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Harnack inequalities for subordinate Brownian motions*

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Abstract

We consider a subordinate Brownian motion X in $\mathbb{R}^d$, $d \geq 1$, where the Laplace exponent ϕ of the corresponding subordinator satisfies some mild conditions. The scale invariant Harnack inequality is proved for X. We first give new forms of asymptotical properties of the Lévy and potential density of the subordinator near zero. Using these results we find asymptotics of the Lévy density and potential density of X near the origin, which is essential to our approach. The examples which are covered by our results include geometric stable processes and relativistic geometric stable processes, i.e. the cases when the subordinator has the Laplace exponent

$$\phi(\lambda) = \log(1 + \lambda^{\alpha/2}) \quad (0 < \alpha \leq 2)$$

and

$$\phi(\lambda) = \log(1 + (\lambda + m^{\alpha/2})^{2/\alpha} - m) \quad (0 < \alpha < 2,\, m > 0)\,.$$

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1 Introduction

Consider a Brownian motion $B = (B_t, \mathbb{P}_x)$ in $\mathbb{R}^d$, $d \geq 1$, and an independent subordinator $S = (S_t : t \geq 0)$. It is known that the stochastic process $X = (X_t, \mathbb{P}_x)$ defined by $X_t = B(S_t)$ is a Lévy process. The process X is called a subordinate Brownian motion.

A non-negative function $h : \mathbb{R}^d \to [0, \infty)$ is said to be harmonic with respect to X in an open set $D \subset \mathbb{R}^d$ if for all open sets $B \subset \mathbb{R}^d$ whose closure is compact and contained in D the following mean value property holds

$$h(x) = \mathbb{E}_x[h(X_{\tau_B})] \quad \text{for all} \quad x \in B\,,$$

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where $\tau_B = \inf\{t > 0 \colon X_t \notin B\}$ denotes the first exit time from the set B .

We say that the Harnack inequality holds for the process X if there exists a constant $c > 0$ such that for any $r \in (0,1)$ and any non-negative function h on $\mathbb{R}^d$ which is harmonic in the ball $B(0,r) = \{z \in \mathbb{R}^d \colon |z| < r\}$ the following inequality is true

$$h(x) \le c\, h(y) \quad \text{for all} \quad x, y \in B(0, \tfrac{r}{2})\,. \tag{1.1}$$

Space homogeneity of Lévy processes implies that the same inequality is true on any ball $B(x_0,r) = \{z \in \mathbb{R}^d \colon |z - x_0| < r\}$. This type of Harnack inequality is sometimes called a scale invariant (or geometric) Harnack inequality, since the constant c in (1.1) stays the same for any $r \in (0,1)$.

The main goal of this paper is to prove the scale invariant Harnack inequality for a class of subordinate Brownian motions. Our most important contribution is that within our framework we can treat subordinate Brownian motions with subordinators whose Laplace exponent

$$\phi(\lambda) := -\log \mathbb{E} e^{-\lambda S_t}$$

varies slowly at infinity. In particular, we are able to give a positive answer for many processes for which only the non-scaling version of the Harnack inequality was known so far.

Here are a few examples of such processes.

Example 1 (Geometric stable processes)

$$\phi(\lambda) = \log(1 + \lambda^{\beta/2}), \quad (0 < \beta \le 2).$$

Example 2 (Iterated geometric stable processes)

$$\begin{aligned} \phi_1(\lambda) &= \log(1 + \lambda^{\beta/2}) \quad (0 < \beta \le 2) \\ \phi_{n+1} &= \phi_1 \circ \phi_n \quad n \in \mathbb{N}. \end{aligned}$$

Example 3 (Relativistic geometric stable processes)

$$\phi(\lambda) = \log\left(1 + \left(\lambda + m^{\beta/2}\right)^{2/\beta} - m\right) \quad (m > 0,\, 0 < \beta < 2).$$

Remark 1.1. *The non-scaling version of the Harnack inequality for geometric stable and iterated geometric stable processes was proved in [17]. It was not known whether scale invariant version of this inequality held. Recently this turned out to be the case in dimension $d = 1$ (see [8]). In [8] the authors used the theory of fluctuation of one-dimensional Lévy processes and it was not clear how to generalize this technique to higher dimensions. Nevertheles, this result suggests that the scale invariant version of the Harnack inequality may hold in higher dimensions.*

Another feature of our approach is that it is unifying in the sense that it covers many classes of subordinate Brownian motions for which the scale invariant Harnack inequality was recently proved. For example we can treat many subordinators whose Laplace exponent varies regularly at infinity. As a special example, rotationally invariant α-stable processes ($\alpha \in (0,2)$) are included in our framework.

Let us be more precise now. In this paper we consider subordinate Brownian motions X in $\mathbb{R}^d$ ($d \ge 1$), for which the Laplace exponent ϕ of the corresponding subordinator S satisfies (see Sections 2 and 3 for details concerning these conditions):

(A-1) the potential measure of S has a decreasing density;

(A-2) the Lévy measure of S is infinite and has a decreasing density;

(A-3) there exist constants $\sigma > 0$, $\lambda_0 > 0$ and $\delta \in (0,1]$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \le \sigma\, x^{-\delta} \text{ for all } x \ge 1 \text{ and } \lambda \ge \lambda_0\,;$$

Our main result is the following scale invariant Harnack inequality.

Theorem 1.2 (Harnack inequality). *Suppose $d \ge 1$ and $X_t = B_{S_t}$ is a subordinate Brownian motion where B_t is a Brownian motion in $\mathbb{R}^d$ and S_t is an independent subordinator whose Laplace exponent is ϕ. We assume that the subordinator S_t satisfies* **(A-1)–(A-3)** *and that the Lévy density $J(x) = j(|x|)$ of X satisfies*

$$j(r+1) \le j(r) \le c' j(r+1),\ r > 1, \tag{1.2}$$

for some constant $c' \ge 1$.

Then there exists a constant $c > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$

$$h(x_1) \le c\, h(x_2) \text{ for all } x_1, x_2 \in B(x_0, \tfrac{r}{2})$$

and for every non-negative function $h\colon \mathbb{R}^d \to [0,\infty)$ which is harmonic in $B(x_0, r)$.

As already mentioned at the beginning, this theorem is a new result for Examples 1–3 above. The condition (0.2) in Theorem 1.2 is implied by the following two conditions on the Lévy measure of the subordinator (see the proof of Proposition 3.5 [12]);
(a) For any $K > 0$, there exists $c_1 = c_1(K) > 1$ such that

$$\mu(r) \le c_1\, \mu(2r), \qquad \forall r \in (0, K). \tag{1.3}$$

(b) There exists $c_2 > 1$ such that

$$\mu(r) \le c_2\, \mu(r+1), \qquad \forall r > 1. \tag{1.4}$$

Note that, when ϕ is a complete Bernstein function, by Lemma 2.1 in [12], (1.4) holds. On the other hand, by Proposition 3.3, under the assumption **(A-2)** and **(A-3)**, (1.3) holds (also see Remark 4.3).

The condition **(A-3)** is implied by the following stronger condition

$$\forall\, x > 0 \qquad \lim_{\lambda\to\infty} \frac{\phi'(\lambda x)}{\phi'(\lambda)} = x^{\frac{\alpha}{2}-1} \quad (0 \le \alpha < 2)\,. \tag{1.5}$$

In other words, (1.5) says that ϕ' varies regularly at infinity with index $\frac{\alpha}{2} - 1$. Examples 1–3 satisfy this condition with $\alpha = 0$.

The following example is also covered by our approach.

Example 4 Assume that ϕ satisfies **(A-1)**, **(A-2)** and

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda),\ \lambda \to \infty \quad (0 < \alpha < 2)$$

where ℓ varies slowly at infinity, i.e.

$$\forall\, x > 0 \quad \lim_{\lambda\to\infty} \frac{\ell(\lambda x)}{\ell(\lambda)} = 1$$

($f(\lambda) \asymp g(\lambda),\ \lambda \to \infty$ means that $f(\lambda)/g(\lambda)$ stays bounded between two positive constants as $\lambda \to \infty$). We can take, for example,

$$\ell(\lambda) = [\log(1+\lambda)]^{1-\alpha/2} \quad \text{or} \quad \ell(\lambda) = [\log(1+\log(1+\lambda))]^{1-\alpha/2}\,.$$

The Harnack inequality in this case has been known before (see [12, 14, 15]).

The main ingredient in our proof of the Harnack inequality is a good estimate of the Green function $G_{B(0,r)}(x,y)$ of the ball $B(0,r)$ when y is near its boundary. To be more precise, we will prove that there are a function $\xi\colon (0,1)\to(0,\infty)$ and constants $c_1,c_2>0$ and $0<\kappa_1<\kappa_2<1$ such that for every $r\in(0,1)$,

$$c_1\xi(r)r^{-d}\,\mathbb{E}_y\tau_{B(0,r)}\le G_{B(0,r)}(x,y)\le c_2\xi(r)r^{-d}\,\mathbb{E}_y\tau_{B(0,r)}, \tag{1.6}$$

for $x\in B(0,\kappa_1 r)$ and $y\in B(0,r)\setminus B(0,\kappa_2 r)$ (see Corollary 5.9).

Depending on considered processes, the function $r\mapsto\xi(r)$ can have two different types of behavior. For example, it turns out that in Example 1

$$\xi(r)\asymp\tfrac{1}{\log(r^{-1})}\ \text{ as }\ r\to 0+\,,$$

while in Example 4

$$\xi(r)\asymp 1\ \text{ as }\ r\to 0+\ .$$

To obtain the mentioned estimates of the Green function we use asymptotical properties of Lévy density $\mu(t)$ and potential density $u(t)$ of the underlying subordinator near zero. It turns out that it is not possible to use Tauberian theorems in each case. In Section 3 we obtain needed asymptotical properties without use of such theorems. More precisely, we show that the asymptotical behavior can be expressed in terms of the Laplace exponent (see Proposition 3.3 and Proposition 3.4):

$$\mu(t)\asymp t^{-2}\phi'(t^{-1})\quad\text{and}\quad u(t)\asymp t^{-2}\frac{\phi'(t^{-2})}{\phi(t^{-2})^2}\quad\text{as}\quad t\to 0+\ .$$

Harnack inequalities for symmetric stable Lévy processes were obtained in [5, 4]. A new technique on Harnack inequalities for stable like jump processes was developed in [3] and generalized in [19]. Similar technique was used for various jump processes in [2, 6, 7]. In [11] the Harnack inequality was proved for truncated stable processes and it was generalized in [14]. Harnack inequality for some classes subordinate Brownian motions was also considered in [12].

