Percolation in an ultrametric space

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Abstract

We study percolation in the hierarchical lattice of order N where the probability of connection between two points separated by distance k is of the form $c_k/N^{k(1+\delta)}$, $\delta > -1$. Since the distance is an ultrametric, there are significant differences with percolation in the Euclidean lattice. We consider three regimes: $\delta < 1$, where percolation occurs, $\delta > 1$, where it does not occur, and $\delta = 1$ which is the critical case corresponding to the phase transition. In the critical case we use an approach in the spirit of the renormalization group method of statistical physics, and connectivity results of Erdős-Rényi random graphs play a key role. We find sufficient conditions on c_k such that percolation occurs, or that it does not occur. An intermediate situation called pre-percolation, which is necessary for percolation, is also considered. In the cases of percolation we prove uniqueness of the constructed percolation clusters. In a previous paper we studied percolation in the $N \to \infty$ limit (mean field percolation), which provided a simplification that allowed finding a necessary and sufficient condition for percolation. For fixed N there are open questions, in particular regarding the behaviour at the critical values of parameters in the definition of c_k. Those questions suggest the need to study *ultrametric random graphs*.

1 Introduction

Percolation theory in a lattice (e.g., the Euclidean lattice $\mathbb{Z}^d$) began with the work of Broadbent and Hammersley in 1957. The principal features of the model are that the space of sites is infinite and its geometry plays an essential role. The main problem is to determine if there is an infinite connected component, in which case it is said that percolation occurs. In the first models the connections (bonds) were only between nearest neighbors. (See [22, 12] for background, and for a physics point of view, [34].) The question of percolation on noneuclidean graphs including nonamenable Cayley graphs

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given by finitely generated groups was formulated in [6]. The study of long range percolation began in the mathematical physics literature (e.g., [1, 7, 27, 32]). In this case connections are allowed between points at any distance from each other with probability depending on the distance. The main problem is the same, and the geometry remains crucial. (See also [6, 8, 9, 15, 35].)

The theory of random graphs started with the work of Erdős and Rényi in 1959. The model consists of a finite number n of vertices with connection probability p_n between pairs of vertices, depending on n in some way, and there is no structure on the set of vertices. The results refer to what happens as $n \to \infty$, for example with the largest connected component. (See [10, 23, 19] for background.) Results from the theory of random graphs have been useful as technical tools in studies on percolation (e.g. [4, 5]).

By introducing a structure on the set of vertices of a random graph or some special form of connection probabilities by means of a kernel which induces a sort of geometry, it is possible to generate a large class of interesting models which share properties of both percolation and/or (classical) random graphs, and also "small world" random graphs (see e.g. the models in [2, 11, 36]).

On the other hand, hierarchical structures arise in the physical, biological and social sciences due to the multiscale organization of many natural objects (see e.g. [3, 29]). In particular the hierarchical Ising model which was introduced by Dyson [20] has played an important role in statistical physics (see [13, 14, 33]) and in population genetics (see e.g. [30]). Important applications of hierarchical structures have also been made by Kleinberg [24, 25] in the area of search algorithms in computer science. A basic model is the *hierarchical group* Ω_N of order N (defined in Section 2), which can be represented as the set of leaves at the top of an infinite regular tree, where the distance between two points is the number of levels from the top to their most recent common node. Such a distance satisfies the strong triangle inequality

$$d(x,y) \le \max\{d(x,z), d(z,y)\} \quad \text{for any} \quad x, y, z,$$

which is the characteristic property of an ultrametric. (See e.g. [31] for background on ultrametric spaces.) The main qualitative difference between Euclidean-type lattices and an ultrametric space such as Ω_N is that in the former case it is possible to go far by a sequence of small steps, while in the latter case that is not possible, and the only way to go far is to make jumps of ever bigger sizes. This has important consequences for random walks for which there are analogies and differences with the Euclidean case (see e.g. [17, 18] and references therein) and percolation in ultrametric spaces [16]. In particular, percolation in Ω_N is possible only in the form of long range percolation, that is, with positive probabilities of connections between vertices separated by arbitrarily large distances.

With these precedents it is natural to investigate percolation in ultrametric spaces such as Ω_N where classical tools do not apply. Our aim is to develop a mathematical framework that might be useful generally for this kind of model, not thinking about specific motivations from or applications to physics or any other field.

In [16] we studied *asymptotic percolation* in Ω_N as $N \to \infty$ (or *mean field percolation*) with connection probabilities of the form c_k/N^{2k-1} between two points separated by distance k, and we obtained a necessary and sufficient condition for percolation. (See Subsection 3.1 for the definition of asymptotic percolation). The Erdős-Rényi results on giant components of random graphs were a useful tool, although there are significant differences between classical random graphs and ultrametric ones (e.g., the average length of paths in the giant component of an ultrametric ball is much longer than in the classical case).

In the present paper we study percolation in Ω_N for fixed N with connection probabilities of the form $c_k/N^{k(1+\delta)}$, $\delta > -1$, between two points separated by distance k. In this case percolation means that there is a positive probability that a given point of Ω_N belongs to an infinite connected component. This is a quite different situation from the asymptotic $N \to \infty$ model, and new methods must be used. However, properties of Erdős-Rényi graphs are again useful, but now it is connectivity results that are of help, specially the result in the Appendix based on Durrett's approach to connectivity [19]. There are three regimes: $\delta < 1$, $\delta > 1$ and $\delta = 1$. Roughly speaking, under certain natural assumptions on c_k, for $\delta > 1$ percolation does not occur, and for $\delta < 1$ percolation occurs and the infinite connected component is unique. The most difficult is the critical case, $\delta = 1$, where we use certain special forms of c_k and percolation may or may not occur. Our aim is to find forms of c_k such that percolation occurs, or that it does not occur, and in the case of percolation it turns out that the infinite connected component is unique.

We emphasize that taking the limit $N \to \infty$ provides a simplification which allowed us to obtain a sharp result, that is, necessary and sufficient condition for percolation in [16]. In the case of finite N that does not seem possible at present, and our work leads to some open problems, in particular regarding the behaviour at the critical values of the parameters in the form of c_k in the critical case $\delta = 1$. A study of those problems suggests the analysis of a class of *ultrametric random graphs* (see Subsection 3.4). We also consider an intermediate situation that we call *pre-percolation*, which is necessary for percolation. Pre-percolation occurs in one of our results.

While we were working on this paper we learned about the manuscript of Koval et al [26] (for which we thank them), where they also study percolation in Ω_N for fixed N, with connection probabilities of the form $1 - \exp\{-\alpha/\beta^k\}$, $\alpha \geq 0, \beta > 0$, between two points separated by distance k. Some of their results for $\beta > 1$ and ours may be compared by setting $\beta = N^{1+\delta}$ (see Remark 3.2).

For the cases where percolation occurs, specially with $\delta = 1$, we use an approach in the spirit of the renormalization group method of statistical physics which has been employed, for example, in [14] for ferromagnetic systems on Dyson's hierarchical lattice Ω_2 and for the study of long range percolation on $\mathbb{Z}^d$ (see [27, 32]).

In Section 2 we describe the model (the hierarchical group Ω_N and the associated random graph $\mathcal{G}_N$). In Section 3 we recall the result on mean field percolation [16] in order to compare it with a result in the present paper, and we state our results for $\delta < 1$ and $\delta > 1$ (Theorem 3.1) and the critical case $\delta = 1$ (Theorems 3.3 and 3.5), and we mention some open problems (Subsection 3.4). Sections 4 and 5 contain the proofs. In an appendix we give a result on connectivity of random graphs derived from [19], which is a key ingredient for the proof of percolation in the critical case.

2 Description of the model

2.1 The hierarchical group Ω_N

For an integer $N \geq 2$, the *hierarchical group of order* N, also called *hierarchical lattice of order* N, is defined as

$$\Omega_N \quad = \quad \{\boldsymbol{x} = (x_1, x_2, \ldots) : x_i \in \{0, 1, \ldots, N-1\}, i = 1, 2, \ldots, \sum_i x_i < \infty\}$$

with addition componentwise mod N; in other words, Ω_N is a countable Abelian group given by the direct sum of a countable number of copies of the cyclic group of order N.

The *hierarchical distance* on Ω_N is defined as

$$d(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 0 & \text{if} \quad \boldsymbol{x} = \boldsymbol{y}, \\ \max\{i : x_i \neq y_i\} & \text{if} \quad \boldsymbol{x} \neq \boldsymbol{y}. \end{cases}$$

It is a translation-invariant metric which satisfies the strong (non-Archimedean) triangle inequality

$$d(\boldsymbol{x}, \boldsymbol{y}) \leq \max\{d(\boldsymbol{x}, \boldsymbol{z}), d(\boldsymbol{z}, \boldsymbol{y})\} \quad \text{for any} \quad \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}.$$

Hence (Ω_N, d) is an ultrametric space, and it is well known that it can be represented as the leaves at the top of an infinite regular tree where N branches emerge up from each node, and the distance between two points of Ω_N is the number of levels from the top to their most recent common node.

For each integer $k \geq 1$, a *k-ball* in Ω_N, denoted by B_k, is a set of all points which are at distance at most k from each other. Any point of a ball can serve as a center. Once a center is chosen, one may speak of the interior and the boundary of the ball. A k-ball contains N^k points, and its boundary contains $N^{k-1}(N-1)$ points. For $k > 1$, a k-ball is the union of N $(k-1)$-balls, which are at distance k from each other. For $j > k \geq 1$, a j-ball is the union of N^{j-k} k-balls, which are at distance at least $k+1$ and at most j from each other. Two balls are either disjoint, or one is contained in the other (this is the reason why connections between nearest neighbours alone cannot produce percolation). For $k \leq j < \ell$, the (j, ℓ)-*annulus* around B_k is the set of all points $\boldsymbol{y}$ such that $j < d(\boldsymbol{x}, \boldsymbol{y}) \leq \ell$, where $\boldsymbol{x}$ is any point in B_k. The (j, ℓ)-annulus is also described as $B_\ell \setminus B_j$ with $B_k \subset B_j \subset B_\ell$, and it contains $N^\ell(1 - N^{j-\ell})$ points. The number of points in a bounded subset $A \subset \Omega_N$ is denoted by $|A|$. The probability that a point chosen at random (uniformly) in B_k belongs to $A \subset B_k$ is $|A|N^{-k}$.

We fix a point of Ω_N which we denote as $\boldsymbol{0}$. Most of our considerations about percolation will refer to balls containing $\boldsymbol{0}$.

