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Branching random walks in time inhomogeneous environments*

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Abstract

We study the maximal displacement of branching random walks in a class of time inhomogeneous environments. Specifically, binary branching random walks with Gaussian increments will be considered, where the variances of the increments change over time macroscopically. We find the asymptotics of the maximum up to an $O_P(1)$ (stochastically bounded) error, and focus on the following phenomena: the profile of the variance matters, both to the leading (velocity) term and to the logarithmic correction term, and the latter exhibits a phase transition.

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1 Introduction

One dimensional branching random walks and their maxima have been studied mostly in space-time homogeneous environments (deterministic or random). For work on the deterministic homogeneous case of relevance to our study we refer to [6] and the recent [1], [2] and [29]. For the random environment case, a sample of relevant papers is [15, 17, 21, 22, 25, 26, 27]. As is well documented in these references, under reasonable hypotheses, in the homogeneous case the maximum grows linearly, with a logarithmic correction, and is tight around its median.

Branching random walks are also studied under some space inhomogeneous environments. A sample of those papers are [4, 10, 12, 16, 18, 20, 23].

Recently, Bramson and Zeitouni [8] and Fang [13] showed that the maxima of branching random walks, recentered around their median, are still tight in time inhomogeneous environments satisfying certain uniform regularity assumptions, in particular, the laws of the increments can vary with respect to time and the walks may have some local dependence. A natural question is to ask, in that situation, what is the asymptotic

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behavior of the maxima. Similar questions were discussed in the context of branching Brownian motion using PDE techniques, see e.g. Nolen and Ryzhik [28], using the fact that the distributions of the maxima satisfy the KPP equation whose solution exhibits a traveling wave phenomenon.

In all these models, while the linear traveling speed of the maxima is a relatively easy consequence of the large deviation principle, the evaluation of the second order correction term, like the ones in Bramson [6] and Addario-Berry and Reed [1], is more involved and requires a detailed analysis of the walks; to our knowledge, it has so far only been performed in the time homogeneous case.

Our goal is to start exploring the time inhomogeneous setup. As we will detail below, the situation, even in the simplest setting, is complex and, for example, the order in which inhomogeneity presents itself matters, both in the leading term and in the correction term. In order to best describe this phenomenon without the burden of inessential technical details, we focus on the simplest case of binary Gaussian branching random walks where the diffusivity of the particles takes two distinct values as a function of time.

We now describe the setup in detail. For $\sigma > 0$, let $N(0,\sigma^2)$ denote the normal distributions with mean zero and variance σ^2. Let n be an integer, and let $\sigma_1^2, \sigma_2^2 > 0$ be given. We start the system with one particle at location 0 at time 0. Suppose that v is a particle at location S_v at time k. Then v dies at time $k+1$ and gives birth to two particles $v1$ and $v2$, and each of the two offspring ($\{vi, i=1,2\}$) moves independently to a new location S_{vi} with the increment $S_{vi} - S_v$ independent of S_v and distributed as $N(0,\sigma_1^2)$ if $k < n/2$ and as $N(0,\sigma_2^2)$ if $n/2 \le k < n$. Let $\mathbb{D}_n$ denote the collection of all particles at time n. For a particle $v \in \mathbb{D}_n$ and $i < n$, we let v^i denote the ith level ancestor of v, that is the unique element of $\mathbb{D}_i$ on the geodesic connecting v and the root. We study the maximal displacement $M_n = \max_{v\in\mathbb{D}_n} S_v$ at time n, for n large.[1]

The analysis we present should extend in a straightforward manner to a wide class of walks with non-Gaussian increments and to more general branching mechanisms. Concerning the former, some of the Gaussian computations need to be replaced by fine asymptotics in the large deviation regime; these require assumptions on the increments (examples where the correction term is not logarithmic are known even in the homogeneous bounded case, see [7]) and a fair amount of technical work, especially in arguments involving conditioning. Concerning other branching mechanisms, the analysis in the k-ary and Galton-Watson setups proceeds as in the binary case. More complicated is the situation where either spatially inhomogeneous branching mechanisms or increment distributions are present, see e.g. [5]; estimating the correction term in the latter setup is challenging and outside the scope of our methods.

In order to describe the results in a concise way, we recall the notation $O_P(1)$ for stochastic boundedness. That is, a sequence of random variables $\{R_n\}_n$ is said to satisfy $R_n = O_P(1)$ if it is tight, i.e. if for any $\epsilon > 0$ there exists an $M = M(\epsilon)$ such that $P(|R_n| > M) < \epsilon$ for all n.

An interesting feature of M_n is that the asymptotic behavior depends on the order relation between σ_1^2 and σ_2^2. That is, while

$$M_n = \left(\sqrt{2\log 2}\,\sigma_{\text{eff}}\right) n - \beta \frac{\sigma_{\text{eff}}}{\sqrt{2\log 2}} \log n + O_P(1) \tag{1.1}$$

is true for some choice of σ_{eff} and β, σ_{eff} and β take different expressions for different orderings of σ_1 and σ_2. Note that (1.1) is equivalent to saying that the sequence $\{M_n -$

[1] Since one can understand a branching random walk as a 'competition' between branching and random walk, one may get similar results by fixing the variance and changing the branching rate with respect to time.

$\text{Med}(M_n)\}_n$ is tight and

$$\text{Med}(M_n) = \left(\sqrt{2\log 2}\,\sigma_{\text{eff}}\right) n - \beta \frac{\sigma_{\text{eff}}}{\sqrt{2\log 2}} \log n + O(1),$$

where $\text{Med}(X) = \sup\{x : P(X \le x) \le \frac{1}{2}\}$ is the median of the random variable X. In the following, we will use superscripts to distinguish different cases, see (1.2), (1.3) and (1.4) below.

A special and well-known case is when $\sigma_1 = \sigma_2 = \sigma$, i.e., all the increments are i.i.d.. In that case, the maximal displacement is described as follows:

$$M_n^{=} = \left(\sqrt{2\log 2}\,\sigma\right) n - \frac{3}{2} \frac{\sigma}{\sqrt{2\log 2}} \log n + O_P(1); \tag{1.2}$$

the proof can be found in [1], and its analog for branching Brownian motion can be found in [6] using probabilistic techniques (see also [29] for a modern streamlined proof) and [24] using PDE techniques. This homogeneous case corresponds to (1.1) with $\sigma_{\text{eff}} = \sigma_{\text{eff}}^{=} := \sigma$ and $\beta = \beta^{=} := \frac{3}{2}$. In this paper, we deal with the extension to the inhomogeneous case. The main results are the following two theorems.

Theorem 1.1. *When $\sigma_1^2 < \sigma_2^2$ (increasing variances), the maximal displacement is*

$$M_n^{\uparrow} = \left(\sqrt{(\sigma_1^2 + \sigma_2^2)\log 2}\right) n - \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{4\sqrt{\log 2}} \log n + O_P(1), \tag{1.3}$$

which is of the form (1.1) *with* $\sigma_{\text{eff}} = \sigma_{\text{eff}}^{\uparrow} := \sqrt{\frac{\sigma_1^2+\sigma_2^2}{2}}$ *and* $\beta = \beta^{\uparrow} := \frac{1}{2}$.

Theorem 1.2. *When $\sigma_1^2 > \sigma_2^2$ (decreasing variances), the maximal displacement is*

$$M_n^{\downarrow} = \frac{\sqrt{2\log 2}(\sigma_1 + \sigma_2)}{2} n - \frac{3(\sigma_1 + \sigma_2)}{2\sqrt{2\log 2}} \log n + O_P(1), \tag{1.4}$$

which is of the form (1.1) *with* $\sigma_{\text{eff}} = \sigma_{\text{eff}}^{\downarrow} := \frac{\sigma_1+\sigma_2}{2}$ *and* $\beta = \beta^{\downarrow} := 3$.

