

Existence of an intermediate phase for oriented percolation*

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Abstract

We consider the following oriented percolation model of $\mathbb{N} \times \mathbb{Z}^d$: we equip $\mathbb{N} \times \mathbb{Z}^d$ with the edge set $\{(n, x), (n + 1, y) \mid n \in \mathbb{N}, x, y \in \mathbb{Z}^d\}$, and we say that each edge is open with probability $pf(y - x)$ where f is a fixed non-negative compactly supported function on \mathbb{Z}^d with $\sum_{z \in \mathbb{Z}^d} f(z) = 1$ and $p \in [0, \inf(1/f)]$ is the percolation parameter. Let p_c denote the percolation threshold and Z_N the number of open oriented-paths of length N starting from the origin. We investigate the growth-rate of Z_N when percolation occurs. We prove that if $d \geq 5$ and the function f is sufficiently spread-out, then there exists a second threshold $p_c^{(2)} > p_c$ such that Z_N/p^N decays exponentially fast for $p \in (p_c, p_c^{(2)})$ and does not so when $p > p_c^{(2)}$. The result should extend to the nearest neighbor-model for high-dimension, and for the spread-out model when $d = 3, 4$. It is known (see [2, 20]) that this phenomenon does not occur in dimension 1 and 2.

Keywords: Percolation; Growth model; Directed Polymers; Phase transition; Random media.

AMS MSC 2010: 82D60, 60K37, 82B44.

Submitted to EJP on February 7, 2012, final version accepted on May 31, 2012.

Supersedes arXiv:1201.4552v2.

1 Introduction

Oriented percolation was introduced by Hammersley [16] as a model for porous-media, the oriented-character of the model can be seen as an attempt to take into account gravity when considering diffusion of liquid the medium (as opposed to ordinary percolation which was introduced at the same period). Its main object is to study connectivity property of an inhomogeneous lattice in a given orientation. In this paper we do not focus on the most common question for percolation, which is existence of infinite open path, but rather on their abundance. The main result we prove is that if we consider a graph with a density of open edges barely sufficient to create infinite open-path, then the number of open path of a given length starting from the origin is much lower than its expected value. This implies (see [26]) that diffusion does not occur in the medium. This contrasts with what happens when there is an high density

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of edge and the transversal dimension is larger than 3, in which case, the number of open path is roughly equal to its expected value when percolation occurs. Although we believe that this phenomenon holds with great generality, with the method developed here, we can prove it only for a spread-out version of the model and possibly for very high-dimensional nearest neighbor model. The proof is based on size biasing argument and path counting.

2 Model and result

2.1 The model

We consider the following oriented independent edge-percolation model on $\mathbb{N} \times \mathbb{Z}^d$:

- Consider $f : \mathbb{Z}^d \rightarrow \mathbb{R}^+$ and with finite support and $p \geq 0$ that satisfies

$$\sum_{z \in \mathbb{Z}^d} f(z) = 1 \text{ and } p \leq \inf_{z \in \mathbb{Z}^d} (1/f(z)) := p_{\max}. \quad (2.1)$$

- We define $(X_{(n,x),(n+1,y)})_{n \geq 0, x, y \in \mathbb{Z}^d}$, our percolation environment, to be a field of independent Bernouilli random variable with parameter

$$\bar{p}(x - y) := pf(x - y), \quad (2.2)$$

(let \mathbb{P}_p and \mathbb{E}_p denote the probability distribution and expectation).

- We say that an edge $((n, x), (n + 1, y))$ is *open* if $X_{(n,x),(n+1,y)} = 1$, and that an oriented path

$$(S_n)_{N_1 \leq n \leq N_2}, \quad N_1 \geq 0, N_2 \in \mathbb{N} \cup \{\infty\}$$

is *open* if all the edges

$$((n, S_n), (n + 1, S_{n+1}))_{N_1 \leq n \leq N_2 - 1}$$

are open.

The usual aim of oriented-percolation is to investigate the existence of infinite open paths and their properties, and more generally, the connectivity properties of the oriented-network formed by the open-edges. And also the evolution of these property for fixed f when p varies. In this paper we focus more specifically on two cases for the function f :

$$\begin{aligned} f(z) &:= \frac{1}{2d} \mathbf{1}_{|z|_1=1}, \\ f(z) &:= \frac{1}{(2L+1)^d} \mathbf{1}_{|z|_\infty \leq L}, \end{aligned} \quad (2.3)$$

(where $|z|_1 := \sum_{i=1}^d |z_i|$, $|z|_\infty := \max_{1 \leq i \leq d} |z_i|$) The first case is called nearest neighbor oriented-percolation and is the one that has received the most interest in the physics literature. The second case is called spread-out oriented-percolation with range L (L is to be thought as a large integer). Spread-out models have been studied by mathematicians for a long time for a technical reason: whereas considering long-range (but finite) interactions instead of nearest-neighbor ones should not change the essential properties of a model, a lot of questions becomes easier to solve for these models when the range L gets large (an example of that is the use of long-range model to make the lace expansion work for all dimensions above the critical one, see [25]). We will study also a generalized version of the spread-out oriented-percolation.

One defines \mathcal{P} to be the event of percolation from the origin

$$\mathcal{P} := \{\exists(S_n)_{n \geq 0}, S_0 = 0, \forall n \geq 0, X_{(n, S_n), (n+1, S_{n+1})} = 1\}. \quad (2.4)$$

One defines the percolation threshold by

$$\begin{aligned} p_c &:= \inf\{p \geq 0 \mid \mathbb{P}_p(\mathcal{P}) > 0\} \\ &= \sup\{p \geq 0 \mid \mathbb{P}_p(\mathcal{P}) = 0\}. \end{aligned} \quad (2.5)$$

The aim of this paper is to discuss the asymptotics of the number of open path of length N starting from the origin when percolation occurs. Define

$$\begin{aligned} Z_N(X) = Z_N &:= \#\{\text{open oriented-paths of length } N \text{ starting from the origin}\} \\ &:= \#\{S : \{0, \dots, N\} \mid S_0 = 0, S \text{ is open}\}. \end{aligned} \quad (2.6)$$

We want to compare the asymptotic behavior of Z_N with the one of its expected value: When percolation does not occur, $Z_N = 0$ for N large enough when percolation occurs from the origin the questions we want to answer are:

- (i) Is Z_N asymptotically equivalent (up to a random positive constant) to $\mathbb{E}_p[Z_N] = p^N$.
- (ii) Does Z_N have the same exponential growth-rate that $\mathbb{E}_p[Z_N]$, i.e. is $\log Z_N$ equal to $N(1 + o(1)) \log p$?

