DOI: 10.1214/EJP.v16-904

Electronic Journal of Probability

Vol. 16 (2011), Paper no. 45, pages 1254–1280.

Journal URL
http://www.math.washington.edu/~ejpecp/

Extremes of Gaussian Processes with Random Variance

Jürg Hüsler*
University of Bern

Vladimir Piterbarg†
Moscow Lomonosov State University

Yueming Zhang‡
University of Bern

Abstract

Let $\xi(t)$ be a standard locally stationary Gaussian process with covariance function $1 - r(t, t + s) \sim C(t)|s|^\alpha$ as $s \to 0$, with $0 < \alpha \leq 2$ and $C(t)$ a positive bounded continuous function. We are interested in the exceedance probabilities of $\xi(t)$ with a random standard deviation $\eta(t) = \eta - \zeta t^\beta$, where η and ζ are non-negative bounded random variables. We investigate the asymptotic behavior of the extreme values of the process $\xi(t)\eta(t)$ under some specific conditions which depends on the relation between α and β.

Key words: Gaussian processes; Locally stationary; Ruin probability; Random variance; Extremes; Fractional Brownian motions.

AMS 2010 Subject Classification: Primary 60G15; Secondary: 60G70, 60F05.

Submitted to EJP on November 5, 2010, final version accepted May 28, 2011.

*Supported by a grant of the Swiss national science foundation; email: juerg.huesler@stat.unibe.ch
†Supported by a grant of the Swiss national scientific foundation; email: piter@mech.math.msu.su
‡Supported by a grant of the Swiss national science foundation; email: yueming.zhang@stat.unibe.ch

1 Introduction and Main Results

Let $(X(t), Y),\ t \in \mathbb{R}$, be a random element, where $X(t)$ is a random process taking values in $\mathbb{R}$, and Y is an arbitrary random element. We say $X(t)$ is a conditionally Gaussian process if the conditional distribution of $X(\cdot)$ given Y is Gaussian. We investigate the probabilities of large extremes,

$$P_u(T) := P(\sup_{t\in[0,T]} X(t) > u),$$

as $u \to \infty$ where $T > 0$. Denote the random mean of X conditioned on Y by

$$m(t,Y) := E(X(t)|Y)$$

and the random covariance by

$$C(s,t,Y) := E((X(s) - m(s,Y))(X(t) - m(t,Y)) \mid Y),$$

so that

$$V^2(t,Y) := C(t,t,Y)$$

is the random variance of X.

Such processes were introduced in applications in finance, optimization and control problems. To our best knowledge, the paper by Adler et al. [1] was the first mathematical work where probabilities of large extremes of conditionally Gaussian processes where considered. The authors considered sub-Gaussian processes as an example of stable processes, that means processes of the type $X(t) = \sqrt{\zeta}\xi(t)$, where $\xi(t)$ is a stationary Gaussian process and ζ is a stable random variable, independent of $\xi(\cdot)$. That is, in our notations, $Y = \sqrt{\zeta}$ and $X(t) = Y\xi(t)$. Therefore we have a Gaussian process with random variance. This paper dealt with the mean of the number of upcrossings of a level u, as in the Rice formula, which can be applied for smooth Gaussian processes. Further results on this problem are dealt with in [9], [2], [7], [8]. For examples, Doucet et al [2] considered to model the behavior of latent variables in neural networks by Gaussian processes with random parameters. Lototsky [7] studied stochastic parabolic equations with solutions of Gaussian processes, where the coefficients are modeled by a dynamic system. We consider in our paper more general Gaussian processes.

The aim of the present paper and subsequent ones which are in preparation, is to develop asymptotic methods for large extremes of conditional Gaussian processes. Our intention is to expand the Gaussian tools to wider class of random processes. The asymptotic theory for large extremes of Gaussian processes and fields is already well developed, see [11], [3], and the references therein.

A good part of this asymptotic theory for large extremes of conditional Gaussian processes is mainly based on the corresponding theory for Gaussian processes. The last one was began from the celebrated Pickands' theorem [10] on large extremes of stationary Gaussian processes and its extension to non-stationary Gaussian processes, as in Hüsler [5] for certain types of non-stationarity, and in Piterbarg and Prisyazhn'uk [12] where the non-stationary process has a non constant variance with a unique point of maximum. In [6] we consider also the type of processes $\xi(t)\eta(t)$, but with smooth processes $\eta(t)$. In this paper we investigate the case of less smooth processes η. Also we let $\xi(t)$ to be a locally stationary Gaussian process $\xi(t)$, instead of a stationary Gaussian process in [6].