For comparison purpose, it is useful to introduce the model of 2^n independent (inhomogeneous) random walks with centered independent Gaussian variables, with variance profile as above. Denote by M_n^{ind} the maximal displacement at time n in this model. Because of the complete independence, it can be easily shown that

$$M_n^{\text{ind}} = \left(\sqrt{(\sigma_1^2 + \sigma_2^2)\log 2}\right) n - \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{4\sqrt{\log 2}} \log n + O_P(1) \tag{1.5}$$

for all choices of σ_1^2 and σ_2^2. Thus, in this case, $\sigma_{\text{eff}} = \sigma_{\text{eff}}^{\text{ind}} := \sqrt{(\sigma_1^2 + \sigma_2^2)/2}$ and $\beta = \beta^{\text{ind}} := 1/2$. Thus, the difference between $M_n^{=}$ and M_n^{ind} when $\sigma_1^2 = \sigma_2^2$ lies in the logarithmic correction. As commented (for branching Brownian motion) in [6], the different correction is due to the intrinsic dependence between particles coming from the branching structure in branching random walks.

Another related quantity is the sub-maximum obtained by a greedy algorithm, which only considers the maximum over all descendants of the maximal particle at time $n/2$. Applying (1.2), we find that the output of such algorithm is

$$\begin{aligned} &\left(\sqrt{2\log 2}\sigma_1 \frac{n}{2} - \frac{3}{2}\frac{\sigma_1}{\sqrt{2\log 2}} \log \frac{n}{2}\right) + \left(\sqrt{2\log 2}\sigma_2 \frac{n}{2} - \frac{3}{2}\frac{\sigma_2}{\sqrt{2\log 2}} \log \frac{n}{2}\right) + O_P(1) \\ &= \frac{\sqrt{2\log 2}(\sigma_1 + \sigma_2)}{2} n - \frac{3(\sigma_1 + \sigma_2)}{2\sqrt{2\log 2}} \log n + O_P(1). \end{aligned} \tag{1.6}$$

Comparing (1.6) with the theorems, we see that the greedy algorithm has $\sigma_{\text{eff}} = \sigma_{\text{eff}}^{\text{gr}} = \sigma_{\text{eff}}^{\downarrow}$ and $\beta = \beta^{\text{gr}} = \beta^{\downarrow}$. That is, in the case of decreasing variances, the greedy algorithm yields the maximum up to an $O_P(1)$ error; this is not the case when variances are either constant or increasing (compare with (1.2) and (1.3)).

A few remarks are now in order.

1. When the variances are increasing, $M_n^{\uparrow}$ is asymptotically (up to $O_P(1)$ error) the same as M_n^{ind}, which is exactly the same as the maximum of independent homogeneous random walks with effective variance $\frac{\sigma_1^2+\sigma_2^2}{2}$.
2. When the variances are decreasing, $M_n^{\downarrow}$ shares the same asymptotic behavior with the sub-maximum (1.6). In this case, a greedy strategy yields the approximate maximum.
3. With the same set of diffusivity constants $\{\sigma_1^2, \sigma_2^2\}$ but different order, $M_n^{\uparrow}$ is greater than $M_n^{\downarrow}$.
4. While the leading order terms in (1.2), (1.3) and (1.4) are continuous in σ_1 and σ_2 (they coincide upon setting $\sigma_1 = \sigma_2$), the logarithmic corrections exhibit a phase transition phenomenon (they are not the same when we let $\sigma_1 = \sigma_2$).

We will prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. Before proving the theorems, we state a tightness result.

Lemma 1.3. *The sequences $\{M_n^{\uparrow} - \mathrm{Med}(M_n^{\uparrow})\}_n$ and $\{M_n^{\downarrow} - \mathrm{Med}(M_n^{\downarrow})\}_n$ are tight.*

This lemma follows from either [8] or [13]. We sketch the proof at the end of the paper.

A remark on notation: throughout, we use C to denote a generic positive constant, possibly depending on σ_1 and σ_2, that may change from line to line.

2 Increasing Variances: $\sigma_1^2 < \sigma_2^2$

In this section, we prove Theorem 1.1. We begin in Subsection 2.1 with a result on the fluctuation of an inhomogeneous random walk. In the short Subsection 2.2 we provide large-deviations based heuristics for our results. While these are not used in the actual proof, these heuristics explain the leading term of the maximal displacement and hint at the derivation of the logarithmic correction term. The actual proof of Theorem 1.1 is provided in subsection 2.3.

2.1 Fluctuation of an Inhomogeneous Random Walk

For each $n \in \mathbb{N}$, let

$$S_n(k) = \begin{cases} \sum_{i=1}^{k} X_i, & k \le n/2, \\ \sum_{i=1}^{n/2} X_i + \sum_{i=n/2+1}^{k} Y_i, & n/2 < k \le n. \end{cases} \tag{2.1}$$

define an inhomogeneous random walk path up to time n, where $X_i \sim N(0, \sigma_1^2)$, $Y_i \sim N(0, \sigma_2^2)$, and X_i and Y_i are independent. We use the shorthand notation $S_n = S_n(n)$ for the endpoint of such an inhomogeneous random walk. Define

$$s_{k,n}(x) = \begin{cases} \dfrac{\sigma_1^2 k}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} x, & 0 \le k \le \dfrac{n}{2}, \\ \dfrac{\sigma_1^2 \frac{n}{2} + \sigma_2^2 (k - \frac{n}{2})}{(\sigma_1^2 + \sigma_2^2)\frac{n}{2}} x, & \dfrac{n}{2} \le k \le n, \end{cases} \tag{2.2}$$

and

$$f_{k,n} = \begin{cases} c_f k^{2/3}, & k \le n/2, \\ c_f (n-k)^{2/3}, & n/2 < k \le n, \end{cases} \tag{2.3}$$

where c_f is some large constant (chosen in Lemma 2.1 below). The following lemma states that conditioned on $\{S_n = x\}$, the path of the walk S_n follows $s_{k,n}(x)$ with fluctuations bounded by $f_{k,n}$ at level $k \le n$.

Lemma 2.1. *There exist constants $C > 0$ and $c_f > 0$ (independent of n) such that*

$$P(S_n(k) \in [s_{k,n}(S_n) - f_{k,n}, s_{k,n}(S_n) + f_{k,n}] \text{ for all } 0 \le k \le n | S_n) \ge C \text{ a.s..}$$

Proof. Let $\tilde{S}_{k,n} = S_n(k) - s_{k,n}(S_n)$. Then, similar to the continuous time setup of Brownian bridges, one can check that $\tilde{S}_{k,n}$ are independent of S_n. To see this, first note that the covariance between $\tilde{S}_{k,n}$ and S_n is

$$Cov(\tilde{S}_{k,n}, S_n) = E\tilde{S}_{k,n}S_n - E\tilde{S}_{k,n}ES_n = E\tilde{S}_{k,n}S_n,$$

since $ES_n = 0$ and $E\tilde{S}_{k,n} = 0$. Now, for $k \le n/2$,

$$\tilde{S}_{k,n} = \left(1 - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right)\sum_{i=1}^{k} X_i - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\sum_{i=k+1}^{n/2} X_i - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\sum_{i=n/2+1}^{n} Y_i.$$

Expand $\tilde{S}_{k,n}S_n$, take expectation, and then all terms vanish except for those containing X_i^2 and Y_i^2. Taking into account that $EX_i^2 = \sigma_1^2$ and $EY_i^2 = \sigma_2^2$, one has

$$\begin{aligned} & Cov(\tilde{S}_{k,n}, S_n) = E\tilde{S}_{k,n}S_n \\ = \; & \left(1 - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right)\sum_{i=1}^{k} EX_i^2 - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\sum_{i=k+1}^{n/2} EX_i^2 - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\sum_{i=n/2+1}^{n} EY_i^2 \\ = \; & \left(1 - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right)k\sigma_1^2 - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}(n/2-k)\sigma_1^2 - \frac{\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}(n/2)\sigma_2^2 \\ = \; & 0. \end{aligned} \tag{2.4}$$