2.2 The random graph $\mathcal{G}_N$

We define an infinite random graph $\mathcal{G}_N$ with the points of Ω_N as vertices, and for each $k \geq 1$ the probability of connection, $p_{\boldsymbol{x}, \boldsymbol{y}}$, between $\boldsymbol{x}$ and $\boldsymbol{y}$ with $d(\boldsymbol{x}, \boldsymbol{y}) = k$ is given by

$$p_{(k)} = p_{\boldsymbol{x}, \boldsymbol{y}} = \min\left(\frac{c_k}{N^{k(1+\delta)}}, 1\right), \tag{2.1}$$

where $\delta > -1$ and $c_k > 0$, all connections being independent. This can be realized in terms of a collection of independent uniform $[0, 1]$ random variables $\{U_{(\boldsymbol{x}, \boldsymbol{y})}\}$ by adding an (undirected) edge between $\boldsymbol{x}$ and $\boldsymbol{y}$ if and only if $U_{(\boldsymbol{x}, \boldsymbol{y})} \leq p_{\boldsymbol{x}, \boldsymbol{y}}$ (see e.g. [23], page 4).

Our aim is to find sufficient conditions on c_k and δ which imply that percolation occurs, or that it does not occur.

We will study separately the cases $\delta > 1$, $\delta < 1$, and $\delta = 1$. As we shall see, the case $\delta = 1$ requires a more delicate analysis. In this case we take c_k of the following special forms:

(i)
$$c_k = C_0 + C_1 \log k + C_2 k^\alpha, \tag{2.2}$$

with constants $C_0 \geq 0, C_1 \geq 0, C_2 \geq 0$ and $\alpha > 0$.

(ii) We consider the hierarchical distances with logarithmic scale

$$k_n = k_n(K) := \lfloor Kn \log n \rfloor, \quad n = 1, 2, \ldots, \tag{2.3}$$

K constant > 0, so that

$$k_{n+1} - k_n \sim K \log n \quad \text{as} \quad n \to \infty, \tag{2.4}$$

(where $\sim$ has the usual meaning, see beginning of the proofs), and we consider the class of connection rates given by (2.1) with $\delta = 1$, c_k satisfying

$$c_{k_n} = C + a \log n \cdot N^{b \log n} = C + a \log n \cdot n^{b \log N}, \tag{2.5}$$

and

$$c_{k_n} \leq c_j \leq c_{k_{n+1}} \quad \text{for} \quad k_n < j < k_{n+1}. \tag{2.6}$$

with constants $C \geq 0$, $a > 0$, $b \geq 0$. The constant K in (2.3) will be chosen suitably.

The reason for considering $k_n(K)$ and these forms for c_{k_n} is that they provide a four parameter family of comparison connection rates (with parameters K, C, a, b) suitable for the renormalization analysis used in the proof of percolation in Theorem 3.5. An intuitive argument for this is given before Theorem 3.5. Results of Theorem 3.5 are used to prove Theorem 3.3 which is our main result.

A set of vertices any two of which are linked by a path of connections is called a *cluster*. By transitivity of (Ω_N, d) we may focus on clusters containing $\mathbf{0}$. In the proofs it is implicitly assumed that we consider a sequence of nested balls $(B_k)_{k \geq 1}$ such that $\mathbf{0}$ belongs to B_1 (or to some B_k). We denote by X_k the largest cluster contained in B_k, considering only connections within B_k and not through points outside of B_k. If there are more than one largest cluster, we choose one of them uniformly from the existing ones. In this way each k-ball B_k has a unique attached cluster X_k. Note that for $B_k \subset B_{k+j}$, either $X_k \cap X_{k+j} = \phi$ or $X_k \subset X_{k+j}$. Those clusters will be used only in the proofs of sufficient conditions for percolation. The assumption of connections only within balls makes the renormalization approach quite practical for percolation, and connections through points outside would add to the possibility of percolation.

Definition 2.1. We say that *percolation occurs* in $\mathcal{G}_N$ if there is a positive probability that a fixed point of $\mathcal{G}_N$ (for example $\mathbf{0}$) belongs to an infinite cluster.

If percolation occurs, the probability that there is an infinite cluster is 1. Indeed, the event that there is an infinite cluster is measurable with respect to the tail σ-algebra generated by the connections involving points outside each k-ball (containing $\mathbf{0}$) for every k, and the connections involving points outside a ball are independent of those inside, so by a 0-1 law the probability that there is an infinite cluster is 0 or 1.

In some cases we consider percolation clusters of positive density, that is, $|X_k| N^{-k}$ does not decrease to 0 as $k \to \infty$.

Remark 2.2. It follows immediately from the construction in terms of the random variables $\{U_{(\boldsymbol{x},\boldsymbol{y})}\}$ that given two families of connection probabilities $p^1_{\boldsymbol{x},\boldsymbol{y}}$, $p^2_{\boldsymbol{x},\boldsymbol{y}}$ with $p^2_{\boldsymbol{x},\boldsymbol{y}} \geq p^1_{\boldsymbol{x},\boldsymbol{y}}$ for any $\boldsymbol{x}, \boldsymbol{y}$, percolation for family 1 implies percolation for family 2.

3 Results

3.1 Mean field percolation

For completeness, we start by recalling the result on asymptotic percolation as $N \to \infty$ [16]. This will also be used for comparison with a result below. The probability of connection between two points separated by distance k is c_k / N^{2k-1}. Note that this corresponds to the critical case $\delta = 1$ with c_k in (2.1) multiplied by N, and this may

be viewed as a normalization required for obtaining the result in the limit. *Asymptotic percolation* is said to occur if

$$P_{\mathrm{perc}} := \inf_k \liminf_{N \to \infty} P(\mathbf{0} \text{ is linked by a path of connections to a point at distance } k) > 0.$$

P_{perc} is the probability of percolation. For each $k \geq 1$, let $\beta_k \in (0,1)$ satisfy

$$\beta_k = 1 - e^{-c_k \beta_{k-1}^2 \beta_k}, \quad \beta_0 = 1, \tag{3.1}$$

where $c_k \beta_{k-1}^2 > 1$. Note that β_k is the well-known survival probability of a Poisson branching process with parameter $c_k \beta_{k-1}^2$. This corresponds to hierarchical level k, and the β_{k-1}^2 comes from the sizes of two connected giant components at the previous level $k-1$. Assume that $c_k \nearrow \infty$ as $k \to \infty$, $c_1 > 2 \log 2$ and $c_2 > 8 \log 2$. The results (see [16], Theorem 2.2 and Lemma 2.1) are that asymptotic percolation occurs if and only if

$$\sum_{k=1}^{\infty} e^{-c_k} < \infty,$$

and when it occurs, the probability of percolation is given by

$$P_{\mathrm{perc}} = \prod_{k=1}^{\infty} \beta_k.$$

(which is strictly positive if and only if the exponential series converges), and percolation takes place through a cascade of clusters (in this case giant components) at consecutive hierarchical distances. For example, if $c_k = a \log k$ for large k, $a > 0$, then asymptotic percolation occurs if and only if $a > 1$. See Remark 3.6(1) for a partially analogous result with fixed N.

3.2 The cases $\delta > 1$ and $\delta < 1$

Theorem 3.1. *(a) If $\delta > 1$ and $\sup_k c_k < \infty$, then percolation does not occur.*
(b) If $\delta < 1$ and $c = \inf_k c_k$ is large enough, then percolation occurs through a chain of clusters in k-balls, and the percolation cluster is unique.

Remark 3.2. Our results and those of [26] can be compared for $\beta > 1$ therein, since in this case their connection probabilities $p_k = 1 - \exp(-\alpha/\beta^k) \sim \alpha/\beta^k$ as $k \to \infty$. If we set $\beta = N^{1+\delta}$ and let $c_k = c$ not depending on k in (2.1), then the decay rates agree with c corresponding their α. The comparison is between Theorem 3.1 above and Theorem 1.1 in [26]. We have that for $\delta > 1, \beta > N^2$, percolation does not occur with any value of c, and for $-1 < \delta < 1, 1 < \beta < N^2$, percolation occurs with c sufficiently large, corresponding to $\alpha > \alpha_c(\beta)$ in [26]. In addition in [26] it is proved that there exists $\alpha_c(\beta) > 0$ such that percolation does not occur for $\alpha < \alpha_c(\beta)$. Our main objective is to investigate the critical case $\delta = 1, \beta = N^2$, for which percolation does not occur for any α in [26]. Our results in this case are stated in the next subsection.

3.3 The case $\delta = 1$

In the previous subsection we have seen that $\delta = 1$ identifies the critical exponential decay rate for percolation. In this subsection we formulate our main results that determine the critical polynomial rate for percolation.

Theorem 3.3. *Let $\delta = 1$ and*

$$c_k = C_0 + C_1 \log k + C_2 k^{\alpha} \text{ with } \alpha \geq 0,$$

where $C_0 \geq 0$, $C_1 \geq 0$, $C_2 \geq 0$.
(a) If $\alpha > 2$, then for any C_1 there exist $C_0^* > 0$ and $C_2^* > 0$ such that if $C_0 > C_0^*$ and $C_2 > C_2^*$, percolation occurs and the percolation cluster is unique.
(b) If $C_2 = 0$ and $C_1 < N$, then percolation does not occur for any C_0.
(c) If $\alpha > 2$, there exists $C_* > 0$ such that if $\max(C_0, C_1, C_2) < C_*$, then percolation does not occur.

The proof of this result is based on a renormalization argument that is formulated using the hierarchical distances $k_n(K)$ defined in (2.3) and the family of connection rates c_{k_n} defined in (2.5) with parameters C, a, b. This is the substance of Theorem 3.5.

In order to express one of the results below we introduce the following notion.

Definition 3.4. We call *pre-percolation* the situation that (with probability 1) there exists n_0 such that there is at least one connection from $(k_n, k_{n+1}]$ to $(k_{n+1}, k_{n+2}]$ for all $n \geq n_0$.

Note that for percolation, in addition to pre-percolation there would have to be paths connecting points in $(k_{n+1}, k_{n+2}]$ which are connected to $(k_n, k_{n+1}]$ to points in $(k_{n+1}, k_{n+2}]$ which are connected to $(k_{n+2}, k_{n+3}]$, etc. This is the central question in one of the comments in open problem (1) in Subsection 3.4.

