}} g(A) dX_A^\theta$. It is enough to show that F^θ and G^θ are measurable, second order stationary and stationary correlated. Using the chaos expansions $F^{\theta+\varphi} = \sum_n I_n^{\theta+\varphi}(f_n)$, $G^\theta = \sum_n I_n^\theta(g_n)$ and the fact that the bracket of the martingales $X^{\theta+\varphi}$ and X^θ is

$$\langle X^{\theta+\varphi}, X^\theta \rangle_t = t \cos\varphi, \tag{2.1}$$

we get $\mathbb{E}[I_n^{\theta+\varphi}(f_n) I_n^\theta(g_n)] = n! \langle f_n, g_n \rangle_{L^2(\lambda_n)} \cos^n \varphi$ and $\mathbb{E}[I_m^{\theta+\varphi}(f_m) I_n^\theta(g_n)] = 0$ if m is different of n. It is then easy to conclude. □

It follows that the stationary process H^θ, defined in the previous proposition, possesses a spectral representation

$$H^\theta = \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \xi_n \tag{2.2}$$

where the c_n are real $\geqslant 0$ and the ξ_n are orthonormal in $L^2_{\mathbb{C}}(\mathbb{P})$. The norm $\|H^\theta\|^2$ which doesn't depend on θ is the total mass of the spectral measure $\sum c_n^2$.

The c_n are linked with the norms of the components of H on the chaos by formulas involving Bessel functions. In the particular case where H is an exponential vector,

$$H = \sum_{k=0}^{\infty} I_k\left(\frac{h^{\otimes k}}{k!}\right) = \exp\left[\int h dB - \frac{1}{2}\int h^2 dt\right]$$

where $h = f + ig$ belongs to $L^2_{\mathbb{C}}(\lambda_1)$, what implies $\|H\|^2_{L^2_{\mathbb{C}}} = \exp \|h\|^2$, we have by (2.1) the covariance

$$\mathbb{E}[H^{\theta+\varphi} \overline{H^\theta}] = \sum_k \frac{1}{k!} \|h\|^{2k}_{L^2_{\mathbb{C}}(\lambda_1)} \cos^k \varphi = \exp(\|h\|^2 \cos\varphi)$$

which is the Fourier transform of the spectral measure hence equal to $\sum_n c_n^2 e^{in\varphi}$.

By the relation defining the Bessel functions J_n (formula of Schlömilch)

$$\exp(iz\sin\varphi) = \sum_{n\in\mathbb{Z}} e^{in\varphi} J_n(z) \qquad z \in \mathbb{C}, \quad z \neq 0,$$

it comes $c_n^2 = i^n J_n(-i\|h\|^2)$ and for $n \geqslant 0$ (cf [15])

$$c_n^2 = c_{-n}^2 = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left(\frac{\|h\|^2}{2}\right)^{2k+n}.$$

The variables $c_n \xi_n$ may be also expressed in terms of Bessel functions using the expression of exponential vectors for X^θ, cf formula (2.4) below.

2.2 Chaotic structure of $L^2(\mathbb{P})$.

This part is independent of the rest of the paper. It is devoted to the study of chaotic representations for X^θ.

Let us first remark that the above considerations don't use the martingale representation property for X^θ which is false if $\sin\theta\cos\theta \neq 0$, as it is well known cf for instance [7]. Let us denote by $L^2(\mathbb{P}_{X^\theta})$ the vector space of $\sigma(X^\theta)$-measurable random variables belonging to $L^2(\mathbb{P})$. That means that, if $\sin\theta\cos\theta \neq 0$, the vector space $C(X^\theta) = \{F^\theta : F \in L^2(\mathbb{P}_1)\}$ which is closed in $L^2(\mathbb{P}_{X^\theta})$ has a non empty complement.

If we consider the simplest example of the square of a functional of the first Brownian chaos $F = \int h dB$ with $h \in L^1 \cap L^\infty$, we have $F^\theta = \int h dX^\theta$ and by Ito formula

$$(F^\theta)^2 = 2\int\int_0^t h(s)dX^\theta_s\, h(t)dX^\theta_t + \int h^2 ds + \sin^2\theta \int h^2 d\tilde{N}$$

since $\tilde{N} = \sin\theta\, X^\theta + \cos\theta\, X^{\theta+\pi/2}$, we see that

$$(F^\theta)^2 = U + \sin^2\theta\cos\theta \int h^2 dX^{\theta+\pi/2}$$

with $U \in C(X^\theta)$ and $\int h^2 dX^{\theta+\pi/2}$ orthogonal to $C(X^\theta)$.

It follows that for $k \in L^2(\mathbb{R}_+)$, $\int k dX^{\theta+\pi/2} \in L^2(\mathbb{P}_{X^\theta})$ and this implies

Proposition 2.3. *Let us suppose $\sin\theta\cos\theta \neq 0$. The σ-fields generated by X^θ and $X^{\theta+\pi/2}$ are the same. The spaces $L^2(\mathbb{P}_{X^\theta})$ do not depend on θ and are equal to $L^2(\mathbb{P})$.*

The intuitive meaning of this proposition is that on a sample path of X^θ it is possible to measurably detect the underlying Brownian and Poisson paths.

