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Some sufficient conditions for infinite collisions of simple random walks on a wedge comb

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Abstract

In this paper, we give some sufficient conditions for the infinite collisions of independent simple random walks on a wedge comb with profile $\{f(n), n \in \mathbb{Z}\}$. One interesting result is that two independent simple random walks on the wedge comb will collide infinitely many times if $f(n)$ has a growth order as $n \log n$. On the other hand, if $\{f(n), n \in \mathbb{Z}\}$ are given by i.i.d. non-negative random variables with finite mean, then for almost all wedge combs with such profile, three independent simple random walks on it will collide infinitely many times.

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1 Introduction

In this paper, we study the number of collisions of two independent simple random walks on an infinite connected graph with finite degrees. Let $X = \{X_n\}$ and $X' = \{X'_n\}$ be independent simple random walks starting from the same vertex. As usual, we say that the graph has the infinite collision property if X and X' collide infinitely often, i.e., $|\{n : X_n = X'_n\}| = \infty$, almost surely. Likewise we say that the graph has the finite collision property if X and X' collide finitely many times almost surely. Krishnapur and Peres [5] first finds the example $\text{Comb}(\mathbb{Z})$ on which two independent simple random walks will collide finitely many times. Notice that $\mathbb{Z} \subset \text{Comb}(\mathbb{Z}) \subset \mathbb{Z}^2$ and both $\mathbb{Z}$ and $\mathbb{Z}^2$ have the infinite collision property. This indicates that the infinite collision property is not simply monotone. So we are interested in probing the subgraphs of $\text{Comb}(\mathbb{Z})$.

Let f be a function mapping $\mathbb{Z}$ into $\mathbb{R}^+$. It induces a graph called wedge comb, $\text{Comb}(\mathbb{Z}, f)$, with the set of vertices

$$\mathbb{V} = \{(n, x) : n, x \in \mathbb{Z}, -f(n) \leq x \leq f(n)\}$$

and the set of edges $\{[(n, x), (n, y)] : |x - y| = 1\} \cup \{[(n, 0), (m, 0)] : |m - n| = 1\}$.

Chen, Wei and Zhang [3] shows that $\text{Comb}(\mathbb{Z}, f)$ has the infinite collision property when $f(n) \leq n^{\frac{1}{5}}$. Recently, Barlow, Peres and Sousi [1] gives a sufficient condition (in terms of Green functions) for the infinite collision property and shows that $\text{Comb}(\mathbb{Z}, f)$ has the infinite collision property when $f(n) \leq n$; while it has the finite collision property when $f(n) = n^\alpha$ for each $\alpha > 1$. Collisions on other graphes, such as random clusters and random trees are studied in [1] and [2]. We will restrict our attention to wedge combs, and give a sufficient condition for a wedge comb to have the infinite collision property.

Theorem 1.1. *Let $\check{f}(n) = 1 \vee \max_{-n \leq i \leq n} f(i)$. If*

$$\sum_{n=1}^{\infty} \frac{1}{\check{f}(n)} = \infty, \tag{1.1}$$

then $\text{Comb}(\mathbb{Z}, f)$ has the infinite collision property.

We like to point out that it is not required that $f(n)$ is monotone in n for $n \geq 0$. This would enable us to deal with a large class of wedge combs. As an application of Theorem 1.1, one can improve the results of [3] and [1].

Corollary 1.2. *If $f(-n) + f(n) = O(n \log n)$, then $\text{Comb}(\mathbb{Z}, f)$ has the infinite collision property.*

Corollary 1.2 should be compared with Theorem 1.3. Unfortunately there is a gap. We still do not know the answer for $1 < \beta \leq 2$.

Theorem 1.3. *Let $f(x) = |x| \log^\beta(|x| \vee 1)$ for $x \in \mathbb{Z}$. If $\beta > 2$, then the total number of collisions by two independent simple random walks on $\text{Comb}(\mathbb{Z}, f)$ is a.s. finite.*

A natural question to ask is what happens if there are more than two independent simple random walks. Suppose that X, X' and X'' are independent simple random walks on a graph. If $|\{n : X_n = X'_n = X''_n\}| = \infty$ almost surely, then we say that the three processes collide together infinitely often and that the graph has the infinitely many triple collisions property. It is shown in [1] that $\mathbb{Z}$ has the infinitely many triple collisions property; while $\text{Comb}(\mathbb{Z}, \alpha)$ has the finitely many triple collisions property for any $\alpha > 0$.

Theorem 1.4. *Let $\{f(n), n \in \mathbb{Z}\}$ are independent and identically distributed random variables with law μ supported in $[0, \infty)$. If μ has finite mean, then almost surely Comb($\mathbb{Z}, f$) has the infinitely many triple collisions property.*

Theorem 1.1 will be proved in the next section. Proofs of Theorems 1.3 and 1.4 will be given in Section 4 and Section 3 respectively. Throughout this paper we use the following notation. For $u, v \in \mathbb{V}$, $\mathbf{P}^u$ is the probability measure of the simple random walk X starting from u, and $\mathbf{P}^{u,v}$ is the joint probability measure of the two random walks X and X' starting from u and v, respectively. $\mathbf{E}^u$ and $\mathbf{E}^{u,v}$ stand for the corresponding expectations. For vertex $x \in \mathbb{V}$, we write x_1 for the first coordinate of x and x_2 for the second coordinate, i.e., $x = (x_1, x_2)$. For $a, b \in \mathbb{R}$, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. For set A, $|A|$ stands for the number of elements of A.

2 Proof of Theorem 1.1

Let $X = \{X_n, n \geq 0\}$ be a simple random walk on Comb($\mathbb{Z}, f$). Write $X_n = (U_n, V_n)$ for $n \geq 0$. Then $\{U_n\}$ is a random process on $\mathbb{Z}$. Define a sequence of stopping times: $T_0 = 0$ and

$$T_{k+1} := \inf\{n > T_k : U_n \neq U_{n-1}\}.$$

Almost surely $T_k < \infty$ for each $k \geq 1$. By definition, X stays in the line segment $\{u\} \times [-f(u), f(u)]$, where $u = U_{T_k}$, during time $[T_k, T_{k+1} - 1]$. Let $W_k = U_{T_k}$. Then $\{W_k, k \geq 0\}$ is a simple random walk on $\mathbb{Z}$ (by the strong Markov property).

For $n \geq 1$, let

$$\mathbb{V}_n = \{(x_1, x_2) \in \mathbb{V} : |x_1| \leq n\},$$

and

$$\theta_n = \inf\{m \geq 0 : X_m \notin \mathbb{V}_{n-1}\}. \tag{2.1}$$

Notice that if $X_0 \in \mathbb{V}_{n-1}$ then θ_n is the hitting time of $\{(-n, 0), (n, 0)\}$ by X.