Before stating the next theorem, let us give an intuitive explanation for the choice of k_n and c_{k_n} in (2.3)-(2.6) with $b > 0$, and the assumption $K < b$ (the assumption $\frac{2}{\log N} < K$ is a technical requirement for the method of proof). For this argument only we use the notation $\approx$ for approximate equality for large n without giving it a rigorous meaning. The idea is that k_n is the right scaling and the form of c_{k_n} is the right one which combines exactly with k_n in order to produce the percolation cluster. We consider the largest clusters X_{k_n} in each one of the $N^{k_{n+1}-k_n} \approx N^{K \log n}$ k_n-balls in a k_{n+1}-ball, and assume that their sizes are $|X_{k_n}| \approx \beta N^{k_n}$ for some $\beta \in (0, 1)$, and that the probability of connection between two points in different clusters is $c_{k_n}/N^{2k_{n+1}}$ (which is a lower bound for the actual probabilities). Let $s_n(\beta)$ denote the probability that two such clusters X_{k_n} and X'_{k_n} in disjoint k_n-balls are connected. Then

$$s_n(\beta) \approx 1 - \left(1 - \frac{c_{k_n}}{N^{2k_{n+1}}}\right)^{|X_{k_n}||X'_{k_n}|}$$

$$\approx 1 - \left(1 - \frac{c_{k_n}}{N^{2k_{n+1}}}\right)^{\beta^2 N^{2k_n}}$$

$$\approx 1 - \exp\left(-\frac{c_{k_n}\beta^2}{N^{2(k_{n+1}-k_n)}}\right)$$

$$\approx \frac{C + a\log n \cdot N^{b\log n}}{N^{2K\log n}}\beta^2$$

$$\approx \frac{\beta^2 a\log n}{N^{(2K-b)\log n}} =: r_n(\beta), \text{ with } b < 2K,$$

(see Lemma 5.7). Consider the E-R random graph

$$G(N^{K\log n}, r_n(\beta)),$$

and write $r_n(\beta)$ as

$$r_n(\beta) = \frac{\beta^2 a\log n}{N^{(K-b)\log n}} \frac{1}{N^{K\log n}}.$$

If $K > b$, then by the E-R theory only order $\log(N^{K\log n})$ of the X_{k_n} are connected, hence the ratio of the size of the largest connected component in the k_{n+1}-ball to the size of the ball decreases to 0 as $n \to \infty$, so there cannot be a percolation cluster (of positive

density). Therefore we choose $K < b$. Now let $K_1 = 2K - b$, then $0 < K_1 < K$. Writing $r_n(\beta)$ as

$$r_n(\beta) = \frac{\beta^2 a}{K_1 \log N} \log N^{K_1 \log n} \frac{1}{N^{K_1 \log n}},$$

then by Theorem 2.8.1 of [19] the probability that the graph $G(N^{K_1 \log n}, r_n(\beta))$ is connected tends to 1 as $n \to \infty$ if $a > \frac{K_1 \log N}{\beta^2}$. Connectivity of that graph whose vertices are the clusters X_{k_n} in the k_n-balls in a k_{n+1}-ball means that the largest cluster in the k_{n+1}-ball contains the largest clusters in all the k_n-balls it contains. If this can be proved for all sufficiently large n, then percolation follows. Note that this argument provides a percolation cluster if $K < b$, but it does not imply that percolation does not occur if $K > b$.

Theorem 3.5. *Let $\delta = 1$.*
(a) Let $C_2 = 0$ in (2.2) (correspondingly $b = 0$ in (2.5)) and $c_{k_n} = C + aN \log n$ in (2.5), with $K > \frac{1}{\log N}$ in (2.3). Then
    (i) for $a < 1$, percolation does not occur,
    (ii) for $a > 1$, pre-percolation occurs.

(b) Assume that $\{c_{k_n}\}$ satisfy (2.5), (2.6) with $b > 0$ and $c_{k_n} = C + a \log n \cdot N^{b \log n}$ where $k_n = k_n(K)$ is given by (2.3) and the pair (K, b) satisfy

$$\frac{2}{\log N} < K < b.$$

(1) Then there exist $C > 0$ and $a_* > 0$ such that for $a > a_*$ there is a sequence $(\beta_n)_n$ such that

$$\liminf_{n \to \infty} \beta_n > 0, \tag{3.2}$$

and for the clusters X_{k_n} in a nested sequence of k_n-balls B_{k_n} containing $\mathbf{0}$,

$$P(\text{there exists } n_{00} \text{ such that } |X_{k_n}| \geq \beta_n N^{k_n} \text{ for all } n \geq n_{00}) = 1, \tag{3.3}$$

percolation occurs, and the percolation cluster is unique.

(2) Assume in addition that $b < 2K - \frac{1}{\log N}$. Then the percolation cluster is given by a "cascade" of clusters at distances k_n, more precisely, there exists a (random) number n_0 such that for $n \geq n_0$ connections between $X_{k_n} \cap (B_{k_n} \backslash B_{k_{n-1}})$ and $X_{k_{n+\ell}} \cap (B_{k_{n+\ell}} \backslash B_{k_{n+\ell-1}})$ occur only for $\ell = 1, 2$.
(c) In the special case with connection probabilities $c_j = \frac{c_{k_n}}{N^{2k_{n+1}}}$ for $k_n + 1 \leq j \leq k_{n+1}$ and $0 < b \leq \frac{2}{\log N} < K$, percolation does not occur.

Remark 3.6.
(1) Note the consistency of Theorem 3.5 (a) with the example of asymptotic percolation recalled in Subsection 3.1. The difference is that in the finite N case we only have pre-percolation, and the connections are at hierarchical distances $k \log k$ rather than k.
(2) Theorem 3.5(a)(ii) implies that in part (b) pre-percolation occurs with any $a > 0$ and $b > 0$.
(3) The cascade of clusters in Theorem 3.5 (b)(2) is analogous to the cascade of giant components in the mean field case [16].
(4) The formulation in Theorem 3.5(b) is used as a technical tool to prove the result in Theorem 3.3, and also provides a setting to give a refined result.

3.4 Open problems and related developments.

(1) In Theorem 3.3(b) we have proved that with $c_k = C_0 + C_1 \log k + C_2 k^\alpha$ percolation does not occur if $C_2 = 0$ and $C_1 < N$. On the other hand in (a) we proved that for $\alpha > 2$ and C_0 and C_2 sufficiently large, percolation occurs. It remains an open question as to whether percolation can occur for all $\alpha > 0$ or even for $C_2 = 0$ and some C_1 sufficiently large. We next explain that to resolve these questions analogues of well-known results for Erdős-Rényi graphs would be needed for a class of ultrametric random graphs. An *ultrametric random graph* URG(M,d) is a random graph on a finite set M with ultrametric d and with connection probabilities $p_{x,y}$ that are random and depend on the ultrametric distance $d(x, y)$.

Consider the case $b = 0$ and $a > 1$ in Theorem 3.5(a). The expected number of in-edges to the annulus $(k_{n+1}, k_{n+2}]$ from $B_{k_{n+1}}$ is of order $O(a(1 - \frac{1}{N}) \log n)$ and the expected number of out-edges to the $(k_{n+2}, k_{n+3}]$-annulus is of order $O(a(1 - \frac{1}{N}) \log(n + 1))$. Regarding the question if pre-percolation already implies percolation, in order to determine the number of in-edges that connect to an out-edge, it would be necessary to determine the probability that two randomly chosen vertices in the $(k_{n+1}, k_{n+2}]$-annulus are connected by a path in the associated ultrametric random graph. This is related to the problem of determining the distribution of sizes of the connected components. These are open problems.

Consider the case $C_2 > 0$. In Theorem 3.5(c) we have proved that the random graph based on a lower bound for the connection probabilities at distances $k_n + 1, \ldots, k_{n+1}$ does not exhibit percolation in the case $0 < b \le \frac{2}{\log N} < K$ (this would correspond to $\alpha < 2$ in Theorem 3.3). In order to refine the argument and determine the behaviour for the actual connection probabilities which arise if $0 < \alpha < 2$ it would be necessary to determine the size of the largest connected component, that is, the number of k_n-balls (more precisely their largest connected components) in the k_{n+1}-ball which are connected and in the largest connected component in the associated ultrametric random graph (as $n \to \infty$). This is an open problem.

(2) It would also be of interest to consider the case $\delta = 1$ with connection probabilities of the form

$$p_{(k_n)} = \frac{C + a(\log n)^h N^{b(\log n)^h}}{N^{2k_n}}, \quad h > 1,$$

which are intermediate between those of Theorems 3.1 and 3.5.

(3) One can ask if there can exist a zero-density infinite cluster for parameter values where positive density clusters do not exist.

(4) Berger [7] has studied the behaviour of random walk on the infinite cluster of long-range percolation in Euclidean lattices of dimensions $d = 1, 2$. It would also be interesting to investigate this behaviour on the infinite clusters obtained in Theorems 3.1(b) and 3.5 (b). Long-range random walks on Ω_N have been studied in [17].

3.5 The renormalization group approach

The basic strategy we employ is in the spirit of the renormalization group method of statistical physics [14], which has been used by Newman and Schulman [27], Section 2, in their study of long range percolation in the Euclidean lattice.

Consider the countable ultrametric space (Ω_N, d). For each integer k we define an equivalence relation on Ω_N by

$$\boldsymbol{x} \equiv_k \boldsymbol{y} \text{ iff } d(\boldsymbol{x}, \boldsymbol{y}) \le k, \text{ that is, } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ belong to the same } k\text{-ball.}$$

Now consider the set of equivalence classes $\mathcal{E}_k$ furnished with the ultrametric

$$d_k(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) := d(\boldsymbol{x}, \boldsymbol{y}) = 0 \text{ if } d(\boldsymbol{x}, \boldsymbol{y}) \leq k, d_k(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) := d(\boldsymbol{x}, \boldsymbol{y}) - k \text{ if } d(\boldsymbol{x}, \boldsymbol{y}) > k,$$

where $\widetilde{\boldsymbol{x}}$ is an equivalence class containing a point $\boldsymbol{x}$. Then the resulting set of equivalence classes with ultrametric d_k can be identified with (Ω_N, d).