Now, let $\xi(t)$, $0 \le t \le T$ be a standard locally stationary Gaussian process with the covariance function $r(t)$ satisfying that uniformly in t

$$r(t,t+s) = 1 - C(t)|s|^{\alpha} + o(|s|^{\alpha}), \text{ as } s \to 0,\ 0 < \alpha \le 2$$

with $C(t)$ a positive continuous function on some interval $[0,T]$ with $T < \infty$. Assume that

$$r(t,t+s) < 1 \text{ for all } s,t > 0.$$

Let $\eta(t)$ be another stochastic process with $\eta(t) > 0$ (a.s.), which is independent of $\xi(t)$. We are interested in the exceedance probabilities of the product process $\xi(t)\eta(t)$, i.e.

$$P\{\xi(t)\eta(t) > u, \text{ for some } t \in [0,T]\} \qquad \text{as } u \to \infty,$$

where $T < \infty$; here $\eta(t)$ can be interpreted as the random standard deviation of the Gaussian process $\xi(t)$. In this paper we further assume that

$$\eta(t) = \eta - \zeta t^{\beta},$$

where η and ζ are non-negative bounded random variables, being independent of $\xi(\cdot)$; and assume that $\eta \ge s_0$ (a.s.) for some $s_0 > 0$.

For keeping the random standard deviation $\eta(t)$ strictly positive, we consider that the time interval is small enough. Hence we study the probability of exceedance occurring in a time interval $[0,T]$, i.e.

$$P_{u,\beta} := P\{\max_{t\in[0,T]} \xi(t)(\eta - \zeta t^{\beta}) > u\},$$

where $0 < T < (\frac{1}{\sigma(\zeta/\eta)})^{1/\beta}$ and $\sigma(\zeta/\eta) = \sup\{x : P(\zeta/\eta \le x) < 1\}$ is assumed to be finite. In general, we will denote by $\sigma(U) = \sup\{x : P(U \le x) < 1\}$ for any random variable U.

We approximate the tail of the standard normal distribution by the well-known relation

$$\Psi(u) := \frac{1}{\sqrt{2\pi}u} e^{-u^2/2} \sim P(\xi(0) > u) \qquad \text{as } u \to \infty.$$

We use Pickands constant H_{α} which is defined by

$$H_{\alpha} = \lim_{T\to\infty} \frac{1}{T} E \exp\big(\max_{t\in[0,T]} \chi(t)\big),$$

where the process $\chi(t)$ is a shifted fractional Brownian motion with expectation $E\chi(t) = -|t|^{\alpha}$ and covariance function $\mathrm{cov}(\chi(t),\chi(s)) = |t|^{\alpha} + |s|^{\alpha} - |t-s|^{\alpha}$.

First we assume that the conditional expectation $E\big[\zeta^{-1/\beta}\,|\,\eta\big]$ is bounded for all η given, which implies ζ is strictly positive with probability one. If the Gaussian process is stationary, we note that for almost all given η and ζ the conditions in Theorem D.3 of Piterbarg [11] also hold for the conditional probability

$$P\{\max_{t\in[0,T]} \xi(t)(1 - \frac{\zeta}{\eta} t^{\beta}) > \frac{u}{\eta}\,|\,\eta,\zeta\}.$$

It can be considered as a ruin probability for Gaussian processes with deterministic variance. In the following theorems, we show that under the condition above, the results can be generalized for locally stationary Gaussian processes with a random variance.

Theorem 1.1. *Let $\xi(t)$ be a standard locally stationary Gaussian process with $\alpha \in (0,2]$. Suppose that the random variable η has a bounded density function $f_\eta(y)$, which is k times continuously differentiable in a neighborhood of $\sigma = \sigma(\eta)$, for some $k = 0,1,2,\ldots$, and satisfies $f_\eta^{(r)}(\sigma) = 0$ for $r = 0,1,\ldots,k-1$ and $f_\eta^{(k)}(\sigma) \neq 0$. Further assume that the function $E^{(\zeta)}(y) := E\left[\zeta^{-1/\beta} \,|\, \eta = y\right]$ is bounded in $[s_0,\sigma]$. For any $T \in (0,(\sigma(\zeta/\eta))^{-1/\beta})$,*