Let us comment what happens when one applies techniques developed for jump processes (as in [3]) to our situation. The proof in this case relied on an estimate of Krylov-Safonov type: there exists a constant $c>0$ such that

$$\mathbb{P}_x(T_A<\tau_{B(0,r)})\ge c\,\frac{|A|}{|B(0,r)|}$$

for any $r\in(0,1)$, $x\in B(0,\frac{r}{2})$ and $A\subset\mathbb{R}^d$ closed, where $T_A=\tau_{A^c}$ denotes the first hitting time of the set A and $|A|$ denotes its Lebesgue measure.

Although this technique is quite general and can be applied to a much larger class of Markov jump processes, there are some situations in our setting which show that it is not applicable even to a rotationally invariant Lévy process. A good example is the proof of the Harnack inequality in [17], where the mentioned Krylov-Safonov type estimate was indispensable. Contrary to the case of stable-like processes, this estimate is not uniform in $r\in(0,1)$.

For example, for a geometric stable process it is possible to find a sequence of radii (r_n) and closed sets $A_n\subset B(0,r_n)$ such that $r_n\to 0$, $\frac{|A_n|}{|B(0,r_n)|}\ge\frac14$ and

$$\mathbb{P}_0(T_{A_n}<\tau_{B(0,r_n)})\to 0,\ \text{ as }\ n\to\infty\,.$$

This non-uniformity does not allow to obtain the scale-invariant Harnack inequality using this technique. In this sense, the investigation of Harnack inequality becomes interesting even in the case of a Lévy process. We have not encountered a technique so far that would cover cases of a more general jump process in this direction.

The paper is organized as follows. In Section 2 we give basic notions which we use in sections that follow. New forms of asymptotical properties of the Lévy and the potential densities of subordinators are obtained in Section 3. Technical lemmas concerning asymptotic inversion of the Laplace transform used in this section are deferred to Appendix A. These results in Appendix A can be also of independent interest, since they represent an alternative to the Tauberian theorems, which were mainly used in previous works.

Using results of the Section 3 we obtain the behavior of the Lévy measure and the Green function (potential) of the process X in Section 4. In Section 5 we obtain pointwise estimates of the Green functions of small balls needed to prove the main result, which is proved in Section 6.

Notation. Throughout the paper we use the notation $f(r) \asymp g(r),\ r \to a$ to denote that $f(r)/g(r)$ stays between two positive constants as $r \to a$. Simply, $f \asymp g$ means that the quotient $f(r)/g(r)$ stays bounded between two positive numbers on their common domain of definition. We say that $f\colon \mathbb{R} \to \mathbb{R}$ is increasing if $s \le t$ implies $f(s) \le f(t)$ and analogously for a decreasing function. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure. We will use ":=" to denote a definition, which is read as "is defined to be". For any $a, b \in \mathbb{R}$, we use the notations $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The values of the constants $c_1, c_2, \cdots$ stand for constants whose values are unimportant and which may change from location to location. The labeling of the constants $c_1, c_2, \ldots$ starts anew in the proof of each result.

2 Preliminaries

A stochastic process $X = (X_t, \mathbb{P}_x)$ in $\mathbb{R}^d$ is said to be a pure jump Lévy process if it has stationary and independent increments, its trajectories are right-continuous with left limits and the characteristic exponent Φ in

$$\mathbb{E}_x\left[\exp\{i\langle \xi, X_t - X_0\rangle\}\right] = \exp\{-t\Phi(\xi)\},\ \xi \in \mathbb{R}^d$$

is of the form

$$\Phi(\xi) = \int_{\mathbb{R}^d} \left(1 - \exp\{i\langle \xi, x\rangle\} + i\langle \xi, x\rangle\, 1_{\{|x|<1\}}\right) \Pi(dx). \tag{2.1}$$

The measure Π in (2.1) is called the Lévy measure of X and it satisfies $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\Pi(dx) < \infty$.

Let $S = (S_t\colon t \ge 0)$ be a subordinator, i.e. a Lévy process taking values in $[0, \infty)$ and starting at 0. It is more convenient to consider the Laplace transform in this case

$$\mathbb{E}\exp\{-\lambda S_t\} = \exp\{-t\phi(\lambda)\}. \tag{2.2}$$

The function ϕ in (2.2) is called the Laplace exponent of S and it is of the form

$$\phi(\lambda) = \gamma t + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)\,, \tag{2.3}$$

where $\gamma \ge 0$ and the Lévy measure μ of S is now a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)}(1 \wedge t)\mu(dt) < \infty$ (see p. 72 in [1]). The function ϕ is an example of a Bernstein function, i.e. $\phi \in C^\infty(0, \infty)$ and $(-1)^n\phi^{(n)} \le 0$ for all $n \in \mathbb{N}$ (see p. 15 in [18]). Here $\phi^{(n)}$ denotes the n-th derivative of ϕ. Conversely, every Bernstein function ϕ satisfying $\phi(0+) = 0$ has a representation (2.3) and there exists a subordinator with the Laplace exponent ϕ .

The potential measure of the subordinator S is defined by

$$U(A) = \int_0^\infty \mathbb{P}(S_t \in A)\, dt. \tag{2.4}$$

The Laplace transform of U is then

$$\mathcal{L}U(\lambda) = \mathbb{E}\int_{(0,\infty)} e^{-\lambda S_t}\, dt = \frac{1}{\phi(\lambda)}, \quad \lambda > 0\,. \tag{2.5}$$

A Bernstein function ϕ is said to be a complete Bernstein function if the Lévy measure μ has a completely monotone density, i.e. $\mu(dt) = \mu(t)\, dt$ with $\mu \in C^\infty(0,\infty)$ satisfying $(-1)^n \mu^{(n)} \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$. In this case we can control large jumps of Lévy density μ in the following way. There exists a constant $c > 0$ such that

$$\mu(t) \leq c\,\mu(t+1) \ \text{ for all } \ t \geq 1 \tag{2.6}$$

(see Lemma 2.1 in [13]). If, in addition, $\mu(0,\infty) = \infty$, the potential measure U has a decreasing density, i.e. there exists a decreasing function $u\colon (0,\infty) \to (0,\infty)$ such that $U(dt) = u(t)\, dt$ (see Corollary 10.7 in [18]).

Let $B = (B_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$ (running with a time clock twice as fast as the standard Brownian motion) and let $S = (S_t\colon t \geq 0)$ be an independent subordinator. We define a new process $X = (X_t, \mathbb{P}_x)$ by $X_t = B(S_t)$ and call it subordinate Brownian motion. This process is a Lévy process with the characteristic exponent $\Phi(\xi) = \phi(|\xi|^2)$. Moreover, Φ has the Lévy measure of the form $\Pi(dx) = j(|x|)\, dx$ and

$$j(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right) \mu(dt),\ r > 0 \tag{2.7}$$

(see Theorem 30.1 in [16]).

If S is not a compound Poisson process, the process X has a transition density $p(t,x,y)$ given by

$$p(t,x,y) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\tfrac{|x-y|^2}{4t}\right) \mathbb{P}(S_t \in ds)\,. \tag{2.8}$$

The process X is said to be transient if $\mathbb{P}_0(\lim_{t\to\infty} |X_t| = \infty) = 1$. Since the characteristic exponent of X is symmetric we have the following Chung-Fuchs type criterion for transience

$$\begin{aligned} X \quad \text{is transient} \quad &\Longleftrightarrow \quad \int_{B(0,R)} \frac{d\xi}{\phi(|\xi|^2)} < \infty \ \text{ for some } \ R > 0 \\ &\Longleftrightarrow \quad \int_0^R \frac{\lambda^{\frac{d}{2}-1}}{\phi(\lambda)}\, d\lambda < \infty \ \text{ for some } \ R > 0 \qquad (2.9) \\ &\Longleftrightarrow \ \mathbb{E}_0\left[\int_0^\infty 1_{\{|X_t|<R\}}\, dt\right] < \infty \quad \text{for every } \ R > 0 \qquad (2.10) \end{aligned}$$

(see Corollary 37.6 and Theorem 35.4 in [16]).

In this case we can define the Green function (potential) by $G(x,y) = \int_0^\infty p(t,x,y)\, dt$. Then (2.4) and (2.8) give us a useful formula $G(x,y) = G(y-x) = g(|y-x|)$, where

$$g(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right) U(dt),\ r > 0\,. \tag{2.11}$$

Note that g and j are decreasing.

Let $D \subset \mathbb{R}^d$ be a bounded open subset. We define a killed process X^D by $X_t^D = X_t$ if $t < \tau_D$ and $X_t^D = \Delta$ otherwise, where Δ is some point adjoined to D (usually called cemetery).

The transition density and the Green function of X^D are given by

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}_x\left[p(t-\tau_D, X(\tau_D), y); \tau_D < t\right]$$

and $G_D(x,y)=\int_0^\infty p_D(t,x,y)\,dt$. In the transient case we have the following formula

$$G_D(x,y)=G(x,y)-\mathbb{E}_x[G(X(\tau_D),y)]\,. \tag{2.12}$$

Also, $G_D(x,y)$ is symmetric and, for fixed $y\in D$, $G_D(\cdot,y)$ is harmonic in $D\setminus\{y\}$. Furthermore, $G_D\colon (D\times D)\setminus\{(x,x)\colon x\in D\}\to[0,\infty)$ and $x\mapsto \mathbb{E}_x\tau_D$ are continuous functions.