Given a graph

$$\mathcal{G}_N = (\Omega_N, \mathcal{E}_N),$$

with edges $\mathcal{E}_N$ given by a symmetric subset of $\Omega_N \times \Omega_N$ we obtain a new graph as follows. The set of vertices is the set of all k-balls, and the set of edges $\mathcal{E}_{N,1}$ are such that $\widetilde{\boldsymbol{x}}$ has a connection to $\widetilde{\boldsymbol{y}}$ if there is a connection in $\mathcal{G}_N$ between the largest connected subset (cluster) of $\widetilde{\boldsymbol{x}}$ and the largest connected subset of $\widetilde{\boldsymbol{y}}$. Using the above identification this defines a new graph $\mathcal{G}_N^1 = \Phi \mathcal{G}_N$ on (Ω_N, d). Iterating this procedure we obtain a sequence

$$\mathcal{G}_N^k = \Phi^k \mathcal{G}_N, \quad k \geq 1,$$

of graphs all having vertex set Ω_N. In addition we assign to each vertex v in $\mathcal{G}_N^k$ the $[0, 1]$-valued random variable

$$Y_k(v) := \frac{|X_k(v)|}{N^k}$$

where $X_k(v)$ denotes the set of vertices in $\mathcal{G}_N$ contained in the connected cluster in the k-ball corresponding to v obtained as the union of the clusters in the $(k-1)$-balls it contains. This construction defines the renormalization mapping $\Phi : \mathcal{G}_N \to \mathcal{G}_N$ such that $\Phi^k : \mathcal{G}_N \to \mathcal{G}_N^k$ for each k. Note that at each iteration the connection probabilities are the probabilities that the largest connected subsets of equivalence classes are connected and that these probabilities are random and dependent (because the connected components have random sizes), and they change at each iteration.

Rather than working directly with the sequence $\mathcal{G}_N^k$ we choose a subsequence k_n and construct a sequence of renormalization maps Φ^{k_n} such that the number of points in a ball of radius 1 (with respect to the new distances d_{k_n}) increases to infinity as $n \to \infty$. In particular, we will show that there exists an increasing sequence of integers (k_n) (see (2.3)) and a sequence of graphs $\mathcal{G}^{k_n}$ with $N^{k_n - k_{n-1}}$ vertices constructed recursively as follows, that is,

$$\mathcal{G}^{k_{n+1}} = \Phi_n \mathcal{G}^{k_n},$$

where Φ_n depends on n since it is a mapping from a graph with vertices $\Omega_{N^{k_n - k_{n-1}}}$ to a graph with vertices $\Omega_{N^{k_{n+1} - k_n}}$ and with connection probabilities between vertices that are a function of the distance between them. Moreover we can identify $\mathcal{G}^{k_n}$ with a subgraph of $\mathcal{G}_N$ and these subgraphs are a decreasing function of n. We establish percolation by showing that the intersection of these subgraphs starting at a given point in Ω_N is non-empty with positive probability. The difference now is that $\Phi^{k_{n+1}}$ is not obtained by iteration but by means of Φ_n:

$$\Phi^{k_{n+1}} = \Phi_n \Phi^{k_n}.$$

Given the sequence (k_n) we can consider the equivalence classes given by

$$\boldsymbol{x} \equiv_{k_n} \boldsymbol{y} \text{ iff } d(\boldsymbol{x}, \boldsymbol{y}) \leq k_n,$$

and define the ultrametric d_{k_n} by

$$d_{k_n}(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) = \ell \text{ iff } k_{n+\ell-1} < d(\boldsymbol{x}, \boldsymbol{y}) \leq k_{n+\ell}, \ \ell \geq 1.$$

We now consider the graph $\mathcal{G}^{k_n}$ whose vertices are the d_{k_n}-equivalence classes. Two points in $\mathcal{G}^{k_n}$ at d_{k_n}-distance $\ell \geq 1$ are connected if there is a $\mathcal{G}_N$-edge joining the largest connected components in these equivalence classes in $\mathcal{G}^{k_n-1}$. Note that there are $N^{k_{n+1}-k_n}$ points in a ball of d_{k_n}-radius 1, $N^{k_{n+2}-k_n}$ in a ball of radius 2, etc. The proof of percolation in the case $\delta < 1$ given in Section 4 involves showing that as $n \to \infty$ the graphs $\mathcal{G}^{k_n}$ occupy a certain portion of the k_n-balls and an increasing sequence can be linked in a cascade with probability approaching 1.

In order to apply these ideas to the more delicate critical case $\delta = 1$ in Section 5 we define $Y_{k_n}(v)$, $v \in \mathcal{G}_{k_n}$ as above. Then given the random graph $\mathcal{G}_N$, the nontriviality, $\liminf_{n\to\infty} Y_{k_n} > 0$, has probability 0 or 1. Our goal is to find a sufficient condition for this to be 1. In order to achieve this our strategy is to look for a pair of sequences $k_n \to \infty$, $\liminf_n \beta_n > 0$, such that the probability that $Y_{k_n} \geq \beta_n$ converges to 1 as $n \to \infty$. This program will be carried out in Subsection 5.2 using as basic tools large deviation estimates for binomial distributions and probability bounds for the connectivity of an Erdős-Rényi graph.

4 Proofs for the cases $\delta > 1$ and $\delta < 1$

We first mention a few notational points. In some places in the proofs in this and the following section where a number appears which should be a non-negative integer and it is not necessarily so, it should be interpreted as its integer part.

$a_n \sim b_n$ means that $\frac{a_n}{b_n} \to 1$ as $n \to \infty$, $a_n >> b_n$ means that $b_n = o(a_n)$, and

$$a_n \lesssim b_n \text{ means that } 0 \leq \liminf_n \frac{a_n}{b_n} \leq \limsup_n \frac{a_n}{b_n} \leq 1.$$

Definition 4.1. For $0 < \gamma < 1$, we say that a k-ball B_k *is γ-good* if its attached cluster X_k satisfies $|X_k| \geq N^{\gamma k}$.

If $\mathbf{0} \in B_k$ and B_k is γ-good, then the probability that $\mathbf{0} \in X_k$ is greater than or equal to $N^{(\gamma-1)k}$.

Proof of Theorem 3.1.
(a) If suffices to show that

$$P(B_k \text{ is connected to its complement for infinitely many } k) = 0. \tag{4.1}$$

For $j \geq k$,

$$P(B_k \text{ is connected to the } (j, j+1]\text{-annulus around it})$$
$$= 1 - \left(1 - \frac{c_{j+1}}{N^{(j+1)(1+\delta)}}\right)^{N^k N^j (N-1)}$$
$$\quad (\text{using } 1 - (1-x/y)^z < xz/y, \quad 0 < x < y, \quad z \geq 2)$$
$$< c_{j+1} \frac{N^k N^j (N-1)}{N^{(j+1)(1+\delta)}}$$
$$\leq M \frac{N^k}{N^{j\delta}}, \quad M \text{ is a constant (since } \sup_k c_k < \infty),$$

hence

$$P(B_k \text{ is connected to its complement}) \leq M \sum_{j \geq k} \frac{N^k}{N^{j\delta}},$$

and since

$$\sum_{k \geq 1} \sum_{j \geq k} \frac{N^k}{N^{j\delta}} < \infty,$$

(4.1) follows by Borel-Cantelli.

(b) For each n, let k_n be given by:

$$k_n = \lfloor n \log n \rfloor, \tag{4.2}$$

hence

$$k_{n+1} - k_n \sim \log n \quad \text{as } n \to \infty. \tag{4.3}$$

Choose γ so that

$$\frac{1+\delta}{2} < \gamma < 1. \tag{4.4}$$

Then the probability of connection between two clusters X_{k_n} and X'_{k_n} in (disjoint) γ-good k_n-balls in a k_{n+1}-ball is bounded below by

$$1 - \left(1 - \frac{c}{N^{k_{n+1}(1+\delta)}}\right)^{|X_{k_n}||X'_{k_n}|} \geq 1 - \left(1 - \frac{c}{N^{k_{n+1}(1+\delta)}}\right)^{N^{2\gamma k_n}}. \tag{4.5}$$

There are $N^{k_{n+1}-k_n}$ k_n-balls in a k_{n+1}-ball. If at least $N^{\gamma(k_{n+1}-k_n)}$ of them are γ-good and their clusters are connected within the k_{n+1}-ball, then the size of the cluster $X_{k_{n+1}}$ in the k_{n+1}-ball is greater than or equal to $N^{\gamma k_{n+1}}$, so the k_{n+1}-ball is γ-good.

For each n, let p_n denote the probability that B_{k_n} is γ-good:

$$p_n = P(|X_{k_n}| \geq N^{\gamma k_n}). \tag{4.6}$$

Then, writing $\text{Bin}(n, p, \geq k) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$ and $\text{Bin}(n, p, < k) = 1 - \text{Bin}(n, p, \geq k)$, we have, by (4.5), (4.6), and independence of connections at different hierarchical distances,

$$p_{n+1} \geq \text{Bin}\left(N^{k_{n+1}-k_n}, p_n, \geq N^{\gamma(k_{n+1}-k_n)}\right) \left[1 - \left(1 - \frac{c}{N^{k_{n+1}(1+\delta)}}\right)^{N^{2\gamma k_n}}\right]^{N^{\gamma(k_{n+1}-k_n)}} \tag{4.7}$$

Now,

$$1 - \left(1 - \frac{c}{N^{k_{n+1}(1+\delta)}}\right)^{N^{2\gamma k_n}}$$

$$\geq 1 - \left(1 - \frac{cN^{2\gamma k_n - (1+\delta)k_{n+1}}}{N^{2\gamma k_n}}\right)^{N^{2\gamma k_n}}$$

$$\text{(using } (1 - x/y)^y < e^{-x}, \quad x > 0, \, y > 0\text{)},$$

$$> 1 - \exp\{-cN^{2\gamma k_n - (1+\delta)k_{n+1}}\}, \tag{4.8}$$

hence

$$\left[1 - \left(1 - \frac{c}{N^{k_{n+1}(1+\delta)}}\right)^{N^{2\gamma k_n}}\right]^{N^{\gamma(k_{n+1}-k_n)}}$$

$$> (1 - \exp\{-cN^{2\gamma k_n - (1+\delta)k_{n+1}}\})^{N^{\gamma(k_{n+1}-k_n)}}. \tag{4.9}$$

(4.2) and (4.4) imply that

$$2\gamma k_n - (1+\delta)k_{n+1} > \varepsilon n \log n \tag{4.10}$$

for some $0 < \varepsilon < 1$ and sufficiently large n. Then, from (4.9),

$$\left[1 - \left(1 - \frac{c}{N^{k_{n+1}(1+\delta)}}\right)^{N^{2\gamma k_n}}\right]^{N^{\gamma(k_{n+1}-k_n)}} > 1 - \varepsilon_n, \tag{4.11}$$

where

$$\varepsilon_n := 1 - \left(1 - \exp\{-cN^{2\gamma k_n - (1+\delta)k_{n+1}}\}\right)^{N^{\gamma(k_{n+1} - k_n)}}, \tag{4.12}$$

and we have that

$$\sum_n \varepsilon_n < \infty, \tag{4.13}$$

because, from (4.12), (4.3), and (4.10),

$$\begin{aligned}
\varepsilon_n &< N^{\gamma(k_{n+1}-k_n)} \exp\{-cN^{2\gamma k_n - (1+\delta)k_{n+1}}\} \\
&\lesssim N^{\gamma \log n} \exp\{-cN^{\varepsilon n \log n}\} \\
&= n^{\gamma \log N} \exp\{-cn^{\varepsilon n \log N}\}
\end{aligned}$$

for all sufficiently large n, which is summable.