Let X' be another simple random walk on Comb($\mathbb{Z}, f$), independent of X, and $U'_n, V'_n, W'_k, T'_k, \theta'_n$ be the corresponding counterparts. Define another sequence of stopping times by setting $\sigma_0 = 0$ and

$$\sigma_{m+1} := \inf\{n > \sigma_m : U_n = U'_n \text{ and } (U_n \neq U_{n-1} \text{ or } U'_n \neq U'_{n-1})\}.$$

Lemma 2.1. *For any $\varepsilon > 0$, there exists $d \in \mathbb{N}$ such that*

$$\mathbf{P}^{u,v}\left(\sigma_N \geq \theta_{dN} \wedge \theta'_{dN}\right) < \varepsilon,$$

for all $N \in \mathbb{N}$ and for all $u, v \in \mathbb{V}_N$ with $u_1 + u_2 + v_1 + v_2$ being even.

Proof. For $n \geq 0$, define

$$Z_{2n} = U_n - U'_n \quad \text{and} \quad Z_{2n+1} = U_{n+1} - U'_n.$$

Define a sequence of stopping times inductively by setting $\tau_0 = 0$ and

$$\tau_m = \inf\{n > \tau_{m-1} : Z_n \neq Z_{n-1}\}.$$

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By the strong Markov property, $\{Z_{\tau_m}, m \geq 0\}$ is a simple random walk on $\mathbb{Z}$. Suppose that $u, v \in \mathbb{V}_N$. Write $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then $|u_1|, |v_1| \leq N$ and

$$Z_{\tau_0} = u_1 - v_1 \in [-2N, 2N]. \tag{2.2}$$

By definition, if $Z_{\tau_m} = 0$ then

$$U_k = U_k', \ U_k \neq U_{k-1}', \ \tau_m = 2k \ \text{ for some } k \in \mathbb{Z}^+; \ \text{ or}$$
$$U_{k+1} = U_k', \ U_k \neq U_k', \ \tau_m = 2k+1 \ \text{ for some } k \in \mathbb{Z}^+.$$

Notice that $U_n + V_n + U_n' + V_n'$ is always even under the assumption that $U_0 + V_0 + U_0' + V_0' = u_1 + u_2 + v_1 + v_2$ is even. This fact, together with that $U_{k+1} = U_k' \neq U_k$, implies that $U_{k+1} = U_{k+1}'$. For each $M \geq 0$, let

$$\xi(0, M) := |\{m : Z_{\tau_m} = 0, 0 \leq m \leq M\}|,$$

the local time of 0 by $\{Z_{\tau_m}, m \geq 0\}$. As a result of the previous argument,

$$\{\xi(0, M) \geq N\} \subseteq \{\sigma_N \leq \tau_M\} \subseteq \{\sigma_N \leq T_M \wedge T_M'\}. \tag{2.3}$$

For each $\widetilde{w} \in \mathbb{Z}$, let $\mathbf{P}^{\widetilde{w}}$ be the probability measure of a simple random walk $\widetilde{W}$ on $\mathbb{Z}$ starting from $\widetilde{w}$. Use $\widetilde{\xi}(\cdot, \cdot)$ to denote the local time by $\widetilde{W}$. By (9.11) of [7], there exists $x \in \mathbb{N}$ such that, for any $\widetilde{w} \in \mathbb{Z}$, $|\widetilde{w}| \leq 2N$,

$$\mathbf{P}^{\widetilde{w}}(\widetilde{\xi}(0, xN^2) \leq N) < \frac{\varepsilon}{2}. \tag{2.4}$$

By (2.2), (2.4) and the argument that $\{Z_{\tau_m}, m \geq 0\}$ is a simple random walk on $\mathbb{Z}$, for any $N \in \mathbb{N}$ and $u, v \in \mathbb{V}_N$

$$\mathbf{P}^{u,v}(\xi(0, xN^2) \leq N) = \mathbf{P}^{u_1 - v_1}(\widetilde{\xi}(0, xN^2) \leq N) < \frac{\varepsilon}{2}. \tag{2.5}$$

Furthermore, by Theorem 2.13 of [7] and the argument that W is a simple random walk on $\mathbb{Z}$, we can find $d \in \mathbb{N}$ such that for any $N \in \mathbb{N}$ and $u \in \mathbb{V}_N$,

$$\mathbf{P}^u(T_{xN^2} \geq \theta_{dN}) = \mathbf{P}^u(\max_{0 \leq k \leq xN^2} |W_k| \geq dN) = \mathbf{P}^{u_1}(\max_{0 \leq k \leq xN^2} |\widetilde{W}_k| \geq dN) \leq \frac{\varepsilon}{4}. \tag{2.6}$$

Combining (2.3), (2.5) with (2.6), we arrive at the desired conclusion.

$$\mathbf{P}^{u,v}\left(\sigma_N \geq \theta_{dN} \wedge \theta_{dN}'\right)$$
$$= \mathbf{P}^{u,v}\left(\sigma_N \geq \theta_{dN} \wedge \theta_{dN}', \sigma_N \geq \tau_{xN^2}\right) + \mathbf{P}^{u,v}\left(\sigma_N \geq \theta_{dN} \wedge \theta_{dN}', \sigma_N < \tau_{xN^2}\right)$$
$$\leq \mathbf{P}^{u,v}\left(\sigma_N \geq \tau_{xN^2}\right) + \mathbf{P}^{u,v}\left(\sigma_N \geq \theta_{dN} \wedge \theta_{dN}', \sigma_N \leq T_{xN^2} \wedge T_{xN^2}'\right)$$
$$\leq \mathbf{P}^{u,v}(\xi(0, xN^2) \leq N) + \mathbf{P}^{u,v}\left(\theta_{dN} \leq T_{xN^2} \ \text{ or } \ \theta_{dN}' \leq T_{xN^2}'\right)$$
$$\leq \mathbf{P}^{u,v}(\xi(0, xN^2) \leq N) + \mathbf{P}^u\left(\theta_{dN} \leq T_{xN^2}\right) + \mathbf{P}^v\left(\theta_{dN} \leq T_{xN^2}\right) \leq \varepsilon$$

$$\square$$

The above lemma shows that, with a small exception, the number of collisions is bounded by departure times θ and θ' of U and U' linearly. In the following applications of Lemma 2.1, $\varepsilon = 1/2$, we can find $d \in \mathbb{N}$, such that for all $N \in \mathbb{N}$ and for all $u, v \in \mathbb{V}_N$ with $u_1 + u_2 + v_1 + v_2$ being even,

$$\mathbf{P}^{u,v}\left(\sigma_N \geq \theta_{dN} \wedge \theta_{dN}'\right) < \frac{1}{2}. \tag{2.7}$$

Obviously, $d \geq 2$. We fix such d throughout this section.

Next, we want to estimate the probability that there is at least one collision of X and X' once U and U' collide. To this end, we define a sequence of events. For $m \geq 0$, let

$$\Psi_m = \{X_n = X'_n, \ |V_n| + |V'_n| \geq |V_{\sigma_m}| + |V'_{\sigma_m}|$$
$$\text{for some } \sigma_m \leq n < \inf\{h > \sigma_m : V_h = 0 \text{ or } V'_h = 0\}\}.$$

Notice that Ψ_m occurs if $X_{\sigma_m} = X'_{\sigma_m}$. Moreover, suppose that $(u, 0), (u, v) \in \mathbb{V}$ with v being even, if $\Psi_m \cap \{X_{\sigma_m} = (u, 0), X'_{\sigma_m} = (u, v)\}$ occurs then X enters the segment $\{u\} \times [-f(u), f(u)]$ and collides with X' at a vertex with height greater than or equal to $v/2$ after time σ_m but before X or X' exits the segment. Here, by height (of a vertex) we mean the distance from the vertex to the x-axis. The next lemma shows that these events have good bounds.