The multiple stochastic integrals w.r. to X^θ are not enough to fill $L^2(\mathbb{P}_{X^\theta})$. In view of the previous example, we may think to add the stochastic integrals w.r. to $X^{\theta+\pi/2}$, i.e. to add $C(X^{\theta+\pi/2})$, which is orthogonal to $C(X^\theta)$ and included in $L^2(\mathbb{P}_{X^\theta})$. But this is not sufficient, we must add also the crossed chaos in the following manner:

Let us consider the vector martingale $\mathbf{X}^\theta = (X^\theta, X^{\theta+\pi/2})$. For $h = (h_1, h_2) \in L^2(\mathbb{R}_+, \mathbb{R}^2)$ we may consider the stochastic integral $\int h.d\mathbf{X}^\theta$, and more generally (notation of [4] Chap II §2)

$$\int_{\Delta_n} f.d^{(n)}\mathbf{X}^\theta \tag{2.3}$$

for $f \in L^2(\Delta_n, (\mathbb{R}^2)^{\otimes n})$. And using (2.1) for $\varphi = \pi/2$ we have

$$\| \int_{\Delta_n} f.d^{(n)}\mathbf{X}^\theta \|_{L^2(\mathbb{P})} = \|f\|_{L^2(\Delta_n,(\mathbb{R}^2)^{\otimes n})}.$$

These stochastic integals define orthogonal sub-spaces of $L^2(\mathbb{P}_{X^\theta}) = L^2(\mathbb{P})$. Now considering the exponential vector

$$\mathcal{E}^\theta(h_1,h_2) = \sum_n \frac{1}{n!} \sum_{i_k\in\{1,2\}} \int h_{i_1}\otimes\cdots\otimes h_{i_n} dX^{\alpha_1}\cdots dX^{\alpha_n}$$

where $\alpha_k = \theta$ or $\theta + \pi/2$ according to $i_k = 1$ or 2, and putting $\mathcal{E}^\theta_t(h_1,h_2)$ for $\mathcal{E}^\theta(h_1 1_{[0,t]}, h_2 1_{[0,t]})$, we see that the following SDE is satisfied

$$\mathcal{E}^\theta_t(h_1,h_2) = 1 + \int_0^t \mathcal{E}^\theta_{s-}(h_1,h_2)(h_1 dX^\theta_s + h_2 dX^{\theta+\pi/2}_s)$$

what gives

$$\mathcal{E}_t^\theta(h_1,h_2)=e^{V_t-\frac{1}{2}[V,V]_t^c}\prod_{s\leqslant t}(1+\Delta V_s)e^{-\Delta V_s} \tag{2.4}$$

with $V_t=\int_0^t h_1dX^\theta+\int_0^t h_2 dX^{\theta+\pi/2}$. We obtain

Proposition 2.4. *For any θ, the stochastic integrals* (2.3) *define a complete orthogonal decomposition of $L^2(\mathbb{P})$.*

Proof. a) Let us suppose first $\sin\theta\cos\theta\neq 0$. Starting with (2.4) an easy computation yields that $\mathcal{E}_t^\theta(h_1,h_2)$ is equal up to a multiplicative constant to $\exp[\int_0^t(h_1\cos\theta-h_2\sin\theta)dB+\int_0^t ud\tilde{N}]$ where we have taken $e^u-1=h_1\sin\theta+h_2\cos\theta$.

If we take a step function $\xi\in L^2(\mathbb{R}_+)$ and choose h_1 and h_2 such that

$$\begin{aligned}h_1\sin\theta+h_2\cos\theta &= e^{\xi\sin\theta}-1\\ h_1\cos\theta-h_2\sin\theta &= \xi\cos\theta\end{aligned}$$

we obtain that $\exp[\int\xi dX^\theta]$ belongs to the space generated by the chaos, and the result follows.

b) Now if $\theta=0$, $\mathbf{X}^\theta=(B,\tilde{N})$. The above argument is still valid and

$$\mathcal{E}_t^0(h_1,h_2)=\exp[\int_0^t h_1dB-\frac{1}{2}\int_0^t h_1^2ds+\int_0^t u_2d\tilde{N}+\int_0^t u_2ds]$$

with $h_2=e^{u_2}-1$. That gives easily the result and the same in the other cases where $\sin\theta\cos\theta\neq 0$. □

In other words $L^2(\mathbb{P})$ is isomorphic to the symmetric Fock space $Fock(L^2(\mathbb{R}_+,\mathbb{R}^2))$. This implies the predictable representation property with respect to $\mathbf{X}^\theta$.

3 Derivative in θ and gradient of Malliavin.

We come back to the setting of subsection 2.1 with stochastic integrals with respect to the real process X^θ. We want to study the behavior near $\theta=0$ using the fact that $X^0=B$.

But since we deal no more with chaotic representation we may replace $\tilde{N}$ by any normal martingale M independent of B (for instance a centered normalized Lévy process) define under a probability that we still denote $\mathbb{P}_2$ and as in subsection 2.1, $\mathbb{P}=\mathbb{P}_1\times\mathbb{P}_2$. Let us put

$$Y_t^\theta=B_t\cos\theta+M_t\sin\theta$$

and consider the chaotic extensions $F\mapsto F^\theta$ with respect to Y^θ i.e. if $F=\sum_{n=0}^\infty I_n(f_n)$, then $F^\theta=\sum_{n=0}^\infty I_n^\theta(f_n)$, where from now on, I_n^θ denotes the multiple stochastic integral with respect to Y^θ.

To see the connection with the Brownian chaos expansion, let us remark that – as in the preceding part – for any $\theta\in\mathbb{R}$, the pair $(Y_1,Y_2)=(Y^\theta,Y^{\theta+\pi/2})$ is a vector normal martingale in the sense of [6] i.e.

$$\langle Y_i,Y_j\rangle_t=\delta_{ij}t \tag{3.1}$$

this allows to prove the following property

Proposition 3.1. *For any F with finite Brownian chaos expansion, $F=\sum_{k=0}^n I_n(f_n)$, f_n symmetric,*

$$\mathbb{E}[(\frac{d}{d\theta}F^\theta)^2]=\sum_{k=0}^n n!n\|f_n\|_{L^2}^2$$

Proof. Our notation is $F^\theta=\sum_{k=0}^n I_k^\theta(f_k)$ and $F^0=F=\sum_{k=0}^n I_k(f_k)$.