By the result of Ikeda and Watanabe (see Theorem 1 in [9])

$$\mathbb{P}_x(X_{\tau_D}\in F)=\int_F\int_D G_D(x,y)j(|z-y|)\,dy\,dz \tag{2.13}$$

for any $F\subset \overline{D}^c$. If we define the Poisson kernel of the set D by

$$K_D(x,z)=\int_D G_D(x,y)j(|z-y|)\,dy, \tag{2.14}$$

then $\mathbb{P}_x(X_{\tau_D}\in F)=\int_F K_D(x,z)\,dz$ for any $F\subset\overline{D}^c$. In other words, the Poisson kernel is the density of the exit distribution.

Since a subordinate Brownian motion is a rotationally invariant Lévy process, it follows that in the case of the subordinator with zero drift

$$\mathbb{P}_x(X_{\tau_{B(x_0,r)}}\in\partial B(x_0,r))=0$$

(see [10, 20]) and thus, for a measurable function $h\colon\mathbb{R}^d\to[0,\infty)$,

$$\mathbb{E}_x[h(X_{\tau_{B(x_0,r)}})]=\int_{\overline{B(x_0,r)}^c} K_{B(x_0,r)}(x,z)h(z)\,dz \tag{2.15}$$

for any ball $B(x_0,r)$.

3 Subordinators

Let $S=(S_t\colon t\geq 0)$ be a subordinator with the Laplace exponent ϕ satisfying the following conditions:

(A-1) the potential measure U of S has a decreasing density u, i.e. there is a decreasing function $u\colon(0,\infty)\to(0,\infty)$ so that $U(dt)=u(t)\,dt$;

(A-2) the Lévy measure μ of S is infinite (i.e. $\mu(0,\infty)=\infty$) and it has a decreasing density μ;

(A-3) there exist constants $\sigma>0$, $\lambda_0>0$ and $\delta\in(0,1]$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)}\leq\sigma\,x^{-\delta}\ \text{ for all }\ x\geq 1\ \text{ and }\ \lambda\geq\lambda_0 \tag{3.1}$$

Remark 3.1. *(i)* **(A-1)** *implies that ϕ is a special Bernstein function, i.e. $\lambda\mapsto\frac{\lambda}{\phi(\lambda)}$ is also a Bernstein function (see pp. 92-93 in [18]).*

(ii) Since $\phi(\lambda)=\gamma\lambda+\int_0^\infty(1-e^{-\lambda t})\mu(dt)$, **(A-3)** *implies $\gamma=0$ (by letting $x\to+\infty$).*

First we prove a simple result that holds for any Bernstein function, which will be used in Section 5.

Lemma 3.2. *Let ϕ be a Bernstein function.*

(i) For every $x\geq 1$, $\phi(\lambda x)\leq x\phi(\lambda)$ for all $\lambda>0$.

(ii) If **(A-3)** *holds, then for every* $\varepsilon > 0$ *there is a constant* $c = c(\varepsilon) > 0$ *so that*

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \le c\, x^{1-\delta+\varepsilon} \quad \textit{for all} \quad \lambda \ge \lambda_0 \quad \textit{and} \quad x \ge 1\,.$$

Proof. (i) Since ϕ' is decreasing and $x \ge 1$,

$$\phi(\lambda x) = \int_0^{\lambda x} \phi'(s)\, ds \le \int_0^{\lambda x} \phi'(\tfrac{s}{x})\, ds = x\phi(\lambda)\,.$$

(ii) Without loss of generality we may assume that $\sigma \ge 2$ in **(A-3)**. Using **(A-3)**, for any $k \ge 2$ the following recursive inequality holds

$$\begin{aligned}\phi(\lambda\sigma^{\frac{k}{\varepsilon}}) - \phi(\lambda\sigma^{\frac{k-1}{\varepsilon}}) &= \int_{\lambda\sigma^{\frac{k-1}{\varepsilon}}}^{\lambda\sigma^{\frac{k}{\varepsilon}}} \phi'(s)\, ds \le \sigma^{1-\frac{\delta}{\varepsilon}} \int_{\lambda\sigma^{\frac{k-1}{\varepsilon}}}^{\lambda\sigma^{\frac{k}{\varepsilon}}} \phi'(s\sigma^{-\frac{1}{\varepsilon}})\, ds \\ &= \sigma^{1+\frac{1-\delta}{\varepsilon}} \left(\phi(\lambda\sigma^{\frac{k-1}{\varepsilon}}) - \phi(\lambda\sigma^{\frac{k-2}{\varepsilon}})\right)\,.\end{aligned}$$

Iteration yields

$$\phi(\lambda\sigma^{\frac{k}{\varepsilon}}) - \phi(\lambda\sigma^{\frac{k-1}{\varepsilon}}) \le \sigma^{(k-1)(1+\frac{1-\delta}{\varepsilon})} \left(\phi(\lambda\sigma^{\frac{1}{\varepsilon}}) - \phi(\lambda)\right) \text{ for every } k \ge 2. \tag{3.2}$$

Let $n \in \mathbb{N}$ be chosen so that $\sigma^{\frac{n-1}{\varepsilon}} \le x < \sigma^{\frac{n}{\varepsilon}}$.

If $n = 1$, then by (i), $\phi(\lambda\sigma^{\frac{1}{\varepsilon}}) \le \sigma^{\frac{1}{\varepsilon}}\phi(\lambda) \le \sigma^{\frac{1}{\varepsilon}+\frac{2\delta}{\varepsilon}} x^{-\delta}\phi(\lambda)$ which, by monotonicity of ϕ, implies that $\frac{\phi(\lambda x)}{\phi(\lambda)} \le \sigma^{\frac{1+2\delta}{\varepsilon}} x^{-\delta}$.

Let us consider now the case $n \ge 2$. Using (3.2) and (i) we deduce

$$\begin{aligned}\phi(\lambda\sigma^{\frac{n}{\varepsilon}}) - \phi(\lambda) &\le \left(\phi(\lambda\sigma^{\frac{1}{\varepsilon}}) - \phi(\lambda)\right) \sum_{k=1}^{n} \sigma^{(k-1)(1+\frac{1-\delta}{\varepsilon})} \\ &\le \left(\phi(\lambda\sigma^{\frac{1}{\varepsilon}}) - \phi(\lambda)\right) \frac{\sigma^{n(1+\frac{1-\delta}{\varepsilon})}}{\sigma^{1+\frac{1-\delta}{\varepsilon}} - 1} \le \sigma^{\frac{1}{\varepsilon}} \phi(\lambda) \sigma^{n(1+\frac{1-\delta}{\varepsilon})}\,.\end{aligned}$$

Therefore

$$\phi(\lambda x) \le \phi(\lambda\sigma^{\frac{n}{\varepsilon}}) \le 2\sigma^{1+\frac{2-\delta}{\varepsilon}} \phi(\lambda) \left(\sigma^{\frac{n-1}{\varepsilon}}\right)^{\varepsilon+1-\delta} \le 2\sigma^{1+\frac{2-\delta}{\varepsilon}} \phi(\lambda) x^{\varepsilon+1-\delta}\,.$$

□

Proposition 3.3. *If* **(A-2)** *and* **(A-3)** *hold, then*

$$\mu(t) \asymp t^{-2}\phi'(t^{-1}),\ t \to 0+\ .$$

Proof. Note that

$$\phi(\lambda + \varepsilon) - \phi(\varepsilon) = \int_0^\infty \left(e^{-\lambda t} - e^{-\lambda(t+\varepsilon)}\right) \mu(t)\, dt$$

for any $\lambda > 0$ and $\varepsilon > 0$ and thus the condition (A.1) in Appendix A holds with $f = \phi$ and $\nu = \mu$. Since ϕ is a Bernstein function, it follows that $\phi' \ge 0$ and ϕ' is decreasing. Now we can apply Lemmas A.1 and A.2. □

Proposition 3.4. *If* **(A-1)** *and* **(A-3)** *hold, then*

$$u(t) \asymp t^{-2} \frac{\phi'(t^{-1})}{\phi(t^{-1})^2},\ t \to 0+\ .$$

Proof. By (2.5), with $\psi(\lambda) = \frac{1}{\phi(\lambda)}$, we have $\int_0^\infty e^{-\lambda t} u(t)\, dt = \psi(\lambda)$. Note that, for $\lambda \geq \lambda_0$ and $x \geq 1$, **(A-3)** implies

$$\left| \frac{\psi'(\lambda x)}{\psi'(\lambda)} \right| = \left(\frac{\phi(\lambda)}{\phi(\lambda x)} \right)^2 \frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq \frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq c x^{-\delta},$$

since ϕ is increasing.

We see that (A.1) in Appendix A is satisfied with $f = \frac{1}{\phi}$ and $\nu = u$. Since ϕ is a Bernstein function, $\phi' \geq 0$ and ϕ' is a decreasing function. Thus $|f'| = \frac{\phi'}{\phi^2}$ is also a decreasing function. The result follows now from Lemmas A.1 and A.2. □

4 Lévy density and potential density

In Section 3 we have established asymptotic behavior of the Lévy and potential density of S near zero. In this section we are going to use these results to give new forms of asymptotic behavior of the Lévy density and potential density of the process X near the origin. Throughout the remainder of the paper, X is the subordinate Brownian motion with the characteristic exponent $\phi(|\xi|^2)$ where ϕ is the Laplace exponent of S.

Lemma 4.1. *Suppose that ϕ is a special Bernstein function, i.e. $\lambda \mapsto \frac{\lambda}{\phi(\lambda)}$ is also a Bernstein function. Then the functions $\eta_1, \eta_2 \colon (0,\infty) \to (0,\infty)$ defined by*

$$\eta_1(\lambda) = \lambda^2 \phi'(\lambda) \ \text{ and } \ \eta_2(\lambda) = \lambda^2 \frac{\phi'(\lambda)}{\phi^2(\lambda)}$$

are increasing.