In order to better formulate these questions we need to introduce some notation and technical results (that we prove in the next Section):

2.2 Upper-growth rate and renormalized partition function

Define

$$\begin{aligned} W_N(X) = W_N &:= Z_N / \mathbb{E}_p[Z_N] = p^{-N} Z_N, \\ \mathcal{X} &:= \limsup_{N \rightarrow \infty} \frac{1}{N} \log Z_N \leq \log p, \end{aligned} \quad (2.7)$$

(take the convention that the \limsup is equal to $-\infty$ when percolation does not occur) that we call respectively the renormalized partition function, and the upper-growth rate for the number of path.

Proposition 2.1 (Properties of the upper-growth rate of Z_N). (i) *On the event \mathcal{P} , \mathcal{X} is a.s. constant and thus we can define $f(p)$ by the relation*

$$\begin{aligned} \mathcal{X} &=: f(p), \quad \mathbb{P}_p(\cdot \mid \mathcal{P}) \text{ a.s.} \quad \text{if } \mathbb{P}_p(\mathcal{P}) > 0, \\ f(p) &:= -\infty, \quad \text{if } \mathbb{P}_p(\mathcal{P}) = 0. \end{aligned} \quad (2.8)$$

(ii) *The function*

$$\begin{aligned} [0, p_{\max}] &\rightarrow \{-\infty\} \cup \mathbb{R} \\ p &\mapsto f(p) - \log p. \end{aligned} \quad (2.9)$$

is non-decreasing so that the threshold

$$p_c^{(2)} := \inf\{p \in [0, p_{\max}] \mid f(p) = \log p\} = \sup\{p \in [0, p_{\max}] \mid f(p) < \log p\}. \quad (2.10)$$

is well defined whenever $\{p \in [0, p_{\max}] \mid f(p) = \log p\}$ is non-empty.

Remark 2.2. It is rather intuitive that some kind of self-averaging should occur and that whenever percolation occurs, one should have

$$f(p) = \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N. \quad (2.11)$$

Thus $e^{f(p)}$ should be understood as some sort of quenched connective constant for the oriented percolation network. However we could not prove this statement, nor find a proof of it in the literature. This is the reason why the less natural lim sup appears in the definition.

Proposition 2.3. *The random sequence $(W_N)_{N \geq 0}$ is a positive martingale with respect to the natural filtration $(\mathcal{F}_N)_{N \geq 0}$ defined by*

$$\mathcal{F}_N := \sigma(X_{(n,x),(n+1,y)}, x, y \in \mathbb{Z}^d, 0 \leq n \leq N-1). \quad (2.12)$$

Thus the limit

$$W_\infty := \lim_{N \rightarrow \infty} W_N. \quad (2.13)$$

exists almost surely. Conditioned on \mathcal{P} , W_∞ satisfies the following zero-one law.

$$\mathbb{P}_p(W_\infty > 0 \mid \mathcal{P}) \in \{0, 1\}. \quad (2.14)$$

Moreover the function

$$p \mapsto \mathbb{P}_p(W_\infty > 0) \quad (2.15)$$

is non-decreasing so that

$$p_c^{(3)} := \inf\{p \in [0, p_{\max}] \mid \mathbb{P}_p(W_\infty > 0) > 0\} = \sup\{p \in [0, p_{\max}] \mid \mathbb{P}_p(W_\infty > 0) = 0\}, \quad (2.16)$$

is well defined whenever $\{p \mid \mathbb{P}_p(W_\infty > 0)\}$ is non-empty.

Remark 2.4. For notational convenience one can set $p_c^{(2)} = p_{\max}$ (resp. $p_c^{(3)} = p_{\max}$) when (2.10) (resp. (2.16)) does not give a definition. This is not of crucial importance, since $\{p \mid \mathbb{P}_p(W_\infty > 0)\}$ is non-empty in all the cases for which we present results.

With these definition, we trivially have

$$p_c \leq p_c^{(2)} \leq p_c^{(3)} \leq p_{\max}, \quad (2.17)$$

and one wants to investigate whether if these inequalities are strict or not.

Remark 2.5. Whether W_N tends to a positive limit or tends to zero also has an interpretation in terms of paths localization: when $p > p_c^{(3)}$ an open path of length N should essentially look like a typical simple random walk path, and, once rescaled, converge in law to Brownian Motion (to make this a rigorous statement, one needs to adapt the proof of Comets and Yoshida [9]). On the contrary, when $p < p_c^{(2)}$, trajectories should behave in a totally different way, having a different scaling exponent and exhibiting localization (in a sense that two independent trajectories chosen at random should intersect many times), see [6] for a rigorous statement about localization for directed polymers, and [27] for oriented percolation.

2.3 Known results from the literature and formulation of the problem

The idea of investigating $f(p)$ appears in the work of Darling [12] where a list of open-questions is present.

A first question is to know what is the condition for having a phase in which $p = f(p)$ (to have $p_c^{(2)} < p_{\max}$). This question has been fully answered for the nearest-neighbor model and it turns out that this occurs only in dimension 3 and higher (on which we focus our attention)

$$p_c < p_c^{(2)} = p_c^{(3)} = p_{\max} \iff d = 1 \text{ or } 2. \quad (2.18)$$

Yoshida [26] gave the answer for the case $d = 1$ (for a more general model, called Linear Stochastic Evolution) adapting a result of Comets and Vargas for directed polymers [8] and also proved $p_c^{(3)} = p_{\max}$ for $d = 2$, also using methods related to directed polymers. Bertin [2] completed the picture by proving $p_c^{(2)} = p_{\max}$ for $d = 2$ (adapting a result proved for directed polymers in [20]). Note also that these result may be extended into more quantitative one by using the same methods as in [20]. We do not give the proof of these statement as the proof is not short and exactly similar to what is done in [20, 3, 21] for directed polymers in discrete a setup, Poissonian environment, or Brownian environment, but we think that they are worth being mentioned:

Proposition 2.6. *We have the following quantitative estimate on $f(p)$ for $d = 1$ and $d = 2$*

- (i) *For the nearest neighbor-model with $d = 1$, there exists a constant c such that for all $p \in [1, 2]$ (recall that $p_{\max} = 2$ in that case)*

$$\frac{1}{c}(p - p_{\max})^2 \leq \log p - f(p) \leq c(p - p_{\max})^2 |\log(p_{\max} - p)|. \quad (2.19)$$

- (ii) *For the nearest neighbor-model with $d = 2$ there constants c and ε such that for all $p \in [4 - \varepsilon, 4]$ (recall that $p_{\max} = 4$ in that case)*

$$\exp\left(-\frac{c}{(p_{\max} - p)^2}\right) \leq \log p - f(p) \leq \exp\left(-\frac{1}{c(p_{\max} - p)}\right). \quad (2.20)$$

Remark 2.7. In analogy with what one has for directed polymer the lower bound in (2.19) should give the right order for $\log p - f(p)$ whereas the upper-bound is supposed to be the true asymptotic in (2.20). More justification on these conjectures is available in [20].