(a) if $\alpha < \beta \in (0,\infty)$, and $E^{(\zeta)}(y)$ is continuous at $y = \sigma$, then

$$P_{u,\beta} \sim (-1)^k \frac{C^{1/\alpha}(0) H_\alpha \Gamma(1/\beta)}{\beta} \sigma^{3/\beta - 2/\alpha + 3k + 3} E^{(\zeta)}(\sigma) \times f_\eta^{(k)}(\sigma) u^{2/\alpha - 2/\beta - 2 - 2k} \Psi(u/\sigma)$$

as $u \to \infty$;

(b) if $\alpha = \beta \in (0,2]$, and $\widetilde{H}_\alpha(y) := E\left[\exp\left(\max_{t\in[0,\infty)}(\chi(t) - \frac{\zeta t^\alpha}{C(0)\eta})\right) \,|\, \eta = y\right]$ is continuous at $y = \sigma$, then

$$P_{u,\beta} \sim (-1)^k \sigma^{3k+3} f_\eta^{(k)}(\sigma) \widetilde{H}_\alpha(\sigma) u^{-2-2k} \Psi(u/\sigma) \qquad \text{as } u \to \infty;$$

(c) if $0 < \beta < \alpha \in (0,2]$, then

$$P_{u,\beta} \sim (-1)^k \sigma^{3k+3} f_\eta^{(k)}(\sigma) u^{-2-2k} \Psi(u/\sigma) \qquad \text{as } u \to \infty.$$

If the conditional expectation $E^{(\zeta)}(y)$ does not exist or is not bounded for $y \in [s_0,\sigma]$, then we have to consider the joint density $f_{\zeta,\eta}(x,y)$ of ζ and η, and restrict appropriately the local behavior of the conditional density $e(x,y) := f_\zeta(x|\eta = y) = f_{\zeta,\eta}(x,y)/f_\eta(y)$. The following results on $P_{u,\beta}$ as $u \to \infty$ depend also on the values α and β. The possible set $(\alpha,\beta) \in (0,2] \times (0,\infty)$ is split into six subsets or cases which are shown and labeled in Figure 1. The results for the cases depend on suitable assumptions.

For the first three cases a), b) and c), we need to assume a particular regularity **condition R**: Assume that $e(0,y)$ is continuous at $y = \sigma = \sigma(\eta)$ and $e(x,y)$ is positive in a neighborhood of $x = 0$, and continuous for any given $y \in [s_0,\sigma]$.

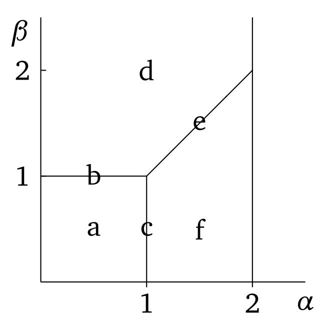

Figure 1: The 6 different domains of α and β dealt with in Theorem 1.2.

Theorem 1.2. *Let $\xi(t)$ be a standard locally stationary Gaussian process with $\alpha \in (0,2]$. Let the density function f_η of η satisfy the same assumptions as in Theorem 1.1. Suppose that ζ and η have a joint density function $f_{\zeta,\eta}(x,y)$, and the conditional density function $e(x,y) := f_\zeta(x|\eta = y) = f_{\zeta,\eta}(x,y)/f_\eta(y)$ is bounded in $[0,\sigma(\zeta)] \times [s_0,\sigma(\eta)]$. Let $T \in (0,(\sigma(\zeta/\eta))^{-1/\beta})$.*

(a) If $\alpha,\beta \in (0,1)$, then assuming condition **R**

$$P_{u,\beta} \sim (-1)^k H_\alpha e(0,\sigma)\Big(\int_0^T \frac{(C(s))^{1/\alpha}}{s^\beta}\,\mathrm{d}s\Big)\sigma^{5+3k-2/\alpha} f_\eta^{(k)}(\sigma) u^{2/\alpha-4-2k}\Psi(u/\sigma).$$

(b) If $\alpha \in (0,1]$ and $\beta = 1$, then assuming condition **R**

$$P_{u,\beta} \sim (-1)^k 2H_\alpha e(0,\sigma)(C(0))^{1/\alpha}\sigma^{5+3k-2/\alpha} f_\eta^{(k)}(\sigma) u^{2/\alpha-4-2k}\log u\Psi(u/\sigma)$$

(c) If $1 = \alpha > \beta$, then assuming condition **R**

$$P_{u,\beta} \sim (-1)^k \sigma^{3+3k}\Big(1 + e(0,\sigma)\int_0^T \frac{C(s)}{s^\beta}\,\mathrm{d}s\Big) f_\eta^{(k)}(\sigma) u^{-2-2k}\Psi(u/\sigma)$$