Proof. It is enough to prove that η_2 is increasing, because $\eta_1 = \eta_2 \cdot \phi^2$ is then a product of two increasing functions.

Since ϕ is a special Bernstein function,

$$\frac{\lambda}{\phi(\lambda)} = \theta + \int_{(0,\infty)} (1 - e^{-\lambda t}) \nu(dt),$$

for some $\theta \geq 0$ and a Lévy measure ν (see pp. 92-93 in [18]). Then

$$\begin{aligned} \lambda^2 \frac{\phi'(\lambda)}{\phi(\lambda)^2} &= \lambda \left(-\frac{\lambda}{\phi(\lambda)} \right)' + \frac{\lambda}{\phi(\lambda)} \\ &= \theta + \int_{(0,\infty)} \left(1 - (1 + \lambda t) e^{-\lambda t} \right) \nu(dt) . \end{aligned}$$

Now the claim follows, since $\lambda \mapsto 1 - (1 + \lambda t) e^{-\lambda t}$ is increasing for any $t > 0$. □

Proposition 4.2. *If* **(A-2)** *and* **(A-3)** *hold, then*

$$j(r) \asymp r^{-d-2} \phi'(r^{-2}), \ r \to 0+ \ .$$

Proof. We use the formula (2.7), i.e.

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t} \right) \mu(t)\, dt .$$

Proposition 3.3 implies that $\mu(t) \asymp t^{-2} \phi'(t^{-1}), \ t \to 0+$.

We are going to use Proposition A.3 in Appendix A with $A = 2$, $\eta = \mu$ and $\psi = \phi'$. In order to do this, we need to check conditions (a), (b) and (c)-(ii). The condition (a) follows from the fact that ϕ is a Bernstein function and Lemma 4.1, while (b) follows from

$$\int_1^\infty t^{-d/2}\mu(t)\,dt \le \int_1^\infty \mu(t)\,dt = \mu(1,\infty) < \infty,$$

since μ is a Lévy measure. Finally the condition (c)-(ii) follows from (4.1)–(4.2). □

Remark 4.3. *If ϕ is a complete Bernstein function, using Lemma 4.2 in [15], (2.6) and Proposition 4.2, we see that* (1.2) *holds.*

Lemma 4.4. *If* **(A-1)** *hold and X is transient, then*

$$\int_1^\infty t^{-d/2}u(t)\,dt < \infty\,.$$

Proof. It follows from (2.5) and (2.9) that for any $t \ge 1$ and $R > 0$

$$\begin{aligned}\infty > \int_0^R \frac{\lambda^{\frac{d}{2}-1}}{\phi(\lambda)}\,d\lambda &= \int_0^R \int_0^\infty \lambda^{\frac{d}{2}-1}e^{-\lambda t}u(t)\,dt\,d\lambda \\ &= \int_0^\infty \int_0^{tR} s^{\frac{d}{2}-1}t^{-\frac{d}{2}}e^{-s}u(t)\,ds\,dt \ge \int_1^\infty \int_0^{tR} s^{\frac{d}{2}-1}t^{-\frac{d}{2}}e^{-s}u(t)\,ds\,dt \\ &\ge \left(\int_0^R s^{\frac{d}{2}-1}e^{-s}\,ds\right)\cdot\left(\int_1^\infty t^{-d/2}u(t)\,dt\right)\,.\end{aligned}$$

□

To handle the case $d \le 2$ in the next proposition and several other places, we will add the following assumption to **(A-3)**. Note that we do not put the next assumption in Theorem 1.2.

(B) When $d \le 2$, we assume that $d + 2\delta - 2 > 0$ where δ is the constant in **(A-3)**, and there are $\sigma' > 0$ and

$$\delta' \in \left(1 - \tfrac{d}{2}, (1 + \tfrac{d}{2}) \wedge (2\delta + \tfrac{d-2}{2})\right) \tag{4.1}$$

such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \ge \sigma'\,x^{-\delta'} \quad \text{for all} \quad x \ge 1 \quad \text{and} \quad \lambda \ge \lambda_0\,; \tag{4.2}$$

Proposition 4.5. *If* **(A-1)**, **(A-3)** *and* **(B)** *hold and X is transient, then*

$$g(r) \asymp r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2},\ r \to 0+\ .$$

Proof. By (2.11) we have

$$g(r) = \int_0^\infty (4\pi t)^{-d/2}\exp\left(-\tfrac{r^2}{4t}\right)u(t)\,dt\,.$$

Proposition 3.4 implies that $u(t) \asymp t^{-2}\frac{\phi'(t^{-1})}{\phi(t^{-1})^2},\ t \to 0+$.

We are going to use Proposition A.3 in Appendix with $A = 2$, $\eta = u$ and $\psi = \frac{\phi'}{\phi^2}$. In order to use it, we need to check conditions (a), (b) and (c)-(ii). The condition (a) follows from the fact that ϕ' and $\frac{1}{\phi^2}$ are decreasing, since ϕ is a Bernstein function. The condition (b) follows from Lemma 4.4.

Now we check the condition (c)-(ii) when $d \le 2$; Note that by (4.1), $1 - \frac{d}{2} < \delta' < 2\delta - 1 + \frac{d}{2}$ (and $\delta \le 1 < 1 + \frac{\delta'}{2}$). Thus $0 < \delta' + 2 - 2\delta < 1 + \frac{d}{2}$. Choose $\varepsilon > 0$ small so that $0 < \delta' + 2 - 2\delta + 2\varepsilon < 1 + \frac{d}{2}$, then applying (4.2) and Lemma 3.2 (ii), we get

$$\frac{\psi(\lambda x)}{\psi(\lambda)} = \frac{\phi'(\lambda x)}{\phi'(\lambda)}\frac{\phi(\lambda)^2}{\phi(\lambda x)^2} \ge c_1 x^{-\delta'}c_2 x^{-2+2\delta-2\varepsilon} = c_1c_2x^{-(\delta'+2-2\delta+2\varepsilon)},$$

Thus (A.7) holds. □

5 Green function estimates

The purpose of this section is to establish pointwise Green function estimates when X is transient. More precisely, we are interested in an estimate of $G_{B(x_0,r)}(x,y)$ for $x \in B(x_0, b_1 r)$ and $y \in A(x_0, b_2 r, r) := \{y \in \mathbb{R}^d : b_2 r \le |y - x_0| < r\}$, for some $b_1, b_2 \in (0,1)$. As a starting point we need an estimate of $G_{B(x_0,r)}(x,y)$ away from the boundary.

In this section, we assume that $S = (S_t : t \ge 0)$ is a subordinator with the Laplace exponent ϕ satisfying **(A-1)–(A-3)**, **(B)** and that $X = (X_t, \mathbb{P}_x)$ is the transient subordinate process defined by $X_t = B(S_t)$, where $B = (B_t, \mathbb{P}_x)$ is a Brownian motion in $\mathbb{R}^d$ independent of S.

Recall that, since X is transient, its potential G is finite.

Lemma 5.1. *There exists $a \in (0, \frac{1}{3})$ and $c_1 > 0$ such that for any $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$G_{B(x_0,r)}(x,y) \ge c_1 \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \quad \text{for all } \; x, y \in B(x_0, ar)\,. \tag{5.1}$$

In particular, there is a constant $c_2 \in (0,1)$ so that

$$G_{B(x_0,r)}(x,y) \ge c_2 g(|x-y|) \quad \text{for all } \; x, y \in B(x_0, ar)\,.$$

Proof. Let $x, y \in B(x_0, ar)$ with $a \in (0,1)$ chosen in the course of the proof. We use (2.12), i.e.

$$G_{B(x_0,r)}(x,y) = g(|x-y|) - \mathbb{E}_x[g(|X(\tau_{B(0,r)}) - y|)].$$

Since $|X(\tau_{B(x_0,r)}) - y| \ge (1-a)r$ and $|x-y| \le 2ar$, we get

$$|X(\tau_{B(x_0,r)}) - y| \ge \tfrac{1-a}{2a}|x-y|.$$

This together with the fact that g is decreasing yields

$$G_{B(x_0,r)}(x,y) \ge g(|x-y|) - g(\tfrac{1-a}{2a}|x-y|)\,. \tag{5.2}$$

By Proposition 4.5 there exist constants $0 < c_1 < c_2$ such that

$$c_1 s^{-d+2}\psi(s^{-2}) \le g(s) \le c_2 s^{-d+2}\psi(s^{-2}), \;\; s \in (0,1)\,, \tag{5.3}$$

with

$$\psi(\lambda) = \lambda^2 \tfrac{\phi'(\lambda)}{\phi(\lambda)^2}, \;\; \lambda > 0\,.$$

Considering only $a < \frac{1}{3}$ it follows that $\frac{2a}{1-a} < 1$. Combining (5.2) and (5.3) we arrive at

$$\begin{aligned} &G_{B(x_0,r)}(x,y) \\ &\ge c_1|x-y|^{-d+2}\psi(|x-y|^{-2}) \left[1 - c_2 c_1^{-1}\left(\tfrac{2a}{1-a}\right)^{d-2} \frac{\psi\left(\left(\frac{2a}{1-a}\right)^2|x-y|^{-2}\right)}{\psi(|x-y|^{-2})}\right]. \end{aligned} \tag{5.4}$$

If $d \ge 3$, we can choose $a < \frac{1}{3}$ small enough so that $c_2 c_1^{-1}\left(\frac{2a}{1-a}\right)^{d-2} \le \frac{1}{2}$.
If $d \le 2$, by (4.1), first we can choose $\varepsilon > 0$ small enough so that $d - 2 - 2\delta' + 4\delta - 4\varepsilon > 0$ and then we can choose $a < \frac{1}{3}$ small enough so that $c_2 c_1^{-1}\left(\frac{2a}{1-a}\right)^{d-2-2\delta'+4\delta-4\varepsilon} \le \frac{1}{2}$.