In dimension $d \geq 3$, one can to get an upper-bound on $p_c^{(3)}$ by using second-moment method and the fact that d -dimensional random walks are transient. This was first noticed by Kesten, who was only interested in that to get an upper-bound on the percolation threshold p_c , his proof appears in [10] for a different oriented percolation model.

We present briefly his argument below adapted to our setup: if $\sup_{N \geq 0} \mathbb{E}_p [W_N^2] < \infty$, then W_N is uniformly integrable so that $\mathbb{E}_p [W_\infty] = 1$ and thus $\mathbb{P}_p(W_\infty > 0) > 0$.

To compute the second moment one considers $(T_n)_{n \geq 0}$ the random walk on \mathbb{Z} starting from zero, with IID increments whose law \mathbf{P} is given by

$$\mathbf{P}(T_1 = x) = f(x). \quad (2.21)$$

Then

$$\mathbb{E}_p[W_N^2] = \mathbf{E}^{\otimes 2} \left[\prod_{\{n \in [0, N-1] \mid T_n^{(1)} = T_n^{(2)} \text{ and } T_{n+1}^{(1)} = T_{n+1}^{(2)}\}} (f(T_{n+1}^{(1)} - T_n^{(1)})p)^{-1} \right]. \quad (2.22)$$

where $\mathbf{E}^{\otimes 2}$ denote expectation with respect to two independent copies of T . Some further calculation then show that

$$\lim_{N \rightarrow \infty} \mathbf{E}_p[W_N^2] < \infty \Leftrightarrow p \geq \frac{1}{1 - \mathbf{P}^{\otimes 2} \left[S_1^{(1)} \neq S_1^{(2)} ; \exists n \geq 2, T_n^{(1)} = T_n^{(2)} \right]}. \quad (2.23)$$

It implies in particular that $p_c^{(3)} < p_c$ when f is uniformly distributed over a set (e.g. for both cases given in (2.3)).

For percolation on the $d + 1$ -regular tree, it is known that $p_c = p_c^{(2)} = p_c^{(3)}$ and this can easily be achieved by performing the two first moment of the number of paths. (see [19] for the best known general result with that flavor on Galton-Watson trees).

We can then precise our question and ask ourselves: when $d \geq 3$ do we have in general $p_c^{(2)} > p_c$, which means, do we have an intermediate phase where percolation occurs, but with much less paths than expected, and this also on the exponential scale. We give a (positive) answer to this question in the case of spread-out percolation, in the limit where the range L is sufficiently large. The reason why we cannot give a full answer for other cases is our lack of knowledge on the value of the percolation threshold p_c , but the method we develop here gives also some results for large d in the nearest-neighbor model and for the spread-out model with $d = 3$ and 4. We discuss the case of nearest-neighbor case with $d \geq 3$ small later in this introduction.

The question of of studying the growth rate $f(p)$ appears in a work of Darling [12], but it seems that it has then been left aside for many years. In [7] a similar problem is raised but concerning the number of path with a density ρ of open edges. More recently [15], Fukushima and Yoshida proved that $f(p) > 1$ when \mathcal{P} occurs with positive probability in the generalized setup of Linear Stochastic Evolution.

2.4 Results

Our main result in this paper is an asymptotic lower-bound for $p_c^{(2)}$ for both:

- (i) The high-dimensional nearest neighbor model (first example in (2.3)), for large d .
- (ii) A spread-out model that generalizes the second example in (2.3) and that we describe below.

Consider F being a continuous function $\mathbb{R}^d \rightarrow \mathbb{R}^+$ with compact support which is invariant under the reflections $(x_1, \dots, x_i, \dots, x_d) \mapsto (x_1, \dots, -x_i, \dots, x_d)$ and such that $\int F(x)dx = 1$, and (large) $L \in \mathbb{N}$ and set

$$f_L(x) := \frac{F(x/L)}{\sum_{y \in \mathbb{Z}^d} F(y/L)}. \quad (2.24)$$

Theorem 2.8. *For the nearest neighbor-model one has, asymptotically when $d \rightarrow \infty$*

$$p_c^{(2)}(d) \geq 1 + \frac{\log 2}{2d^2} + O(d^{-3}). \quad (2.25)$$

For the generalized spread-out model one has, for every $d \geq 3$, when $L \rightarrow \infty$

$$p_c^{(2)}(d, L) \geq 1 + \log 2 \sum_{k=2}^{\infty} f_L^{*2k}(0) + O(L^{-3d/2}). \quad (2.26)$$

where $*$ stands for discrete convolution.

For the spread-out model in dimension $d > 4$ the above results gives a positive answer to the question raised earlier concerning the existence of an intermediate phase :

- The bound we have on $p_c^{(2)}$ can also read

$$p_c^{(2)} \geq 1 + \frac{\log 2}{L^d} \sum_{k=2}^{\infty} F^{*2k}(0) + o(L^{-d}). \quad (2.27)$$

- In [18] lace expansion has been used to prove asymptotic in L of p_c for the spread-out model for $d > 4$. Their result (Theorem 1.1) implies

$$p_c = 1 + \frac{1}{2L^d} \sum_{k=2}^{\infty} F^{*2k}(0) + o(L^{-d}). \quad (2.28)$$

Thus we have

Corollary 2.9. *For any $d > 4$, for all L large enough one has*

$$p_c^{(2)}(d, L) > p_c(d, L). \quad (2.29)$$

In [18], it is also mentioned that (2.28) should still hold for $d = 3, 4$ (and thus also the above Corollary). Indeed, the analogue of (2.28) has been proved to hold for the contact process in [14], and to many respect this model is very similar to oriented percolation.