Lemma 2.2. *There exist $c_1, c_2 > 0$, such that for all $(u, v) \in \mathbb{V}$ with v being even,*

$$\frac{c_1}{|v| \vee 1} \leq \mathbf{P}^{(u,0),(u,v)}(\Psi_0) \leq \frac{c_2}{|v| \vee 1}.$$

Proof. We first examine the case that $f(u) \geq 2v > 0$ and v is even. For $x \in \mathbb{Z}$, define

$$\tau_x = \inf\{n > 0 : X_n = (u, x)\},$$

and τ'_x similarly. If $\tau_{2v} < \tau_0$ and X' stays in $\{u\} \times [v/2, 3v/2]$ before time τ_{2v}, then X and X' must collide before time $\tau_{2v} < \tau_0 \wedge \tau'_0$ at a vertex whose height is greater than or equal to $v/2$. Therefore

$$\mathbf{P}^{(u,0),(u,v)}(\Psi_0) \geq \mathbf{P}^{(u,0),(u,v)}\left(X_1 = (u, 1), \ \tau_{2v} \leq v^2 \wedge \tau_0, \ \tau'_{v/2} \vee \tau'_{3v/2} \geq v^2\right)$$

$$= \mathbf{P}^{(u,0)}(X_1 = (u, 1))\mathbf{P}^{(u,1)}\left(\tau_{2v} \leq v^2, \tau_{2v} < \tau_0\right)\mathbf{P}^{(u,v)}\left(\tau_{v/2} \vee \tau_{3v/2} \geq v^2\right)$$

$$= \frac{1}{4}\mathbf{P}^{(u,1)}\left(\tau_{2v} \leq v^2, \tau_{2v} < \tau_0\right)\mathbf{P}^{(u,v)}\left(\tau_{v/2} \vee \tau_{3v/2} \geq v^2\right). \tag{2.8}$$

Obviously, if X starts from vertex $(u, 1)$ then X will stay at $\{u\} \times [1, 2v-1] \cap \mathbb{Z}^2$ before time $\tau_0 \wedge \tau_{2v}$. Consider a simple random walk on $\mathbb{Z}$. Let $\{\eta_i\}$ be i.i.d. random variables with $\mathbf{P}(\eta_1 = 1) = \mathbf{P}(\eta_1 = -1) = 1/2$, and $S_k = 1 + \sum_{i=1}^{k} \eta_i$ for $k \geq 0$. Define stopping times

$$\widetilde{\tau}_{2v} = \inf\{k > 0 : S_k \geq 2v\} \quad \text{and} \quad \widetilde{\tau}_0 = \inf\{k > 0 : S_k \leq 0\}.$$

Both $\{\widetilde{\tau}_{2v} \leq v^2\}$ and $\{\widetilde{\tau}_{2v} < \widetilde{\tau}_0\}$ are increasing events of $\{\eta_i, i \geq 1\}$. By the FKG inequality and the gambler's ruin problem (or refer to Lemma 3.1 of [7]),

$$\mathbf{P}\left(\widetilde{\tau}_{2v} \leq v^2, \widetilde{\tau}_{2v} < \widetilde{\tau}_0\right) \geq \mathbf{P}(\widetilde{\tau}_{2v} \leq v^2)\mathbf{P}(\widetilde{\tau}_{2v} < \widetilde{\tau}_0) = \mathbf{P}(\widetilde{\tau}_{2v} \leq v^2)\frac{1}{2v}. \tag{2.9}$$

It can be verified that $\{X_n : 0 \leq n \leq \tau_0 \wedge \tau_{2v}\}$ conditioned on $\{X_0 = (u, 1)\}$ has the same law as $\{S_n : 0 \leq n \leq \widetilde{\tau}_{2v} \wedge \widetilde{\tau}_0\}$. So

$$\mathbf{P}^{(u,1)}\left(\tau_{2v} \leq v^2, \tau_{2v} < \tau_0\right) = \mathbf{P}\left(\widetilde{\tau}_{2v} \leq v^2, \widetilde{\tau}_{2v} < \widetilde{\tau}_0\right). \tag{2.10}$$

By Theorem 2.13 of [7] again, there exists $c_1 > 0$ independent of u and v, such that

$$\mathbf{P}(\widetilde{\tau}_{2v} \leq v^2) \geq c_1 \quad \text{and} \quad \mathbf{P}^{(u,v)}\left(\tau_{v/2} \vee \tau_{3v/2} \geq v^2\right) \geq c_1. \tag{2.11}$$

Taking (2.8)-(2.11) together, we obtain the first inequality of the lemma.

$$\mathbf{P}^{(u,0),(u,v)}(\Psi_0) \geq \frac{c_1^2}{8v}. \tag{2.12}$$

We now turn to the proof of the second inequality. Define

$$H = \sum_{n=0}^{\infty} 1_{\{X_n = X'_n,\ n < \tau_0 \wedge \tau'_0\}},$$

the number of collisions of X and X' before one of them exits $\{u\} \times [1, f(u)]$. Then

$$
\begin{aligned}
\mathbf{E}^{(u,0),(u,v)}(H) &= \sum_{n=0}^{\infty}\sum_{x=1}^{f(u)} \mathbf{P}^{(u,0),(u,v)}(X_n = X'_n = (u,x),\ n < \tau_0 \wedge \tau'_0) \\
&= \sum_{n=0}^{\infty}\sum_{x=1}^{f(u)} \mathbf{P}^{(u,0)}(X_n = (u,x),\ n < \tau_0)\mathbf{P}^{(u,v)}(X_n = (u,x),\ n < \tau_0) \\
&\leq 2\sum_{n=0}^{\infty}\sum_{x=1}^{f(u)} \mathbf{P}^{(u,0)}(X_n = (u,x),\ n < \tau_0)\mathbf{P}^{(u,x)}(X_n = (u,v),\ n < \tau_0) \\
&= 2\sum_{n=0}^{\infty} \mathbf{P}^{(u,0)}(X_{2n} = (u,v),\ 2n < \tau_0) \\
&= 2\mathbf{E}^{(u,0)}(\text{ number of visits to } (u,v) \text{ by } X \text{ before returning to } (u,0)).
\end{aligned}
$$

In the previous arguments, the inequality follows from the property of reversible Markov chain and the fact that the reversible measure for $x \in [1, f(u)] \cap \mathbb{Z}$ is just the degree of the vertex. Therefore

$$\mathbf{P}^{(u,v)}(X_n = (u,x),\ n < \tau_0) \leq 2\mathbf{P}^{(u,x)}(X_n = (u,v),\ n < \tau_0).$$

By Theorem 9.7 of [7],

$$\mathbf{E}^{(u,0),(u,v)}(H) \leq 2. \tag{2.13}$$

The second inequality of the lemma will follow once we show that there exists $c_2 > 0$ independent of u, v such that

$$\mathbf{E}^{(u,0),(u,v)}(H|\Psi_0) \geq c_2 v. \tag{2.14}$$

When the event Ψ_0 occurs, there is a collision at vertex (u,w) for some w with $w \geq v/2$. Conditioned on this event, the total number of collisions in the set $\{u\} \times [v/3, f(u)]$ will be greater than the number of collisions that take place before the first time that one of the random walks exits this interval. A lower bound could be obtained by considering two independent simple random walks in an interval, starting at $v/2$. Before hitting either $v/3$ or $2v/3$, the average number of collisions is the average number of returning to the starting point before exiting the interval, which is exactly the Green function of a simple random walk starting at $v/2$, before exiting the interval $(v/3, 2v/3)$. This is of order v.