Then using the fact that $\lambda \to \psi(\lambda)$ is increasing (Lemma 4.1) when $d \geq 3$, and using (3.1), (4.2) and Lemma 3.2 (ii) when $d \leq 2$, we get

$$c_2 c_1^{-1} \left(\tfrac{2a}{1-a}\right)^{d-2} \frac{\psi\left(\left(\tfrac{2a}{1-a}\right)^2 |x-y|^{-2}\right)}{\psi(|x-y|^{-2})}$$
$$\leq \begin{cases} c_2 c_1^{-1} \left(\tfrac{2a}{1-a}\right)^{d-2} & \text{when } d \geq 3 \\ c_2 c_1^{-1} \left(\tfrac{2a}{1-a}\right)^{d+2} \frac{\phi'\left(\left(\frac{2a}{1-a}\right)^2 |x-y|^{-2}\right) \phi(|x-y|^{-2})^2}{\phi'(|x-y|^{-2}) \phi\left(\left(\frac{2a}{1-a}\right)^2 |x-y|^{-2}\right)^2} \leq \left(\tfrac{2a}{1-a}\right)^{(d+2)-2\delta'-4+4\delta-4\varepsilon} & \text{when } d \leq 2 \end{cases}$$
$$\leq \frac{1}{2}. \tag{5.5}$$

Finally, (5.4) and (5.5) yield

$$G_{B(x_0,r)}(x,y) \geq \tfrac{c_1}{2c_2} |x-y|^{-d+2} \psi(|x-y|^{-2}) \text{ for all } \ x, y \in B(x_0, ar)\,.$$

□

Proposition 5.2. *There exists a constant $c > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$\mathbb{E}_x \tau_{B(x_0,r)} \geq \frac{c}{\phi(r^{-2})} \ \textit{ for all } \ x \in B(x_0, \tfrac{ar}{2})\,,$$

with $a \in (0, \frac{1}{3})$ as in Lemma 5.1.

Proof. Take a as in Lemma 5.1 and set $b = \frac{a}{2}$. For any $x \in B(x_0, br)$ we have $B(x, br) \subset B(x_0, ar)$ and so it follows from Lemma 5.1 that

$$\begin{aligned} \mathbb{E}_x \tau_{B(x_0,r)} &\geq \int_{B(x,br)} G_{B(x_0,r)}(x,y)\, dy \\ &\geq c_1 \int_{B(x,br)} \frac{|x-y|^{-d-2} \phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}\, dy = \frac{c_2}{\phi(b^{-2} r^{-2})} \geq \frac{b^2 c_2}{\phi(r^{-2})}\,. \end{aligned}$$

The last inequality follows Lemma 3.2, since $b < 1$. □

Remark 5.3. *Note that, by (2.12), for any $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$ we have*

$$G_{B(x_0,r)}(x,y) \leq g(|x-y|) \ \textit{ for all } \ x, y \in B(x_0, r)$$

and, consequently, $\mathbb{E}_x \tau_{B(x_0,r)} \leq \frac{c}{\phi(r^{-2})}$ for any $x \in B(x_0, r)$.

Our approach in obtaining pointwise estimates of Green function of balls uses maximum principle for certain operators (in a similar way as in [4]).

To be more precise, define, for $r > 0$ a Dynkin-like operator $\mathcal{U}_r$ by

$$(\mathcal{U}_r f)(x) = \frac{\mathbb{E}_x[f(X(\tau_{B(x,r)}))] - f(x)}{\mathbb{E}_x \tau_{B(x,r)}}$$

for measurable functions $f \colon \mathbb{R}^d \to \mathbb{R}$ whenever it is well-defined.

Example 5.4. *Let $x \in \mathbb{R}^d$ and $r > 0$. Define*

$$\eta(z) := \mathbb{E}_z \tau_{B(x,r)}, \ z \in \mathbb{R}^d\,.$$

By the strong Markov property, for any $y \in B(x,r)$ and $s < r - |y-x|$

$$\eta(y) = \mathbb{E}_y[\tau_{B(y,s)} + \tau_{B(x,r)} \circ \theta_{\tau_{B(y,s)}}] = \mathbb{E}_y \tau_{B(y,s)} + \mathbb{E}_y \eta(X(\tau_{B(y,s)}))\,.$$

Therefore

$$(\mathcal{U}_s \eta)(y) = -1 \quad \textit{ for any } y \in B(x,r) \textit{ and } s < r - |y-x|. \tag{5.6}$$

Remark 5.5. *Let* $h\colon \mathbb{R}^d \to [0,\infty)$ *be a non-negative function that is harmonic in bounded open set* $D \subset \mathbb{R}^d$*. Then for* $x \in D$ *and* $s < \mathrm{dist}(x, \partial D)$ *we have* $h(x) = \mathbb{E}_x[h(X(\tau_{B(x,s)}))]$*. Thus*

$$(\mathcal{U}_s h)(x) = 0 \ \text{ for all } \ x \in D.$$

Proposition 5.6 (Maximum principle)**.** *Assume that there exist* $x_0 \in \mathbb{R}^d$ *and* $r > 0$ *such that* $(\mathcal{U}_r f)(x_0) < 0$*. Then*

$$f(x_0) > \inf_{x\in\mathbb{R}^d} f(x)\,. \tag{5.7}$$

Proof. If (5.7) is not true, then $f(x_0) \le f(x)$ for all $x \in \mathbb{R}^d$. This implies $(\mathcal{U}_r f)(x_0) \ge 0$, which is in contradiction with the assumption. □

Proposition 5.7. *There exists a constant* $c > 0$ *such that for all* $r \in (0,1)$ *and* $x_0 \in \mathbb{R}^d$

$$G_{B(x_0,r)}(x,y) \le c\,\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\,\mathbb{E}_y\tau_{B(x_0,r)}$$

for all $x \in B(x_0, \frac{br}{2})$ *and* $y \in A(x_0, br, r)$*, where* $b = \frac{a}{2}$ *with the* $a \in (0, \frac{1}{3})$ *from Lemma 5.1.*

Proof. Take $a \in (0, \frac{1}{3})$ as in Proposition 5.2 (which is the same one as in Lemma 5.1), set $b = \frac{a}{2}$ and let $x \in B(x_0, \frac{br}{2})$ and $y \in A(x_0, br, r)$. Define functions

$$\eta(z) := \mathbb{E}_z\tau_{B(x_0,r)} \ \text{ and } \ h(z) := G_{B(x_0,r)}(x,z)$$

and choose $s < (r - |y|) \wedge \frac{br}{4}$. Note that h is harmonic in $B(x_0,r) \setminus \{x\}$.

Since $h(z) \le g(\frac{br}{8})$ for $z \in B(x, \frac{br}{8})^c$ and $y \in A(x_0, br, r) \subset B(x, \frac{br}{8})^c$, we can use (2.13), (2.15) and Remark 5.5 to get

$$\begin{aligned}
&\mathcal{U}_s(h \wedge g(\tfrac{br}{8}))(y) = \mathcal{U}_s(h \wedge g(\tfrac{br}{8}) - h)(y)\\
&\quad= \frac{1}{\mathbb{E}_y\tau_{B(y,s)}}\int_{\overline{B(y,s)}^c}\int_{B(y,s)} G_{B(y,s)}(y,v)j(|z-v|)(h(z)\wedge g(\tfrac{br}{8}) - h(z))\,dv\,dz\\
&\quad= \frac{1}{\mathbb{E}_y\tau_{B(y,s)}}\int_{B(x,\frac{br}{8})}\int_{B(y,s)} G_{B(y,s)}(y,v)j(|z-v|)(h(z)\wedge g(\tfrac{br}{8}) - h(z))\,dv\,dz\\
&\quad\ge -\frac{1}{\mathbb{E}_y\tau_{B(y,s)}}\int_{B(x,\frac{br}{8})}\int_{B(y,s)} G_{B(y,s)}(y,v)j(|z-v|)h(z)\,dv\,dz.
\end{aligned}$$

Note that $|z-v| \ge |x-y| - |x-z| - |y-v| \ge \frac{br}{8}$ for $z \in B(x, \frac{br}{8})$ and $v \in B(y,s)$ implies $-j(|z-v|) \ge -j(\frac{br}{8})$. Thus

$$\begin{aligned}
&\mathcal{U}_s(h \wedge g(\tfrac{br}{8}) - h)(y)\\
&\quad\ge -\frac{j(\frac{br}{8})}{\mathbb{E}_y\tau_{B(y,s)}}\left(\int_{B(x,\frac{br}{8})} G_{B(x_0,r)}(x,z)\,dz\right)\cdot\left(\int_{B(y,s)} G_{B(y,s)}(y,v)\,dv\right)\\
&\quad\ge -\frac{j(\frac{br}{8})}{\mathbb{E}_y\tau_{B(y,s)}}\left(\int_{B(x_0,r)} G_{B(x_0,r)}(x,z)\,dz\right)\mathbb{E}_y\tau_{B(y,s)}\\
&\quad= -j(\tfrac{br}{8})\eta(x) \ge -c_1\left(\tfrac{b}{8}\right)^{-d-2}\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})},
\end{aligned}\tag{5.8}$$

where in the last inequality we have used Proposition 4.2, Remark 5.3 and the fact that ϕ' is decreasing.

Similarly, by Proposition 4.5 and Proposition 5.2 we see that there is a constant $c_2 > 0$ such that

$$g(\tfrac{br}{8}) \le c_2 \left(\tfrac{b}{8}\right)^{-d-2} \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\eta(z) \quad \text{for all } z \in B(x_0, br)\,.$$

Setting $c_3 := (c_1 \vee c_2)\left(\frac{b}{8}\right)^{-d-2} + 1$ we obtain

$$c_3\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\eta(z) - h(z) \wedge g(\tfrac{br}{8}) \ge c_3\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\eta(z) - g(\tfrac{br}{8}) \ge 0$$

for all $z \in B(x_0, br)$. Therefore, the function

$$u(\cdot) := c_3\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\eta(\cdot) - h(\cdot) \wedge g(\tfrac{br}{8})$$

is nonnegative in $B(x_0, br)$, vanishes on $B(x_0, r)^c$ and, by (5.6) and (5.8),

$$\mathcal{U}_s u(y) \le -c_3\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} + c_1\left(\tfrac{b}{8}\right)^{-d-2}\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} < 0 \ \text{ for } \ y \in A(x_0, br, r)\,.$$

If it would hold $\inf_{y\in\mathbb{R}^d} u(y) < 0$, then by continuity of u on $B(x_0, r)$ there would exist $y_0 \in A(x_0, br, r)$ such that $u(y_0) = \inf_{y\in\mathbb{R}^d} u(y)$. But then $\mathcal{U}_s u(y_0) \ge 0$, by the maximum principle (see Proposition 5.6), which is not true. Therefore $\inf_{y\in\mathbb{R}^d} u(y) \ge 0$.