Let us consider first $I_n^\theta(f_n)$ in the case of an elementary tensor $f_n=g_1\otimes\cdots\otimes g_n$. We can write this multiple integral (with the notation of Bouleau-Hirsch [4] p79)

$$I_n^\theta(f_n)=n!\int_{\Delta_n} f_n\,d^{(n)}Y^\theta=n!\int_0^t[\int_{\Delta_{n-1}(s)} g_1\otimes\cdots\otimes g_{n-1}d^{(n-1)}Y^\theta]g_n(s)dY_s^\theta$$

so that

$$\begin{aligned}\tfrac{d}{d\theta}I_n^\theta(f_n)= \quad & n!\textstyle\int_0^t \tfrac{d}{d\theta}[\int_{\Delta_{n-1}(s)} g_1\otimes\cdots\otimes g_{n-1}d^{(n-1)}Y^\theta]g_n(s)dY_s^\theta\\ &+n!\textstyle\int_0^t[\int_{\Delta_{n-1}(s)} g_1\otimes\cdots\otimes g_{n-1}d^{(n-1)}Y^\theta]g_n(s)dY_s^{\theta+\pi/2}\end{aligned}$$

hence by (3.1)

$$\begin{aligned}\mathbb{E}[(\tfrac{d}{d\theta}I_n^\theta(f_n))^2]= \quad & (n!)^2\textstyle\int_0^t \mathbb{E}[(\tfrac{d}{d\theta}[\int_{\Delta_{n-1}(s)} g_1\otimes\cdots\otimes g_{n-1}d^{(n-1)}Y^\theta])^2]g_n^2(s)ds\\ &+(n!)^2\textstyle\int_0^t \mathbb{E}[(\int_{\Delta_{n-1}(s)} g_1\otimes\cdots\otimes g_{n-1}d^{(n-1)}Y^\theta)^2]g_n^2(s)ds\end{aligned}$$

what gives, running the induction down,

$$\mathbb{E}[(\frac{d}{d\theta}I_n^\theta(f_n))^2]=n(n!)^2\int_{\Delta_n}(g_1\otimes\cdots\otimes g_{n-1}\otimes g_n)^2d\lambda_n=n.n!\|f_n\|^2.$$

This extends to general tensors and similarly we can show that if $k\neq\ell$

$$\mathbb{E}[(\frac{d}{d\theta}I_k^\theta(f_k))(\frac{d}{d\theta}I_\ell^\theta(f_\ell))]=0$$

what yields the proposition. □

We denote by $\mathbb{D}$, the domain of the Ornstein-Uhlenbeck form. We recall that an element $F=\sum_{n=0}^{+\infty}I_n(f_n)\in L^2(\mathbb{P}_1)$ belongs to $\mathbb{D}$ iff

$$\sum_{n=0}^{+\infty} nn!\parallel f_n\parallel_{L^2}^2<+\infty.$$

Let us take now an $F\in\mathbb{D}$, the random variables $\sum_{k=0}^n I_k^\theta(f_k)$ converge in $L^2(\mathbb{P})$ to F^θ uniformly in θ. Their derivatives – because $F\in\mathbb{D}$ – form a Cauchy sequence and converge also uniformly in θ. This implies that F^θ is differentiable and that the derivatives of the $\sum_{k=0}^n I_k^\theta(f_k)$ converge to the derivative of F^θ. So we have

Proposition 3.2. *$\forall F\in\mathbb{D}$ the process $\theta\mapsto F^\theta$ is differentiable in $L^2(\mathbb{P})$ and*

$$\forall\theta\qquad \mathbb{E}[(\frac{d}{d\theta}F^\theta)^2]=\sum_{n\in\mathbb{Z}}n^2c_n^2=\mathbb{E}\Gamma[F]=2\mathcal{E}[F]$$

where $\mathcal{E}$ is the Ornstein-Uhlenbeck form and Γ the associated carré du champ operator.

We also have the converse property:

Proposition 3.3. *Let $F\in L^2(\mathbb{P}_1)$. If the map $\theta\mapsto F^\theta$ is differentiable in $L^2(\mathbb{P})$ at a certain point $\theta_0\in\mathbb{R}$ then F belongs to $\mathbb{D}$.*

Proof. We write $F = \sum_n I_n(f_n)$ and consider a sequence $(\theta_k)_{k\geqslant 1}$ which converges to θ_0 and such that $\theta_k \neq \theta_0$, for all $k \geqslant 1$. As

$$\lim_{k\to+\infty} \frac{F^{\theta_k} - F^{\theta_0}}{\theta_k - \theta_0}$$

exists in $L^2(\mathbb{P})$ we deduce that there exists a constant $C > 0$ such that

$$\forall k \geqslant 1,\ \left\|\frac{F^{\theta_k} - F^{\theta_0}}{\theta_k - \theta_0}\right\|^2_{L^2(\mathbb{P})} = \sum_n \left\|\frac{I_n^{\theta_k}(f_n) - I_n^{\theta_0}(f_n)}{\theta_k - \theta_0}\right\|^2_{L^2(\mathbb{P})} \leqslant C.$$

By the Fatou's Lemma and the previous Proposition, we get

$$\sum_n \lim_{k\to+\infty} \left\|\frac{I_n^{\theta_k}(f_n) - I_n^{\theta_0}(f_n)}{\theta_k - \theta_0}\right\|^2_{L^2(\mathbb{P})} = \sum_n n!n\|f_n\|^2_{L^2} \leqslant C,$$

which yields the result. □

This provides the following result :

Proposition 3.4. *For all $F \in \mathbb{D}$ with chaotic representation $F = \int_{\mathcal{P}} f(A)dB_A$*

$$\frac{dF^\theta}{d\theta}|_{\theta=0} = \frac{d}{d\theta}\int_{\mathcal{P}} f(A)d(B\cos\theta + M\sin\theta)_A\ |_{\theta=0} = \int D_sF\ dM_s$$

the righthand term is a gradient for the Ornstein-Uhlenbeck form that we may denote $F^\sharp$, so we have in the sense of $L^2(\mathbb{P}) = L^2(\mathbb{P}_1 \times \mathbb{P}_2)$

$$F^\sharp = \lim_{\theta\to 0} \frac{1}{\theta}(F^\theta - F).$$

Proof. Let be $F \in \mathbb{D}$. We assess the distance between $\frac{1}{\theta}(F^\theta - F)$ and $\int D_sF\ dM_s$ by steps : distance between $\frac{1}{\theta}(F^\theta - F)$ and $\frac{1}{\theta}(F_n^\theta - F_n)$; between $\frac{1}{\theta}(F_n^\theta - F_n)$ and $\int D_sF_n\ dM_s$; then between $\int D_sF_n\ dM_s$ and $\int D_sF\ dM_s$.