Remark 2.10. In a work of Blease [5], a heuristic power-expansion of p_c as a function of $1/d$ is given for a different oriented percolation model closely related to this one. Adapted to our setup, it tells us that the conjectured asymptotic for p_c is is

$$p_c = 1 + \frac{1}{4d^2} + O\left(\frac{1}{d^3}\right). \quad (2.30)$$

Asymptotic for p_c to even higher order have been obtained rigorously for (unoriented) nearest neighbor percolation (see [17]), using lace-expansion techniques, and it is quite reasonable to think that a similar work for this model (which to many respect is simpler to handle than usual percolation) would turn (2.30) into a rigorous statement, and yield $p_c^{(2)} > p_c$ in high-enough dimension.

Remark 2.11. We believe that $p_c^{(2)} > p_c$ in every lattice model of oriented percolation, when the transversal dimension d is larger than 2. This conjecture is supported by the fact that we are able to prove the result for the spread out model for any profile function F . However there are several obstacles to prove this for the nearest-neighbor model when d is not very large (e.g. $d = 3$ or 4). We develop this point in the open-question section.

2.5 Open questions

Finally we present open questions or possible direction for research:

2.5.1 Concerning oriented percolation

:

- Equation (2.23) gives an upper-bound for $p_c^{(3)}$, which for large L gives, (in the spread-out case)

$$p_c^{(3)} \leq 1 + \sum_{k=2}^{\infty} f_L^{*2k}(0) + O(L^{-2d}). \quad (2.31)$$

This makes us wonder what is the correct asymptotic behavior of $p_c^{(3)}$ at the second order when L goes to infinity.

- In analogy with a conjecture for directed polymer, it is natural that one should have $p_c^{(2)} = p_c^{(3)}$ in general, which means that when W_N decays to zero, it does so exponentially fast, except maybe at the critical point. Whether $W_\infty = 0$ at $p_c^{(3)}$ is more difficult to conjecture and this may depend on the dimension.
- Proposition 4.3, gives a way to get a lower bound on $p_c^{(2)}$ that is quite general (if $\mathbb{E}_p \log \bar{Z}_N > N \log p$, then $p \geq p_c^{(2)}$). This bound should get quite acute when N takes large value. However $\mathbb{E}_p \log \bar{Z}_N$ is quite heavy to compute by brute force as it involves $O(N^d)$ Bernoulli variables. An interesting perspective would be get method to compute the above expectation in an effective way (e.g. combining computer calculus and concentration-like theoretical results) and and see how the obtained bound compares with conjectured values for p_c ([11, Table |B.1]). To get an answer to the question “ $p_c^{(2)} > p_c^{(1)}$?” one also have to find an good upper-bound on $p_c^{(1)}$. Indeed the only rigorous upper-bound for $p_c^{(1)}$ we have seen in the literature is the one from [10] and it is also an upper-bound for $p_c^{(3)}$ (and thus can be of no use for our purpose).

2.5.2 Concerning percolation

: In analogy with what we do here for oriented-percolation, it is a quite natural question to study the *quenched* connective constant of percolated lattices, that is to say: the asymptotic growth-rate (with N) of the number of edge (or site) self-avoiding paths of length N starting from a given point of the graph. In particular one would like to understand how its compares with the growths rate of its expected value (the *annealed* connective constant which is trivially equal to $p\nu$ where ν is the connective constant for the original lattice). The problem seems considerably more difficult that in the directed case, as self-avoiding walks are involved instead of directed walks (see the monograph of Madras and Slade for a rather complete account on Self Avoiding Walk. [23]). This issue has been studied in the physics literature (see e.g. [1] and reference therein although we would not quite agree with the conjecture that are present there, see below), but to our knowledge, no rigorous result has been established so far on the mathematical side:

- To begin with it would be nice to prove the existence of the quenched connectivity constant (something we were not able to do in the directed case see Remark 2.2).
- Another question would be the existence of a phase where the number of open self-avoiding path starting from the origin behave likes its expected value (like here when $d \geq 3$ and p large). This could happens on \mathbb{Z}^d if the dimension is large enough, (say e.g. $d > 4$ as 4 is the critical dimension for self-avoiding walk, but the author has far from enough evidence to state this as a conjecture), and one could try to prove that using second moment method similar to (2.22): The question reduces then to know whether or not the Laplace transform of the overlap of two infinite self avoiding walk of is bounded in a neighborhood of zero. This seems quite a difficult question to tackle but maybe not hopeless as there have been quite a lot of tools developed to study and understand the self-avoiding walk in high-dimension (see [25]).

- A challenging issue is to prove that this never happens in low dimension (say $d = 2$ and $d = 3$), and that for whatever small edge-dilution, the quenched connectivity constant is strictly smaller than the annealed one. As in the directed case, there are heuristic evidences that this occurs in dimension 2 and 3, but there is a need of a better (i.e. rigorous) understanding the behavior of the self-avoiding walk to transform that into a proof. The recent important result obtained on the hexagonal lattice [13] gives some hopes that more about that will be known in the future at least in some special two dimensional case.
- Another question is to settle whether if, in high dimension, just above the critical point, the number of path is exponentially smaller than its expected value. For this, the techniques used in this paper might adapt and some precise asymptotics in d are available for the value of p_c is available in the literature (see [17]).

Remark 2.12. After the present paper was written, the question for percolation in dimension 2 has been solved by the author: it has been proved in [22] that for every $p > p_c$, the number of self-avoiding paths of length n starting from a point of the supercritical percolation cluster has a smaller growth rate than its expectation.

3 Technical preliminaries

This Section is devoted to the proof of Proposition 2.1 and 2.3. We let the reader check that W_N is a martingale, and prove all the other statements. The Section is divided into two parts, one for the proof of the 0 – 1 law statements, and the other for the proof of monotonicities in p by coupling.

3.1 Zero-One laws

In this Section we prove

- (i) $\mathbb{P}_p[W_\infty > 0 \mid \mathcal{P}] \in \{0, 1\}$.
- (ii) \mathcal{X} is as constant on the event \mathcal{P} .

The proof of the two statement use exactly the same ideas thus we prove (i) in full detail and then explain how to adapt the proof to get (ii).