(d) if $\alpha < \beta \in (1,\infty)$ and $E^{(\zeta)}(y)$ is continuous at $y = \sigma$, then

$$P_{u,\beta} \sim (-1)^k \frac{(C(0))^{1/\alpha} H_\alpha \Gamma(1/\beta)}{\beta} \times \sigma^{3/\beta-2/\alpha+3k+3} E^{(\zeta)}(\sigma) f_\eta^{(k)}(\sigma) u^{2/\alpha-2/\beta-2-2k}\Psi(u/\sigma) \quad \text{as } u \to \infty;$$

(e) if $\alpha = \beta \in (1,2]$ and $\widetilde{H}_\alpha(y) := E\big[\exp\big(\max_{t\in[0,\infty)}(\chi(t) - \frac{\zeta t^\alpha}{C(0)\eta})\big)\,|\,\eta = y\big]$ is continuous at $y = \sigma$, then

$$P_{u,\beta} \sim (-1)^k \sigma^{3k+3} f_\eta^{(k)}(\sigma)\widetilde{H}_\alpha(\sigma) u^{-2-2k}\Psi(u/\sigma) \qquad \text{as } u \to \infty;$$

(f) if $\alpha > \beta$ and $\alpha > 1$, then

$$P_{u,\beta} \sim (-1)^k \sigma^{3k+3} f_\eta^{(k)}(\sigma) u^{-2-2k}\Psi(u/\sigma) \qquad \text{as } u \to \infty.$$

These results show that the exact asymptotic behavior of the ruin probability $P_{u,\beta}$ depends on the local behavior of the marginal density $f_\eta(y)$ at $\sigma(\eta)$ and on the relation between α and β: $\beta < \alpha$, $= \alpha$ and $> \alpha$. We also notice that the impact of the function $C(t)$ of the locally stationary Gaussian process is restricted in some cases on $C(0)$, if $\alpha \le \beta \ge 1$, and that the whole function $C(t)$ plays a role only in the case $\beta < 1$.

Remark: In case $C(t) \equiv C$, as for stationary Gaussian processes $\xi(t)$, the integral on the $C(t)$ function in (a) and (c) simplifies to $C^{1/\alpha}T^{1-\beta}/(1-\beta)$ in the case $\beta < 1$.

In the next section we introduce some necessary lemmas, and prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

2 Lemmas

For our derivations, some useful lemmas are stated in this section.

The first lemma is a reformulation of Lemma 6.1 of Piterbarg [11] for the case of a stationary Gaussian process with general $C(t) = C > 0$, by use of a time transformation.

Lemma 2.1. *For any $z > 0$ and $h > 0$, as $u \to \infty$,*

$$P\{\max_{t \in [0, hu^{-2/\alpha}]} \xi(t)(1 - zt) > u\} \sim \Psi(u) E \exp\big(\max_{[0, hC^{1/\alpha}]} \chi(t) - zC^{-1/\alpha} u^{2-2/\alpha} t\big).$$

The more general random variance case with $(1 - \zeta t^{\beta})^2$ is dealt with in Lemma 3.1. For the derivation of the asymptotic behavior, we state the common result based on saddle-point approximation, in the following proposition.

Proposition 2.2. *Let $g(x)$, $x \in [0, \sigma]$, be a bounded function, which is k times continuously differentiable in a neighborhood of σ and satisfies that $g^{(r)}(\sigma) = 0$ for $r = 0, 1, \ldots, k-1$, and $g^{(k)}(\sigma) \neq 0$. Then for any $\epsilon \in (0, \sigma)$*

$$\int_{\epsilon}^{\sigma} g(x)\Psi(u/x)\mathrm{d}x = (-1)^k \sigma^{3k+3} g^{(k)}(\sigma) u^{-2-2k} \Psi(u/\sigma)(1 + o(1)) \text{ as } u \to \infty. \tag{1}$$

If $g(x) = g_1(x)g_2(x)$, $g_1(x)$ is continuous at σ with $g_1(\sigma) > 0$, and $g_2(x)$ satisfies the above conditions on g, one can change $g^{(k)}(\sigma)$ in (1) to $g_1(\sigma)g_2^{(k)}(\sigma)$.

To prove this, we make the variable change $y = u^2(x - \sigma)$ in the integral and use the saddle-point approximation, or simply see Fedoruk [4].

Another asymptotic approximation concerns a particular case of "delta-wise" sequences.