By the preceding propositions

$$\|\frac{1}{\theta}(F^\theta - F) - \frac{1}{\theta}(F_n^\theta - F_n)\|^2 \leqslant \|F - F_n\|^2_{\mathbb{D}}$$

We may choose n so that the first one and the third one be both small independently of θ. And n being fixed we do $\theta \to 0$ in the second one and apply the argument of the proof of Proposition 3.1. □

The classical integration by part formula, i.e. the property that the divergence operator, dual of D, can be expressed by a stochastic integral on predictable processes, is a consequence of propositions 3.2 and 3.4 by derivation in θ.

Indeed let us denote $\mathcal{A}$ the closed sub-vector space of $L^2(\mathbb{P}, L^2(\mathbb{R}_+, dt))$ generated by the processes of the form $\int_{\Delta(t)} f_n d^{(n)}B$ with $f_n \in L^2(\mathbb{R}^n)$.

Lemma 3.5. *For $F \in L^2(\mathbb{P}_1)$, $G \in \mathcal{A}$, we have*

$$\mathbb{E}[F^\theta \int G_t^\theta dY_t^{\theta+\pi/2}] = 0.$$

Proof. By relation (3.1) the property is true if F has a finite expansion on the chaos hence also if $F \in L^2$. □

Let us denote now $\mathbb{D}_{\mathcal{A}}$ the closed vector space generated by the processes $\int_{\Delta(t)} f_n d^{(n)} B$ with $f_n \in L^2(\mathbb{R}^n)$ for the norm of $\mathbb{D}$ with values in $L^2(\mathbb{R}_+)$.

Proposition 3.6. *Let be $F \in \mathbb{D}$ and $G \in \mathbb{D}_{\mathcal{A}}$, we have*

$$\mathbb{E}[F^\theta \int G_u^\theta dY_u^\theta] = \mathbb{E}[\frac{dF^\theta}{d\theta} \int G_u^\theta dY_u^{\theta+\pi/2}]$$

so that taking $\theta = 0$

$$\mathbb{E}[F \int G_u dB_u] = \mathbb{E}[\int D_u F dM_u \int G_u dM_u] = \mathbb{E}[\int D_u F\, G_u du].$$

Proof. We differentiate $F^\theta \int G_t^\theta dY_t^{\theta+\pi/2}$ taking in account the lemma and the fact that $Y^{\theta+\pi} = -Y^\theta$. □

Remark 3.7. *Taking anew $\tilde{N}$ for M, we may apply the previous reasoning at the point $\theta = \pi/2$. Denoting $D^{(N)}$ the operator of Nualart-Vivès [12] which acts on the Poisson chaos as D acts on the Brownian ones, Proposition 3.4 says that for (f_n) such that $\sum n!n\|f_n\|^2 < \infty$ the Poisson functional $F = \sum I_n^{\pi/2}(f_n)$ is such that $\frac{d}{d\theta} F^\theta|_{\theta=\pi/2} = \int D^{(N)} F dB$.*

And by Proposition 3.6 we obtain that the finite difference operator $D^{(N)}$ of the Ornstein-Uhlenbeck structure on the Poisson space satisfies an integration by part formula (cf Øksendal and al [9] Thm 12.10) despite its non local character.

Remark 3.8. *In the case of another standard Brownian motion $\hat{B}$ for M, Proposition 3.4 gives exactly the derivation operator in the sense of Feyel-La Pradelle cf [4] Chap. II §2.*

$$\frac{d}{d\theta} F^\theta|_{\theta=0} = F' = \int D_u F d\hat{B}_u$$

In that case the situation is quite different from the one we had in Section 2. Indeed $Y^\theta = B\cos\theta + \hat{B}\sin\theta$ does satisfy the chaotic representation property, so that the space $\{F^\theta : F \in L^2(\mathbb{P}_1)\}$ is $L^2(\mathbb{P}_{Y^\theta})$. It is not possible to measurably detect the paths of B and $\hat{B}$ on those of Y^θ. But the concept of chaotic extension becomes simpler because it is compatible with the composition of the functions. It is valid to write in this case

$$F^\theta = F(B\cos\theta + \hat{B}\sin\theta).$$

Indeed, it is correct for $F = \Phi(\int h_1 dB, \ldots, \int h_k dB)$ with Φ a polynomial by Ito formula and induction (what was false in the case of the Poisson process), and then for general F in L^2 by approximation. As a consequence, Proposition 3.4 gives a formula of Mehler type without integration for the gradient

$$\forall F \in \mathbb{D} \qquad F' = \int D_u F d\hat{B}_u = \frac{d}{d\theta} F(B\cos\theta + \hat{B}\sin\theta)|_{\theta=0} \tag{3.2}$$

and with integration for the carré du champ

$$\Gamma[F] = \hat{\mathbb{E}}[(\frac{d}{d\theta} F(B\cos\theta + \hat{B}\sin\theta))^2|_{\theta=0}] \tag{3.3}$$

where $\hat{\mathbb{E}}$ acts on $\hat{B}$ as usual. By the change of variable $\cos\theta = e^{-t/2}$ this may be also written in a form similar to Mehler formula:

$$F' = \lim_{t\to 0} \frac{1}{\sqrt{t}} (F(B\sqrt{e^{-t}} + \sqrt{1-e^{-t}}\hat{B}) - F(B)) \tag{3.4}$$

what gives denoting P_t the Ornstein-Uhlenbeck semi-group

$$\Gamma[F] = \lim_{t\to 0}\frac{1}{t}(P_t(F^2) - 2FP_tF + F^2) \tag{3.5}$$

well known formula when F and F^2 are in the domain of the generator and that we obtain here for $F \in \mathbb{D}$.
To our knowledge, formulae (3.2), (3.3) *and* (3.4) *seem to be new under these hypotheses.*