Finally, since $h \le g(\frac{br}{8})$ on $A(x_0, br, r)$ it follows that

$$G_{B(x_0,r)}(x, y) \le c_4\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\eta(y) \ \text{ for all } \ y \in A(x_0, br, r)\,.$$

□

Proposition 5.8. *There exist constants $c > 0$ and $b \in (0, 1)$ such that for any $r \in (0, 1)$ and $x_0 \in \mathbb{R}^d$*

$$G_{B(x_0,r)}(x, y) \ge c\,\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\,\mathbb{E}_y\tau_{B(x_0,r)}$$

for all $x \in B(x_0, br)$ and $y \in B(x_0, r)$.

Proof. Choose $a \in (0, \frac{1}{3})$ as in Lemma 5.1. Then

$$G_{B(x_0,r)}(x, v) \ge c_1\frac{|x-v|^{-d-2}\phi'(|x-v|^{-2})}{\phi(|x-v|^{-2})^2} \ \text{ for } \ x, v \in B(x_0, ar)\,. \tag{5.9}$$

By Proposition 5.7 we know that there exists a constant $c_2 > 0$ so that

$$G_{B(x_0,r)}(x, v) \le c_2\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\mathbb{E}_v\tau_{B(x_0,r)} \ \text{ for } \ x \in B(x_0, \tfrac{ar}{4}),\ v \in A(x_0, \tfrac{ar}{2}, r)\,. \tag{5.10}$$

Also, by Remark 5.3 there is a constant $c_3 > 0$ such that

$$\mathbb{E}_v\tau_{B(x_0,r)} \le \frac{c_3}{\phi(r^{-2})} \ \text{ for } \ v \in B(x_0, r)\,. \tag{5.11}$$

We take

$$b \le \min\left\{\tfrac{1}{4}\left(\tfrac{c_1}{2c_2c_3}\right)^{1/d}, \tfrac{a}{8}\right\}$$

and fix it. Then $c_2c_3 \le \frac{c_1}{2}(4b)^{-d}$ and so by Lemma 4.1 we deduce

$$c_2c_3\frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})^2} \le \tfrac{c_1}{2}\,\frac{(4br)^{-d-2}\phi'((4b)^{-2}r^{-2})}{\phi((4b)^{-2}r^{-2})^2}\,.$$

Now, by (5.9) and (5.11), for all $x \in B(x_0, br)$ and $v \in B(x, br)$ we get

$$c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} \leq \tfrac{1}{2} G_{B(x_0,r)}(x,v)\,. \tag{5.12}$$

For the rest of the proof, we fix $x \in B(x_0, br)$ and define a function

$$h(v) = G_{B(x_0,r)}(x,v) \wedge \left(c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} \right)\,.$$

Let $y \in A(x_0, \frac{ar}{2}, r)$ and take $s < (r - |y|) \wedge \frac{br}{8}$. Note that, by (5.12),

$$h(v) \leq \tfrac{1}{2} G_{B(x_0,r)}(x,v) \ \text{ for } \ v \in B(x, br).$$

Therefore, (2.13) and Remark 5.5 yield

$$\begin{aligned}
(\mathcal{U}_s h)(y) &= \mathcal{U}_s \left(h - G_{B(x_0,r)}(x,\cdot)\right)(y) \\
&= \frac{1}{\mathbb{E}_y \tau_{B(y,s)}} \int_{\overline{B(y,s)}^c} \int_{B(y,s)} G_{B(y,s)}(y,v) j(|z-v|) \left(h(z) - G_{B(x_0,r)}(x,z)\right) dv\, dz \\
&\leq -\frac{1}{2\mathbb{E}_y \tau_{B(y,s)}} \int_{B(x,br)} \int_{B(y,s)} G_{B(y,s)}(y,v) j(|z-v|) G_{B(x_0,r)}(x,z)\, dv\, dz\,.
\end{aligned}$$

Note that in the second equality we have used that $h(y) = G_{B(x_0,r)}(x,y)$, which follows from (5.10). Since $|z-v| \leq 2r$, it follows

$$(\mathcal{U}_s h)(y) \leq -\frac{j(2r)}{2\mathbb{E}_y \tau_{B(y,s)}} \left(\int_{B(x,br)} G_{B(x_0,r)}(x,z)\, dz \right) \mathbb{E}_y \tau_{B(y,s)}$$

By Proposition 4.2, (5.9) and the fact that $\lambda \mapsto \frac{\lambda}{\phi(\lambda)}$ is increasing (by Lemma 3.2) and ϕ' decreasing we finally arrive at

$$(\mathcal{U}_s h)(y) \leq -c_4 \frac{r^{-d-2}\phi'(r^{-2})}{\phi\left((br)^{-2}\right)} \leq -c_5 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\,.$$

Define $u = h - \kappa\eta$, where $\eta(v) = \mathbb{E}_v \tau_{B(x_0,r)}$ and

$$\kappa = \min\left\{ \frac{c_5}{2}, \frac{c_1}{2c_3}, \frac{c_2}{2} \right\} \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\,.$$

For $y \in A(x_0, \frac{ar}{2}, r)$ we have by (5.6),

$$(\mathcal{U}_s u)(y) \leq -c_5 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} + \kappa \leq -\tfrac{c_5}{2} \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} < 0\,.$$

On the other hand, by (5.9) and (5.11), for all $v \in B(x_0, \frac{ar}{2})$,

$$\begin{aligned}
u(v) &\geq \left(\tfrac{c_1}{c_3} \wedge c_2\right) \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} - \kappa\, \mathbb{E}_v \tau_{B(x_0,r)} \\
&\geq \left(\tfrac{c_1}{2c_3} \wedge \tfrac{c_2}{2}\right) \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} \geq 0\,.
\end{aligned}$$

Similarly as in Proposition 5.7, by using the maximum principle we finally obtain

$$u(y) \geq 0 \ \text{ for all } \ y \in B(x_0, r)\,.$$

□

Combining Propositions 5.7 and 5.8 we obtain an important estimate for the Green function.

Corollary 5.9. *There exist constants $c_1, c_2 > 0$ and $b_1, b_2 \in (0, \frac{1}{2})$, $2b_1 < b_2$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)} \le G_{B(x_0,r)}(x,y) \le c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)}$$

for all $x \in B(x_0, b_1 r)$ and $y \in A(x_0, b_2 r, r)$. □

6 Poisson kernel and Harnack inequality

The first goal of this section is to estimate the Poisson kernel of X in a ball given by

$$K_{B(x_0,r)}(x,z) = \int_{B(x_0,r)} G_{B(x_0,r)}(x,y) j(|z-y|)\, dy\,, \tag{6.1}$$

where $x \in B(x_0, br)$ and $z \notin B(x_0, r)$.

Proposition 6.1. *Suppose that ϕ satisfies* **(A-1)–(A-3)**, **(B)** *and that the corresponding subordinate Browninan motion $X = (X_t, \mathbb{P}_x)$ is transient. Then there exist constants $c > 0$ and $b \in (0,1)$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$K_{B(x_0,r)}(x_1, z) \le c K_{B(x_0,r)}(x_2, z)$$

for all $x_1, x_2 \in B(x_0, br)$ and $z \in B(x_0, r)^c$.

Proof. Take $b_1, b_2 \in (0, \frac{1}{2})$ as in Corollary 5.9, and let $x_0 \in \mathbb{R}^d$, $x_1, x_2 \in B(x_0, b_1 r)$ and $z \in B(x_0, r)^c$.

We split the integral in (6.1) into two parts

$$\begin{aligned} K_{B(x_0,r)}(x_1,z) &= \int_{B(x_0,b_2 r)} G_{B(x_0,r)}(x_1,y) j(|z-y|)\, dy \\ &\quad + \int_{A(x_0,b_2 r,r)} G_{B(x_0,r)}(x_1,y) j(|z-y|)\, dy \ =: \ I_1 + I_2\,. \end{aligned}$$

To estimate I_2 we use Corollary 5.9 to get that for $y \in A(x_0, b_2 r, r)$

$$c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)} \le G_{B(x_0,r)}(x_i,y) \le c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)}, \quad i = 1,2.$$

Therefore

$$\begin{aligned} I_2 &= \int_{A(x_0,b_2 r,r)} G_{B(x_0,r)}(x_1,y) j(|z-y|)\, dy \\ &\le c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \int_{A(x_0,b_2 r,r)} \mathbb{E}_y\, \tau_{B(x_0,r)} j(|z-y|)\, dy \\ &\le \frac{c_1}{c_2} \int_{A(x_0,b_2 r,r)} G_{B(x_0,r)}(x_2,y) j(|z-y|)\, dy \ \le \ \frac{c_1}{c_2} K_{B(x_0,r)}(x_2,z)\,. \end{aligned}$$

To handle I_1 we consider two cases. If $z \in A(x_0, r, 2)$, then

$$(1-b_2)|z-x_0| \le |z-y| \le (1+b_2)|z-x_0| \ \text{ for all } \ y \in B(x_0, b_2 r)\,.$$

Since $1 - b_2 \ge \frac{1}{2}$ and $1 + b_2 \le 2$ we obtain

$$j\left(2|z-x_0|\right) \le j(|z-y|) \le j\left(\tfrac{1}{2}|z-x_0|\right)\,. \tag{6.2}$$