Set $\kappa = \mathbb{P}_p[W_\infty > 0]$. For $x \in \mathbb{Z}^d$, $N \in \mathbb{N}$ set $Z_N(x)$ to be the number of open path from $(0, 0)$ to (N, x)

$$Z_N(x) := \#\{(S_n)_{n \in [0, N]} \mid S_0 = 0, S_N = x, S \text{ is open}\}. \quad (3.1)$$

We define also $\theta_{N,x}$ to be the shift operator on the environment X (which determines the set of open edges) by

$$\theta_{N,x}(X)_{(n,u),(n+1,v)} := X_{(n+N,u+x),(n+N+1,v+x)} \quad (3.2)$$

and $\theta_{N,x}W_\infty$ to be limit of the renormalized partition function constructed from the shifted environment $\theta_{N,x}(X)$ instead of X . One has

$$W_\infty := \sum_{x \in \mathbb{Z}^d} p^{-N} Z_N(x) \theta_{N,x}W_\infty. \quad (3.3)$$

Then, as $(\theta_{N,x}W_\infty)_{x \in \mathbb{Z}^d}$ is independent of \mathcal{F}_N (recall (2.12)),

$$\mathbb{P}_p(W_\infty > 0 \mid \mathcal{F}_N) = \mathbb{P}_p[\exists x, Z_N(x) > 0, \theta_{N,x}W_\infty > 0 \mid \mathcal{F}_N] \geq \kappa \mathbf{1}_{Z_N > 0}. \quad (3.4)$$

Thus

$$\mathbb{P}_p(W_\infty > 0 \mid \mathcal{F}_N) \geq \kappa \mathbf{1}_{Z_N > 0}, \quad (3.5)$$

so that making N tends to infinity, one gets

$$\mathbf{1}_{\{W_\infty > 0\}} \geq \kappa \mathbf{1}_{\mathcal{P}}, \quad (3.6)$$

which is enough to conclude that either $\kappa = 0$ or $W_\infty > 0$ on the event of percolation.

We now turn to (ii). Given r define $\mathbb{P}_p[\mathcal{X} > r] := \kappa'$. Then by the same reasoning as for the proof of (i)

$$\mathbb{P}_p[\mathcal{X} > r | \mathcal{F}_N] \geq \kappa' \mathbf{1}_{\{Z_N > 0\}}. \quad (3.7)$$

Thus making N tends to infinity one get that either $\kappa' = 0$ or that $\mathcal{X} > r$ a.s. when \mathcal{P} occurs, and this for all r . □

3.2 Existence of the threshold

We prove now that the function in (2.9) and (2.15) are non-decreasing (and thus that $p_c^{(2)}$ and $p_c^{(3)}$ are well defined) by a coupling argument. The idea of the proof is quite common in percolation, it is to couple the realization of the process for different p in a monotone way. We consider a field of random variables $(U_{(n,x),(n+1,y)})_{n \geq 0, x, y \in \mathbb{Z}^d}$ uniformly distributed on $[0, 1]$ (denote its law by \mathbb{P}). Then one sets

$$X_{(n,x),(n+1,y)}(p) := \mathbf{1}_{\{U_{(n,x),(n+1,y)} \leq f(y-x)p\}}. \quad (3.8)$$

It is immediate to check that $X_{(n,x),(n+1,y)}(p)$ has distribution \mathbb{P}_p . This construction implies immediately that $\mathbb{P}_p(\mathcal{P})$ is a non-decreasing function of p and existence of p_c (but this is an already well established fact). In this section we write $W_N(p)$ for $W_N(X(p))$.

Set

$$\mathcal{F}_p := \sigma(X_{(n,x),(n+1,y)}(p), n \in \mathbb{N}, x, y \in \mathbb{Z}^d) \quad (3.9)$$

Our key observation is that for $p' \leq p$

$$\mathbb{E}[W_N(p') | \mathcal{F}_p] = W_N(p). \quad (3.10)$$

Indeed the probability that a path is open for $X(p')$ knowing that it is open for $X(p)$ is equal to $(p'/p)^N$. This gives immediately

$$\mathbb{P}[W_\infty(p') > 0] \leq \mathbb{P}[W_\infty(p) > 0]. \quad (3.11)$$

To check that $f(p') - \log p' \leq f(p) - \log p$, it is sufficient to show that for all ε , for all sufficiently large N

$$\log W_N(p') \leq N(f(p) - \log p + \varepsilon). \quad (3.12)$$

For N large enough $W_N(p) \leq N(f(p) - \log p + \varepsilon/2)$, by the definition of $f(p)$ and furthermore by (3.10) and the Borel-Cantelli Lemma.

$$\log W_N(p') \leq \log W_N(p) + N\varepsilon/2, \quad (3.13)$$

so that (3.12) holds. □

4 Size biasing

A key element of our proof is to consider the system under a law which has been tilted by W_N : the size-biased version of \mathbb{P}_p . In this section we present a nice way to describe the law of the environment under the size biased law, encountered in a paper of Birkner on directed polymers [4], and much related to the so-called spine techniques used in the study of branching structures (see e.g. [24]). Then we use this construction to get some operational condition under which the inequality $f(p) < \log p$ holds. The argument developed here are completely general and can be used for any kind of directed percolation model.

4.1 A description of the distribution of the environment under the size-biased law

We define $\tilde{\mathbb{P}}_p^N$ the so-called size-biased measure on the edge-environment X , as a measure absolutely continuous w.r.t to \mathbb{P}_p and whose Radon-Nikodym derivative is given by

$$\frac{d\tilde{\mathbb{P}}_p^N}{d\mathbb{P}_p}(X) := W_N(X), \quad (4.1)$$

and then one studies the behavior of W_N under this new measure. There are some reason for doing so, e.g. the sequence of W_N converges to zero in law if and only if $W_N \rightarrow \infty$ in law under the size biased measure $\tilde{\mathbb{P}}_p^N$.