4 Functional calculus of class $\mathcal{C}^1 \cap Lip$.

Proposition 4.1. *Let us suppose that the process H_s be in $\mathbb{D}_{\mathcal{A}}$* (cf proposition 3.6) *then*

$$(\int H_s dB_s)^\theta = \int (H_s)^\theta dY_s^\theta.$$

Proof. The functional $F = \int H_s dB_s$ is in $\mathbb{D}$ and has a chaotic expansion $F = \int_{\mathcal{P}} f(A)dB_A$. Following the short notation of [6] (p203) if we put for $E \in \mathcal{P}$

$$\dot{f}_t(E) = f(E \cup \{t\}) \text{ if } E \subset [0,t[, \quad = 0 \text{ otherwise},$$

and if $g_t = \int_{\mathcal{P}} \dot{f}_t(E)dB_E$, then

$$F = \int_{\mathcal{P}} f(A)dB_A = f(\emptyset) + \int g_t dB_t.$$

Hence we have $F^\theta = \int_{\mathcal{P}} f(A)dY_A^\theta$ and $(g_t)^\theta = \int_{\mathcal{P}} \dot{f}_t(E)dY_E^\theta$ and

$$F^\theta = f(\emptyset) + \int (g_t)^\theta dY_t^\theta,$$

what proves the proposition. □

Let be $F = (F_1, \ldots, F_k) \in \mathbb{D}^k$ and $\Phi \in \mathcal{C}^1 \cap Lip(\mathbb{R}^k, \mathbb{R})$, where in a natural way $Lip(\mathbb{R}^k, \mathbb{R})$ denotes the set of uniformly Lipschitz real-valued functions defined on $\mathbb{R}^k$. It comes from Proposition 3.4 and from the functional calculus in local Dirichlet structures the following result

Proposition 4.2. *When $\theta \to 0$, we have in $L^2(\mathbb{P})$*

$$\frac{1}{\theta}[(\Phi(F_1, \ldots, F_k))^\theta - \Phi(F_1^\theta, \ldots, F_k^\theta)] \to 0.$$

Proof. We have indeed in the sense of L^2, the function Φ being Lipschitz and $\mathcal{C}^1$

$$\begin{aligned}\lim \tfrac{1}{\theta}[\Phi(F_1^\theta, \ldots, F_k^\theta) - \Phi(F_1, \ldots, F_k)] &= \textstyle\sum_{i=1}^k \Phi_i'(F_1, \ldots, F_k)F_i^\sharp \\ &= \lim \tfrac{1}{\theta}[(\Phi(F_1, \ldots, F_k))^\theta - \Phi(F_1, \ldots, F_k)].\end{aligned}$$

□

It follows that we may replace $(\Phi(F_1, \ldots, F_k))^\theta$ by $\Phi(F_1^\theta, \ldots, F_k^\theta)$ in applying the method.

Let us define an *equivalence relation* denoted $\cong$ in the set of functionals in $L^2(\mathbb{P})$ depending on θ and differentiable in L^2 at $\theta = 0$ by

$$F(\omega_1, \omega_2, \theta) \cong G(\omega_1, \omega_2, \theta) \text{ if } \left(\frac{d}{d\theta}F|_{\theta=0} = \frac{d}{d\theta}G|_{\theta=0} \text{ and } F|_{\theta=0} = G|_{\theta=0}\right).$$

Let us also define a weaker equivalence relation denoted $\simeq$ for functionals in $L^0(\mathbb{P})$ depending on θ and differentiable in probability at $\theta = 0$ by

$$F(\omega_1, \omega_2, \theta) \simeq G(\omega_1, \omega_2, \theta) \text{ if } \left(\frac{d}{d\theta}F|_{\theta=0} = \frac{d}{d\theta}G|_{\theta=0} \text{ and } F|_{\theta=0} = G|_{\theta=0}\right)$$

the limits in the derivations being in probability.

Proposition 4.3. *If $H_t(\theta) \cong K_t(\theta)$ for all t then*

$$\int_0^t H_s(\theta) dY_s^\theta \cong \int_0^t G_s(\theta) dY_s^\theta.$$

Proof. The equality of the value at zero of the two terms is evident, and differentiating the lefthand term in zero gives

$$\int_0^t \frac{dH_s(\theta)}{d\theta}|_{\theta=0} dB_s + \int_0^t H_s(0) dM_s$$

which is equal to the derivative of the righthand term. □

Let us consider a stochastic differential equation (SDE) with $\mathcal{C}^1 \cap Lip$ coefficients with respect to the argument x

$$X_t = x + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds. \tag{4.1}$$

Let us recall that we are considering chaotic extensions $F \mapsto F^\theta$ with respect to $Y^\theta = B\cos\theta + M\sin\theta$.
The following proposition shows that we can calculate the Malliavin gradient of the diffusion by perturbing the Brownian trajectories using an independent normal martingale such as a compensated Poisson process.