By (2.13) and (2.14), we conclude that $\mathbf{P}^{(u,0),(u,v)}(\Psi_0) \leq 2/(c_2 v)$. This completes the proof of the case that $f(u) \geq 2v > 0$ and v is even. The proof can be modified to treat other cases and is omitted here. $\square$

Now we are ready to state a key lemma. To be concise, we set

$$\breve{f}(n) = 1 \vee \max_{-n \le i \le n} f(i).$$

As a result, $\breve{f}$ is a strictly positive and increasing function on $\mathbb{Z}^+$.

Lemma 2.3. *There exists $c > 0$ such that, for all $N \in \mathbb{N}$ and for all $u, v \in \mathbb{V}_N$ with $u_1 + u_2 + v_1 + v_2$ being even,*

$$\mathbf{P}^{u,v}\left(X_n = X'_n \text{ for some } n \in [0, \theta_{dN} \wedge \theta'_{dN})\right) \ge \frac{cN}{\breve{f}(dN) + N}.$$

Proof. Let

$$H = \sum_{m=1}^{N} \left(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1\right) 1_{\Psi_m} 1_{\{\sigma_m < \theta_{dN} \wedge \theta'_{dN}\}}.$$

If $H > 0$ then X and X' collide before time $\theta_{dN} \wedge \theta'_{dN}$. We shall use the second moment method to estimate the probability of $\{H > 0\}$. Calculate directly as follows.

$$\mathbf{E}^{u,v}(H) = \sum_{m=1}^{N} \mathbf{E}^{u,v}\left(\left(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1\right) 1_{\Psi_m} 1_{\{\sigma_m < \theta_{dN} \wedge \theta'_{dN}\}}\right)$$

$$= \sum_{m=1}^{N} \mathbf{E}^{u,v}\left[\mathbf{E}^{u,v}\left(\left(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1\right) 1_{\Psi_m} 1_{\{\sigma_m < \theta_{dN} \wedge \theta'_{dN}\}} \mid X_i, X'_i, 0 \le i \le \sigma_m\right)\right]$$

$$= \sum_{m=1}^{N} \mathbf{E}^{u,v}\left[\left(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1\right) \mathbf{P}^{u,v}\left(\Psi_m \mid X_i, X'_i, 0 \le i \le \sigma_m\right); \sigma_m < \theta_{dN} \wedge \theta'_{dN}\right]$$

$$= \sum_{m=1}^{N} \mathbf{E}^{u,v}\left[\left(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1\right) \mathbf{P}^{X_{\sigma_m}, X'_{\sigma_m}}\left(\Psi_0\right); \sigma_m < \theta_{dN} \wedge \theta'_{dN}\right] \tag{2.15}$$

$$\ge \sum_{m=1}^{N} \mathbf{E}^{u,v}\left(c_1; \sigma_m < \theta_{dN} \wedge \theta'_{dN}\right) \qquad \text{by Lemma 2.2}$$

$$\ge c_1 N \mathbf{P}^{u,v}(\sigma_N < \theta_{dN} \wedge \theta'_{dN}) \ge \frac{c_1}{2} N \qquad \text{by (2.7)}.$$

Here the equation (2.15) is by the strong Markov property. Applying Lemma 2.2 and the strong Markov property again, we have the following estimates.

$$\mathbf{E}^{u,v}(H^2)$$

$$= \mathbf{E}^{u,v}\left(\sum_{m=1}^{N}(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1) 1_{\Psi_m} 1_{\{\sigma_m < \theta_{dN} \wedge \theta'_{dN}\}} \sum_{n=1}^{N}(|V_{\sigma_n}| \vee |V'_{\sigma_n}| \vee 1) 1_{\Psi_n} 1_{\{\sigma_n < \theta_{dN} \wedge \theta'_{dN}\}}\right)$$

$$= \mathbf{E}^{u,v}\left(\sum_{m=1}^{N}(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1)^2 1_{\Psi_m} 1_{\{\sigma_m < \theta_{dN} \wedge \theta'_{dN}\}}\right)$$

$$+ 2\mathbf{E}^{u,v}\left(\sum_{m=1}^{N}(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1) 1_{\Psi_m} 1_{\{\sigma_m < \theta_{dN} \wedge \theta'_{dN}\}} \sum_{n>m}^{N}(|V_{\sigma_n}| \vee |V'_{\sigma_n}| \vee 1) 1_{\Psi_n} 1_{\{\sigma_n < \theta_{dN} \wedge \theta'_{dN}\}}\right)$$

$$\leq 2\breve{f}(dN)\mathbf{E}^{u,v}\left(\sum_{m=1}^{N}(|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1)1_{\Psi_m}1_{\{\sigma_m<\theta_{dN}\wedge\theta'_{dN}\}}\right)$$

$$+2\sum_{m=1}^{N}\sum_{n>m}^{N}\mathbf{E}^{u,v}\left((|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1)1_{\Psi_m}1_{\{\sigma_m<\theta_{dN}\wedge\theta'_{dN}\}}(|V_{\sigma_n}| \vee |V'_{\sigma_n}| \vee 1)1_{\Psi_n}1_{\{\sigma_n<\theta_{dN}\wedge\theta'_{dN}\}}\right)$$

$$=2\breve{f}(dN)\mathbf{E}^{u,v}(H)+2\sum_{m=1}^{N}\sum_{n>m}^{N}\mathbf{E}^{u,v}\left((|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1)1_{\Psi_m}1_{\{\sigma_m<\theta_{dN}\wedge\theta'_{dN}\}}\right.$$

$$\left.\times\ (|V_{\sigma_n}| \vee |V'_{\sigma_n}| \vee 1)\mathbf{P}^{X_{\sigma_n},X'_{\sigma_n}}(\Psi_0)1_{\{\sigma_n<\theta_{dN}\wedge\theta'_{dN}\}}\right)$$

$$\leq 2\breve{f}(dN)\mathbf{E}^{u,v}(H)+2c_2\sum_{m=1}^{N}\sum_{n>m}^{N}\mathbf{E}^{u,v}\left((|V_{\sigma_m}| \vee |V'_{\sigma_m}| \vee 1)1_{\Psi_m}1_{\{\sigma_m<\theta_{dN}\wedge\theta'_{dN}\}}\right)$$

$$\leq(2\breve{f}(dN)+2c_2N)\mathbf{E}^{u,v}(H).$$

By the Hölder inequality,

$$\mathbf{P}^{u,v}(H>0)\geq\frac{[\mathbf{E}^{u,v}(H)]^2}{\mathbf{E}^{u,v}[H^2]}\geq\frac{\mathbf{E}^{u,v}(H)}{2\breve{f}(dN)+2c_2N}\geq\frac{c_1N}{4\breve{f}(dN)+4c_2N}.$$