Using Proposition 4.2 and the fact that ϕ' is decreasing we see that

$$j\left(\tfrac{1}{2}|z-x_0|\right) \le c_3 j\left(2|z-x_0|\right) . \tag{6.3}$$

Then Lemma 5.1, Remark 5.3, (6.2) and (6.3) yield

$$\begin{aligned} I_1 &\le j\left(\tfrac{1}{2}|z-x_0|\right) \int_{B(x_0,b_2r)} G_{B(x_0,r)}(x_1,y)\,dy \\ &\le c_3 j\left(2|z-x_0|\right) \frac{c_4}{\phi((b_2r)^{-2})} \\ &\le c_5 \int_{B(x_0,b_2r)} G_{B(x_0,r)}(x_2,y) j(|z-y|)\,dy \;\le\; c_5 K_{B(x_0,r)}(x_2,z) . \end{aligned}$$

When $z \in B(x_0,2)^c$ we use $|z-x_0| - br \le |z-y| \le |z-x_0| + br$ for all $y \in B(x_0,br)$. Since $b_2 \in (0,\frac{1}{2})$ and $r \in (0,1)$ we have

$$j(|z-x_0| + \tfrac{1}{2}) \le j(|z-y|) \le j(|z-x_0| - \tfrac{1}{2}) . \tag{6.4}$$

By (1.2) we have

$$j(|z-x_0| - \tfrac{1}{2}) \le c_6 j(|z-x_0|+1) \le j(|z-x_0| + \tfrac{1}{2}) . \tag{6.5}$$

Similar to the previous case, by (1.2), Lemma 5.1, (6.4) and (6.5) we have $I_2 \le c_7 K_{B(x_0,r)}(x_2,z)$. Therefore, $I_2 \le (c_5 \vee c_7) K_{B(x_0,r)}(x_2,z)$ which implies that

$$K_{B(x_0,r)}(x_1,z) \le \max\{c_5, c_7, \tfrac{c_1}{c_2}\} K_{B(x_0,r)}(x_2,z) .$$

□

Now we are ready to prove our main result.

Proof of Theorem 1.2. Suppose $d \ge 3$. Then X is always transient. Take $b > 0$ as in Proposition 6.1 and set $a = \frac{b}{4}$. Suppose that $h\colon \mathbb{R}^d \to [0,\infty)$ is harmonic in $B(x_0,r)$. Using representation

$$h(x) = \mathbb{E}_x[h(X_{\tau_{B(x_0,2ar)}})] = \int_{\overline{B(x_0,2ar)}^c} K_{B(x_0,2ar)}(x_1,z) h(z)\,dz$$

and Proposition 6.1 we have

$$\begin{aligned} h(x_1) &= \int_{\overline{B(x_0,2ar)}^c} K_{B(x_0,2ar)}(x_1,z) h(z)\,dz \\ &\le c \int_{\overline{B(x_0,2ar)}^c} K_{B(x_0,2ar)}(x_2,z) h(z)\,dz = c\, h(x_2). \end{aligned}$$

Then, using a standard Harnack chain argument, we prove the theorem for $d \ge 3$.

To handle the lower dimensional case, we use the following notation: for $x = (x^1,\ldots,x^{d-1},x^d) \in \mathbb{R}^d$ we set $\tilde{x} = (x^1,\ldots,x^{d-1})$. Let $X = ((\widetilde{X}_t, X_t^d), \mathbb{P}_{(\tilde{x},x^d)})$ be a d-dimensional subordinate Brownian motion with the characteristic exponent

$$\Phi(\xi) = \phi(|\xi|^2),\ \xi \in \mathbb{R}^d .$$

By checking the characteristic functions, it follows that, for every $x^d \in \mathbb{R}$, $\widetilde{X} = (\widetilde{X}_t, \mathbb{P}_{\tilde{x}})$ is a $(d-1)$-dimensional subordinate Brownian motion with characteristic exponent

$$\widetilde{\Phi}(\tilde{\xi}) = \phi(|\tilde{\xi}|^2),\ \tilde{\xi} \in \mathbb{R}^{d-1} .$$

Suppose that the theorem is true for for some $d \geq 2$. Let $h\colon \mathbb{R}^{d-1} \to [0,\infty)$ be a function that is harmonic in $B(\widetilde{x}_0, r)$.

Since

$$\tau_{\mathcal{B}(\widetilde{x}_0,s)\times\mathbb{R}} = \inf\{t > 0 : \widetilde{X}_t \notin \mathcal{B}(\widetilde{x}_0, s)\},$$

the strong Markov property implies that the function $f\colon \mathbb{R}^d \to [0,\infty)$ defined by $f(\widetilde{x}, x^d) = h(\widetilde{x})$ is harmonic in $B(\widetilde{x}_0, r) \times \mathbb{R}$.

In particular, f is harmonic in $B((\widetilde{x}_0, 0), r)$. By applying the result to f, we see that there exists a constant $c > 0$ such that for all $\widetilde{x}_0 \in \mathbb{R}^{d-1}$ and $r \in (0,1)$

$$h(\widetilde{x}_1) = f((\widetilde{x}_1, 0)) \leq c\, f((\widetilde{x}_2, 0)) = c\, h(\widetilde{x}_2) \;\text{ for all }\; \widetilde{x}_1, \widetilde{x}_2 \in B(\widetilde{x}_0, \tfrac{r}{2}).$$

Applying this argument first to $d = 3$ and then to $d = 2$ we finish the proof of the theorem. □

Since $K_{B(x_0,r)}(x,\cdot)$ is continuous on $\overline{B(x_0,r)}^{\,c}$ for every $x \in B(x_0, r)$, Theorem 1.2 implies Proposition 6.1 without the conditions **(B)** and X being transient.

Corollary 6.2. *Suppose that ϕ satisfies* **(A-1)–(A-3)**. *Then for every $b \in (0,1)$, there exists a constant $c = c(b) > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$K_{B(x_0,r)}(x_1, z) \leq c K_{B(x_0,r)}(x_2, z)$$

for all $x_1, x_2 \in B(x_0, br)$ and $z \in B(x_0, r)^c$.

We we omit the proof since it is the same as the proof of Proposition 1.4.11 in [12].

A Asymptotical properties

In this section we always assume that $f\colon (0,\infty) \to (0,\infty)$ is a differentiable function satisfying

$$|f(\lambda+\varepsilon) - f(\lambda)| = \int_0^\infty \left(e^{-\lambda t} - e^{-(\lambda+\varepsilon)t}\right) \nu(t)\, dt\,, \tag{A.1}$$

for all $\lambda > 0$, $\varepsilon \in (0,1)$ and a decreasing function $\nu\colon (0,\infty) \to (0,\infty)$.

Lemma A.1. *For all $t > 0$,*

$$\nu(t) \leq (1 - 2e^{-1})^{-1}\, t^{-2} |f'(t^{-1})|\,.$$

Proof. Let $\varepsilon \in (0,1)$. Then

$$\begin{aligned} |f(\lambda+\varepsilon) - f(\lambda)| &= \int_0^\infty \left(e^{-\lambda t} - e^{-\lambda t - \varepsilon t}\right) \nu(t)\, dt \\ &= \lambda^{-1} \int_0^\infty e^{-z} \left(1 - e^{-\varepsilon\lambda^{-1} z}\right) \nu(\lambda^{-1} z)\, dz\,. \end{aligned}$$

Since ν is decreasing, for any $r > 0$ we conclude

$$\begin{aligned} |f(\lambda+\varepsilon) - f(\lambda)| &\geq \lambda^{-1} \int_0^r e^{-z} \left(1 - e^{-\varepsilon\lambda^{-1} z}\right) \nu(\lambda^{-1} z)\, dz \\ &\geq \lambda^{-1} \nu(\lambda^{-1} r) \int_0^r e^{-z} \left(1 - e^{-\varepsilon\lambda^{-1} z}\right)\, dz. \end{aligned}$$

Therefore

$$\left|\frac{f(\lambda+\varepsilon) - f(\lambda)}{\varepsilon}\right| \geq \lambda^{-2} \nu(\lambda^{-1} r) \int_0^r z\, e^{-z}\, \frac{1 - e^{-\varepsilon\lambda^{-1} z}}{\varepsilon\lambda^{-1} z}\, dz. \tag{A.2}$$

By Fatou's lemma and (A.2) we obtain

$$\begin{aligned}|f'(\lambda)| &= \lim_{\varepsilon\to 0+}\left|\frac{f(\lambda+\varepsilon)-f(\lambda)}{\varepsilon}\right| \geq \lambda^{-2}\nu(\lambda^{-1}r)\int_0^r ze^{-z}\,dz \\ &= \lambda^{-2}\nu(\lambda^{-1}r)\left(1-e^{-r}(r+1)\right)\,.\end{aligned}$$

In particular, for $r=1$ we deduce

$$\nu(t) \leq \left(1-2e^{-1}\right)^{-1} t^{-2}|f'(t^{-1})|,\ t>0\,.$$

□

Lemma A.2. *Assume that $|f'|$ is decreasing and there exist constants $c_1>0$, $\lambda_0>0$ and $\delta>0$ such that*

$$\left|\frac{f'(\lambda x)}{f'(\lambda)}\right| \leq c_1 x^{-\delta} \ \text{ for all } \ \lambda\geq\lambda_0 \ \text{ and } \ x\geq 1\,. \tag{A.3}$$

Then there is a constant $c_2=c_2(c_1,\lambda_0,\delta)>0$ such that

$$\nu(t)\geq c_2 t^{-2}|f'(t^{-1})| \ \text{ for any } \ t\leq 1/\lambda_0\,.$$

Proof. Let $\varepsilon\in(0,1)$. For $r\in(0,1]$ we have

$$\begin{aligned}|f(\lambda+\varepsilon)-f(\lambda)| &= \lambda^{-1}\int_0^\infty e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\nu(\lambda^{-1}z)\,dz \\ &= I_1(\varepsilon)+I_2(\varepsilon)\,,\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}I_1(\varepsilon) &= \lambda^{-1}\int_0^r e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\nu(\lambda^{-1}z)\,dz \\ I_2(\varepsilon) &= \lambda^{-1}\int_r^\infty e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\nu(\lambda^{-1}z)\,dz\,.\end{aligned}$$