For $n/2 < k \le n$, one can calculate $Cov(\tilde{S}_{k,n}, S_n) = 0$ similarly as follows. First,

$$\tilde{S}_{k,n} = \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\sum_{i=1}^{n/2} X_i + \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\sum_{i=n/2+1}^{k} Y_i - \left(1 - \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right)\sum_{i=k+1}^{n} Y_i.$$

Then, expanding $\tilde{S}_{k,n}S_n$ and taking expectation, one has

$$\begin{aligned} & Cov(\tilde{S}_{k,n}, S_n) = E\tilde{S}_{k,n}S_n \\ = \; & \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\sum_{i=1}^{n/2} EX_i^2 + \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\sum_{i=n/2+1}^{k} EY_i^2 - \left(1 - \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right)\sum_{i=k+1}^{n} EY_i^2 \\ = \; & \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}(n/2)\sigma_1^2 + \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}(k-n/2)\sigma_2^2 - \left(1 - \frac{\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)\frac{n}{2}}\right)(n-k)\sigma_2^2 \\ = \; & 0 \end{aligned}$$

Therefore, $\tilde{S}_{k,n}$ are independent of S_n since they are Gaussian. Using this independence,

$$\begin{aligned} & P(S_n(k) \in [s_{k,n}(S_n) - f_{k,n}, s_{k,n}(S_n) + f_{k,n}] \text{ for all } 0 \le k \le n | S_n) \\ = \; & P(\tilde{S}_{k,n} \in [-f_{k,n}, f_{k,n}] \text{ for all } 0 \le k \le n | S_n) \\ = \; & P(\tilde{S}_{k,n} \in [-f_{k,n}, f_{k,n}] \text{ for all } 0 \le k \le n). \end{aligned}$$

By a calculation similar to (2.4), $\tilde{S}_{k,n}$ is a Gaussian sequence with mean zero and variance $k\sigma_1^2\frac{((\sigma_1^2+\sigma_2^2)n-2\sigma_1^2 k)}{(\sigma_1^2+\sigma_2^2)n}$ for $k \le n/2$ and $(n-k)\sigma_2^2\frac{((\sigma_1^2+\sigma_2)n-2\sigma_2^2(n-k))}{(\sigma_1^2+\sigma_2^2)n}$ for $n/2 < k \le n$. The above quantity is

$$1 - P(|\tilde{S}_{k,n}| > f_{k,n}, \text{ for some } 0 \le k \le n) \ge 1 - \sum_{k=1}^{n} P(|\tilde{S}_{k,n}| > f_{k,n}).$$

Using a standard Gaussian estimate, e.g. [11, Theorem 1.4], the above quantity is at least,

$$1 - \sum_{k=1}^{n} \frac{c_0}{\sqrt{k}} e^{-\frac{f_{k,n}^2}{k} c_1} \ge 1 - 2\sum_{k=1}^{\infty} \frac{c_0}{\sqrt{k}} e^{-c_f^2 c_1 k^{1/3}} := C > 0$$

where c_0, c_1 are constants depending on σ_1 and σ_2, and $C > 0$ can be realized by choosing the constant c_f large enough. This proves the lemma. □

2.2 Sample Path Large Deviation Heuristics

We explain (without giving a proof) what we expect for the order n term of $M_n^{\uparrow}$, by means of large deviation heuristics. Note that these heuristics apply also to the non-Gaussian setup. The actual proof of Theorem 1.1 is postponed to the next subsection.

Consider the inhomogeneous random walk $S_n(k)$ as defined in (2.1) and a function $\phi(t)$ defined on $[0,1]$ with $\phi(0) = 0$. Let $s \in [0,1]$. A sample path large deviation result, see [9, Theorem 5.1.2], tells us that the probability for $S_{\lfloor rn \rfloor}$ to be roughly $\phi(r)n$ for all $r \in [0,s]$ is roughly $\exp\{-nI_s(\phi)\}$, where

$$I_s(\phi) = \int_0^s \Lambda_r^*(\dot{\phi}(r))dr, \tag{2.5}$$

$\dot{\phi}(r) = \frac{d}{dr}\phi(r)$, and

$$\Lambda_r^*(x) = \begin{cases} \dfrac{x^2}{2\sigma_1^2}, & 0 \le r \le 1/2, \\ \dfrac{x^2}{2\sigma_2^2}, & 1/2 < r \le 1. \end{cases}$$

A first moment argument would yield a necessary condition for a particle that roughly follows the path $\phi(r)n$ to exist in the branching random walks,

$$I_s(\phi) \le s \log 2, \quad \text{for all } \ 0 \le s \le 1. \tag{2.6}$$

This is equivalent to

$$\begin{cases} \displaystyle\int_0^s \frac{\dot{\phi}^2(r)}{2\sigma_1^2} dr \le s \log 2, & 0 \le s \le \dfrac{1}{2}, \\ \displaystyle\int_0^{\frac{1}{2}} \frac{\dot{\phi}^2(r)}{2\sigma_1^2} dr + \int_{\frac{1}{2}}^s \frac{\dot{\phi}^2(r)}{2\sigma_2^2} dr \le s \log 2, & \dfrac{1}{2} \le s \le 1. \end{cases} \tag{2.7}$$

Otherwise, if (2.6) is violated for some s_0, i.e., $I_{s_0}(\phi) > s_0 \log 2$, there will be no path seen in the n limit following $\phi(r)n$ to $\phi(s_0)n$, since the expected number of such paths is $2^{sn}e^{-nI_s(\phi)} = e^{-(I_s(\phi) - s\log 2)n}$, which decreases exponentially.

Our goal is then to maximize $\phi(1)$ under the constraints (2.7). By Jensen's inequality and convexity, one sees that this problem is equivalent to maximizing $\phi(1)$ subject to

$$\frac{\phi^2(1/2)}{\sigma_1^2} \le \frac{1}{2}\log 2, \ \frac{\phi^2(1/2)}{\sigma_1^2} + \frac{(\phi(1) - \phi(1/2))^2}{\sigma_2^2} \le \log 2. \tag{2.8}$$

Note that the above argument does not necessarily require $\sigma_1^2 < \sigma_2^2$.

Under the assumption that $\sigma_1^2 < \sigma_2^2$, the solution to the optimization problem is the optimal curve

$$\phi(s) = \begin{cases} \frac{2\sigma_1^2\sqrt{\log 2}}{\sqrt{(\sigma_1^2+\sigma_2^2)}}s, & 0 \le s \le \frac{1}{2}, \\ \frac{2\sigma_1^2\sqrt{\log 2}}{\sqrt{(\sigma_1^2+\sigma_2^2)}}\frac{1}{2} + \frac{2\sigma_2^2\sqrt{\log 2}}{\sqrt{(\sigma_1^2+\sigma_2^2)}}(s-\frac{1}{2}), & \frac{1}{2} \le s \le 1. \end{cases} \tag{2.9}$$

If we plot this optimal curve and the suboptimal curve leading to (1.6) as in Figure 1, it is easy to see that the ancestor at time $n/2$ of the actual maximum at time n is not a maximum at time $n/2$, since $\frac{2\sigma_1^2\sqrt{\log 2}}{\sqrt{(\sigma_1^2+\sigma_2^2)}} < \sqrt{2\sigma_1^2\log 2}$. A further rigorous calculation as in the next subsection shows that, along the optimal curve (2.9), the branching random walks have an exponential decay of correlation. Thus a fluctuation between $n^{1/2}$ and n that is larger than the typical fluctuation of a random walk is admissible. This is consistent with the naive observation from Figure 1. This kind of behavior also occurs in the independent random walks model, explaining why $M_n^{\uparrow}$ and M_n^{ind} have the same asymptotic expansion up to an $O_P(1)$ error, see (1.3) and (1.5).