We give a nice representation of the size-biased measure, adapted from the work of Birkner [4] on directed polymers. First we sample a random walk T chosen according to probability measure \mathbf{P} given by (2.21). Then given a realization of X under \mathbb{P}_p , we consider \tilde{X} an alternative percolation environment whose definition is given by

$$\tilde{X}_{(n,x),(n+1,y)} := \begin{cases} 1 & \text{if } T_n = x, T_{n+1} = y, \\ X_{(n,x),(n+1,y)} & \text{otherwise.} \end{cases} \quad (4.2)$$

Let \tilde{Z}_N, \tilde{W}_N be defined analogously to Z_N and W_N but using environment \tilde{X} instead of X . Then \tilde{W}_N under the law $\mathbb{P}_p \otimes \mathbf{P}$ as the same law that W_N under the size biased measure $\tilde{\mathbb{P}}_p$. More precisely

Proposition 4.1. *For any function $F : \mathbb{R} \rightarrow \mathbb{R}$ one has*

$$\mathbb{E}_p \mathbf{E} \left[F(\tilde{Z}_N) \right] = \tilde{\mathbb{E}}_p^N [F(Z_N)] := \mathbb{E}_p [W_N F(Z_N)]. \quad (4.3)$$

(the same result being obviously valid when Z_N is replaced by W_N)

Proof. See [4] Lemma 1. □

4.2 A link between the size-biased law and the original law

Some important properties of the law W_N under the original measure can be recovered from its property under the size biased measure. For example if W_N tends to 0 whereas its expectation is equal to one for every N , it means that W_N must be large on some atypical event that carries most of the expectation. Under the size-biased measure, this event must become typical.

Proposition 4.2. *We have the following properties concerning \tilde{W}_N ,*

(i) *If under $\mathbb{P}_p \otimes \mathbf{P}$,*

$$\lim_{N \rightarrow \infty} \tilde{W}_N = \infty, \quad (4.4)$$

in law, then

$$W_\infty = 0, \mathbb{P}_p \text{ a.s.} \tag{4.5}$$

(ii) If there exists a constant $c > 0$ such that for all N large enough

$$\mathbb{E}_p \mathbf{E} \left[\tilde{W}_N \leq N \right] \leq e^{-cN}, \tag{4.6}$$

then

$$f(p) < \log p - c/2. \tag{4.7}$$

Proof. The first point is classic. If W_∞ is non degenerate, there exists constants $a > 0$ and $A > a$ such that $\mathbb{P}_p [W_N \in (a, A)] > c$ uniformly for all N . Then $\tilde{\mathbb{P}}_p^N (W_N \in [a, A]) \geq ac$ so that in can't converge to ∞ in law.

For the second point, if (4.6) is valid then $W_\infty = 0$ from the first point, so that once N is large enough $W_N \leq 1$. We just have to check that $W_N \in (e^{-cN/2}, 1)$ does not happen infinitely often. One has

$$\begin{aligned} \mathbb{P}_p \left[W_N \in (e^{-cN/2}, 1) \right] &= \tilde{\mathbb{E}}_p^N \left[\frac{1}{W_N} \mathbf{1}_{W_N \in (e^{-cN/2}, 1)} \right] \\ &\leq e^{cN/2} \tilde{\mathbb{P}}_p \left[W_N \in (e^{-cN/2}, 1) \right] \leq e^{-cN/2}. \end{aligned} \tag{4.8}$$

Thus by Borel-Cantelli Lemma \mathbb{P}_p -a.s. eventually for all N

$$W_N \leq e^{-cN/2}. \tag{4.9}$$

□

4.3 Reduction to a finite volume criterion

The criterion given by (4.6), will be satisfied if one can bound $\log \tilde{W}_N$ from below by a sum of independent variables which have positive mean. This way of doing gives us a very simple criterion for having $f(p) < \log p$.

Set

$$\bar{Z}_N := \#\{S : \{0, \dots, N\} \mid S_0 = 0, S_N = T_N, S \text{ is open for } \tilde{X}\}. \tag{4.10}$$

It is straight-forward to see that

$$\bar{Z}_{N+M} \geq \bar{Z}_N \times \bar{Z}_M^{(1)}, \tag{4.11}$$

where $\bar{Z}_M^{(1)}$ is independent of \bar{Z}_N and has the same law that \bar{Z}_M . Hence we have

Proposition 4.3. *If for some value of N_0 one has*

$$\mathbb{E}_p \mathbf{E} \log \bar{Z}_{N_0} > N_0 \log p, \tag{4.12}$$

then there exists a constant c such that (4.6) holds (and thus so does (4.7)).

Proof. For $N \geq 0$, set $N =: nN_0 + r$ be the Euclidean division of N by N_0 . One has

$$\log \bar{Z}_N \geq \sum_{i=1}^n \log \bar{Z}_{N_0}^{(i)}, \tag{4.13}$$

where $\bar{Z}_{N_0}^{(i)}$ are independent copies of \bar{Z}_{N_0} . As a sum of i.i.d. bounded random variables, the probability of large deviation of the r.h.s. of (4.13) below its average is exponentially small in n (thus in N) and this ends the proof. □

5 Proof of Theorem 2.8

Now we want to use the criterion provided by Proposition 4.3 to get lower bounds for $p_c^{(2)}$ for specific models, either high-dimensional nearest-neighbor or spread-out. The particularity of these model is that all the $p_c^{(i)}$ are really close to one.

Our strategy is just to estimate the probability that $\bar{Z}_N \geq 2$ to get a lower-bound on $\frac{1}{N} \mathbf{E} \mathbf{E}_p \log \bar{Z}_N$.

We start with the spread-out model

Lemma 5.1. *For the spread-out model with $d \geq 3$ fixed, for all $p \geq 1$,*

$$\frac{1}{N} \mathbf{E} \mathbf{E}_p \log \bar{Z}_N \geq \log 2 \sum_{k=2}^N f_L^{*2k}(0) + O(N^{-1}L^{-d}) + O(NL^{-2d}). \tag{5.1}$$

In particular choosing $N_L := L^{d/2}$ one gets

$$\frac{1}{N_L} \mathbf{E} \mathbf{E}_p \log \bar{Z}_{N_L} \geq \log 2 \sum_{k=2}^N f_L^{*2k}(0) + O(L^{-3d/2}) \tag{5.2}$$

As a consequence of the Lemma, one gets that (4.12) holds as soon as

$$p \leq \exp \left(\frac{1}{N_L} \mathbf{E} \mathbf{E}_1 Z_{N_L} \right) = 1 + \log 2 \sum_{k=2}^N f_L^{*2k}(0) + O(L^{-3d/2}). \tag{5.3}$$

and this gives the wanted lower-bound for $p_c^{(2)}$ of Theorem 2.8.

