

A comment on the Wigner-Dyson-Mehta bulk universality conjecture for Wigner matrices*

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Abstract

Recently we proved [3, 4, 6, 7, 9, 10, 11] that the eigenvalue correlation functions of a general class of random matrices converge, weakly with respect to the energy, to the corresponding ones of Gaussian matrices. Tao and Vu [15] gave a proof that for the special case of Hermitian Wigner matrices the convergence can be strengthened to vague convergence at any fixed energy in the bulk. In this article we show that this theorem is an immediate corollary of our earlier results. Indeed, a more general form of this theorem also follows directly from our work [2].

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Consider an $N \times N$ Wigner ensemble of random matrices $H \equiv H_N = (h_{ij})$ with matrix elements having mean zero and variance $1/N$, i.e.,

$$\mathbb{E} h_{ij} = 0, \quad \mathbb{E} |h_{ij}|^2 = \frac{1}{N}, \quad i, j = 1, 2, \dots, N.$$

A long-standing conjecture of Mehta [13] (also known as the universality conjecture) was to prove that the local correlation functions of the eigenvalues depend only on the symmetry class of the ensemble (symmetric, Hermitian or self-dual quaternion). Tao and Vu recently published a paper [15] where they renamed *the special Hermitian case of this conjecture* as the “Wigner-Dyson-Mehta conjecture”. They further describe a key objective of their paper (page 4 of [15]) as “to provide an almost complete solution for vague convergence” of the “Wigner-Dyson-Mehta conjecture”.

Mehta’s conjecture, open for almost half a century, was finally solved for all symmetry classes in [6, 7, 9, 10, 11]. In these works, convergence was proved in a weak sense in the energy parameter. Due to a special formula of Brézin-Hikami [1] and Johansson [12], the Hermitian case is much easier and was resolved earlier in [3, 14, 4], in some cases in a somewhat stronger sense of convergence. The convergence type, however,

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was not considered a major issue in Mehta’s formulation of the conjecture [13]. In [15] the authors single out the much simpler Hermitian case as the “Wigner-Dyson-Mehta conjecture”. They further subdivide this conjecture by different types of convergence and address the technical point of strengthening it to vague convergence at a fixed energy. Their one-page proof of this extension is a simple combination of previous facts from [14] and our key results from [3] and [11].

We remark that our general theory can very easily be applied to obtain universality results at a fixed energy for the Hermitian case. For example, one main result of [15] (Theorem 5), stated here as Theorem 1.1, is an immediate corollary of two of our previous theorems, see below. We also present a more general result, Theorem 1.4, for generalized Hermitian Wigner matrices. The proofs of Theorem 1.1 and Theorem 1.4 are only a few lines each, given the key inputs from our earlier works.

We first recall the result stated as Theorem 5 in [15].

Theorem 1.1. *Let $p_N^{(k)}$ be the k -point eigenvalue correlation function for the Hermitian Wigner ensemble H_N . Suppose that a sufficiently high moment of the rescaled matrix elements is finite, i.e.,*

$$\mathbb{E}|\sqrt{N}h_{ij}|^M \leq C \tag{1.1}$$

with some large M and C , uniformly in N, i, j . Let $O : \mathbb{R}^k \rightarrow \mathbb{R}$ be a compactly supported bounded continuous function. Then for any $|E| < 2$ we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_{\mathbb{R}^k} O(\alpha_1, \dots, \alpha_k) \frac{1}{[\varrho(E)]^k} p_N^{(k)}\left(E + \frac{\alpha_1}{N\varrho(u)}, \dots, E + \frac{\alpha_k}{N\varrho(u)}\right) d\alpha_1 \dots d\alpha_k \\ = \int_{\mathbb{R}^k} O(\alpha_1, \dots, \alpha_k) \det \left(\frac{\sin \pi(\alpha_i - \alpha_j)}{\pi(\alpha_i - \alpha_j)} \right)_{i,j=1}^k d\alpha_1 \dots d\alpha_k. \end{aligned} \tag{1.2}$$

Here $\varrho(E) = \frac{1}{2\pi} \sqrt{4 - E^2}$ is the limiting eigenvalue density at the energy E .

We first recall our general theory for universality [5, 3, 6, 9, 10, 11] (see [8] for a review). It consists of three steps: 1. the local semicircle law, 2. universality for Gaussian divisible ensembles and 3. approximation by Gaussian divisible ensembles via a perturbation argument. This strategy was first introduced in [3] and has subsequently led to a recent surge of activity in the subject. In particular, the bulk universality for Hermitian Wigner matrices with smooth distributions, at a fixed energy, was first proved [3]. Notice that all results in Steps 1 and 3 hold and were originally stated for a fixed energy. Step 2 can be achieved via two different routes:

2a Proposition 3.1 of [3] (explained below).

2b local ergodicity of Dyson Brownian motion (DBM) (Theorem 4.1 of [7])

The first approach, 2a, which uses an extension of Johansson’s formula [12], is valid for fixed energy. However, it only works in the Hermitian case. This prompted us to develop the approach 2b which is very general and conceptually appealing; it, however, requires an energy average of size $N^{-1+\varepsilon}$. To emphasize that our theory is general in the spirit of universality, we chose to state our results [9, 10, 2] for the general cases covering all symmetry classes. Thus an average in energy was needed. If we restricted ourselves to the Hermitian case, we can revert to Step 2a and all these results are valid for fixed energy. In particular, Theorem 1.1 is valid for generalized Hermitian matrices with finite $4 + \varepsilon$ moments, see Theorem 1.4 below. We now prove Theorem 1.1 and demonstrate how to replace Step 2b by Step 2a.