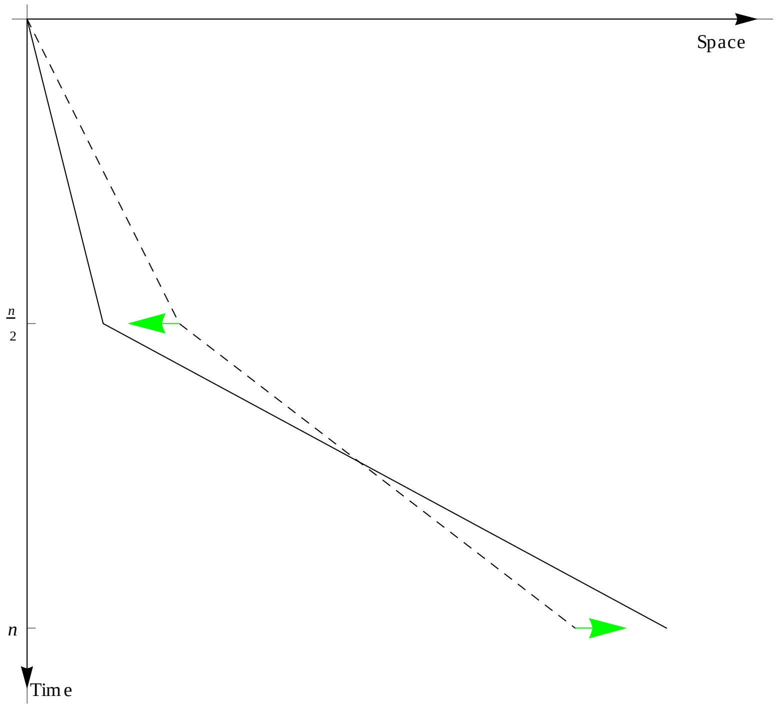


Figure 1: $\sigma_1^2 < \sigma_2^2$. Dashed: path leading to maximum at time n of BRW starting from maximum at time $n/2$ (the greedy algorithm). Solid: path leading to maximum at time n of BRW starting from time 0. Arrows show the displacement of the optimal path from the greedy algorithm.

2.3 Proof of Theorem 1.1

With Lemma 2.1 and the observation from Section 2.2, we can now provide a proof of Theorem 1.1, applying the standard first and second moment methods (see e.g. [3]) to the appropriate sets. In our setup, this essentially coincides with using the so called many-to-one and many-to-two lemmas, see [19, 29], and goes back to Bramson's original work [6]. Recall $S_n(k)$ and S_n as defined in (2.1).

Proof of Theorem 1.1. *Upper bound.* Let $a_n = \left(\sqrt{(\sigma_1^2+\sigma_2^2)\log 2}\right) n - \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{4\sqrt{\log 2}}\log n$. Let $N_{1,n} = \sum_{v\in\mathbb{D}_n} 1_{\{S_v > a_n + y\}}$ be the number of particles in D_n whose displacements are greater than $a_n + y$. Then

$$EN_{1,n} = 2^n P(S_n \geq a_n + y) \leq c_2 e^{-c_3 y}$$

where c_2 and c_3 are constants independent of n and the last inequality is due to the fact that $S_n \sim N(0, \frac{\sigma_1^2+\sigma_2^2}{2} n)$. So we have, by Chebyshev's inequality,

$$P(M_n^{\uparrow} > a_n + y) = P(N_1 \geq 1) \leq EN_{1,n} \leq c_2 e^{-c_3 y}. \tag{2.10}$$

Therefore, this probability can be made as small as we wish by choosing a large y.

Lower bound. Consider the walks which are at $s_n \in I_n = [a_n, a_n+1]$ at time n and follow $s_{k,n}(s_n)$, defined by (2.2), at intermediate times with fluctuation bounded by $f_{k,n}$, defined by (2.3). Let $I_{k,n}(x) = [s_{k,n}(x) - f_{k,n}, s_{k,n}(x) + f_{k,n}]$ be the 'admissible' interval at time k given $S_n = x$, and let

$$N_{2,n} = \sum_{v\in\mathbb{D}_n} 1_{\{S_v \in I_n, S_{v^k} \in I_{k,n}(S_v) \text{ for all } 0 \leq k \leq n\}}$$

be the number of such walks. By Lemma 2.1,

$$\begin{aligned} EN_{2,n} &= 2^n P(S_n \in I_n, S_n(k) \in I_{k,n}(S_n) \text{ for all } 0 \leq k \leq n) \\ &= 2^n E(1_{\{S_n \in I_n\}} P(S_n(k) \in I_{k,n}(S_n) \text{ for all } 0 \leq k \leq n | S_n)) \\ &\geq 2^n C P(S_n \in I_n) \geq c_4. \end{aligned} \tag{2.11}$$

Next, we bound the second moment $EN_{2,n}^2$. By considering the location of any pair $v_1, v_2 \in \mathbb{D}_n$ of particles at time n and at their common ancestor $v_1 \wedge v_2$, we have

$$\begin{aligned} &EN_{2,n}^2 = E\sum_{v_1,v_2\in\mathbb{D}_n} 1_{\{S_{v_i}\in I_n,\ S_{(v_i)^j}\in I_{j,n}(S_{(v_i)^j}) \text{ for all } 0\leq j\leq n, i=1,2\}} \\ = &\sum_{k=0}^{n} \sum_{\substack{v_1,v_2\in\mathbb{D}_n \\ v_1\wedge v_2\in\mathbb{D}_k}} E1_{\{S_{v_i}\in I_n,\ S_{(v_i)^j}\in I_{j,n}(S_{(v_i)^j}) \text{ for all } 0\leq j\leq n, i=1,2\}} \\ \leq &\sum_{k=0}^{n} \sum_{\substack{v_1,v_2\in\mathbb{D}_n \\ v_1\wedge v_2\in\mathbb{D}_k}} P(S_{v_1}\in I_n,\ S_{(v_1)^j}\in I_{j,n}(S_{(v_1)^j}) \text{ for all } 0\leq j\leq n) \\ &\qquad \cdot P(S_{v_2} - S_{v_1\wedge v_2} \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x\in I_n), \end{aligned}$$

where we use the independence between $S_{v_2} - S_{v_1\wedge v_2}$ and $S_{(v_1)^j}$ in the last inequality. The last expression (double sum) in the above display equals

$$\begin{aligned} &\sum_{k=0}^{n} 2^{2n-k} P(S_n \in I_n, S_n(j) \in I_{j,n}(S_n) \text{ for all } 0\leq j\leq n) \\ &\qquad \cdot P(S_n - S_n(k) \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x\in I_n) \\ \leq\ & EN_{2,n}\sum_{k=0}^{n} 2^{n-k} P(S_n - S_n(k) \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x\in I_n). \end{aligned}$$

The above probabilities can be estimated separately when $k \le n/2$ and $n/2 < k \le n$. For $k \le n/2$, $S_n - S_n(k) \sim N(0, \frac{n}{2}(\sigma_1^2 + \sigma_2^2) - k\sigma_1^2)$. Thus,

$$\begin{aligned} & P(S_n - S_n(k) \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x \in I_n) \\ \le\ & 2f_{k,n} \frac{1}{\sqrt{\pi((\sigma_1^2+\sigma_2^2)n - 2k\sigma_1^2)}} \exp\left(-\frac{\left((1 - \frac{2\sigma_1^2 k}{(\sigma_1^2+\sigma_2^2)n})a_n - f_{k,n}\right)^2}{(\sigma_1^2+\sigma_2^2)n - 2k\sigma_1^2} \right) \\ \le\ & 2^{-n + \frac{2\sigma_1^2}{\sigma_1^2+\sigma_2^2}k + o(k)}. \end{aligned}$$