Proof. One can focus on the case $p = 1$ with no loss of generality. Moreover one can bound the expectation of $\log \bar{Z}_N$ as follows,

$$\mathbf{E} \mathbf{E}_1 \log \bar{Z}_N \geq \mathbf{E} \mathbf{P}_1 [\bar{Z}_N \geq 2] \log 2. \tag{5.4}$$

Then notice that $Z_N \geq 2$ if and only if there exists $a < b$ and an open path linking (a, T_a) and (b, T_b) that does not use any edge nor meet any sites on the path T (except the starting and ending sites). Given a fixed T we want to estimate $\mathbf{P}_1 [\bar{Z}_N \geq 2]$. We call a path linking (a, T_a) and (b, T_b) a bridge (see Fig. 1), we call $\mathcal{B}_T = \mathcal{B}$ the set of "bridges" on $(T_n)_{0 \leq n}$

$$\begin{aligned} \mathcal{B}_T := \{ (B_n)_{a \leq n \leq b} \mid \exists 0 \leq a < b \leq n, \\ B_a = T_a, B_b = T_b, \forall c \in (a, b), B_c \neq T_c, f(B_{c+1} - B_c) > 0 \} \end{aligned} \tag{5.5}$$

One has

$$\begin{aligned} \mathbf{P}_1 [\bar{Z}_N \geq 2] &= \mathbf{P}_1 [\exists B \in \mathcal{B} ; B \text{ is open}] \\ &\geq \sum_{B \in \mathcal{B}} \mathbf{P}_1 [B \text{ is open}] - \sum_{B \in \mathcal{B}} \mathbf{P}_1 [B \text{ is open} ; \exists \tilde{B} \in \mathcal{B} \setminus \{B\}, \tilde{B} \text{ is open}] \\ &\geq \left(\sum_{B \in \mathcal{B}} \mathbf{P}_1 [B \text{ is open}] \right) \left(1 - \max_{B \in \mathcal{B}} \mathbf{P}_1 [\exists \tilde{B} \in \mathcal{B} \setminus \{B\}, \tilde{B} \text{ is open} \mid B \text{ is open}] \right). \end{aligned} \tag{5.6}$$

We first control the term $\sum_{B \in \mathcal{B}} \mathbf{P}_1 [B \text{ is open}]$ and its expected value with respect to T . It is larger than

$$\sum_{a=0}^{N-2} \sum_{b=a+2}^N \left(f_L^{*(b-a)}(T_b - T_a) - \sum_{c=a+1}^{b-1} f_L^{*(c-a)}(T_c - T_a) f_L^{*(b-c)}(T_c - T_b) \right), \tag{5.7}$$

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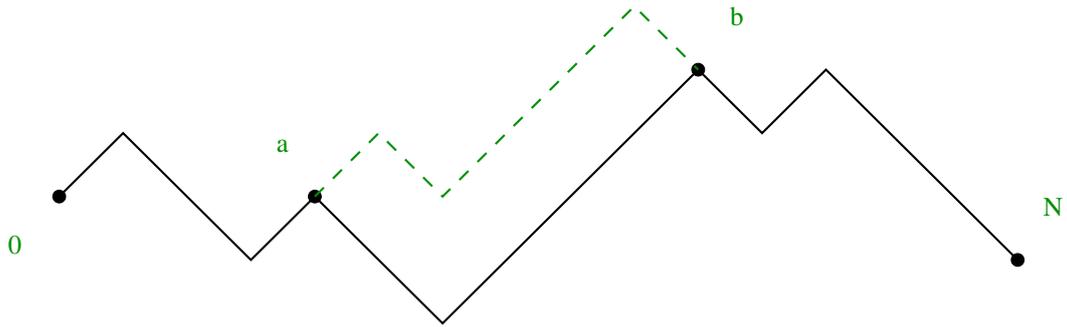


Figure 1: Illustration of a trajectory T of length $N = 15$ (full line), together with a bridge between $a = 4$ and $b = 10$ for the nearest-neighbor model in dimension $1 + 1$. The bridge does not meet the trajectory T (site-wise) except at the starting and ending points.

(in the first term, some paths between (a, T_a) and (b, T_b) intersect T at intermediate point, i.e. that are not bridges, have been counted, the second term is subtracting the contribution of all those path: all path intersecting T at time c with varying c . Some contribution are subtracted more than once and that is the reason why we get an inequality).

Using the fact that f is symmetric we can compute the expected value of the first term in (5.7) (averaging over T that has law \mathbf{P}). It is equal to

$$\sum_{a=0}^{N-2} \sum_{b=a+2}^N f_L^{*2(b-a)}(0) = \sum_{k=2}^N (N-k) f_L^{*2k}(0). \quad (5.8)$$

The expected value with respect to \mathbf{P} of the second term in (5.7) is smaller than

$$\sum_{a=0}^N \left(\sum_{b=0}^{\infty} f_L^{*(2b)}(0) \right)^2 = O(NL^{-2d}) \quad (5.9)$$

Now it remains to show that for all choices of B and T ,

$$\mathbb{P}_1 \left[\exists \tilde{B} \in \mathcal{B} \setminus \{B\}, \tilde{B} \text{ is open} \mid B \text{ is open} \right] \text{ is small.} \quad (5.10)$$

To get an additional bridge on T between a and b knowing that B is open, one can either have an open path not using edges of T and B that links (a, T_a) to (b, T_b) or use open edges of B to construct a new open bridge say by having an open path that links say (a, B_a) and (b, T_b) (or (a, T_a) and (b, B_b) , or (a, B_a) and (b, B_b)). We can use union bound to get that

$$\begin{aligned} & \mathbb{P}_1 \left[\exists \tilde{B} \in \mathcal{B} \setminus \{B\} \tilde{B} \text{ is open} \mid B \text{ is open} \right] \\ & \leq \sum_{a=1}^{N-1} \sum_{b=a+1}^N \sum f_L^{*(b-a)}(T_b - T_a) + f_L^{*(b-a)}(B_b - T_a) \\ & \quad + f_L^{*(b-a)}(T_b - B_a) + f_L^{*(b-a)}(B_b - B_a) \\ & \leq 4 \sum_{a=0}^N \sum_{n=1}^{\infty} \max_z f_L^{*n}(z) \leq O(NL^{-d}). \quad (5.11) \end{aligned}$$