From (4.7) and (4.11),

$$p_{n+1} > \mathrm{Bin}(N^{k_{n+1}-k_n}, p_n, \geq N^{\gamma(k_{n+1}-k_n)})(1 - \varepsilon_n). \tag{4.14}$$

Now we assume that there exists n_0 such that

$$N^{k_{n_0+1}-k_{n_0}} p_{n_0} >> N^{\gamma(k_{n_0+1}-k_{n_0})}, \tag{4.15}$$

(this assumption will be verified below), and we suppose that $p_{n_0} > \eta > 0$. Then, by a large deviation inequality [23] (p. 26, (2.6)),

$$\begin{aligned}
&\mathrm{Bin}\left(N^{k_{n_0+1}-k_{n_0}}, p_{n_0}, \geq N^{\gamma(k_{n_0+1}-k_{n_0})}\right) \tag{4.16} \\
&= 1 - \mathrm{Bin}\left(N^{k_{n_0+1}-k_{n_0}}, p_{n_0}, < N^{k_{n_0+1}-k_{n_0}} p_{n_0} - (N^{k_{n_0+1}-k_{n_0}} p_{n_0} - N^{\gamma(k_{n_0+1}-k_{n_0})})\right) \\
&\geq 1 - \exp\left\{-\frac{(N^{k_{n_0+1}-k_{n_0}} p_{n_0} - N^{\gamma(k_{n_0+1}-k_{n_0})})^2}{2N^{k_{n_0+1}-k_{n_0}} p_{n_0}}\right\},
\end{aligned}$$

hence, by (4.15),

$$\mathrm{Bin}\left(N^{k_{n_0+1}-k_{n_0}}, p_{n_0}, \geq N^{\gamma(k_{n_0+1}-k_{n_0})}\right) \geq 1 - \delta_{n_0}, \tag{4.17}$$

where we may take

$$0 < \delta_{n_0} < (1 - \varepsilon_{n_0} - \eta)/(1 - \varepsilon_{n_0}) \tag{4.18}$$

(with ε_{n_0} and η small enough so that $\varepsilon_{n_0} + \eta < 1$). Then, from (4.14), (4.17) and (4.18),

$$p_{n_0+1} > (1 - \delta_{n_0})(1 - \varepsilon_{n_0}) > \eta, \tag{4.19}$$

so, iterating the argument,

$$p_n > \eta \quad \text{for all} \quad n \geq n_0, \tag{4.20}$$

and therefore (4.15) will in fact hold for all sufficiently large n_0.

From (4.19),

$$1 - p_{n_0+1} \leq \delta_{n_0} + \varepsilon_{n_0}. \tag{4.21}$$

Choosing $\delta_n, n \geq n_0$, so that $\sum_{n \geq n_0} \delta_n < \infty$, we then have from (4.21), together with (4.13),

$$\sum_{n \geq n_0} (1 - p_n) < \infty.$$

Therefore, by Borel-Cantelli,

$$P(\text{all but finitely many } B_{k_n} \text{ are } \gamma\text{-good}) = 1. \tag{4.22}$$

On the other hand, the probability that the clusters X_{k_n} and $X_{k_{n+1}}$ of the γ-good balls B_{k_n} and $B_{k_{n+1}}, B_{k_n} \subset B_{k_{n+1}}$, are not connected is, for large n, by (4.2) and (4.10), bounded above by

$$
\begin{aligned}
\left(1 - \frac{c_{k_n}}{N^{k_{n+1}(1+\delta)}}\right)^{|X_{k_n}||X_{k_{n+1}}|}
&\leq \left(1 - \frac{c}{N^{k_{n+1}(1+\delta)}}\right)^{N^{\gamma(k_n+k_{n+1})}} \\
&< \exp\{-cN^{\gamma(k_n+k_{n+1})-k_{n+1}(1+\delta)}\} \\
&\lesssim \exp\{-cN^{(2\gamma k_n-(1+\delta)k_{n+1})}\} \\
&< \exp\{-cN^{\varepsilon n \log n}\} \\
&= \exp\{-cn^{\varepsilon n \log N}\},
\end{aligned}
$$

which is summable. Therefore, by Borel-Cantelli,

$$P(\text{all but finitely many pairs } X_{k_n} \text{ and } X_{k_{n+1}} \text{ are connected}) = 1. \tag{4.23}$$

Since there is a positive probability that $\mathbf{0}$ belongs to some X_{k_n} (the probability is greater than or equal to $N^{(\gamma-1)k_n}$) in the chain of γ-good B_{k_n}'s with connected clusters X_{k_n}, then by (4.22) and (4.23) there is an infinite cluster that contains $\mathbf{0}$ with positive probability. So, percolation occurs.

It remains to verify assumption (4.15). We do this by means of a connectivity result for E-R random graphs. Writing $c_k/N^{k(1+\delta)}$ in (2.1) as $\widetilde{c}_k/N^k, \widetilde{c}_k = c_k/N^{\delta k}$, we have that for all $k \leq n$,
$$\widetilde{c}_k > \overline{c}_n \log N^n \quad \text{with} \quad \overline{c}_n = \frac{c}{N^{\delta n} \log N^n}.$$

We consider the E-R random graph $G(N^n, \overline{c}_n/N^n)$ whose vertices are the points of an n-ball. If

$$c > N^{\delta n} \log N^n \tag{4.24}$$

for some very large n, then the probability that $G(N^n, \overline{c}/N^n)$ is connected is close to 1, by Theorem 2.8.1 in [19]. It is possible to choose c large enough so that (4.24) holds because the only restriction on c is $c < N^{(1+\delta)n}$. This implies that the probability that all the points in the n-ball are connected is close to 1. Then, taking $n = k_{n_0}$, (4.15) is true.

Finally, the uniqueness follows from Theorem 1.2 in [26] (see Remark 4.2).

<div style="text-align: right">□</div>

Remark 4.2. Uniqueness of the infinite cluster is proved in [26] (Theorem 1.2) by showing that (Ω_N, d) can be embedded into $\mathbb{Z}$ such that any ball of radius r will be represented by N^r consecutive integers, and the collection of balls of radius r partitions $\mathbb{Z}$, and that the embedding is stationary and ergodic. The uniqueness then follows from Gandolfi et al. [21] (Theorem 0), on the uniqueness of the infinite cluster for long range percolation on $\mathbb{Z}$ satisfying the positive finite energy condition. The proof uses only the properties that the connection probabilities between vertices $\boldsymbol{x}, \boldsymbol{y}$ are strictly positive and depend only on $d(\boldsymbol{x}, \boldsymbol{y})$, and therefore is also applicable to our case. An intuitive argument for uniqueness of the percolation cluster (of positive density), using the clusters in the proof of Theorem 3.1, is that two chains of nested balls will eventually intersect, and by ultrametricity from then on they coincide, so, if their largest clusters occupy a sizeable part of the balls, they will eventually be the same.

5 Proofs for the critical case $\delta = 1$

5.1 Lemmas for the proof of Theorem 3.5.

Lemma 5.1. *Let $b = 0$ and with aN instead of a in (2.5), and $K > \frac{1}{\log N}$ in (2.3).*

(a) Let $A_{n,j}$ denote the event that a k_n-ball B_{k_n} (containing $\mathbf{0}$) is connected to the $(k_{n+1} + j - 1, k_{n+1} + j]$-annulus around B_{k_n}, $j = 1, \ldots, k_{n+2} - k_{n+1}$. Then

$$P(A_{n,j}) \sim \frac{aN(1 - N^{-1}) \log n}{N^j N^{K \log n}} \quad \text{as } n \to \infty. \tag{5.1}$$

(b) Let $\widehat{A}_{n,j}$ denote the event that there is no connection between the k_{n+1}-ball $B_{k_{n+1}}$ and the $(k_{n+1} + j - 1, k_{n+1} + j]$-annulus around $B_{k_{n+1}}$, $j = 1, \ldots, k_{n+2} - k_{n+1}$. Then

$$P(\widehat{A}_{n,j}) \sim n^{-aN(1 - N^{-1})/N^j} \quad \text{as} \quad n \to \infty. \tag{5.2}$$

(c) Let A_n denote the event that there is no connection between the k_{n+1}-ball $B_{k_{n+1}}$ and the $(k_{n+1}, k_{n+2}]$-annulus around $B_{k_{n+1}}$. Then

$$P(A_n) \sim n^{-a} \quad \text{as} \quad n \to \infty, \tag{5.3}$$

and this implies that

(i) if $a < 1$, then with probability 1 there are infinitely many pairs $((k_n, k_{n+1}], (k_{n+1}, k_{n+2}])$ of successive annuli that are not connected,

(ii) if $a > 1$, then with probability 1 there are at most finitely many pairs $((k_n, k_{n+1}], (k_{n+1}, k_{n+2}])$ of successive annuli that are not connected.

(d) Let $A_{n,j,\ell}$ denote the event that a k_n-ball B_{k_n} is connected to the $(k_{n+\ell} + j - 1, k_{n+\ell} + j]$-annulus around B_{k_n}, $j = 1, \ldots, k_{n+\ell+1} - k_{n+\ell}$. Then for $\ell \geq 1$,

$$P(A_{n,j,\ell}) \sim \frac{aN(1 - N^{-1}) \log(n + \ell)}{N^j (n + \ell)^{\ell K \log N}} \quad \text{as} \quad n \to \infty. \tag{5.4}$$

(Note that (5.1) is a special case with $\ell = 1$.)

(e) Let $\widetilde{A}_n$ denote the event that there are connections from a k_n-ball B_{k_n} to the complement of a k_{n+1}-ball $B_{k_{n+1}}$, with $B_{k_n} \subset B_{k_{n+1}}$. Then

$$P(\widetilde{A}_n) \lesssim a \sum_{\ell=1}^{\infty} \frac{\log(n + \ell)}{(n + \ell)^{\ell K \log N}} \quad \text{as} \quad n \to \infty. \tag{5.5}$$

(f) Let A denote the event that there are no connections from a k_n-ball B_{k_n} to the complement of a k_{n+1}-ball $B_{k_{n+1}}$, with $B_{k_n} \subset B_{k_{n+1}}$, for all but finitely many n. Then for any $a > 0$,

$$P(A) = 1. \tag{5.6}$$

Proof.
(a) By (2.5) and (2.6), $c_{k_{n+1}+j} \sim c_{k_n}$ as $n \to \infty$, hence

$$
\begin{aligned}
P(A_{n,j}) &= 1 - \left(1 - \frac{c_{k_{n+1}+j}}{N^{2(k_{n+1}+j)}}\right)^{N^{k_n} N^{k_{n+1}+j}(1 - N^{-1})} \\
&\sim 1 - \exp\left\{-\frac{aN(1 - N^{-1}) \log(n + 1)}{N^j N^{k_{n+1}-k_n}}\right\} \\
&\sim 1 - \exp\left\{-\frac{aN(1 - N^{-1}) \log(n + 1)}{N^j N^{K \log n}}\right\},
\end{aligned}
$$

and then (5.1) follows.