Proposition 4.4. *The chaotic extension X_t^θ of the solution of* (4.1) *is equivalent (relation $\cong$) to the solution $Z_t(\theta)$ of the SDE*

$$Z_t(\theta) = x + \int_0^t \sigma(s, Z_s(\theta)) dY_s^\theta + \int_0^t b(s, Z_s(\theta)) ds. \tag{4.2}$$

Proof. Let us denote $(X^n)_{n\in\mathbb{N}}$ (resp. $(Z^n)_{n\in\mathbb{N}}$) the approximating sequence in the Picard iteration applied to equation satisfied by X (resp. Z). We have first $X_t^0 = Z_t^0 = x$ and

$$X_t^1 = x + \int_0^t \sigma(s, x) dB_s + \int_0^t b(s, x) ds.$$

By Propositions 4.1 and 4.2

$$(X_t^1)^\theta = \left(x + \int_0^t \sigma(s, x) dB_s + \int_0^t b(s, x) ds\right)^\theta \cong x + \int_0^t \sigma(s, x) dY_s^\theta + \int_0^t b(s, x) ds = Z_t^1(\theta)$$

Then,

$$X_t^2 = x + \int_0^t \sigma(s, X_s^1) dB_s + \int_0^t b(s, X_s^1) ds,$$

so that

$$(X_t^2)^\theta \cong x + \int_0^t \sigma(s, (X_s^1)^\theta) dY_s^\theta + \int_0^1 b(s, (X_s^1)^\theta) ds$$

what gives by Propositions 4.3 and 4.2:

$$(X_t^2)^\theta \cong x + \int_0^t \sigma(s, Z_s^1(\theta)) dY_s^\theta + \int_0^t b(s, Z_s^1(\theta)) ds = Z_t^2(\theta).$$

By induction, we get easily that for all $n \in \mathbb{N}$, $(X_t^n)^\theta \cong Z_t^n(\theta)$. But we know that X_t^n converges to X_t as n tend to $+\infty$ not only in L^2 but in $\mathbb{D}$ since the coefficients of the SDE are Lipschitz (cf [4] Chap IV), and this implies that $(X_t^n)^\theta$ converges to X_t^θ and that $\frac{d}{d\theta}(X_t^n)^\theta$ converges to $\frac{d}{d\theta} X_t^\theta$.

Now, $Z_t^n(\theta)$ converges to $Z_t(\theta)$. and its derivative converges to a solution of

$$Z_t'(\theta) = \int_0^t \sigma_x'(s, Z_s(\theta)) Z_s'(\theta) dY_s^\theta + \int_0^t \sigma(s, Z_s(\theta)) dY_s^{\theta+\pi/2} + \int_0^t b_x'(s, Z_s(\theta)) Z_s'(\theta) ds$$

equation which has a unique explicit solution as linear equation in $Z'(\theta)$ which is the derivative of $Z(\theta)$. That proves the proposition. □

5 The unit jump on the interval $[0,1]$.

In order to express the preceding results on $[0,1]$ with a single jump, we propose two different approaches.

5.1 First approach

We come back to the case $M = \tilde{N}$ and to express the preceding results on the interval $[0,1]$, we are conditioning by $\{N_1 = 1\}$. This amounts to reasoning on $\Omega_1 \times \{N_1 = 1\}$ under the measure $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$ which gives it the mass e^{-1}. Then the unique jump is uniformly distributed on $[0,1]$. For a functional $F \in L^2$ with expansion $F = \sum_n I_n(f_n)$, the expansion of the chaotic extension $F^\theta = \sum_n I_n^\theta(f_n)$ considered on the event $\{N_1 = 1\}$ is the same sum of stochastic integrals but with integrant the semi-martingale $V_t(\theta) = B_t \cos\theta + (1_{\{t \geqslant U\}} - t)\sin\theta$, in other words

$$F^\theta = \int_{\mathcal{P}} f(A) dV_A(\theta) \ \text{ on } \ \{N_1 = 1\}.$$

In the sequel, we use the fact that the notation

$$\int_{\mathcal{P}} f(A) dS_A$$

makes sense for any semi-martingale S which admits the decomposition

$$S_t = M_t + L_t,$$

where M is a local martingale whose skew bracket is absolutely continuous w.r.t. the Lebesgue measure and L an absolutely continuous process.
For example if U is uniform on $[0,1]$, then

$$1_{\{t \geqslant U\}} = M_t - \log(1 - t \wedge U),$$

with $M_t = 1_{\{t \geqslant U\}} + \log(1 - t \wedge U)$ and $\langle M, M\rangle_t = -\log(1 - t \wedge U)$.
By absolutely continuous change of probability measure we may remove the term in $-t\sin\theta$:

Proposition 5.1. *Let U be uniform on $[0,1]$ independent of B. We put $R_t(\theta) = B_t \cos\theta + 1_{\{t \geqslant U\}} \sin\theta$. Let be $F \in \mathbb{D}$, $F = \int_{\mathcal{P}} f(A) dB_A$. we have*

$$\lim_{\theta\to 0} \frac{1}{\theta}\Big(\int_{\mathcal{P}} f(A) dR_A(\theta) - F\Big) = D_U F \qquad \textit{in probability.}$$

For the proof let us state the

Lemma 5.2. *Let be ξ be an element in the Cameron-Martin space.*
a) If $F \in L^p$ for $p > 2$ then

$$\lim_{t\to 0} \mathbb{E}[(F(B+t\xi) - F(B))^2] = 0.$$

b) If $F \in L^0$ then $F(B+t\xi)$ converges to F in probability as t tends to 0.

Proof. a) We develop the square. The first term is

$$\mathbb{E}\left[exp[t\int \dot{\xi} dB + \frac{t^2}{2}\|\dot{\xi}\|^2]F^2\right]$$

as $F \in L^p$ $p > 2$ it is uniformly integrable and it converges to $\mathbb{E}F^2$.

For the rectangle term, it is easily seen by change of probability measure that

$$\mathbb{E}[F(B+t\xi)G(B)]$$

converges to $\mathbb{E}[FG]$ for G bounded and continuous.

And we can reduce to this case by the above argument.

b) We truncate F. If $A_n = \{B : |F| \geqslant n\}$ by uniform intégrability we can find n such that the probability of $A_n(B+t\xi)$ be $\leqslant \varepsilon$ for all t. The result comes now from part a). □

proof of proposition 5.1 :

Puting $C = \{N_1 = 1\}$, we are working under the probability measure $\mathbb{Q} = e \times \mathbb{P}_1 \times (\mathbb{P}_2|_C)$.