The conclusion of all this is that

$$\mathbf{EP}_1 [\bar{Z}_N \geq 2] = \left(\sum_{k=2}^N (N-k) f_L^{*2k}(0) + O(NL^{-2d}) \right) (1 - O(NL^{-d})), \quad (5.12)$$

and hence

$$\mathbf{EE}_1 [\log \bar{Z}_N] \geq \log 2 \left(\sum_{k=2}^N (N-k) f_L^{*2k}(0) + O(N^2L^{-2d}) \right). \quad (5.13)$$

This gives us the result once on has checked (using local central limit theorem) that

$$\sum_{k=0}^N k f^{*2k}(0) = O(L^{-d}). \quad (5.14)$$

□

We can turn to the nearest neighbor case which is a bit simpler

Lemma 5.2. *For the nearest neighbor we have the following lower asymptotic in d , valid for all $p \geq 1$, and for all N*

$$\mathbf{EE}_p [\log \bar{Z}_N] \geq \log 2 \left[1 - \left[1 - \frac{1}{2d^2} + O(d^{-3}) \right]^N \right]. \quad (5.15)$$

In particular choosing $N = d$ one gets

$$\frac{1}{d} \mathbf{EE}_p \log \bar{Z}_d \geq \frac{\log 2}{2d^2} + O(d^{-3}). \quad (5.16)$$

As a consequence of the Lemma, by monotonicity in p one gets that (4.12) holds as soon as

$$p < \exp \left(\frac{1}{d} \mathbf{EE}_1 \log \bar{Z}_d \right). \quad (5.17)$$

and this gives the wanted lower-bound for $p_c^{(2)}$ of Theorem 2.8.

Proof. We consider only the case $p = 1$.

We use the same strategy as for the spread-out model, but here we only need to consider the bridges of length 2. Set (e_1, \dots, e_d) the canonical base of \mathbb{Z}^d ($e_i := (0, \dots, 0, 1, 0, \dots, 0)$ where 1 lies in the i -th position) and $(e_{d+1}, \dots, e_{2d}) := -(e_d, \dots, e_d)$

Given a realization of T (which in this case is a nearest neighbor simple random walk on \mathbb{Z}^d), the number of possibly to have a bridge of length 2 between $a - 1$ and $a + 1$ depends on the local configuration of T (see Fig. 2):

- (i) If $T_a - T_{a-1} = T_{a+1} - T_a$ then there is no possibility.
- (ii) If $T_a - T_{a-1} = -T_{a+1} + T_a$ (i.e. if $T_{a-1} := T_{a+1}$) then there are $2d - 1$ possibilities for having a bridge, each has probability $(1/2d)^2$: opening the edges $[(a-1, T_{a-1}), (a, T_a)]$ and $[(a, T_a + e_i), (a+1, T_{a+1})]$ where i is such that $e_i \neq T_a - T_{a-1}$.
- (iii) In all other cases, there is only one possibility which is opening the edges $[(a-1, T_{a-1}), (a, T_{a-1} + e_i)]$ and $[(a, T_{a-1} + e_i), (a+1, T_{a+1})]$ where $e_i = T_{a+1} - T_a$.

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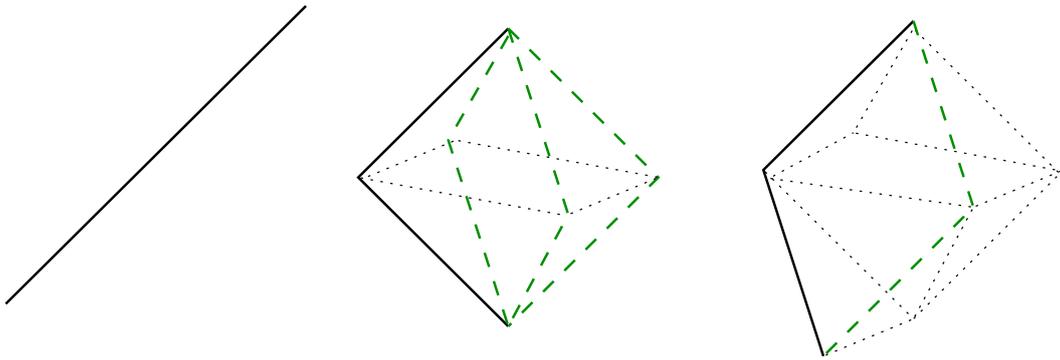


Figure 2: Schematic representation of the three different possibilities listed in the $1 + 2$ dimensional case, trajectories are oriented along the vertical direction. The full line represent the portion of trajectory and the dashed ones, the potential bridges of length 2. From left to right: (i) If the two increments for the walk are equal, there is no possibility to build a bridge, (ii) if they are opposite, one has $3 = 2 * 2 - 1$ options for bridges, (iii) if the two increments are along different dimension, there is only one way to build a bridge (by inverting the order of these increments).

Then we note that all the bridges over T length 2 are pairwise edge-disjoint, so that given T each bridges are open independently with probability $(2d)^{-2}$ (recall that $p = 1$).

Thus

$$\begin{aligned} \mathbb{P}_1[\bar{Z}_N = 1] &\leq \mathbb{P}_1[\text{All bridges of length 2 are closed}] \\ &\leq (1 - (2d)^{-2})^{(2d-1)\#\{a \in [1, N-1] \mid T_{a-1} = T_{a+1}\} + \#\{a \in [1, N-1] \mid T_{a+1} - T_a \neq \pm(T_a - T_{a-1})\}}. \end{aligned} \quad (5.18)$$

Averaging with respect to T gives

$$\begin{aligned} \mathbb{E}\mathbb{P}_1[\bar{Z}_N = 1] &\leq \left[\frac{1}{2d} (1 - (2d)^{-2})^{(2d-1)} + \frac{d-1}{d} (1 - (2d)^{-2}) + \frac{1}{2d} \right]^{N-1} \\ &= \left[1 - \frac{1}{2d^2} + O(d^{-3}) \right]^N. \end{aligned} \quad (5.19)$$

Thus

$$\mathbb{E}\mathbb{E}_1[\log \bar{Z}_N] \geq \log 2 \left[1 - \left[1 - \frac{1}{2d^2} + O(d^{-3}) \right]^N \right]. \quad (5.20)$$

□

Acknowledgments. The author is very grateful to N.Yoshida for suggesting this problem, and enlightening discussions, and to D.Villemonais for valuable comments on the manuscript. This work was written during the author's long-term stay in IMPA, he acknowledges kind hospitality and the support of CNPq.

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