For $n/2 < k \le n$, $S_n - S_n(k) \sim N(0, (n-k)\sigma_2^2)$. Thus,

$$\begin{aligned} & P(S_n - S_n(k) \in [x - s_{k,n}(x) - f_{k,n}, x - s_{k,n}(x) + f_{k,n}], x \in I_n) \\ \le\ & 2f_{k,n} \frac{1}{\sqrt{2\pi(n-k)\sigma_2^2}} \exp\left(-\frac{\left(\frac{2\sigma_2^2(n-k)}{(\sigma_1^2+\sigma_2^2)n}a_n - f_{k,n}\right)^2}{2(n-k)\sigma_2^2} \right) \\ \le\ & 2^{-\frac{2\sigma_2^2}{\sigma_1^2+\sigma_2^2}(n-k) + o(n-k)}. \end{aligned}$$

Therefore,

$$EN_{2,n}^2 \le EN_{2,n} \left(\sum_{k=0}^{n/2} 2^{\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2}k + o(k)} + \sum_{k=n/2+1}^{n} 2^{\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2}(n-k) + o(n-k)} \right) \le c_5 EN_{2,n}, \quad (2.12)$$

where $c_5 = 2\sum_{k=0}^{\infty} 2^{\frac{\sigma_1^2-\sigma_2^2}{\sigma_1^2+\sigma_2^2}k + o(k)}$. By the Cauchy-Schwarz inequality,

$$P(M_n^{\uparrow} \ge a_n) \ge P(N_{2,n} > 0) \ge \frac{(EN_{2,n})^2}{EN_{2,n}^2} \ge c_4/c_5 > 0. \quad (2.13)$$

The upper bound (2.10) and lower bound (2.13) imply that there exists a large enough constant y_0 such that

$$P(M_n^{\uparrow} \in [a_n, a_n + y_0]) \ge \frac{c_4}{2c_5} > 0.$$

Lemma 1.3 tells us that the sequence $\{M_n^{\uparrow} - \mathrm{Med}(M_n^{\uparrow})\}_n$ is tight, so $M_n^{\uparrow} = a_n + O_P(1)$ a.s.. That completes the proof. □

3 Decreasing Variances: $\sigma_1^2 > \sigma_2^2$

We will again separate the proof of Theorem 1.2 into two parts, the lower bound and the upper bound. Fortunately, we can apply (1.2) directly to get a lower bound so that we can avoid repeating the second moment argument. However, we do need to reproduce (the first moment argument) part of the proof of (1.2) in order to get an upper bound.

3.1 An Estimate for Brownian Bridge

We need the following analog of Bramson [6, Proposition 1']. The original proof in Bramson's used the Gaussian density and reflection principle of continuous time Brownian motion, which also hold for the discrete time version. The proof extends without much effort to yield the following estimate for the Brownian bridge $B_k - \frac{k}{n}B_n$, where B_n is a random walk with standard normal increments.

Lemma 3.1. *Let*

$$L(k) = \begin{cases} 0 & \textit{if } s = 0, n, \\ 100 \log k & \textit{if } k = 1, \ldots, n/2, \\ 100 \log(n-k) & \textit{if } k = n/2, \ldots, n-1. \end{cases}$$

Then, there exists a constant C such that, for all $y > 0$,

$$P(B_k - \frac{k}{n} B_n \le L(k) + y \textit{ for } 0 \le k \le n) \le \frac{C(1+y)^2}{n}.$$

The coefficient 100 before $\log$ is chosen large enough to be suitable for later use, and is not crucial in Lemma 3.1.

3.2 Proof of Theorem 1.2

Before proving the theorem, we discuss the equivalent optimization problems (2.7) and (2.8) under our current setting $\sigma_1^2 > \sigma_2^2$. It can be solved by employing the optimal curve

$$\phi(s) = \begin{cases} \sqrt{2\log 2}\sigma_1 s, & 0 \le s \le \frac{1}{2}, \\ \sqrt{2\log 2}\sigma_1 \frac{1}{2} + \sqrt{2\log 2}\sigma_2 (s - \frac{1}{2}), & \frac{1}{2} \le s \le 1. \end{cases} \tag{3.1}$$

If we plot the curve $\phi(s)$ and the suboptimal curve leading to (1.6) as in Figure 2, these two curves coincide with each other up to order n. Figure 2 seems to indicate that the maximum at time n for the branching random walk starting from time 0 comes from the maximum at time $n/2$. As will be shown rigorously, if a particle at time $n/2$ is left significantly behind the maximum, its descendants will not be able to catch up by time n. The difference between Figure 1 and Figure 2 explains the difference in the logarithmic correction between $M_n^{\uparrow}$ and $M_n^{\downarrow}$.

Proof of Theorem 1.2. Lower bound. For each $i = 1, 2$, the formula (1.2) implies that there exist y_i (possibly negative) such that, for branching random walk at time $n/2$ with variance σ_i^2,

$$P\left(M_{n/2} > \left(\frac{\sqrt{2\log 2}\sigma_i}{2}\right) n - \frac{3\sigma_i}{2\sqrt{2\log 2}} \log\left(\frac{n}{2}\right) + y_i\right) \ge \frac{1}{2}.$$

By considering a branching random walk starting from a particle at time $n/2$, whose location is greater than $\sqrt{2\log 2}\sigma_1 n/2 - \frac{3\sigma_1}{2\sqrt{2\log 2}} \log(n/2) + y_1$, and applying the above inequality with $i = 1$ and 2, we get that

$$P\left(M_n^{\downarrow} > \left(\frac{\sqrt{2\log 2}(\sigma_1 + \sigma_2)}{2}\right) n - \frac{3(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}} \log\left(\frac{n}{2}\right) + y_1 + y_2\right) \ge \frac{1}{4}. \tag{3.2}$$

Upper bound. We will use a first moment argument to prove that there exists a constant y (large enough) such that

$$P\left(M_n^{\downarrow} > \left(\frac{\sqrt{2\log 2}(\sigma_1 + \sigma_2)}{2}\right) n - \frac{3(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}} \log\left(\frac{n}{2}\right) + y\right) < \frac{1}{10}. \tag{3.3}$$

Similarly to the last argument in the proof of Theorem 1.1, the upper bound (3.3) and the lower bound (3.2), together with the tightness result from Lemma 1.3, prove Theorem 1.2. So it remains to show (3.3).

Toward this end, we define a polygonal line (piecewise linear curve) leading to $\sqrt{2\log 2}(\sigma_1+\sigma_2)n/2 - \frac{3(\sigma_1+\sigma_2)}{2\sqrt{2\log 2}} \log\left(\frac{n}{2}\right)$ as follows: for $1 \le k \le n/2$,

$$M(k) = \frac{k}{n/2}\left(\frac{\sqrt{2\log 2}\sigma_1}{2} n - \frac{3\sigma_1}{2\sqrt{2\log 2}} \log\left(\frac{n}{2}\right)\right);$$

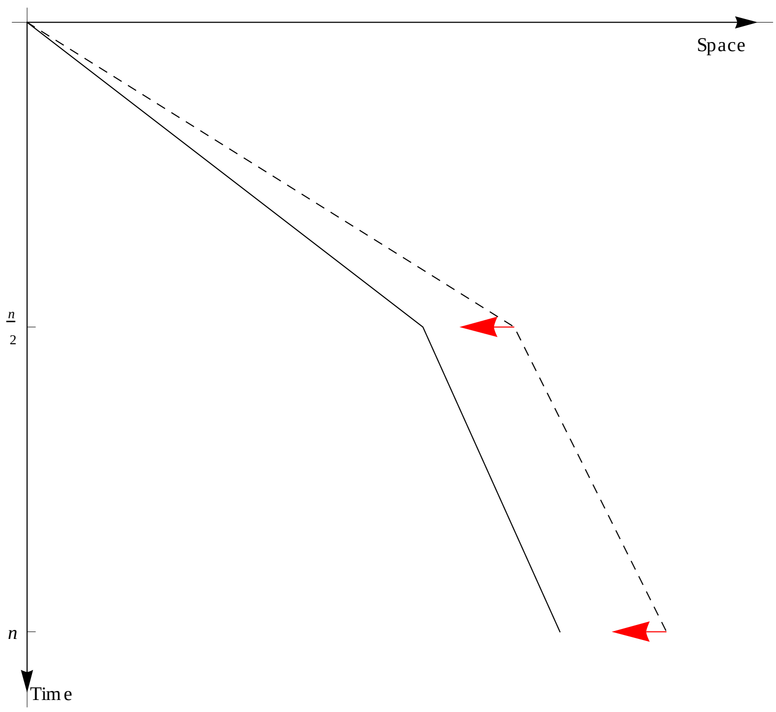


Figure 2: $\sigma_1^2 > \sigma_2^2$. Dashed: the optimal path leading to the maximum at time n which coincides with the greedy algorithm. Solid: the path to the maximal (rightmost) descendant of particles at time $n/2$ that are significantly (of order $\log n$) behind the maximum then, as marked by arrows.