We have reached the conclusion of the lemma. $\qquad\square$

Proof of Theorem 1.1. It suffices to consider the case that two independent simple random walks on $\text{Comb}(\mathbb{Z},f)$ starting from the same vertex $(0,0)$. Notice that as X,X' start from $(0,0)$, almost surely $\theta_d\wedge\theta'_d<\theta_{d^2}\wedge\theta'_{d^2}<\theta_{d^3}\wedge\theta'_{d^3}<\cdots$. Define for $m\geq 1$,

$$\Upsilon_m=\{X_n=X'_n\quad\text{for some}\quad n\in[\theta_{d^m}\wedge\theta'_{d^m},\theta_{d^{m+1}}\wedge\theta'_{d^{m+1}})\}.$$

We write $\mathbf{P}=\mathbf{P}^{(0,0),(0,0)}$ and $t=\theta_{d^m}\wedge\theta'_{d^m}$ for short. By the strong Markov property and Lemma 2.3, there exists $c>0$ such that for all $m\in\mathbb{N}$

$$\mathbf{P}(\Upsilon_m|\ 1_{\Upsilon_i},1\leq i<m,\ X_n,X'_n,\ n\leq\theta_{d^m}\wedge\theta'_{d^m})$$
$$=\mathbf{P}^{X_t,X'_t}(X_n=X'_n\quad\text{for some}\quad n\in[0,\theta_{d^{m+1}}\wedge\theta'_{d^{m+1}}))$$
$$\geq\frac{cd^m}{\breve{f}(d^{m+1})+d^{m+1}}.$$

We will show below that (1.1) implies that

$$\sum_{m=1}^{\infty}\frac{d^m}{\breve{f}(d^m)+d^m}=\infty.\tag{2.16}$$

Consequently, $\mathbf{P}(\Upsilon_m\text{ infinitely often})=1$ by the second Borel-Cantelli Lemma (extended version, Page 237 of [4]). Furthermore,

$$\mathbf{P}(X_n=X'_n\ \text{infinitely often})\geq\mathbf{P}(\Upsilon_m\ \text{infinitely often})=1.$$

We now prove (2.16). If $\breve{f}(d^m)\leq d^m$ for infinitely many m, then it is trivial and we have nothing to do. Otherwise, there exists m_0 such that $\breve{f}(d^m)>d^m$ for all $m\geq m_0$. Hence it suffices to prove

$$\sum_{m=1}^{\infty}\frac{d^m}{\breve{f}(d^m)}=\infty.\tag{2.17}$$

Set $D_m = d + d^2 + \cdots + d^m$. Then $d^m \leq D_m \leq d^{m+1}$, and

$$\frac{d^m}{\check{f}(d^m)} \geq \frac{1}{d} \sum_{l=D_m+1}^{D_{m+1}} \frac{1}{\check{f}(l)}.$$

So

$$\sum_{m=1}^{\infty} \frac{d^m}{\check{f}(d^m)} \geq \frac{1}{d} \sum_{m=1}^{\infty} \sum_{l=D_m+1}^{D_{m+1}} \frac{1}{\check{f}(l)} = \frac{1}{d} \sum_{l=d+1}^{\infty} \frac{1}{\check{f}(l)} = \infty.$$

Hence (2.16) holds in either case and the proof of the theorem is completed. $\qquad\square$

3 Proof of Theorem 1.4

In this section we will show a slightly more general result from which Theorem 1.4 will follow.

Theorem 3.1. *If $\sum_{i=-n}^{n} f(i) = O(n)$, then $Comb(\mathbb{Z}, f)$ has the infinitely many triple collisions property.*

To prove the theorem we introduce yet another independent simple random walk X'' on $Comb(\mathbb{Z}, f)$. For any $u, v, w \in \mathbb{V}$, we write $\mathbf{P}^{u,v,w}$ for the joint probability measure of the three independent simple random walks X, X' and X'' starting from u, v and w, respectively. Let $\mathbf{E}^{u,v,w}$ be the corresponding expectation. By the assumption of Theorem 3.1, there exists $c > 4$, such that for all $n \in \mathbb{N}$,

$$\sum_{i=-n}^{n} f(i) \leq \left(\frac{c}{2} - 2\right) n. \tag{3.1}$$

Hence we fix f and c which satisfy (3.1) throughout this section. Since

$$([-n, n] \times \{0\}) \cap \mathbb{Z}^2 \subseteq \mathbb{V}_n = \left(\bigcup_{i=-n}^{n} \{i\} \times [-f(i), f(i)]\right) \cap \mathbb{Z}^2,$$

therefore

$$2n \leq |\mathbb{V}_n| \leq cn. \tag{3.2}$$

Recall that θ_n is defined in (2.1).

Lemma 3.2. *For any $\varepsilon > 0$, there exists $d \in \mathbb{N} \setminus \{1\}$ such that $\mathbf{P}^u(\theta_{dn} \leq n^2) \leq \varepsilon$ for all $n \in \mathbb{N}$ and for all $u \in \mathbb{V}_n$.*

Proof. For each $\widetilde{w} \in \mathbb{Z}$, let $\mathbf{P}^{\widetilde{w}}$ be the probability measure of a simple random walk $\widetilde{W}$ on $\mathbb{Z}$ starting from $\widetilde{w}$. By Theorem 2.13 of [7] and the argument that W is a simple random walk on $\mathbb{Z}$, there exists $d \in \mathbb{N} \setminus \{1\}$ which depends only on ε, such that for all $n \in \mathbb{N}$ and for all $u \in \mathbb{V}_n$

$$\mathbf{P}^u(\theta_{dn} \leq n^2) \leq \mathbf{P}^u\left(\max_{0 \leq k \leq n^2} |W_k| \geq dn\right) = \mathbf{P}^{u_1}\left(\max_{0 \leq k \leq n^2} |\widetilde{W}_k| \geq dn\right)$$

$$\leq \mathbf{P}^0\left(\max_{0 \leq k \leq n^2} |\widetilde{W}_k| \geq (d-1)n\right) \leq \varepsilon.$$

$\qquad\square$

For $x \in \mathbb{V}$, let $\tau_x = \inf\{m \geq 0 : X_m = x\}$ be the hitting time of x by X.

Lemma 3.3. *For $\varepsilon > 0$, there exists $c_1 \in \mathbb{N}$ such that $\mathbf{P}^u(\tau_v > c_1 n^2) \le \varepsilon$ for all $n \in \mathbb{N}$ and for all $u, v \in \mathbb{V}_n$.*

Proof. Fix $0 < \varepsilon < 1$, $n \in \mathbb{N}$ and $u \in \mathbb{V}_n$. Let $h, c_1 \in \mathbb{N}$ such that

$$h > \frac{8(c+1)}{\varepsilon} \quad \text{and} \quad c_1 > \frac{6c^2 h^2}{\varepsilon}. \tag{3.3}$$

Suppose that the random walk X starts from vertex $u \in \mathbb{V}_n$. Since $\theta_{hn} = \tau_{(hn,0)} \wedge \tau_{(-hn,0)}$ for $u \in \mathbb{V}_n$, then

$$\begin{aligned}
\mathbf{P}^u(\tau_v > c_1 n^2) &\le \mathbf{P}^u(\tau_v > \theta_{hn}) + \mathbf{P}^u(\theta_{hn} > c_1 n^2) \\
&\le \mathbf{P}^u(\tau_v > \tau_{(hn,0)}) + \mathbf{P}^u(\tau_v > \tau_{(-hn,0)}) + \mathbf{P}^u(\theta_{hn} > c_1 n^2).
\end{aligned} \tag{3.4}$$

Hence we are going to estimate probabilities of the three events above.