Lemma 2.3. *Let $g(x)$ be a non-negative and bounded function on $[0, b]$, $b > 0$, which is positive and continuous at 0. Then for any $h > 1$, $a > 0$ and any $\alpha \in (0, 1]$,*

$$\lim_{u \to \infty} \frac{1}{u^{2/\alpha - 2} \log u} \int_0^b \frac{1 - \exp(-axu^2)}{1 - \exp(-hxu^{2-2/\alpha})} g(x)\, \mathrm{d}x = \frac{2g(0)}{h}.$$

Proof: Choose $\epsilon > 0$ arbitrarily small, and let $\delta > 0$ be such that $|g(x) - g(0)| \leq \epsilon$ and $1 - e^{-x} \lesseqgtr (1 \pm \epsilon)x$, for all $x \in [0, \delta]$. Then we have

$$\int_0^b \frac{1 - \exp(-axu^2)}{1 - \exp(-hxu^{2-2/\alpha})} g(x)\, \mathrm{d}x \lesseqgtr (g(0) \pm \epsilon) \frac{u^{2/\alpha - 2}}{h(1 \mp \epsilon)} \int_0^{\delta/h} \frac{1 - \exp(-axu^2)}{x}\, \mathrm{d}x$$
$$+ \int_{\delta/h}^b \frac{1 - \exp(-axu^2)}{1 - \exp(-hxu^{2-2/\alpha})} g(x)\, \mathrm{d}x.$$

Change variable $y = axu^2$ in the first integral, and check that

$$\lim_{u \to \infty} \frac{1}{2 \log u} \int_0^{\frac{\delta a u^2}{h}} \frac{1 - e^{-y}}{y}\, \mathrm{d}y = 1.$$

If $\alpha < 1$, we can bound the second integral by

$$C\int_{\delta/h}^{b} \frac{1}{1-\exp(-hxu^{2-2/\alpha})}\,\mathrm{d}x \le \frac{Cb}{1-\exp(-\delta u^{2-2/\alpha})} \sim \frac{Cb}{\delta}u^{2/\alpha-2},$$

where $C \ge \max_{[\delta/h,b]} g(x)$ is some constant. If $\alpha = 1$, it is easy to see that the second integral is bounded by a constant. Then the statement follows by letting $\epsilon \to 0$. □

We need in the proof of Theorem 1.2 a result which is necessary in other cases too. It is an extention of Lemma 6.3 in Piterbarg [11] for a stationary zero mean Gaussian process $\xi(t)$, $t \in [0,T]$ with the usual correlation assumption: $r(t) = 1 - C|t|^\alpha + o(|t|^\alpha)$ with $C > 0$ and $\alpha \in (0,2]$.

Then there exists some $\varepsilon > 0$ such that

$$1 - 2|C^{1/\alpha}t|^\alpha \le r(t) \le 1 - (1/2)|C^{1/\alpha}t|^\alpha,\ t \in [0,\varepsilon].$$

Lemma 6.3. in [11] assumes $C = 1$, which means that we have to apply a time change. Let $a = \min(1, 2^{\alpha-1})$.

Lemma 2.4. *Let $\xi(t)$, $t \in [0,T]$, be a stationary Gaussian process. Let the functions $s_i = s_i(u)$, $t_i = t_i(u)$, $i = 1,2$ of $u > 0$, with values in $[0,T)$, be such that $0 \le s_1 < t_1 - 2u^{-2/\alpha} < s_2 - 4u^{-2/\alpha} < t_2 - 6u^{-2/\alpha}$. Suppose that $t_2 - s_1 \to 0$ as $u \to \infty$. Then for all $u \ge u_0 := \inf\{u :\ t_2 - s_1 \le C^{-1/\alpha}\varepsilon/4\}$,*

$$P(\max_{[s_i,t_i]} \xi(t) > u, i = 1,2) \le C_0 C_1 \Psi(u) \exp\left(-\frac{aC}{8}((s_2 - t_1)u^{2/\alpha} - 1)^\alpha\right),$$

where C_0 is the absolute constant from Lemma 6.3 of [11] and

$$C_1 = \sum_{i=0}^{K_1}\sum_{j=0}^{K_2} \exp\left(a(i+j)^\alpha\right)$$

where $K_l := [(t_l - s_l)u^{2/\alpha}]$.