and for $n/2+1 \le k \le n$,

$$M(k) = M(n/2) + \frac{k-n/2}{n/2}\left(\frac{\sqrt{2\log 2}\sigma_2}{2}n - \frac{3\sigma_2}{2\sqrt{2\log 2}}\log\left(\frac{n}{2}\right)\right).$$

Note that $\frac{k}{n}\log n \le \log k$ for $k \le n$. Also define

$$f(k) = \begin{cases} y & k = 0, \frac{n}{2}, n, \\ y + \frac{5\sigma_1}{2\sqrt{2\log 2}}\log k & 1 \le k \le n/4, \\ y + \frac{5\sigma_1}{2\sqrt{2\log 2}}\log(\frac{n}{2}-k) & \frac{n}{4} \le k \le \frac{n}{2}-1, \\ y + \frac{5\sigma_2}{2\sqrt{2\log 2}}\log(k-\frac{n}{2}) & \frac{n}{2}+1 \le k \le \frac{3n}{4}, \\ y + \frac{5\sigma_2}{2\sqrt{2\log 2}}\log(n-k) & \frac{3n}{4} \le k \le n-1. \end{cases}$$

We will use $f(k)$ to denote the allowed offset (deviation) from $M(k)$ in the following argument.

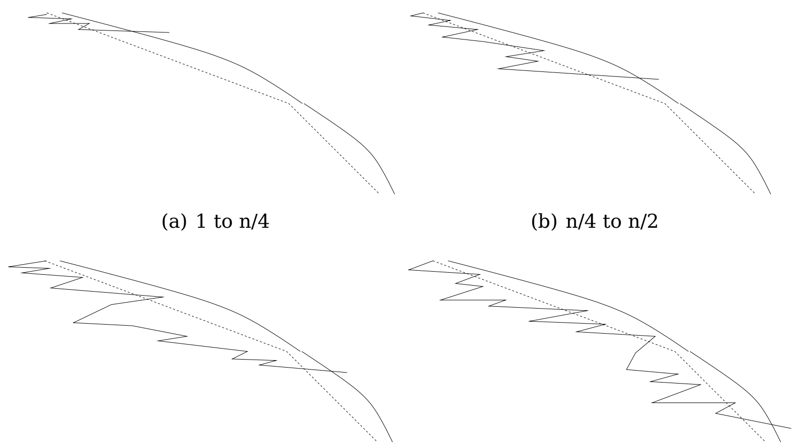
The probability on the left side of (3.3) is equal to

$$P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n)+y).$$

For each $v \in \mathbb{D}_n$, we define $\tau_v = \inf\{k : S_{v^k} > M(k)+f(k)\}$; then (3.3) is implied by

$$\sum_{k=1}^{n} P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n)+y, \tau_v = k) < 1/10. \tag{3.4}$$

We will split the sum into four regimes: $[1, n/4]$, $[n/4, n/2]$, $[n/2, 3n/4]$ and $[3n/4, n]$, corresponding to the four parts of the definition of $f(k)$. The sum over each regime, corresponding to the events in the four pictures in Figure 3, can be made small. The first two are the discrete analog of the upper bound argument in Bramson [6]. We will present a complete proof for the first two cases, since the argument is not too long and the argument (not only the result) is used in the latter two cases.



(a) 1 to n/4 (b) n/4 to n/2

(c) n/2 to 3n/4 (d) 3n/4 to n

Figure 3: Four small probability events. Dashed line: $M(k)$. Solid line: $M(k)+f(k)$. Polygonal line: a random walk.

(i). When $1 \le k \le n/4$, we have, by Chebyshev's inequality,

$$\begin{aligned} & P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n)+y, \tau_v = k) \\ \le\ & P(\exists v \in \mathbb{D}_k, \text{ such that } S_v > M(k)+f(k)) \le E\left(\sum_{v\in\mathbb{D}_k} 1_{\{S_v > M(k)+f(k)\}}\right). \end{aligned}$$

The above expectation is less than or equal to

$$\begin{aligned}\frac{C2^k}{\sqrt{k}}e^{-\frac{(M(k)+f(k))^2}{2\sigma_1^2}} &\le \frac{C2^k}{\sqrt{k}}\exp\left(-\frac{\left(\sqrt{2\log 2}\sigma_1 k+\frac{\sigma_1}{\sqrt{2\log 2}}\log k+y\right)^2}{2k\sigma_1^2}\right)\\ &\le Ck^{-3/2}e^{-\frac{\sqrt{2\log 2}}{\sigma_1}y}.\end{aligned}\tag{3.5}$$

Summing these upper bounds over $k\in[1,n/4]$, we obtain that

$$\sum_{k=1}^{n/4}P(\exists v\in\mathbb{D}_n \text{ such that } S_v>M(n)+y,\tau_v=k)\le Ce^{-\frac{\sqrt{2\log 2}}{\sigma_1}y}\sum_{k=1}^{\infty}k^{-3/2}.\tag{3.6}$$

The right side of the above inequality can be made as small as we wish, say at most $\frac{1}{100}$, by choosing y large enough.

(ii). When $n/4\le k\le n/2$, we again have, by Chebyshev's inequality,

$$\begin{aligned}&P(\exists v\in\mathbb{D}_n \text{ such that } S_v>M(n)+y,\tau_v=k)\\ \le\ &P(\exists v\in\mathbb{D}_k,\ \text{such that } S_v>M(k)+f(k),\ \text{and } S_{v^i}\le M(i)+f(i) \text{ for } 1\le i\le k)\\ \le\ &E\left(\sum_{v\in\mathbb{D}_k}1_{\{S_v>M(k)+f(k),\ \text{and } S_{v^i}\le M(i)+f(i) \text{ for } 1\le i<k\}}\right).\end{aligned}$$

Letting S_k be a copy of the random walks before time $n/2$, then the above expectation is equal to

$$\begin{aligned}&2^kP(S_k>M(k)+f(k),\ \text{and } S_i\le M(i)+f(i) \text{ for } 1\le i<k)\\ \le\ &2^kP(S_k>M(k)+f(k),\ \text{and } \frac{1}{\sigma_1}(S_i-\frac{i}{k}S_k)\le\frac{1}{\sigma_1}(f(i)-\frac{i}{k}f(k)) \text{ for } 1\le i\le k).\end{aligned}\tag{3.7}$$

$\frac{1}{\sigma_1}(S_i-\frac{i}{k}S_k)$ is a discrete Brownian bridge and is independent of S_k. Because of this independence, the above quantity is less than or equal to

$$2^kP(S_k>M(k)+f(k))\cdot P(\frac{1}{\sigma_1}(S_i-\frac{i}{k}S_k)\le\frac{1}{\sigma_1}(f(i)-\frac{i}{k}f(k)) \text{ for } 1\le i<k).$$