Because $\mathrm{Comb}(\mathbb{Z}, f)$ is a tree, there exists a unique simple path from v to $(hn, 0)$. If u is a vertex on the path, then by the gambler's ruin problem and the fact that X is a recurrent process,

$$\mathbf{P}^u(\tau_v > \tau_{(hn,0)}) = \frac{\mathrm{dist}(u, v)}{\mathrm{dist}(v, (hn, 0))}$$

where $\mathrm{dist}(\cdot, \cdot)$ is the graph distance. By the assumption (3.3),

$$\mathbf{P}^u(\tau_v > \tau_{(hn,0)}) \le \frac{2n + \max_{-n \le k \le n} f(k)}{hn - n} \le \frac{2 + c}{h - 1} < \frac{\varepsilon}{3}. \tag{3.5}$$

If u is not on the path, then with probability one X will reach the path first at $v, (v_1, 0)$ or $(u_1, 0)$. A slight change in the proof actually shows that (3.5) holds for these cases. So in all cases, (3.5) gives an upper bound of the probability of the first event. Similarly, the probability of the second event is also small.

$$\mathbf{P}^u(\tau_v > \tau_{(-hn,0)}) < \frac{\varepsilon}{3}. \tag{3.6}$$

We now study the third event. Let $\tilde{\mathbb{G}}$ be the spanning tree of $\mathbb{V}_{hn}$, which is a connected subgraph of $\mathrm{Comb}(\mathbb{Z}, f)$. The process X before time θ_{hn} is also a simple random walk on $\tilde{\mathbb{G}}$. Regard $\tilde{\mathbb{G}}$ as an electric network that each edge is assigned a unit conductance. Then by Proposition 10.6 of [6],

$$\mathbf{E}^u(\theta_{hn}) \le c_{\tilde{\mathbb{G}}} \Re(u \longleftrightarrow \{-hn, hn\}),$$

where $c_{\tilde{\mathbb{G}}}$ is twice of the total conductance and $\Re(u \longleftrightarrow \{-hn, hn\})$ is the effective resistance between u and $\{-hn, hn\}$. Since $\tilde{\mathbb{G}}$ is a tree, we can easily get that

$$c_{\tilde{\mathbb{G}}} = 2|\mathbb{V}_{hn}| - 2 \quad \text{and} \quad \Re(u \longleftrightarrow \{-hn, hn\}) \le |\mathbb{V}_{hn}|.$$

Consequently, $\mathbf{E}^u(\theta_{hn}) \le 2|\mathbb{V}_{hn}|^2 \le 2c^2 h^2 n^2$. By the Markov inequality and (3.3), we get the probability of the third event

$$\mathbf{P}^u(\theta_{hn} > c_1 n^2) \le \frac{\varepsilon}{3}. \tag{3.7}$$

Combining (3.4)-(3.7) together we complete the proof of the lemma. $\qquad\square$

By Lemmas 3.2 and 3.3, we can find $c_1 \in \mathbb{N}$ and $d \in \mathbb{N} \setminus \{1\}$ such that

$$\mathbf{P}^u(\tau_v > c_1 n^2) \leq \frac{1}{4} \quad \text{and} \quad \mathbf{P}^u(\theta_{dn} \leq c_1 n^2) \leq \frac{1}{4}. \tag{3.8}$$

for all $n \in \mathbb{N}$ and for all $u, v \in \mathbb{V}_n$. Fix such c_1 and d throughout this section. We are now ready to proceed to a similar key argument as Lemma 2.3.

Lemma 3.4. *There exists $c_2 > 0$ such that for all $N \in \mathbb{N} \setminus \{1\}$ and for all $u, v, w \in \mathbb{V}_N$ with $(-1)^{u_1 + u_2} = (-1)^{v_1 + v_2} = (-1)^{w_1 + w_2}$,*

$$\mathbf{P}^{u,v,w}\left(X_n = X'_n = X''_n \text{ for some } n \in [0, \Theta)\right) \geq \frac{c_2}{\log N},$$

where $\Theta = \theta_{dN} \wedge \theta'_{dN} \wedge \theta''_{dN} = \inf\{m \geq 0 : \{X_m, X'_m, X''_m\} \not\subseteq \mathbb{V}_{dN-1}\}$.

Proof. Fix $N \in \mathbb{N} \setminus \{1\}$ and $u, v, w \in \mathbb{V}_N$. Notice that Θ is the first time that one of X, X', X'' exits $\mathbb{V}_{dN-1}$. Let

$$H = \sum_{n=0}^{2c_1 N^2} \mathbf{1}_{\{X_n = X'_n = X''_n \in \mathbb{V}_N, \ n < \Theta\}}.$$

We will prove that

$$\mathbf{E}^{u,v,w}(H) \geq c_1^* \quad \text{and} \quad \mathbf{E}^{u,v,w}(H^2) \leq c_2^* \mathbf{E}^{u,v,w}(H) \log N. \tag{3.9}$$

Then by the Hölder inequality,

$$\mathbf{P}^{u,v,w}(X_n = X'_n = X''_n \text{ for some } n < \Theta) \geq \mathbf{P}^{u,v,w}(H > 0) \geq \frac{\left[\mathbf{E}^{u,v,w}(H)\right]^2}{\mathbf{E}^{u,v,w}(H^2)} \geq \frac{c_1^*}{c_2^* \log N}.$$

Now we begin to prove (3.9). For $x \in \mathbb{V}_N$, let $q_n(u, x) = \mathbf{P}^u(X_n = x, \theta_{dN} > n)$. Then $q_{2n}(x, x)$ is decreasing in n by the spectral theory (or referring to [1]). By the strong Markov property, for any $c_1 N^2 \leq n \leq 2c_1 N^2$ with $u_1 + u_2 + x_1 + x_2 + n$ and $k + n$ being even,

$$\mathbf{P}^u(X_n = x, \theta_{dN} > n) = \sum_{k=0}^{n} \mathbf{P}^u(X_n = x, \tau_x = k, n < \theta_{dN})$$

$$= \sum_{k=0}^{n} \mathbf{P}^u(\tau_x = k < \theta_{dN}) \mathbf{P}^u(X_n = x, n < \theta_{dN} \mid \tau_x = k < \theta_{dN})$$

$$= \sum_{k=0}^{n} \mathbf{P}^u(\tau_x = k < \theta_{dN}) q_{n-k}(x, x)$$

$$\geq \sum_{k=0}^{n} \mathbf{P}^u(\tau_x = k < \theta_{dN}) q_{2c_1 N^2}(x, x)$$