Proof: Split the intervals $[s_l,t_l]$, $l = 1,2$ into subintervals $\Delta_{l,i} = [s_l + iu^{-2/\alpha},\ s_l + (i+1)u^{-2/\alpha}]$, $l = 1,2$, $i = 0,1,2,\dots,K_l := [(t_l - s_l)u^{2/\alpha}]$, $l = 1,2$, where the intervals $\Delta_{l,K_l}, l = 1,2$, cover the points t_1 and t_2, respectively. We have

$$\begin{aligned} P(\max_{[s_l,t_l]} \xi(t) > u, l = 1,2) \le& \qquad (2)\\ \le \sum_{i=0}^{K_1}\sum_{j=0}^{K_2} P\{\max_{\Delta_{1,i}} \xi(t) > u,\ \max_{\Delta_{2,j}} \xi(t) > u\} =: \sum_i\sum_j p_{ij}.\end{aligned}$$

Apply Lemma 6.3 in [11] to any term p_{ij} in the sum. The distance between $\Delta_{1,i}$ and $\Delta_{2,j}$ is at least $s_2 - t_1 + ju^{-2/\alpha} + (K_1 - i - 1)u^{-2/\alpha} = s_2 - t_1 + (j - i + K_1 - 1)u^{-2/\alpha}$. We have,

$$\begin{aligned} p_{ij} &\le C_0\Psi(u)\exp\left(-\frac{1}{8}(C^{1/\alpha}((s_2 - t_1)u^{2/\alpha} + j - i + K_1 - 1))^\alpha\right)\\ &\le C_0\Psi(u)\exp\left(-\frac{Ca}{8}((s_2 - t_1)u^{2/\alpha} - 1)^\alpha\right)\exp\left(a(j - i + K_1)^\alpha\right)\end{aligned}$$

for all u sufficiently large. To get the second inequality we use the inequality $(x+y)^\alpha \ge a(x^\alpha + y^\alpha)$, valid for all positive x,y. By summing the bounds, we get the stated assertion. □

3 Proof of Theorem 1.1

We are approximating the locally stationary Gaussian process in small intervals by stationary Gaussian processes. Since $C(t)$ is positive and continuous at 0, we have for any small $\epsilon > 0$ and δ sufficiently small that $\sup_{[0,\delta]} |C(t) - C(0)| \le \epsilon$. Let $X^+(t)$ and $X^-(t)$ be two standard stationary Gaussian processes with covariance functions $r^+(t)$ and $r^-(t)$ respectively, where for all $t \neq s \ge 0$, $r^+(|t-s|) \le r(t,s) \le r^-(|t-s|) < 1$ and

$$r^+(t) = 1 - (C(0)+\epsilon)|t|^\alpha + o(|t|^\alpha),\ r^-(t) = 1 - (C(0)-\epsilon)|t|^\alpha + o(|t|^\alpha) \text{ as } t \to 0.$$

Such stationary Gaussian processes exist. We apply Slepian's lemma (cf. Theorem C.1 of Piterbarg [11]) to derive the bounds

$$\begin{aligned} P\{\max_{[0,\delta]} X^-(t)(1-\zeta t^\beta) > u \mid \zeta\} &\le P\{\max_{[0,\delta]} \xi(t)(1-\zeta t^\beta) > u \mid \zeta\} \\ &\le P\{\max_{[0,\delta]} X^+(t)(1-\zeta t^\beta) > u \mid \zeta\} \end{aligned}$$

In the same way we define further stationary Gaussian processes $X_k^+(t)$ and $X_k^-(t)$ on intervals $I_k = [ku^{-2/\beta}, (k+1)u^{-2/\beta})$ with $C_{\min,k} = \min_{t\in I_k} \le C(t) \le \max_{t\in I_k} C(t) = C_{\max,k}$. These processes approximate the locally stationary Gaussian process $\xi(t)$ in the intervals I_k with Slepian's inequality.

Lemma 3.1. *Let $\xi(t), 0 \le t \le T$, be a locally stationary Gaussian process. Suppose that ζ is a bounded nonnegative random variable which is independent of $\xi(\cdot)$, with $E\zeta^{-1/\beta} < \infty$. For any $T \in (0, (\sigma(\zeta))^{-1/\beta})$,*

(a) if $\alpha < \beta \in (0,\infty)$, then as $u \to \infty$

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \sim \frac{H_\alpha C^{1/\alpha}(0)\Gamma(1/\beta)}{\beta} u^{2/\alpha - 2/\beta}\Psi(u)E\zeta^{-1/\beta}\ ;$$

(b) if $\alpha = \beta \in (0,2]$, then as $u \to \infty$,

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \sim E(H_\alpha^{\zeta/C(0)})\Psi(u),$$

where $0 < H_\alpha^\zeta := E\exp\big(\max_{[0,\infty)}(\chi(t) - \zeta t^\alpha)|\zeta\big) < \infty$;

(c) if $2 \ge \alpha > \beta > 0$, then as $u \to \infty$

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \sim \Psi(u)\,.$$

Proof: (a) We use the intervals I_k as partition of the interval $[0,T]$. Since an interval with length smaller than $u^{-2/\beta}$ has here no asymptotic effect on the probability, we assume without loss of generality that $k \le n := [Tu^{2/\beta}] = Tu^{2/\beta}$.