Proof. Recall that Gaussian divisible Hermitian matrices are matrices of the form

$$H(a) := \sqrt{1 - a^2}H_0 + aV \tag{1.3}$$

where H_0 is an arbitrary Hermitian Wigner matrix, V is an independent standard GUE matrix, and $0 < a < 1$ is a real parameter. With this notation, Proposition 3.1 of [3] takes the following form:

Proposition 1.2. *Suppose that the matrix H_0 satisfies the condition (1.1). Then (1.2) holds for the correlation functions of $H(a)$ if $a = N^{-1/2+\varepsilon}$, for any $\varepsilon > 0$.*

The correlation function comparison theorem, Theorem 4.2 of [8] (which is a slight extension of the Green function comparison theorem, Theorem 2.3 of [9]), compares correlation functions of two Wigner ensembles provided the first four moments of the matrix elements are sufficiently close.

Theorem 1.3 (Correlation function comparison). *[8, Theorem 4.2] Suppose that we have two $N \times N$ Wigner matrices, $H^{(v)}$ and $H^{(w)}$, with matrix elements h_{ij} given by the random variables $N^{-1/2}v_{ij}$ and $N^{-1/2}w_{ij}$, respectively, such that v_{ij} and w_{ij} satisfy the high moment condition (1.1). We assume that the first four moments of v_{ij} and w_{ij} satisfy, for some $\delta > 0$, that*

$$|\mathbb{E}(\operatorname{Re} v_{ij})^a (\operatorname{Im} v_{ij})^b - \mathbb{E}(\operatorname{Re} w_{ij})^a (\operatorname{Im} w_{ij})^b| \leq N^{-\delta-2+(a+b)/2}, \quad 1 \leq a + b \leq 4. \tag{1.4}$$

Let $p_{v,N}^{(k)}$ and $p_{w,N}^{(k)}$ be the k -point correlation functions of the eigenvalues w.r.t. the probability law of the matrix $H^{(v)}$ and $H^{(w)}$, respectively. Then for any $|E| < 2$, any $k \geq 1$ and any compactly supported continuous test function $O : \mathbb{R}^k \rightarrow \mathbb{R}$ we have

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}^k} d\alpha_1 \dots d\alpha_k O(\alpha_1, \dots, \alpha_k) \left(p_{v,N}^{(k)} - p_{w,N}^{(k)} \right) \left(E + \frac{\alpha_1}{N}, \dots, E + \frac{\alpha_k}{N} \right) = 0. \tag{1.5}$$

Theorem 1.3 concerns Hermitian matrices under the high moment decay condition (1.1). On the other hand, Theorem 4.2 of [8] was stated for real symmetric matrices under a subexponential decay condition for the probability law of its rescaled matrix elements. The modification needed in the proof to cover the Hermitian case is obvious. The subexponential condition was used only to obtain subexponential decay in N for certain large deviation events. Since decay of order N^{-C} for C large enough is sufficient for the purpose of proving Theorem 1.3, the proofs in [8, 9] carry through except changing everywhere "subexponential small probability" to "with probability N^{-C} for sufficiently large C ."

Theorem 1.3 allows us to compare local correlation functions of H_0 with $H(a)$ if ε is sufficiently small. This concludes the proof of Theorem 1.1. \square

We emphasized that all our general results are valid for a fixed energy when restricted to Hermitian ensembles. As an example, we now state the bulk universality for generalized Hermitian Wigner ensembles at a fixed energy. Recall that a *generalized Hermitian Wigner matrix* $H = (h_{ij})$ has independent centered entries with variances $\sigma_{ij}^2 = \mathbb{E}|h_{ij}|^2$ satisfying

$$\sum_j \sigma_{ij}^2 = 1, \quad \delta < N\sigma_{ij}^2 < \delta^{-1}$$

for all i, j and for some $\delta > 0$ independent of N .

Theorem 1.4. *Suppose that $H = (h_{ij})$ is a generalized Hermitian Wigner matrix. Assume that for some $\gamma > 4$ we have*

$$\mathbb{E} \left| \frac{h_{ij}}{\sigma_{ij}} \right|^\gamma \leq C, \quad (1.6)$$

for some constant C , independent of i, j , and N . Then (1.2) holds for any fixed energy $|E| < 2$.

Proof. This theorem is the same as the Hermitian version of Theorem 7.2 in [2] except that the weak convergence in energy is now replaced by convergence for any fixed energy in the bulk. The main idea behind the proof of Theorem 7.2 is a cutoff argument which compares the original heavy-tailed ensemble to one with a subexponential decay. This cutoff argument holds for any fixed energy. The energy average was needed in Theorem 7.2 because the universality of the comparison ensemble with subexponential decay was stated in the weak sense in energy so as to apply it to both symmetry classes. In order to prove Theorem 1.4, we only have to prove the universality of generalized Hermitian Wigner ensembles with subexponential decay at a fixed energy. Inspecting the proof of Theorem 1.1 just given above, we only have to check Proposition 1.2 (i.e., Proposition 3.1 in [3]) is valid for generalized Wigner matrices as well.

Along the proof of Proposition 3.1 in [3], the properties of the original Wigner ensemble were used only in the estimate (3.9) which is a direct consequence of a version of the local semicircle law. Since this law was proved for generalized Wigner ensembles (see, e.g., Theorem 2.1 in [9]), this extends Proposition 3.1 of [3] to the generalized Hermitian Wigner case. Together with the cutoff argument in [2], this implies Theorem 1.4. \square

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