The first probability can be estimated similarly to (3.5),

$$\begin{aligned}&P(S_k>M(k)+f(k))\\ \le\ &\frac{C}{\sqrt{k}}\exp\left(-\frac{\left(\sqrt{2\log 2}\sigma_1 k-\frac{3\sigma_1}{2\sqrt{2\log 2}}\log k+\frac{5\sigma_1}{2\sqrt{2\log 2}}\log(\frac{n}{2}-k)+y\right)^2}{2k\sigma_1^2}\right)\\ \le\ &C2^{-k}k(\frac{n}{2}-k)^{-5/2}e^{-\frac{\sqrt{2\log 2}}{\sigma_1}y}.\end{aligned}\tag{3.8}$$

To estimate the second probability, we first estimate $\frac{1}{\sigma_1}(f(i)-\frac{i}{k}f(k))$. It is less than or equal to $\frac{1}{\sigma_1}f(i)=\frac{y}{\sigma_1}+\frac{5}{2\sqrt{2\log 2}}\log i$ for $i\le k/2<n/4$, and, for $k/2\le i<k$, it is less than or equal to

$$\begin{aligned}&\frac{5}{2\sqrt{2\log 2}}\log(n/2-i)-\frac{i}{k}\frac{5}{2\sqrt{2\log 2}}\log(n/2-k)+\frac{y}{\sigma_1}(1-\frac{i}{k})\\ =\ &\frac{5}{2\sqrt{2\log 2}}\left(\log(n/2-i)-\log(n/2-k)+\frac{k-i}{k}\log(n/2-k)\right)+\frac{y}{\sigma_1}(1-\frac{i}{k})\\ \le\ &\frac{5}{2\sqrt{2\log 2}}\left(\log(k-i)+\frac{k-i}{k}\log k\right)+\frac{y}{\sigma_1}\le 100\log(k-i)+\frac{y}{\sigma_1}.\end{aligned}$$

Therefore, applying Lemma 3.1, we have

$$
\begin{aligned}
& P\left(\frac{1}{\sigma_1}(S_i-\frac{i}{k}S_k)\le\frac{1}{\sigma_1}(f(i)-\frac{i}{k}f(k))\text{ for }1\le i\le k\right)\\
\le\ & P\left(\frac{1}{\sigma_1}(S_i-\frac{i}{k}S_k)\le 100\log i+\frac{y}{\sigma_1}\text{ for }1\le i\le k/2,\text{ and }\frac{1}{\sigma_1}(S_i-\frac{i}{k}S_k)\le\right.\\
& \left.100\log(k-i)+\frac{y}{\sigma_1}\text{ for }k/2\le i\le k\right)\le C(1+y)^2/k, \qquad (3.9)
\end{aligned}
$$

where C is independent of n, k and y.

By all the above estimates (3.7), (3.8) and (3.9),

$$\sum_{k=n/4}^{n/2}P(\exists v\in\mathbb{D}_n\text{ such that }S_v>M(n)+y,\tau_v=k)\le C(1+y)^2e^{-\frac{\sqrt{2\log 2}}{\sigma_1}y}\sum_{k=1}^{\infty}k^{-5/2}. \qquad (3.10)$$

This can again be made as small as we wish, say at most $\frac{1}{100}$, by choosing y large enough.

(iii). When $n/2\le k\le 3n/4$, we have

$$
\begin{aligned}
& P(\exists v\in\mathbb{D}_n\text{ such that }S_v>M(n)+y,\tau_v=k)\\
\le\ & P(\exists v\in\mathbb{D}_k\text{ such that }S_v>M(k)+f(k)\text{ and }S_{v^i}\le M(i)+f(i)\text{ for }1\le i\le n/2)\\
\le\ & E\left(\sum_{v\in\mathbb{D}_k}1_{\{S_v>M(k)+f(k),\text{ and }S_{v^i}\le M(i)+f(i)\text{ for }1\le i<n/2\}}\right).
\end{aligned}
$$

The above expectation is, by conditioning on $\{S_{v^{n/2}}=M(n)+x\}$,

$$
\begin{aligned}
& 2^k\int_{-\infty}^{y}P(S'_{k-n/2}>M(k)-M(n/2)+f(k)-x)\cdot\\
& \cdot P(S_i-\frac{i}{n/2}S_{n/2}\le f(i)-\frac{i}{k}x\text{ for }1\le i<n/2)\cdot\\
& \cdot p_{S_{n/2}}(M(n/2)+x)dx, \qquad (3.11)
\end{aligned}
$$

where S and S' are two copies of the random walks before and after time $n/2$, respectively, and $p_{S_{n/2}}(x)$ is the density of $S_{n/2}\sim N(0,\frac{\sigma_1^2 n}{2})$.

We then estimate the three factors of the integrand separately. The first one, which is similar to (3.5), is bounded above by

$$
\begin{aligned}
& P(S'_{k-n/2}>M(k)-M(n/2)+f(k)-x)\le\frac{C}{\sqrt{k-n/2}}e^{-\frac{(M(k)-M(n/2)+f(k)-x)^2}{2(k-n/2)\sigma_2^2}}\\
\le\ & C2^{-(k-n/2)}(k-\frac{n}{2})^{-3/2}e^{-\frac{\sqrt{2\log 2}}{\sigma_2}(y-x)}.
\end{aligned}
$$

The second one, which is similar to (3.9), is estimated using Lemma 3.1,

$$P(S_i-\frac{i}{n/2}S_{n/2}\le f(i)-\frac{i}{k}x\text{ for }1\le i<n/2)\le C(1+2y-x)^2/n. \qquad (3.12)$$

The third one is simply the normal density

$$p_{S_{n/2}}(M(n/2)+x)=\frac{C}{\sqrt{n}}e^{-\frac{(M(n/2)+x)^2}{n\sigma_1^2}}\le C2^{-n/2}ne^{-\frac{\sqrt{2\log 2}}{\sigma_1}x}. \qquad (3.13)$$

Therefore, the integral term (3.11) is no more than

$$C(k-n/2)^{-3/2}e^{-\frac{\sqrt{2\log 2}}{\sigma_2}y}\int_{-\infty}^{y}(1+2y-x)^2e^{(\frac{\sqrt{2\log 2}}{\sigma_2}-\frac{\sqrt{2\log 2}}{\sigma_1})x}dx,$$

which is less than or equal to $C(1+y)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_1}y}(k-n/2)^{-3/2}$ since $\sigma_2 < \sigma_1$.

Summing these upper bounds together, we obtain that

$$\sum_{k=n/2}^{3n/4} P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n)+y, \tau_v = k) \le C(1+y)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_1}y} \sum_{k=1}^{\infty} k^{-3/2}. \quad (3.14)$$

This can again be made as small as we wish, say at most $\frac{1}{100}$, by choosing y large enough.

(iv). When $3n/4 < k \le n$, we have

$$\begin{aligned} & P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n)+y, \tau_v = k) \\ \le\ & P(\exists v \in \mathbb{D}_k \text{ such that } S_v > M(k)+f(k), \text{ and } S_{v^i} \le M(i)+f(i) \text{ for } 1 \le i < k) \\ \le\ & E\left(\sum_{v\in\mathbb{D}_k} 1_{\{S_v > M(k)+f(k), \text{ and } S_{v^i} \le M(i)+f(i), \text{ for } 1\le i<k\}}\right). \end{aligned}$$

The above expectation is, by conditioning on $\{S_{v^{n/2}} = M(n)+x\}$,

$$\begin{aligned} 2^k \int_{-\infty}^{y} & P(S'_{k-n/2} > M(k) - M(n/2) + f(k) - x, \\ & S'_i < M(i) - M(n/2) + f(i) - x, \text{ for } n/2 < i \le k) \\ & \cdot P(S_i - \frac{i}{n/2} S_{n/2} \le f(i) - \frac{i}{k} x \text{ for } 1 \le i < n/2) \cdot p_{S_{n/2}}(M(n/2)+x)dx \end{aligned}$$

where S and S' are copies of the random walks before and after time $n/2$, respectively.