(b) Since there are $N^{k_{n+1}-k_n} \sim N^{K \log n}$ k_n-balls in the k_{n+1}-ball, then from (5.1),

$$
\begin{aligned}
P(\widehat{A}_{n,j}) &\sim \left(1 - \frac{aN(1 - N^{-1}) \log n}{N^j N^{K \log n}}\right)^{N^{K \log n}} \\
&\sim \exp\left\{-\frac{aN(1 - N^{-1}) \log n}{N^j}\right\} \\
&= n^{-aN(1-N^{-1})/N^j}.
\end{aligned}
$$

(c) By (5.2) and independence,

$$
\begin{aligned}
P(A_n) &\sim \prod_{j=1}^{k_{n+2}-k_{n+1}} n^{-aN(1-N^{-1})/N^j} \\
&= n^{-aN(1-N^{-1}) \sum_{j=1}^{K \log(n+1)} N^{-j}} \\
&\sim n^{-a(1 - N^{-K \log n})} \\
&\sim n^{-a}.
\end{aligned}
$$

Since $\sum_n n^{-a} < \infty$ if and only if $a > 1$, then by independence and the (second) Borel-Cantelli lemma, for $a < 1$, with probability 1 there are infinitely many successive $(k_n, k_{n+1}]$-annuli that are not connected, and by the (first) Borel-Cantelli lemma, for $a > 1$, with probability 1 there are at most finitely many successive $(k_n, k_{n+1}]$-annuli that are not connected.

(d) By (2.5) and (2.6),

$$
\begin{aligned}
P(A_{n,j,\ell}) &\sim 1 - \left(1 - \frac{C + aN \log(n + \ell)}{N^{2(k_{n+\ell}+j)}}\right)^{N^{k_n} N^{k_{n+\ell}+j}(1 - N^{-1})} \\
&\sim 1 - \exp\left\{-\frac{aN(1 - N^{-1}) \log(n + \ell)}{N^j N^{k_{n+\ell}-k_n}}\right\} \\
&\sim \frac{aN(1 - N^{-1}) \log(n + \ell)}{N^j N^{\ell K \log(n+\ell)}} \\
&= \frac{aN(1 - N^{-1}) \log(n + \ell)}{N^j (n + \ell)^{\ell K \log N}}.
\end{aligned}
$$

(e) By (5.4),

$$
\begin{aligned}
P(\widetilde{A}_n) &\sim \sum_{\ell=1}^{\infty} \sum_{j=1}^{k_{n+\ell+1}-k_{n+\ell}} \frac{aN(1 - N^{-1}) \log(n + \ell)}{N^j (n + \ell)^{\ell K \log N}} \\
&\sim aN \sum_{\ell=1}^{\infty} (N^{-1} - N^{-K \log(n+\ell)}) \frac{\log(n + \ell)}{(n + \ell)^{\ell K \log N}} \\
&\lesssim a \sum_{\ell=1}^{\infty} \frac{\log(n + \ell)}{(n + \ell)^{\ell K \log N}}.
\end{aligned}
$$

(f) By (5.5),

$$
\begin{aligned}
\sum_{n=1}^{\infty} P(\widetilde{A}_n) &\lesssim a \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\log(n + \ell)}{(n + \ell)^{\ell K \log N}} \\
&\lesssim a \sum_{j=2}^{\infty} \frac{\log j}{j^{K \log N - 1}} < \infty,
\end{aligned}
$$

and since $K \log N > 1$ the result follows by Borel-Cantelli. $\qquad\square$

Lemma 5.2. *Let K and b be as in part (b)(2) of Theorem 3.5, that is, $0 < b < 2K - \frac{1}{\log N}$. Let $A_{n,j}$ denote the event that the cluster X_{k_n} in a k_n-ball B_{k_n} is connected to the $(k_{n+j}, k_{n+j+1}]$-annulus around B_{k_n}, $j \geq 2$. Then there is a positive constant M such that*

$$P(A_{n,j}) \lesssim \frac{M \log n}{n^{(Kj-b)\log N}},$$

as $n \to \infty$, and

$$\sum_{n=1}^{\infty} \sum_{j=2}^{\infty} P(A_{n,j}) < \infty.$$

Hence with probability 1 there exists a (random) number n_0 such that for all $n \geq n_0$ the connections between the clusters X_{k_n} restricted to the $(k_n, k_{n+1}]$-annuli do not skip over two successive annuli, that is, there are no connections between the annulus $(k_{n-1}, k_n]$ and the annuli $(k_{n+2}, k_{n+3}]$, $(k_{n+3}, k_{n+4}]$, etc.

Proof. By (2.5) and (2.6),

$$
\begin{aligned}
P(A_{n,j}) &= \sum_{\ell=k_{n+j}+1}^{k_{n+j+1}} \left[1 - \left(1 - \frac{c_\ell}{N^{2\ell}}\right)\right]^{|X_{k_n}|N^\ell(1-N^{-1})} \\
&\leq (C + a\log(n+1+j) \cdot N^{b\log(n+1+j)})N^{k_n} \sum_{\ell=k_{n+j}+1}^{k_{n+j+1}} \frac{1}{N^\ell} \\
&\lesssim M\frac{\log n \cdot N^{b\log n}}{N^{k_{n+j}-k_n}} \lesssim M\frac{\log n}{N^{(Kj-b)\log n}} = M\frac{\log n}{n^{(Kj-b)\log N}}.
\end{aligned}
$$

It is easy to show that the assumptions on K and b imply that

$$\sum_{n=1}^{\infty} \sum_{j=2}^{\infty} \frac{\log n}{n^{(Kj-b)\log N}} < \infty,$$

and then the result follows by Borel-Cantelli. $\square$

The main tools for proving part (b) of the theorem are a large deviation inequality for the binomial distribution and a connectivity result for an E-R random graph. Recall that a graph is said to be connected if it has only one connected component and no isolated vertices.

We first recall the large deviation bound for the binomial distribution [23] (Corollary 2.4).

Lemma 5.3. *Let Y_n be $Bin(n, 1-p)$ and $c > 1$. Then for $x \geq cn(1-p)$,*

$$P(Y_n \geq x) \leq e^{-h(c)x},$$

where $h(c) = \log c - 1 + 1/c > 0$.

Corollary 5.4. *There exist $\kappa > 0$ and $\varepsilon > 0$ such that for $0 < \sigma < \min(\frac{p}{1-p}, \varepsilon)$,*

$$P\left(1 - \frac{Y_n}{n} \leq p - \sigma(1-p)\right) \leq e^{-\kappa\sigma^2(1-p)n}. \tag{5.7}$$

Proof. It is easy to see that there exist $\kappa > 0$ and $\varepsilon > 0$ such that $h(c) \geq \kappa(c-1)^2$ for $1 < c < 1 + \varepsilon$. Then the result follows from the lemma putting $c = 1 + \sigma$. $\square$

Definition 5.5. For $0 < \beta < 1$, we say that a k-ball B_k is *β-good* if its cluster X_k satisfies $|X_k| \geq \beta N^k$.

Note that if $\mathbf{0} \in B_k$ and B_k is *β-good*, then the probability that $\mathbf{0} \in X_k$ is greater than or equal to β.

Lemma 5.6. *Assume that for some $\beta > 0$ and some n_0,*

$$P(|X_{k_n}| \geq \beta N^{k_n} \quad \text{for all} \quad n \geq n_0) = p_* > 0. \tag{5.8}$$

Then percolation occurs.

Proof. By transitivity we may assume that $\mathbf{0}$ belongs to the k_n-ball whose largest cluster is X_{k_n}. Then by the assumption (5.8) we have that

$$\liminf_{k \to \infty} P(\mathbf{0} \in X_{k_n}) \geq \beta p_* > 0,$$

which implies percolation. $\qquad\square$

Lemma 5.7. *Let $0 < b < 2K$ in (2.5) with k_n as in (2.3), and let $0 < \beta < 1$. Let X_{k_n} and X'_{k_n} be the largest clusters in two (disjoint) β-good k_n-balls in a k_{n+1}-ball. Then*

$$P(X_{k_n} \text{ and } X'_{k_n} \text{ are connected within the } k_{n+1}\text{-ball}) \gtrsim r_n(\beta) \text{ as } n \to \infty, \tag{5.9}$$

where

$$r_n(\beta) = \frac{\beta^2 a \log n}{N^{(2K-b)\log n}}. \tag{5.10}$$

Proof. By (2.6) and (2.5), $k_n < d(X_{k_n}, X'_{k_n}) \leq k_{n+1}$, so

$$P(X_{k_n} \text{ and } X'_{k_n} \text{ are connected within the } k_{n+1}\text{-ball})$$

$$\geq 1 - \left(1 - \frac{c_{k_n}}{N^{2k_{n+1}}}\right)^{|X_{k_n}||X'_{k_n}|}$$

$$\geq 1 - \left(1 - \frac{c_{k_n}}{N^{2k_{n+1}}}\right)^{\beta^2 N^{2k_n}}$$

$$> 1 - \exp\left\{-\frac{c_{k_n}\beta^2}{N^{2(k_{n+1}-k_n)}}\right\}$$

$$\sim 1 - \exp\left\{-\frac{\beta^2 (C + a \log n \cdot N^{b\log n})}{N^{2K\log n}}\right\}$$

$$\sim r_n(\beta).$$

$\qquad\square$

5.2 Proof of Theorem 3.5

(a) (i) Lemma 5.1(f) guarantees that with probability 1 there exists n_0 such that if $n > n_0$ there are no connections between the $(k_n, k_{n+1}]$-annulus and the complement of $B_{k_{n+2}}$. Moreover, if $a < 1$, by Lemma 5.1(c)(i) there are infinitely many n such that the $(k_n, k_{n+1}]$-annulus and the $(k_{n+1}, k_{n+2}]$-annulus are not connected. This implies that with probability 1 there there exists some $n \geq n_0$ such that there are no connections from $B_{k_{n+1}}$ to the exterior.