The conditioning explained above yields the following relation in probability

$$\lim_{\theta\to 0} \frac{1}{\theta}(\int_{\mathcal{P}} f(A) dV_A(\theta) - F) = D_U F - \int_0^1 D_s F ds, \tag{5.1}$$

whose second member is $\int_0^1 D_s F d\tilde{N}_s$ restricted to $\{N_1 = 1\}$. In order to have

$$\lim_{\theta\to 0} \frac{1}{\theta}(\int_{\mathcal{P}} f(A) dR_A(\theta) - F) = D_U F \qquad \text{in probability,}$$

we use that the identity map j is a Cameron-Martin function and that $\int_0^1 D_s F ds = \langle DF, \frac{dj}{ds}\rangle_{L^2(ds)}$.

If we change of measure and take $\exp(-B_1 \sin\theta - \frac{\sin^2\theta}{2}).\mathbb{Q}$ relation (5.1) writes saying that, for all $\varepsilon > 0$,

$$\mathbb{Q}\left[\exp(-B_1 \sin\theta - \frac{\sin^2\theta}{2}) 1_{C_\theta^\varepsilon}\right]$$

tends to zero, where we denote

$$C_\theta^\varepsilon = \{|\lim_{\theta\to 0} \frac{1}{\theta}[\int_{\mathcal{P}} f(A) dR_A(\theta) - F(B + j\sin\theta)]$$

$$-D_U F(B + j\sin\theta) - \int_0^1 D_s F(B + j\sin\theta) ds| \geqslant \varepsilon\}.$$

1) Let us observe that

$$\mathbb{Q}\left[(\exp(-B_1 \sin\theta - \frac{\sin^2\theta}{2}) - 1) 1_{C_\theta^\varepsilon}\right]$$

tends to zero. What reduces to study $\mathbb{Q}[C_\theta^\varepsilon]$.

2) We know that under $\mathbb{Q}$, $\frac{1}{\theta}[F(B+j\sin\theta)-F(B)]$ converges in probability to $\int_0^1 D_sFds$. The other two terms are processed by the lemma.

We obtain indeed that under $\mathbb{Q}$, $\frac{1}{\theta}[\int_{\mathcal{P}} f(A)d(B\cos\theta+1_{\{.\,\geqslant\,U\}}\sin\theta)_A-F]$ converges in probability to D_UF. □

If we are conditioning by the event $\{N_1=1\}$ the result of Proposition 4.4, the equation satisfied by $Z(\theta)$ may be written

$$Z_t(\theta)=x+\int_0^t\sigma(s,Z_s(\theta))d(B_s\cos\theta+(1_{\{s\,\geqslant\,U\}}-s)\sin\theta)+\int_0^t b(s,Z_s(\theta))ds$$

As in the proof of proposition 5.1 an absolutely continuous change of probability measure allows to remove the term in $-s\sin\theta$ if we replace the result $D_UZ_t-\int_0^t D_sZ_tds$ by D_UZ_t.

This change being done, the value at $\theta=0$ and the derivative at $\theta=0$ of $Z(\theta)$ are the same as those of $\eta(\theta)$ solution of the SDE

$$\eta_t(\theta)=x+\int_0^t\sigma(s,\eta_s(\theta))(dB_s+1_{\{s\,\geqslant\,U\}}\theta)+\int_0^t b(s,\eta_s(\theta))ds.$$

In other words we obtain the following result which might have been easily directly verified

Proposition 5.3. *The gradient of Malliavin $D_uX_t^x$ of the solution of the SDE*

$$X_t^x=x+\int_0^t\sigma(s,X_s^x)dB_s+\int_0^t b(s,X_s^x)ds$$

may be computed by considering the solution of the equation

$$X_t^x(\theta)=x+\int_0^t\sigma(s,X_s^x(\theta))d(B_s+\theta 1_{\{s\,\geqslant\,u\}})+\int_0^t b(s,X_s^x(\theta))ds$$

and taking the derivative in θ at $\theta=0$.

Let us remark that since u is defined du-almost surely, we may in the equation defining $X_t^x(\theta)$ put either $\sigma(s,X_s^x(\theta))$ or $\sigma(s,X_{s-}^x(\theta))$.

Let us perform the calculation suggested in the proposition. That gives for $u<t$

$$X_t^x(\theta)=x+\int_0^u\sigma(s,X_s^x)dB_s+\theta\sigma(u,X_u^x)+\int_u^t\sigma(s,X_s^x(\theta)dB_s+\int_0^u b(s,X_s^x)ds$$
$$+\int_u^t b(s,X_s^x(\theta))ds$$
$$X_t^x(\theta)=X_u^x+\theta\sigma(u,X_u^x)+\int_u^t\sigma(s,X_s^x(\theta))dB_s+\int_u^t b(s,X_s^x(\theta))ds. \tag{5.2}$$

Then, derivating with respect to θ at $\theta=0$ we obtain

$$D_uX_t^x=\sigma(u,X_u^x)+\int_u^t\sigma'_X(s,X_s^x)D_uX_s^xdB_s+\int_u^t b'_X(X(s,X_s^x)D_uX_s^xds.$$

We now introduce the process $(Y_t^x)_t$ which is the derivate of the flow generated by X_t^x: $Y_t^x=\dfrac{\partial X_t^x}{\partial x}$, it is the solution of the linear SDE:

$$Y_t^x=1+\int_0^t\sigma'_X(s,X_s^x)Y_s^xdB_s+\int_0^t b'_X(s,X_s^x)Y_s^xds$$

and it is standard that:

$$D_u X_t^x = \sigma(u, X_u^x)\frac{Y_t^x}{Y_u^x}. \tag{5.3}$$

This is a fast way of obtaining this classical result (transfert principle by the flow of Malliavin cf [11] Chap VIII).

Proposition 5.3 is the lent particle formula for the Brownian motion.