The second and third probabilities in the integral are already estimated in (3.12) and (3.13). It remains to bound the first probability. Similar to (3.7), it is bounded above by

$$\begin{aligned} P\Big(& S'_{k-n/2} > M(k) - M(n/2) + f(k) - x, S'_i < M(i) - M(n/2) + f(i) - x, \\ & \text{for } n/2 < i \le k\Big) \le C(1+2y-x)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_2}(2y-x)}(n-k)^{-5/2}. \end{aligned}$$

With these estimates, we obtain in this case, in the same way as in (iii), that

$$\sum_{k=3n/4}^{n} P(\exists v \in \mathbb{D}_n \text{ such that } S_v > M(n)+y, \tau_v = k) \le C(1+y)^2 e^{-\frac{\sqrt{2\log 2}}{\sigma_1}y} \sum_{k=1}^{\infty} k^{-5/2}. \quad (3.15)$$

This can again be made as small as we wish, say at most $\frac{1}{100}$, by choosing y large enough.

Summing (3.6), (3.10), (3.14) and (3.15), then (3.4) and thus (3.3) follow. This concludes the proof of Theorem 1.2. □

4 Further Remarks

We state several immediate generalization and open questions related to binary branching random walks in time inhomogeneous environments where the diffusivity of the particles takes more than two distinct values as a function of time and changes macroscopically.

Extensions to monotone profiles involving a finite number of variances can be obtained similarly to the results on two variances in the previous sections. Specifically, let $k \ge 2$ (constant) be the number of inhomogeneities, let $0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ be given, and set $t_i = s_i - s_{i-1}$ for $i = 1, \ldots, k$. With $\{\sigma_i^2 > 0 : i = 1, \ldots, k\}$, we consider binary branching random walk up to time n, where, for $i = 1, \dots, k$, the increments during the time interval $[s_{i-1}n, s_i n)$ are $N(0, \sigma_i^2)$. That is, during the ith interval, whose duration is $t_i n$, the variances of the increments are σ_i^2. The analogue of Theorems 1.1 and 1.2 is the following.

Theorem 4.1. *a. In the strictly increasing setup* $\sigma_1^2 < \sigma_2^2 < \cdots < \sigma_k^2$*,*

$$M_n = \sqrt{2(\log 2) \sum_{i=1}^{k} t_i \sigma_i^2 n} - \frac{1}{2} \frac{\sqrt{\sum_{i=1}^{k} t_i \sigma_i^2}}{\sqrt{2 \log 2}} \log n + O_P(1).$$

b. In the strictly decreasing setup $\sigma_1^2 > \sigma_2^2 > \cdots > \sigma_k^2$*,*

$$M_n = \sqrt{2 \log 2} (\sum_{i=1}^{k} t_i \sigma_i) n - \frac{3}{2} (\sum_{i=1}^{k} \frac{\sigma_i}{\sqrt{2 \log 2}}) \log n + O_P(1).$$

The proof in the strictly increasing setup is similar to the case $k = 2$ described in Section 2, and M_n behaves asymptotically like the maximum of independent random walks with effective variance $\sum_{i=1}^{k} t_i \sigma_i^2$. In the strictly decreasing setup, the proof follows the argument detailed in Section 3, and M_n behaves asymptotically like the outcome of a greedy algorithm. We omit further details.

Results on other inhomogeneous environments are open and are subjects of further study. We only discuss some of the non rigorous intuition in the rest of this section.

In the general case of finitely many variances, when $\{\sigma_i^2 : i = 1, \dots, k\}$ are not monotone in i, the variational problem consisting of maximizing $\phi(1)$ subject to the constraint (2.6) will give the leading order (velocity) term in M_n. However, the solution to this variational problem may have several intervals along which the constraint is satisfied with equality, and the number of such intervals is expected to influence the second order correction term. An analysis of this general case is not covered by our arguments.

One may also consider situations where the variance profile changes continuously, in a macroscopic way. A general description of the correction term is a challenge. After the current paper was completed, the current authors studied the particular case of a strictly monotone decreasing variance, and described a rather surprising $n^{1/3}$ correction term, see [14]. The general setting remains open and intriguing.

Appendix: Sketch of the Proof of Lemma 1.3

Proof of Lemma 1.3 (sketch). We describe how to fit our model into the framework of [8] and [13], by deriving appropriate recursions for the distribution of the maximum of a branching random walk that, at time n, will coincide with the distribution of M_n. Once this is done, the tightness result in Lemma 1.3 follows directly from the argument in those two papers.

We begin by writing the variance profile for each $n \in \mathbb{N}$ as

$$\sigma_{n,i}^2 = \begin{cases} \sigma_1^2, & \text{when } i \le n/2, \\ \sigma_2^2, & \text{when } n/2 < i \le n. \end{cases}$$

With this profile, we consider a sequence of branching random walks in a leaves-to-root perspective. That is, for each $n \in \mathbb{N}$ and each $0 \le i \le n$, we consider a branching random walk up to time i, with increments at the jth level ($0 \le j < i$) distributed as $N(0, \sigma_{n,n-i+j}^2)$. Denote such a branching random walk by $\mathrm{BRW}_i^{(n)}$ and its maximum at level i by $M_i^{(n)}$ with a distribution function $F_i^{(n)}$. Note that $\mathrm{BRW}_n^{(n)}$ is equivalent to the model we introduced in the beginning of the paper (and to which Lemma 1.3 refers) and that $M_n^{(n)} = M_n$.

For each fixed n, we have the recursions in i

$$F_{i+1}^{(n)}(x) = \left(G_i^{(n)} * F_i^{(n)}(x)\right)^2,\ i = 0, 1, \dots, n-1,$$

where $G_i^{(n)}$ is the distribution function of $N(0, \sigma_{n-i}^2)$. The initial conditions are $F_0^{(n)}(x) = 1_{\mathbf{x}\geq\mathbf{0}}$. Let $H_i^{(n)}(x) = G_i^{(n)} * F_i^{(n)}(x)$ and $\bar{H}_i^{(n)}(x) = 1 - H_i^{(n)}(x)$. Then the recursions for $\bar{H}_i^{(n)}$ are

$$\bar{H}_{i+1}^{(n)} = \bar{G}_{i+1}^{(n)} * \left(Q(\bar{H}_i^{(n)})\right),\ i = 0, 1, \dots, n-1, \tag{4.1}$$

where $Q(x) = 2x - x^2$, and $\bar{H}_0^{(n)} = 1 - G_0^{(n)}$.

The above recursions on $\bar{H}_i^{(n)}$ are exactly the recursions [8, (2.3)], except for the superscript (n). The argument from [8] will apply here, since $G_i^{(n)}$ is either $N(0, \sigma_1^2)$ or $N(0, \sigma_2^2)$ and thus satisfies the uniform (in both n and i) tail assumptions in [8].

The tightness of $\{M_n - \mathrm{Med}(M_n)\}_n$ is derived as an immediate consequence of the tightness of $\{\bar{H}_n^{(n)} - \mathrm{Med}(\bar{H}_n^{(n)})\}_n$. The latter tightness is a consequence of the estimate

$$\sup_n L(\bar{H}_n^{(n)}) < \infty \tag{4.2}$$

with the Lyapunov function $L(\cdot)$ defined in [8, (2.12)], due to [8, Proposition 2.9]. Thus, one only has to prove (4.2). This follows from the recursions (4.1) in i and the fact that $L_0 := L(\bar{H}_0^{(n)})$ is finite and independent of n (since $\bar{H}_0^{(n)}$ is one minus the distribution of $N(0, \sigma_2^2)$), as in the proof of [8, Theorem 2.7]. Indeed, if $\sup_n L(\bar{H}_n^{(n)}) = \infty$, then we can find one large constant $C > L_0$ and $L(\bar{H}_{n_0}^{(n_0)}) > C$ for some n_0. The uniform tail conditions of $G_i^{(n)}$ and [8, Theorem 3.1] then imply that $L_0 = L(\bar{H}_0^{(n_0)}) > C$, which is a contradiction. □

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