(ii) The pre-percolation statement follows from Lemma 5.1(c)(ii).

(b) (1) We begin by indicating the main ideas of the proof. We consider a sequence of nested balls B_{k_n} (containing $\mathbf{0}$) and their largest clusters X_{k_n}. Recall that each k_{n+1}-ball is comprised of $N^{k_{n+1}-k_n}$ (disjoint) k_n-balls.

At each stage we will focus on the subset of the k_n-balls in a k_{n+1}-ball that are β_n-*good* (i.e., $|X_{k_n}| \geq \beta_n N^{k_n}$, see Definition 5.5), where $(\beta_n)_n$ is a sequence of numbers in $(0,1)$ to be determined below. By construction the events that different k_n-balls are β_n-good are independent, and by transitivity they all have the same probability

$$p_n^G(\beta_n) = P(|X_{k_n}| \geq \beta_n N^{k_n}). \tag{5.11}$$

Let $\mathcal{N}_n$ denote the number of β_n-good k_n-balls and recall (5.9) and (5.10).

Now we consider the E-R random graph $G(\mathcal{N}_n, r_n(\beta_n))$ whose vertices are the $\mathcal{N}_n$ β_n-good k_n-balls in the k_{n+1}-ball with connection probability $r_n(\beta_n)$.

The key idea of the proof is to establish that with probability 1 there is a (random) number n_0 such that for $n \geq n_0$,

$$G(\mathcal{N}_n, r_n(\beta_n)) \quad \text{is connected,} \tag{5.12}$$

which implies that

$$|X_{k_{n+1}}| \geq \mathcal{N}_n \beta_n N^{k_n}. \tag{5.13}$$

We denote by E_n the event

$$E_n = \{G(\mathcal{N}_n, r_n(\beta_n)) \quad \text{is connected}\} \tag{5.14}$$

and

$$p_n^A(\beta_n) = P(E_n). \tag{5.15}$$

In order to prove (5.12) for all large n, by Borel-Cantelli it suffices to show that

$$\sum_n (1 - p_n^A(\beta_n)) < \infty. \tag{5.16}$$

We denote by F_n the event

$$F_n = \{\mathcal{N}_n \geq (1 - \varepsilon_n) p_n^G(\beta_n) N^{k_{n+1}-k_n}\}, \tag{5.17}$$

where $\varepsilon_n \in (0,1)$ and $p_n^G(\beta_n)$ is given by (5.11), and

$$p_n^B(\beta_n, \varepsilon_n) = P(F_n), \tag{5.18}$$

The next key idea is to choose a sequence of numbers $\varepsilon_n \in (0,1)$ of the form $\varepsilon_n = n^{-(1+\theta)}$, θ to be chosen below, with $\sum_n \varepsilon_n < \infty$ such that

$$\sum_n (1 - p_n^B(\beta_n, \varepsilon_n)) < \infty. \tag{5.19}$$

If both events E_n and F_n occur, then by (5.13)

$$|X_{k_{n+1}}| \geq (1 - \varepsilon_n) p_n^G(\beta_n) \beta_n N^{k_{n+1}},$$

so the k_{n+1}-ball is β_{n+1}-good with

$$\beta_{n+1} = (1 - \varepsilon_n) p_n^G(\beta_n) \beta_n. \tag{5.20}$$

Therefore, since E_n and F_n are independent (because E_n is defined in terms of distance k_{n+1}, and F_n in terms of distance k_n), then

$$p_{n+1}^G(\beta_{n+1}) \geq p_n^B(\beta_n, \varepsilon_n) p_n^A(\beta_n). \tag{5.21}$$

We will show that for sufficiently large values of C and a there exists a sequence β_n such that

$$\liminf_n \beta_n > 0. \tag{5.22}$$

and

$$\sum_n (1 - p_n^G(\beta_n)) < \infty, \tag{5.23}$$

in order to obtain the results (3.2) and (3.3).

Since the quantities involved in the scheme described above are interdependent, we need to overcome the interaction among them. We proceed as follows:

- We set $\varepsilon_n = n^{-(1+\theta)}$ for some $0 < \theta < \frac{K \log N}{2} - 1$ (recall that $K > \frac{2}{\log N}$), hence

$$K \log N > 2(1 + \theta). \tag{5.24}$$

- In Steps 1 and 2 below we will obtain estimates

$$\sum_{n \geq n_0} \left(1 - p_n^B\left(\beta_n, \frac{1}{n^{1+\theta}}\right)\right) < \infty \quad \text{if } p_n^G(\beta_n) \geq \frac{1}{2} \text{ for all } n \geq n_0, \tag{5.25}$$

and

$$\sup_{\beta \geq \frac{1}{5}} \sum_{n \geq n_0} (1 - p_n^A(\beta)) < \infty. \tag{5.26}$$

We will then show that we can choose n_0, C and a such that $\beta_n \geq \frac{1}{5}$ and $p_n^G(\beta_n) \geq \frac{1}{2}$ for all $n \geq n_0$.

To complete the proof we proceed step by step. We first verify (5.25) and (5.26). **Step 1.** Assume that $p_n^G(\beta_n) \geq \frac{1}{2}$ for all $n \geq n_0$. We first focus on one k_{n+1}-ball. It contains $N^{k_{n+1}-k_n}$ k_n-balls. The probability that a k_n-ball is β_n-good is given by $p_n^G(\beta_n)$ (see (5.11)). The number $\mathcal{N}_n$ of β_n-good k_n-balls in the k_{n+1}-ball is $\mathrm{Bin}(N^{k_{n+1}-k_n}, p_n^G(\beta_n))$. From (5.17), (5.18) and Corollary 5.4,

$$
\begin{aligned}
1 - p_n^B(\beta_n, \varepsilon_n) &= P\left(\frac{\mathcal{N}_n}{N^{k_{n+1}-k_n}} < (1 - \varepsilon_n) p_n^G(\beta_n)\right) \\
&= P\left(1 - \frac{N^{k_{n+1}-k_n} - \mathcal{N}_n}{N^{k_{n+1}-k_n}} < p_n^G(\beta_n) - \sigma_n(1 - p_n^G(\beta_n))\right) \\
&\leq \exp\{-\kappa \sigma_n^2 (1 - p_n^G(\beta_n)) N^{k_{n+1}-k_n}\},
\end{aligned}
$$

where

$$\sigma_n = \varepsilon_n \frac{p_n^G(\beta_n)}{1 - p_n^G(\beta_n)},$$

hence

$$1 - p_n^B(\beta_n, \varepsilon_n) \leq \exp\left\{-\kappa \frac{(p_n^G(\beta_n))^2}{1 - p_n^G(\beta_n)} \varepsilon_n^2 N^{k_{n+1}-k_n}\right\}.$$

Since we assumed that $p := p_n^G(\beta_n) \geq \frac{1}{2}$, then $p^2 \geq \frac{1}{2}(1 - p)$, so

$$1 - p_n^B(\beta_n, \varepsilon_n) \leq \exp\left\{-\frac{\kappa}{2} \varepsilon_n^2 N^{k_{n+1}-k_n}\right\}. \tag{5.27}$$

Using $\varepsilon_n = n^{-(1+\theta)}$ from (5.27) and (2.4),

$$1 - p_n^B\left(\beta_n, \frac{1}{n^{1+\theta}}\right) \lesssim \exp\left\{-\frac{\kappa}{2}\frac{1}{n^{2(1+\theta)}}N^{K\log n}\right\} = \exp\left\{-\frac{\kappa}{2}n^{(K\log N - 2(1+\theta))}\right\}. \quad (5.28)$$

We can then conclude from (5.28) and (5.24) that

$$\sum_n \left(1 - p_n^B\left(\beta_n, \frac{1}{n^{1+\theta}}\right)\right) < \infty,$$

if $p_n^G(\beta_n) \geq \frac{1}{2}$ for all $n \geq n_0$, so (5.25) will be verified.

Step 2. We prove the result for a fixed b satisfying

$$\frac{2}{\log N} < K < b < 2K - \frac{1}{\log N}, \quad (5.29)$$

and then recalling Remark 2.2 observe that a simple coupling argument shows that the result remains true for all larger values of b.

Assume that $\beta_n \geq \frac{1}{5}$ for $n \geq n_0$. This assumption will be verified in Step 3. Define another constant

$$K_1 = 2K - b, \quad (5.30)$$

so that

$$\frac{1}{\log N} < K_1 < K. \quad (5.31)$$

Recalling (5.10), (5.14), (5.15), and conditioning on the event $\{\mathcal{N}_n \geq N^{K_1 \log n}\}$, we have

$$p_n^A(\beta_n) \geq P\left(G\left(\mathcal{N}_n, r_n(\beta_n)\right) \text{ is connected}|\, \mathcal{N}_n \geq N^{K_1 \log n}\right) P(\mathcal{N}_n \geq N^{K_1 \log n}),$$

and therefore

$$1 - p_n^A(\beta_n) \leq P\left(G\left(\mathcal{N}_n, r_n(\beta_n)\right) \text{ is not connected}|\, \mathcal{N}_n \geq N^{K_1 \log n}\right) \quad (5.32)$$
$$+ 1 - P(\mathcal{N}_n \geq N^{K_1 \log n}).$$

Since $p_n^G(\beta_n) \geq \frac{1}{2}$ for large n, $\varepsilon_n \to 0$ and $K_1 < K$ by (5.31), then, for large n,

$$N^{K_1 \log n} < (1 - \varepsilon_n)p_n^G(\beta_n)N^{K\log n},$$

so, by Step 1,

$$\sum_n (1 - (P(\mathcal{N}_n \geq N^{K_1 \log n})) < \infty. \quad (5.33)$$

Using $N^{K_1 \log n} \leq \mathcal{N}_n \lesssim N^{K\log n}$, $\beta_n \geq \frac{1}{5}$, taking $a > 25K\log N$, and applying the inequality in the Appendix we obtain

$$P\left(G\left(\mathcal{N}_n, r_n(\beta_n)\right) \text{ is not connected}|\, \mathcal{N}_n \geq N^{K_1 \log n}\right)$$
$$\leq M[(\log n)^{13}n^{K_1\log N \cdot (1 - a/25K\log N)} + n^{-K_1\log N} + e^{-L(\log n)^3 n^{2K_1 \log N}}],$$

where M and L are positive constants (different from those in the Appendix). Since $K_1 \log N > 1$, the sum of the second terms converges, and the sum of the third term also converges. The sum of the first terms converges if

$$a > 25\left(K\log N + \frac{K}{K_1}\right) =: a_*.$$

Hence for $a > a_*$, together with (5.32), (5.33) we have

$$\sum_n (1 - p_n^A(\beta_n)) < \infty$$

uniformly for $\beta_n \geq \frac{1}{5}$, which implies (5.26).