$$= q_{2c_1 N^2}(x, x) \mathbf{P}^u(\tau_x \leq n, \theta_{dN} > \tau_x)$$

$$\geq q_{2c_1 N^2}(x, x)\left(\mathbf{P}^u(\tau_x \leq c_1 N^2) + \mathbf{P}^u(\theta_{dN} > c_1 N^2) - 1\right)$$

$$\geq \frac{1}{2} q_{2c_1 N^2}(x,x) \quad \text{by (3.8)}.$$

As a result,

$$\mathbf{E}^{u,v,w}(H) = \sum_{x \in \mathbb{V}_N} \sum_{n=0}^{2c_1 N^2} \mathbf{P}^{u,v,w}(X_n = X'_n = X''_n = x, \Theta > n)$$

$$= \sum_{x \in \mathbb{V}_N} \sum_{n=0}^{2c_1 N^2} \mathbf{P}^u(X_n = x, \theta_{dN} > n)\mathbf{P}^v(X_n = x, \theta_{dN} > n)\mathbf{P}^w(X_n = x, \theta_{dN} > n)$$

$$\geq \frac{1}{8} \sum_{x \in \mathbb{V}_N} \sum_{\substack{c_1 N^2 \leq n \leq 2c_1 N^2 \\ u_1 + u_2 + x_1 + x_2 + n \text{ even}}} \left[q_{2c_1 N^2}(x,x)\right]^3$$

$$\geq \frac{c_1 N^2}{16} \sum_{x \in \mathbb{V}_N} \left[q_{2c_1 N^2}(x,x)\right]^3$$

$$\geq \frac{c_1 N^2}{16|\mathbb{V}_N|^2} \left[\sum_{x \in \mathbb{V}_N} q_{2c_1 N^2}(x,x)\right]^3 \geq \frac{c_1}{16c^2} \left[\sum_{x \in \mathbb{V}_N} q_{2c_1 N^2}(x,x)\right]^3.$$

The third inequality is by the Hölder inequality. Applying the Hölder inequality and (3.8) again, we have the following estimates.

$$\sum_{x \in \mathbb{V}_N} q_{2c_1 N^2}(x,x) \geq \sum_{x \in \mathbb{V}_N} \sum_{y \in \mathbb{V}_{dN}} q_{c_1 N^2}(x,y) q_{c_1 N^2}(y,x)$$

$$\geq \frac{1}{4} \sum_{x \in \mathbb{V}_N} \sum_{y \in \mathbb{V}_{dN}} [q_{c_1 N^2}(x,y)]^2$$

$$\geq \frac{1}{4|\mathbb{V}_{dN}|} \sum_{x \in \mathbb{V}_N} \left(\sum_{y \in \mathbb{V}_{dN}} q_{c_1 N^2}(x,y)\right)^2$$

$$\geq \frac{|\mathbb{V}_N|}{4|\mathbb{V}_{dN}|} \min_{x \in \mathbb{V}_N} \left[\sum_y \mathbf{P}^x(X_{c_1 N^2} = y, \theta_{dN} > c_1 N^2)\right]^2$$

$$\geq \frac{1}{2cd} \min_{x \in \mathbb{V}_N} \left[\mathbf{P}^x(\theta_{dN} > c_1 N^2)\right]^2 \geq \frac{1}{4cd}.$$

Combining the two estimates together, we obtain that

$$\mathbf{E}^{u,v,w}(H) \geq \frac{c_1}{16c^2} \cdot \left(\frac{1}{4cd}\right)^3 = \frac{c_1}{1024c^5 d^3}.$$

The first part of (3.9) is proved by choosing $c_1^* = c_1 c^{-5} d^{-3}/1024$.

We now turn to the second moment, i.e., the second part of (3.9). Since that $\text{Comb}(\mathbb{Z}, f)$ is a graph with uniformly bounded degree, by Corollary 14.6 of [8], there exists $c_0^* > 0$ such that $\mathbf{P}^x(X_k = y) \leq c_0^*/\sqrt{k}$ for all $x, y \in \mathbb{V}$. Hence,

$$\mathbf{P}^{x,x,x}\left(X_k = X'_k = X''_k\right) = \sum_{y \in \mathbb{V}} \mathbf{P}^{x,x,x}\left(X_k = X'_k = X''_k = y\right)$$

$$= \sum_{y \in \mathbb{V}} \left[\mathbf{P}^x(X_k = y) \right]^3 \leq \frac{(c_0^*)^2}{k} \sum_{y \in \mathbb{V}} \mathbf{P}^x(X_k = y) = \frac{(c_0^*)^2}{k}.$$

By the inequality above and the strong Markov property,

$$\mathbf{E}^{u,v,w} \left(H^2 \right) = \mathbf{E}^{u,v,w} \left(\sum_{n=0}^{2c_1 N^2} \sum_{x \in \mathbb{V}_N} 1_{\{X_n = X_n' = X_n'' = x, \Theta > n\}} \sum_{k=0}^{2c_1 N^2} \sum_{y \in \mathbb{V}_N} 1_{\{X_k = X_k' = X_k'' = y, \Theta > k\}} \right)$$

$$\leq \mathbf{E}^{u,v,w}(H) + 2\mathbf{E}^{u,v,w} \left(\sum_{n=0}^{2c_1 N^2} \sum_{x \in \mathbb{V}_N} 1_{\{X_n = X_n' = X_n'' = x, \Theta > n\}} \sum_{k > n}^{2c_1 N^2} 1_{\{X_k = X_k' = X_k''\}} \right)$$

$$= \mathbf{E}^{u,v,w}(H) + 2 \sum_{n=0}^{2c_1 N^2} \sum_{x \in \mathbb{V}_N} \sum_{k > n}^{2c_1 N^2} \mathbf{P}^{u,v,w} \left(X_n = X_n' = X_n'' = x, \Theta > n, X_k = X_k' = X_k'' \right)$$

$$= \mathbf{E}^{u,v,w}(H)$$

$$+ 2 \sum_{n=0}^{2c_1 N^2} \sum_{x \in \mathbb{V}_N} \sum_{k > n}^{2c_1 N^2} \mathbf{P}^{u,v,w} \left(X_n = X_n' = X_n'' = x, \Theta > n \right) \mathbf{P}^{x,x,x} \left(X_{k-n} = X_{k-n}' = X_{k-n}'' \right)$$

$$\leq \mathbf{E}^{u,v,w}(H)$$

$$+ 2 \sum_{n=0}^{2c_1 N^2} \sum_{x \in \mathbb{V}_N} \mathbf{P}^{u,v,w} \left(X_n = X_n' = X_n'' = x, \Theta > n \right) \sum_{k=1}^{2c_1 N^2} \mathbf{P}^{x,x,x} \left(X_k = X_k' = X_k'' \right)$$

$$\leq \mathbf{E}^{u,v,w}(H) \left(1 + 2 \max_{x \in \mathbb{V}_N} \sum_{k=1}^{2c_1 N^2} \mathbf{P}^{x,x,x} \left(X_k = X_k' = X_k'' \right) \right)$$

$$\leq \mathbf{E}^{u,v,w}(H) \left(1 + 2 \max_{x \in \mathbb{V}_N} \sum_{k=1}^{2c_1 N^2} \frac{(c_0^*)^2}{k} \right)$$

$$\leq \left(1 + 2(c_0^*)^2 \log(2c_1) + 4(c_0^*)^2 \log N \right) \mathbf{E}^{u,v,w}(H).$$