We see that the method of proof allows to obtain this formula without sinus nor cosinus for general F in $\mathbb{D}$ provided that we be able to find a functional regular in θ equivalent to the chaotic extension of F. In particular the example of the introduction generalises in the following way : if

$$F = \Phi\left(\int_{s_1<\cdots<s_{k_1}} f_{k_1} dB_{s_1}\cdots dB_{s_{k_1}}, \ldots, \int_{s_1<\cdots<s_{k_n}} f_{k_n} dB_{s_1}\cdots dB_{s_{k_n}}\right)$$

with $f_{k_i} \in L^2(\lambda_{k_i})$ and Φ of class $\mathcal{C}^1 \cap Lip$, we have

$$\lim \frac{1}{\theta}\left[\Phi\left(\int_{s_1<\cdots<s_{k_1}} f_{k_1} d(B_{s_1}+\theta 1_{\{s\geqslant u\}})\cdots d(B_{s_{k_1}}+\theta 1_{\{s\geqslant u\}}), \ldots\right) - F\right] = D_u F.$$

The limit is in probability as in Proposition 5.1.

5.2 Second approach

Instead of performing a change of probability measure to remove the term $-t\sin\theta$ as in the previous approach, we consider M a Lévy process with Lévy measure $\frac{1}{2}(\delta_{-1}(dx)+\delta_1(dx))$ in place of $\tilde{N}$. Let us remark that we might have considered any Lévy process whose Lévy measure has mean 0 and variance 1. M can be expressed as

$$\forall t \geqslant 0,\ M_t = \sum_{n=1}^{N_t} J_n,$$

where N is a Poisson process with intensity 1 and $(J_n)_n$ a sequence of i.i.d. variables, independent of N such that

$$\mathbb{P}_2(J_1 = 1) = \mathbb{P}_2(J_1 = -1) = \frac{1}{2}.$$

Proposition 5.4. *Let U be uniform on $[0,1]$ independent of B. We put $R_t(\theta) = B_t\cos\theta + 1_{\{t\geqslant U\}}\sin\theta$. Let be $F \in \mathbb{D}$, $F = \int_{\mathcal{P}} f(A)dB_A$. we have*

$$\lim_{\theta\to 0}\frac{1}{\theta}\left(\int_{\mathcal{P}} f(A)dR_A(\theta) - F\right) = D_U F \qquad \textit{in } L^2(\mathbb{P}).$$

Proof. We denote by U_1 the time of the first jump and by $\widetilde{\mathbb{P}}_2$ the conditional law $e\mathbf{1}_{\{N_1=1\}}\mathbb{P}_2$. We consider

$$R_t^1(\theta) = B_t\cos\theta + 1_{\{t\geqslant U_1\}}\sin\theta \text{ and } \widetilde{R}_t^1(\theta) = B_t\cos\theta + J_1 1_{\{t\geqslant U_1\}}\sin\theta.$$

The chaotic extension related to $Y_t(\theta) = B_t\cos\theta + M_t\sin\theta$ satisfies

$$\forall\theta,\ F^\theta = \int_{\mathcal{P}} f(A)dY_A(\theta) = \int_{\mathcal{P}} f(A)d\widetilde{R}_A^1(\theta)\ \mathbb{P}_1\times\widetilde{\mathbb{P}}_2 \text{ a.e.}$$

As a consequence of Proposition 3.4, we have in the sense of $L^2(\mathbb{P}_1 \times \widetilde{\mathbb{P}}_2)$:

$$\lim_{\theta\to 0} \frac{1}{\theta}(F^\theta - F) = \int D_s F\, dM_S = J_1 D_{U_1} F.$$

Then, we remark that

$$\int_{\mathcal{P}} f(A) dR^1_A(\theta) = J_1 F^\theta + (1-J_1)\cos\theta \int_{\mathcal{P}} f(A) dB_A,$$

and obtain

$$\lim_{\theta\to 0} \frac{1}{\theta}(\int_{\mathcal{P}} f(A) dR^1_A(\theta) - F) = D_{U_1} F \qquad \text{in } L^2(\mathbb{P}_1 \times \widetilde{\mathbb{P}}_2).$$

□

5.3 Another example

Remark 5.5. *When we enlarge the field of validity of the calculus, by using the equivalence relation $\simeq$ instead of $\cong$ to functionals in $L^0(\mathbb{P})$ depending on θ and differentiable in probability at $\theta = 0$, the authorized functional calculus becomes C^1 instead of $C^1 \cap Lip$.*

For example let us consider a càdlàg process K independent of B. And let us put $M(B) = \sup_{s\leqslant 1}(B_s + K_s)$.

Proposition 5.6. $M^\theta \simeq M(B + \theta 1_{\{.\geqslant U\}}) \simeq M(B\cos\theta + 1_{\{.\geqslant U\}}\sin\theta)$.

Proof. For the first equivalence, the value at $\theta = 0$ is of course correct, and about the derivative we have

$$\begin{aligned} M(B+\ \theta 1_{\{.\geqslant U\}}) & \\ &= \sup_{s\leqslant 1}((B_s+K_s)1_{\{s<U\}} + (B_s+\theta+K_s)1_{\{s\geqslant U\}}) \\ &= \max(\sup_{s<U}(B_s+K_s), \sup_{s\geqslant U}(B_s+\theta+K_s)) \end{aligned}$$

Thus we have the convergence in probability

$$\lim_{\theta\to 0}\frac{1}{\theta}(M(B+\theta 1_{\{.\geqslant U\}}) - M(B)) = 1_{\{\sup_{s\geqslant U}(B_s+K_s)\geqslant \sup_{s<U}(B_s+K_s)\}}$$

This result is correct cf [13] for the gradient of M. The second equivalence is similar what shows the proposition. This implies that M possesses a density since $\Gamma[M] = 0$ cannot be true except if $\sup(B_s + K_s) < K_0$ what is impossible since K is independent of B. □

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