Step 3. We must show that the assumptions on β_n and $p_n^G(\beta_n)$ used in steps 1 and 2 are self-consistent, that is, we can choose n_0, β_{n_0}, and $p_{n_0}^G(\beta_{n_0})$ such that

$$p_n^G(\beta_n) \geq \frac{1}{2} \text{ for all } n \geq n_0, \tag{5.34}$$

$$\beta_n \geq \frac{1}{5} \text{ for all } n \geq n_0. \tag{5.35}$$

We proceed as follows. Given θ satisfying (5.24) and recalling (5.25), (5.26) we can choose n_0 such that the following products satisfy:

$$\prod_{n \geq n_0} \left(1 - \frac{1}{n^{1+\theta}}\right) \geq \left(\frac{4}{5}\right)^{1/3}, \tag{5.36}$$

and for any $k \in \mathbb{N}$

$$\prod_{n=n_0}^{n_0+k} p_n^B\left(\beta_n, \frac{1}{n^{1+\theta}}\right) \geq \left(\frac{4}{5}\right)^{1/3} \tag{5.37}$$

provided that $p_n^G(\beta_n) \geq \frac{1}{2}$ for $n = n_0, \ldots, n_0 + k$

$$\prod_{n=n_0}^{n_0+k} p_n^A(\beta_n) \geq \left(\frac{4}{5}\right)^{1/3} \tag{5.38}$$

provided that $\beta_n \geq \frac{1}{5}$ for $n = n_0 \ldots, n_0 + k$.

Now choose

$$\beta_{n_0} \geq \frac{1}{2} \tag{5.39}$$

and choose C in (2.5) sufficiently large so that (see (5.11))

$$p_{n_0}^G(\beta_{n_0}) \geq \frac{2}{3}. \tag{5.40}$$

Then by (5.20) and (5.36)

$$\beta_{n_0+1} = \left(1 - \frac{1}{n_0^{1+\theta}}\right) p_{n_0}^G(\beta_{n_0}) \beta_{n_0} \tag{5.41}$$

$$\geq \left(\frac{4}{5}\right)^{1/3} \cdot \frac{2}{3} \cdot \frac{1}{2} > \frac{1}{5},$$

and by (5.21), (5.37) and (5.38), we have

$$p_{n_0+1}^G(\beta_{n_0+1}) \geq p_{n_0}^B\left(\beta_{n_0}, \frac{1}{n_0^{1+\theta}}\right) p_{n_0}^A(\beta_{n_0}) \tag{5.42}$$

$$\geq \left(\frac{4}{5}\right)^{2/3} \geq \frac{1}{2}.$$

Now assume that

$$\prod_{\ell=1}^{k-1} p_{n_0+\ell}^G(\beta_{n_0+\ell}) \geq \left(\frac{4}{5}\right)^{2/3}, \tag{5.43}$$

and $\beta_{n_0+\ell} \geq \frac{1}{5}$ for $\ell = 1, \ldots, k-1$. Then by (5.21), (5.37) and (5.38) we have

$$\prod_{\ell=1}^{k} p_{n_0+\ell}^{G}(\beta_{n_0+\ell}) \tag{5.44}$$

$$\geq \prod_{\ell=0}^{k-1} p_{n_0+\ell}^{B}\left(\beta_{n_0+\ell}, \frac{1}{(n_0+\ell)^{1+\delta}}\right) p_{n_0+\ell}^{A}(\beta_{n_0+\ell})$$

$$\geq \left(\frac{4}{5}\right)^{2/3} \geq \frac{1}{2}.$$

Moreover by (5.20), (5.36), (5.40) and (5.43),

$$\beta_{n_0+k} = \beta_{n_0} \prod_{j=1}^{k} \frac{\beta_{n_0+j}}{\beta_{n_0+j-1}}$$

$$= \beta_{n_0} \prod_{j=0}^{k-1} \left(1 - \frac{1}{(n_0+j)^{1+\theta}}\right) \prod_{j=0}^{k-1} p_{n_0+j}^{G}(\beta_{n_0+j})$$

$$\geq \frac{1}{2} \cdot \left(\frac{4}{5}\right)^{1/3} \cdot \frac{2}{3} \cdot \left(\frac{4}{5}\right)^{2/3} > \frac{1}{5}.$$

Therefore by induction we have

$$\prod_{n \geq n_0+1} p_n^{G}(\beta_n) \geq \left(\frac{4}{5}\right)^{2/3} \tag{5.45}$$

and

$$\beta_n \geq \frac{1}{5} \text{ for all } n \geq n_0. \tag{5.46}$$

Step 4. For $a > a_*$, n_0 and C chosen above we then have

$$\beta_n \geq \frac{1}{5} \quad \text{and} \quad p_n^{G}(\beta_n) \geq \frac{1}{2} \quad \text{for all} \quad n \geq n_0.$$

This together with (5.36), (5.38) yields the estimate (5.23) and

$$\prod_{n \geq n_0} p_n^{G}(\beta_n) > 0, \tag{5.47}$$

which implies the assumption of Lemma 5.6. Then (5.23) implies (3.3) (for some $n_{00} \geq n_0$) and (5.46) implies (3.2). Percolation then follows from (3.2), (3.3) and Lemma 5.6. The uniqueness of the infinite cluster again follows from [26] (Theorem 1.2).

(2) The proof follows immediately from Lemma 5.2.

(c) Consider the case $0 < b \leq \frac{2}{\log N} < K$ but modifying the model by replacing the actual connection probabilities at distances $k_n + 1, \ldots, k_{n+1}$ with the lower bound $\frac{c_{k_n}}{N^{2k_{n+1}}}$. In this case the lower bound on the connection probabilities in Lemma 5.7 can be replaced by the upper bound

$$P(X_{k_n} \text{ and } X'_{k_n} \text{ are connected within the } k_{n+1}\text{-ball}) \leq \widetilde{r}_n \quad \text{as } n \to \infty,$$

where

$$\widetilde{r}_n = \frac{a \log n}{N^{(K-2/\log N)\log n}} \frac{1}{N^{K \log n}}.$$

We can then consider the E-R graph $G(\mathcal{N}_n, \widetilde{r}_n)$. Assuming that $\mathcal{N}_n$ is of order $N^{K \log n}$, in this case by E-R theory the resulting random graph has only of order $\log(N^{K \log n})$ good

k_n-balls in the largest connected component in the k_{n+1}-ball . This would imply that the limiting density of the largest connected component in the k_n-balls decreases to 0 as $n \to \infty$ so that percolation does not occur.

To make this precise consider $G(\mathcal{N}_n, \lambda/\mathcal{N}_n)$ with $0 < \lambda < 1$ and $|\mathcal{C}_n|$ the size of the largest connected component. We have

$$\alpha(\lambda) = \lambda - 1 - \log \lambda > 0,$$

and there exists n_0 such that for $n \geq n_0$, $\frac{a \log n}{N^{(K-2/\log N) \log n}} < \lambda$. Then, given $\varepsilon > 0$, for $n \geq n_0$,

$$P(|\mathcal{C}_n| \geq (1+\varepsilon)(\log \mathcal{N}_n)/\alpha(\lambda)) \leq \mathcal{N}_n^{-(1+\varepsilon)}/\lambda,$$

(see [19], page 39).

We then have that the probability that there are more than $\frac{(1+\varepsilon)}{\alpha(\lambda)} \log \mathcal{N}_n$ good k_n-balls in the largest connected component in the k_{n+1}-ball for infinitely many n is 0. Therefore there cannot be an infinite connected component (with positive density). □

5.3 Proof of Theorem 3.3

(a) Note that if $c_k = C_0 + C_1 \log k + C_2 k^\alpha$, with $\alpha > 2$, then in Theorem 3.5 we can choose b, K such that

$$\frac{2}{\log N} < K < b < \frac{\alpha}{\log N}.$$

Then

$$c_{k_n} = C_0 + C_1 \log\lfloor Kn \log n \rfloor + C_2 \lfloor K^\alpha n^\alpha (\log n)^\alpha \rfloor \geq C + a \log n \cdot n^{b \log N}$$

for sufficiently large C_0 and C_2, where C and a are as in Theorem 3.5(b). The proof follows then from Theorem 3.5(b) and Remark 2.2.

(b) If $C_2 = 0$ and $C_1 < N$, then $c_{k_n} \leq C_0 + C_1(\log K + \log n + \log \log n) \leq \widetilde{C}_0 + aN \log n$ for some $0 < a < 1$ and $\widetilde{C}_0 > C_0$. The result then follows from Theorem 3.5(a)(i).

(c) The existence of C_* follows by the argument in [26] (Theorem 1.1(b)) as follows. The expected number of edges from a given vertex is (see (2.1))

$$\sum_{k=1}^\infty (N-1)N^{k-1} p_{(k)} \leq \sum_{k=1}^\infty (N-1)N^{k-1} \frac{(C_0 + C_1 \log k + C_2 k^\alpha)}{N^{2k}}$$

which is less than 1 for sufficiently small C_0, C_1, C_2. The result follows by coupling the largest connected cluster containing a given point with a subcritical branching process (see e.g. [23], page 109). □

Appendix. Connectivity of a random graph

Consider the E-R random graph $G(n, \frac{a \log n}{n})$, $a > 0$. Using a random walk approximation for cluster growth in a susceptible-infected-removed epidemic model, Durrett [19] proves the known result that $P(G(n, \frac{a \log n}{n})$ is connected$) \to 1$ as $n \to \infty$ if $a > 1$. Putting together the parts of the proof one obtains (see p. 64) the lower bound for $a > 1$,

$$P\left(G\left(n, \frac{a \log n}{n}\right) \quad \text{is connected}\right)$$
$$\geq \quad \left[\left(1 - \frac{14(a \log n)^{13} e^{(13 a \log n)/n}}{n^a}\right)\left(1 - \frac{1}{n^{2.1}}\right)\left(1 - \frac{1}{n^2}\right)\right]^n$$
$$\cdot (1 - e^{-(\log n)^3/100})^{n(n-1)}.$$

Then using the inequalities $1 - x > e^{-2x}$, $0 < x < 0.7968$, and $1 - e^{-x} < x$, $x > 0$ it follows that for $a > 1$,

$$P\left(G\left(n, \frac{a \log n}{n}\right) \text{ is not connected}\right)$$
$$\leq M[(\log n)^{13} n^{1-a} + n^{-1} + \exp(-L(\log n)^3 n^2)],$$

where M and L are positive constants.

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