For any given ζ, by Theorem D.2 of Piterbarg [11], the stationarity of $\xi(t)$ and the time transformation such that $C = 1$ as in Lemma 2.1, we get the upper bound of the conditional probability, which is used for the dominating convergence.

$$
\begin{aligned}
P\{\max_{t\in[0,T]}\xi(t)(1-\zeta t^{\beta})>u\,|\,\zeta\} &\le \sum_{k=0}^{n-1}P\{\max_{[0,u^{-2/\beta}]}X_k^{+}(t)>u/(1-\zeta(ku^{-2/\beta})^{\beta})\,|\,\zeta\}\\
&\le \sum_{k=0}^{n-1}C_{\max,k}^{1/\alpha}H_{\alpha}u^{-2/\beta}(\frac{u}{1-\zeta k^{\beta}u^{-2}})^{2/\alpha}\Psi(\frac{u}{1-\zeta k^{\beta}u^{-2}})\\
&\le \sum_{k=0}^{n-1}C_1H_{\alpha}u^{2/\alpha-2/\beta}\Psi(u)\exp(-\zeta k^{\beta})\\
&\le C_1H_{\alpha}u^{2/\alpha-2/\beta}\Psi(u)[1+\beta^{-1}\zeta^{-1/\beta}\int_0^{\infty}v^{1/\beta-1}\exp(-v)\,\mathrm{d}v]\\
&= C_1H_{\alpha}u^{2/\alpha-2/\beta}\Psi(u)[1+\Gamma(1/\beta)\beta^{-1}\zeta^{-1/\beta}],
\end{aligned}
$$

where C_1 is some constant, not depending on ζ and k.

By a reformulation of Theorem D.3 (i) in Piterbarg [11] for stationary Gaussian processes, (in which the author considered that the variance reaches its maximum at an interior point of the segment $[0,\delta]$ (with some $\delta>0$); here our variance attains its maximum at 0 which is the boundary point of $[0,\delta]$, which implies the factor 2 is replaced by the factor 1 in that theorem), and with the time transformation to standardize $C(0)+\varepsilon$ to 1 as in Lemma 2.1, we know that for any given $\zeta>0$,

$$
\lim_{u\to\infty}\frac{P\{\max_{t\in[0,\delta]}X^{+}(t)(1-\zeta t^{\beta})>u\,|\,\zeta\}}{u^{2/\alpha-2/\beta}\Psi(u)}=H_{\alpha}(C(0)+\varepsilon)^{1/\alpha}\Gamma(1/\beta)\beta^{-1}\zeta^{-1/\beta}. \tag{3}
$$

The analogous result holds for the $X^{-}(t)$ processes with $C(0)-\varepsilon$. With Slepian's inequality we get the bounds for the conditional probability of the analogous event with $\xi(t)$, for any $\zeta>0$. Similar inequalities hold for the other processes $X_k^{+}(t)$ and $X_k^{-}(t)$ as mentioned. This implies that for the upper bound

$$
\begin{aligned}
P\{\max_{t\in[0,T]}\xi(t)(1-\zeta t^{\beta})>u\,|\,\zeta\} &\le P\{\max_{t\in[0,\delta]}X^{+}(t)(1-\zeta t^{\beta})>u\,|\,\zeta\}\\
&\quad+\sum_{k=n_0}^{n}P\{\max_{t\in[0,u^{-2/\beta}]}X_k^{+}(t)>u/(1-\zeta(ku^{-2/\beta})^{\beta})\,|\,\zeta\}
\end{aligned} \tag{4}
$$

with $n_0=[\delta u^{2/\beta}]$. The first term is approximated in (3). Each term of the sum can be approximated by the upper bounds used as in the domination argument above.