Since ν is decreasing,

$$\frac{I_2(\varepsilon)}{\varepsilon} \leq \lambda^{-2}\nu(\lambda^{-1}r)\int_r^\infty ze^{-z}\,\frac{1-e^{-\varepsilon\lambda^{-1}z}}{\varepsilon\lambda^{-1}z}\,dz\,,$$

and so by the dominated convergence theorem we deduce

$$\limsup_{\varepsilon\to 0+}\frac{I_2(\varepsilon)}{\varepsilon} \leq \lambda^{-2}\nu(\lambda^{-1}r)\int_r^\infty ze^{-z}\,dz = (r+1)e^{-r}\lambda^{-2}\nu(\lambda^{-1}r)\,. \tag{A.5}$$

On the other hand, by Lemma A.1 and (A.3) we have

$$\begin{aligned}\frac{I_1(\varepsilon)}{\varepsilon} &\leq \frac{\lambda^{-2}}{1-2e^{-1}}\int_0^r ze^{-z}\,\frac{1-e^{-\varepsilon\lambda^{-1}z}}{\varepsilon\lambda^{-1}z}\,\frac{|f'(\lambda z^{-1})|}{\lambda^{-2}z^2}\,dz \\ &\leq \frac{c_1}{1-2e^{-1}}\,|f'(\lambda)|\int_0^r e^{-z}\,\frac{1-e^{-\varepsilon\lambda^{-1}z}}{\varepsilon\lambda^{-1}z}z^{\delta-1}\,dz\,.\end{aligned}$$

The dominated convergence implies

$$\limsup_{\varepsilon\to 0+}\frac{I_1(\varepsilon)}{\varepsilon} \leq \frac{c_1}{1-2e^{-1}}\,|f'(\lambda)|\int_0^r e^{-z}z^{\delta-1}\,dz \quad \text{for any } \ \lambda\geq\lambda_0. \tag{A.6}$$

Combining (A.4), (A.5) and (A.6) we deduce

$$|f'(\lambda)| \le \frac{c_1}{1-2e^{-1}}\,|f'(\lambda)| \int_0^r e^{-z} z^{\delta-1}\,dz + (r+1)e^{-r}\lambda^{-2}\nu(\lambda^{-1}r)$$

for all $\lambda \ge \lambda_0$. Choosing $r_0 \in (0,1]$ so that

$$\frac{c_1}{1-2e^{-1}} \int_0^{r_0} e^{-z} z^{\delta-1}\,dz \le \frac{1}{2}\,.$$

we have

$$\nu(\lambda^{-1}r_0) \ge \frac{e^{r_0}}{2(r_0+1)}\lambda^2|f'(\lambda)| \;\text{ for all }\; \lambda \ge \lambda_0.$$

Since $|f'|$ is decreasing, we see that

$$\begin{aligned}\nu(t) &\ge \frac{e^{r_0}}{2(r_0+1)}\frac{|f'(r_0/t)|}{(t/r_0)^2}\\ &\ge \frac{r_0^2 e^{r_0}}{2(r_0+1)} t^{-2}|f'(t^{-1})| \;\text{ for all }\; t \le r_0\lambda_0^{-1}.\end{aligned}$$

On the other hand, for $r_0\lambda_0^{-1} \le t \le \lambda_0^{-1}$ we have

$$\nu(t) \ge \nu(\lambda_0^{-1}) \ge \nu(\lambda_0^{-1})t^{-2}|f'(t^{-1})|\frac{(r_0/\lambda_0)^2}{|f'(\lambda_0)|},$$

since ν and $|f'|$ are decreasing.

Setting

$$c_2 = \frac{r_0^2 e^{r_0}}{2(r_0+1)} \wedge \frac{\nu(\lambda_0^{-1})\lambda_0^{-2}r_0^2}{|f'(\lambda_0)|}$$

we get

$$\nu(t) \ge c_2 t^{-2}|f'(t^{-1})| \;\text{ for all }\; t \le \lambda_0^{-1}\,.$$

□

Proposition A.3. *Let $A > 0$ and $\eta\colon (0,\infty) \to (0,\infty)$ be a decreasing function satisfyng the following conditions:*

- *(a) there exists a decreasing function $\psi\colon (0,\infty) \to (0,\infty)$ such that $\lambda \mapsto \lambda^2\psi(\lambda)$ is increasing and satisfies*
$$\eta(t) \asymp t^{-A}\psi(t^{-1}),\; t \to 0+\,;$$
- *(b) $\int_1^\infty t^{-d/2}\eta(t)\,dt < \infty$*
- *(c) either (i) $A > 3 - \frac{d}{2}$ or (ii) $A > 3 - \frac{d}{2}$ when $d \ge 3$ and in the case $d \le 2$ there exist $\delta > 0$ and $c > 0$ such that $A - \delta > 1 - \frac{d}{2}$ and*

$$\frac{\psi(\lambda x)}{\psi(\lambda)} \ge c x^{-\delta} \;\textit{ for all }\; x \ge 1 \;\textit{ and }\; \lambda \ge 1\,. \tag{A.7}$$

If

$$I(r) = \int_0^\infty (4\pi t)^{-d/2}\exp\left(-\frac{r^2}{4t}\right)\eta(t)\,dt$$

exists for $r \in (0,1)$ small enough, then

$$I(r) \asymp r^{-d-2A+2}\psi(r^{-2}),\; r \to 0+\,.$$

Proof. Write for $r > 0$

$$\begin{aligned} I(r) &= \int_0^{r^2} (4\pi t)^{-d/2} \exp\Big(-\tfrac{r^2}{4t}\Big)\eta(t)\,dt + \int_{r^2}^{\infty} (4\pi t)^{-d/2} \exp\Big(-\tfrac{r^2}{4t}\Big)\eta(t)\,dt \\ &= I_1(r) + I_2(r)\,. \end{aligned} \tag{A.8}$$

By condition (a),

$$\begin{aligned} I_1(r) &\le c_1 \int_0^{r^2} (4\pi t)^{-d/2} \exp\Big(-\tfrac{r^2}{4t}\Big) t^{-A}\psi(t^{-1})\,dt \\ &\le c_2\psi(r^{-2}) \int_0^{r^2} t^{-\frac{d}{2}-A} \exp\Big(-\tfrac{r^2}{4t}\Big)\,dt \\ &= c_3 r^{-d-2A+2}\psi(r^{-2}) \int_{\frac14}^{\infty} t^{A-2+\frac{d}{2}} e^{-t}\,dt\ . \end{aligned} \tag{A.9}$$

Similarly,

$$\begin{aligned} I_2(r) &\le c_1 \int_{r^2}^{1} (4\pi t)^{-d/2} \exp\Big(-\tfrac{r^2}{4t}\Big) t^{-A}\psi(t^{-1})\,dt + \int_1^{\infty} (4\pi t)^{-d/2} \exp\Big(-\tfrac{r^2}{4t}\Big)\eta(t)\,dt \\ &\le c_1 \int_{r^2}^{1} (4\pi t)^{-d/2} t^{-A}\psi(t^{-1})\,dt + \int_1^{\infty} (4\pi t)^{-d/2}\eta(t)\,dt\,. \end{aligned}$$

The following inequality holds

$$\int_{r^2}^{1} (4\pi t)^{-d/2} t^{-A}\psi(t^{-1})\,dt \le c_4 r^{-d-2A+2}\psi(r^{-2})\,, \tag{A.10}$$

since

(1) if condition (c)-(i) holds, then by conditions (a) and (c)-(i)

$$\begin{aligned} \int_{r^2}^{1} (4\pi t)^{-d/2} t^{-A}\psi(t^{-1})\,dt &\le r^{-4}\psi(r^{-2}) \int_{r^2}^{1} (4\pi t)^{-d/2} t^{2-A}\,dt \\ &\le c_5 r^{-d-2A+2}\psi(r^{-2})\,; \end{aligned}$$

(2) if condition (c)-(ii) holds and $d \le 2$, (A.7) implies

$$\begin{aligned} \int_{r^2}^{1} (4\pi t)^{-d/2} t^{-A}\psi(t^{-1})\,dt &\le \psi(r^{-2}) r^{-2\delta} \int_{r^2}^{1} (4\pi t)^{-\frac{d}{2}} t^{-A+\delta}\,dt \\ &\le c_6 r^{-d-2A+2}\psi(r^{-2})\,. \end{aligned}$$

In particular, (A.10) implies

$$r^{-d-2A+2}\psi(r^{-2}) \ge c_7 > 0 \ \text{ for all } \ r \in (0,1)$$

and thus

$$I_2(r) \le c_6 r^{-d-2A+2}\psi(r^{-2}) + c_8 \le c_9 r^{-d-2A+2}\psi(r^{-2})\,. \tag{A.11}$$

Combining (A.8), (A.9) and (A.11) we get the upper bound

$$I(r) \le c_7 r^{-d+2-2A}\psi(r^{-2}) \ \text{ for all } \ r \in (0,1)\,.$$

To get the lower bound we estimate $I(r)$ from below by $I_1(r)$ and use (a) to get

$$
\begin{aligned}
j(r) \geq I_1(r) &\geq c_8 \int_0^{r^2} (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right) t^{-A} \psi(t^{-1})\, dt \\
&\geq c_8 r^{-4} \psi(r^{-2}) \int_0^{r^2} (4\pi t)^{-d/2} t^{2-A} \exp\left(-\tfrac{r^2}{4t}\right) dt \\
&= c_9 r^{-d-2A+2} \psi(r^{-2}) \int_{\frac{1}{4}}^{\infty} s^{-\frac{d}{2}+A-4} e^{-s}\, ds \\
&= c_{10} r^{-d-2A+2} \psi(r^{-2}) \ \text{ for all } \ r \in (0,1).
\end{aligned}
$$

□

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