We have thus verified the second part of (3.9) and finished the proof. $\square$

Proof of Theorem 3.1. It suffices to consider the case that three independent simple random walks on $\mathrm{Comb}(\mathbb{Z}, f)$ starting from the same vertex $(0, 0)$. For each $m \geq 1$, denote by Υ_m the event

$$\{X_n = X_n' = X_n'' \quad \text{for some} \quad n \in [\theta_{d^m} \wedge \theta_{d^m}' \wedge \theta_{d^m}'', \theta_{d^{m+1}} \wedge \theta_{d^{m+1}}' \wedge \theta_{d^{m+1}}'')]\}.$$

Then by the strong Markov property and Lemma 3.4 there exists $c_2 > 0$ such that

$$\mathbf{P}(\Upsilon_m | 1_{\Upsilon_i}, 1 \leq i < m, X_n, X_n', X_n'', n \leq \theta_{d^m} \wedge \theta_{d^m}' \wedge \theta_{d^m}'')$$

$$= \mathbf{P}^{X_t, X_t', X_t''} \left(X_n = X_n' = X_n'' \text{ for some } n \in [\theta_{d^m} \wedge \theta_{d^m}' \wedge \theta_{d^m}'', \theta_{d^{m+1}} \wedge \theta_{d^{m+1}}' \wedge \theta_{d^{m+1}}'') \right)$$

$$\geq \frac{c_2}{\log d^m} = \frac{c_2}{m \log d},$$

where $\mathbf{P} = \mathbf{P}^{(0,0),(0,0),(0,0)}$ and $t = \theta_{d^m} \wedge \theta'_{d^m} \wedge \theta''_{d^m}$. By the second Borel-Cantelli Lemma, $\mathbf{P}(\Upsilon_m$ occurs infinitely often$) = 1$. Furthermore,

$$\mathbf{P}(X_n = X'_n = X''_n \text{ infinitely often}) \geq \mathbf{P}(\Upsilon_m \text{ infinitely often}) = 1.$$

This completes the proof of Theorem 3.1. $\qquad\square$

4 A Related Result

The proof of Theorem 1.3 is almost the same as the case $f(x) = x^\alpha$ for $1 < \alpha < 2$ in [1]. So we just outline the changes needed to run the proof. Fix $\beta > 0$ and $f(x) = |x| \log^\beta(|x| \vee 1)$ for $x \in \mathbb{Z}$.

Lemma 4.1. *Let $k \geq 2$ and $y = (k, h) \in \mathbb{V}$. Let $t \in \mathbb{N}, s \in \mathbb{R}$ with $s^3 \log^\beta s = t$. Then there exists $c > 0$ such that the transition density q of the simple random walk on Comb$(\mathbb{Z}, f)$ satisfies:*

$$q_t(0, y) \leq \begin{cases} c \, s^{-2}(\log s)^{-\beta} & \text{if } s \geq k, \\ c \, k^{-2}(\log k)^{-\beta} & \text{if } s < k. \end{cases}$$

Proof. Following the proof of Lemma 5.1 of [1], we consider the simple random walk X starting from $(0,0)$. The first coordinate makes roughly k^2 steps to go from $k/2$ to k. At each step of the horizontal coordinate we perform an independent experiment. We say the experiment is successful if we spend time greater than $k^2 \log^{2\beta} k$ in the vertical line segment in this step of the first coordinate. The probability of success is then bounded below by $c_1/(k \log^\beta k)$ and in the k^2 experiments with high probability there will be a success and the expected number of successes is $k(\log k)^{-\beta}$, thus the total time needed to reach k will be of order $k^3 \log^\beta k$. We omit the details of the proof here. $\qquad\square$

Assume that $1 \leq h \leq k \log^\beta k$. Let $Q_{k,h} = \{(k, w) : 0 \leq w \leq h\}$, and $Z_{k,h}$ be the number of collisions of the two random walks in $Q_{k,h}$. We also define $\widetilde{Z}_{k,h} = Z_{k,2h/3} - Z_{k,h/3}$ to be the number of collisions that happen in the set $\widetilde{Q}_{k,h} = \{(k, w) : h/3 \leq w \leq 2h/3\}$.

Lemma 4.2. *There exist $c', c'' > 0$ such that for each $k \geq 2$ and $1 \leq h \leq k \log^\beta k$, $\mathbf{E}Z_{k,h} \leq c'h/(k \log^\beta k)$; $\mathbf{E}(Z_{k,h}|\widetilde{Z}_{k,h} > 0) \geq c''h$.*

Proof. Let $s = g(t)$ be the inverse function of $t = s^3 \log^\beta s$. By Lemma 4.1,

$$\mathbf{E}Z_{k,h} = \sum_t \sum_{x \in Q_{k,h}} q_t(0,x)^2 \leq \sum_{t < k^3 \log^\beta k} \frac{c^2 h}{k^4 \log^{2\beta} k} + \sum_{t \geq k^3 \log^\beta k} \frac{c^2 h}{s^4 \log^{2\beta} s}$$

$$\leq \frac{c^2 h}{k \log^\beta k} + c^2 h \int_{k^3 \log^\beta k}^\infty \frac{1}{(g(t))^4 \log^{2\beta} g(t)} dt$$

$$= \frac{c^2 h}{k \log^\beta k} + c^2 h \int_k^\infty \frac{1}{s^4 \log^{2\beta} s} d(s^3 \log^\beta s)$$

$$= \frac{c^2 h}{k \log^\beta k} + c^2 h \int_k^\infty \frac{3}{s^2 \log^\beta s} ds + c^2 h \int_k^\infty \frac{\beta}{s^2 \log^{\beta+1} s} ds \leq \frac{c'h}{k \log^\beta k}.$$

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The second inequality is proved similarly as Lemma 5.2 of [1]. $\qquad\square$

Proof of Theorem 1.3. By Lemma 4.2, $\mathbf{P}(\widetilde{Z}_{k,h} > 0) \le \frac{c'}{c'' k \log^\beta k}$ for each $k \ge 2$ and $1 \le h \le k \log^\beta k$. Summing over all $k \ge 2$ and over all h ranging over powers of 2 and satisfying $1 \le h \le k \log^\beta k$, we get that

$$\sum_k \sum_{h \text{ power of } 2} \mathbf{P}(\widetilde{Z}_{k,h} > 0) \le \frac{c'}{c''} \sum_k \frac{\log_2(k \log^\beta k)}{k \log^\beta k} < \infty, \text{ since } \beta > 2.$$

Hence the total number of sets $\widetilde{Q}_{k,h}$ with at least one collision is almost surely finite. Since $\mathbf{E}(\widetilde{Z}_{k,h}) \le \mathbf{E}(Z_{k,h}) < \infty$, the number of collisions in each set $\widetilde{Q}_{k,h}$ is a.s. finite. So, the total number of collisions in $\{(k, w) \in \mathbb{V} : k \ge 2, w \ge 1\} \subseteq \bigcup_{k,h} \widetilde{Q}_{k,h}$ is a.s. finite. We omit the details of the rest of the proof here. $\qquad\square$

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