$$
\begin{aligned}
&\sum_{k=n_0}^{n-1}P\{\max_{t\in[0,u^{2/\beta}]}X_k^{+}(t)>u/(1-\zeta(ku^{-2/\beta})^{\beta})\,|\,\zeta\}\\
&\qquad\le \sum_{k=n_0}^{n-1}C_1H_{\alpha}u^{2/\alpha-2/\beta}\Psi(u)\exp(-\zeta k^{\beta})\\
&\qquad\le C_1H_{\alpha}u^{2/\alpha-2/\beta}\Psi(u)\int_{n_0-1}^{n-1}\exp(-\zeta z^{\beta})dz\\
&\qquad\le C_1H_{\alpha}u^{2/\alpha-2/\beta}\Psi(u)\zeta^{-1/\beta}\int_{\zeta(n_0-1)^{\beta}}^{\zeta n^{\beta}}\exp(-v)v^{1/\beta-1}dv/\beta
\end{aligned}
$$

Taking the expectation on ζ, the integral term with the factor $\zeta^{-1/\beta}$ is dominated by $E(\zeta^{-1/\beta}) < \infty$. Furthermore, since the integral is converging point wise to 0 for $\zeta > 0$ (as $u \to \infty$), we have that the sum is bounded by $o(u^{2/\alpha - 2/\beta}\Psi(u))$. Hence, the first term in (4) is dominating.

For the lower bound we use

$$
\begin{aligned}
P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u \,|\, \zeta\} &\geq P\{\max_{t\in[0,\delta]} X^-(t)(1-\zeta t^\beta) > u \,|\, \zeta\} \\
&\sim H_\alpha (C(0)-\varepsilon)^{1/\alpha}\Gamma(1/\beta)\beta^{-1}\zeta^{-1/\beta}u^{2/\alpha-2/\beta}\Psi(u)
\end{aligned}
$$

Since $E\zeta^{-1/\beta} < \infty$, we get the stated result by dominated convergence and letting $\varepsilon \to 0$.

(b) Split the interval $[0,T]$ into subintervals with length $u^{-2/\beta}$, with again $n := [Tu^{2/\beta}] = Tu^{2/\beta} = Tu^{2/\alpha}$. The proof follows the steps of the proof in a). However, since $\alpha = \beta$, we need to apply Lemma D.1 of Piterbarg [11] for any given ζ, to show the domination. Here we use that $E\zeta^{-1/\alpha} = E\zeta^{-1/\beta} < \infty$.

By a reformulation of Theorem D.3 (ii) in Piterbarg [11], and with the time transformation to standardize $C(0)+\varepsilon$ to 1, as above, we know that for any given ζ,

$$
\lim_{u\to\infty} \frac{P\{\max_{t\in[0,\delta]} X^+(t)(1-\zeta t^\alpha) > u \,|\, \zeta\}}{\Psi(u)} = H_\alpha^{\zeta/(C(0)+\varepsilon)}.
$$

where $H_\alpha^\zeta := E\big[\exp\big(\max_{[0,\infty)} \chi(t) - \zeta t^\alpha\big)|\zeta\big] \in (0,\infty)$. The analogous result holds for the lower approximation with $X^-(t)$ and $C(0)-\varepsilon$. The approximation for the maximum of the process in the interval $[\delta, T]$ of part a) can be used again. Since $E\zeta^{-1/\alpha} = E\zeta^{-1/\beta} < \infty$, we get the stated result by dominated convergence and letting $\varepsilon \to 0$. The domination shows also that $EH_\alpha^{\zeta/C} < \infty$, since $E\zeta^{1/\beta} < \infty$.

(c) For this case, we split the interval $[0,T]$ into subintervals $I_k = [k\delta, (k+1)\delta]$ of length δ with $0 < \delta < \min(1,T)$, and define new standard stationary Gaussian processes $X_k^+(t)$ with $C_k^+ = \max_{t\in I_k} C(t)$. Then with the result of part b) we get by stationarity

$$
\begin{aligned}
& P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u \,|\, \zeta\} \\
\leq\;& \sum_{k=0}^{[T/\delta]} P\{\max_{[0,\delta]} X_k^+(t)(1-\zeta(t+k\delta)^\beta) > u \,|\, \zeta\} \\
\leq\;& \sum_{k=0}^{[T/\delta]} P\{\max_{[0,\delta]} X_k^+(t)(1-\zeta t^\beta) > u \,|\, \zeta\} \\
\leq\;& ([T/\delta]+1)\cdot P\{\max_{[0,\delta]} X_K^+(t)(1-\zeta t^\alpha) > u \,|\, \zeta\} \\
\leq\;& ([T/\delta]+1) C H_\alpha^{\zeta/C_K}\Psi(u),
\end{aligned}
$$

where $X_K^+(t)$ is that one of the $X_k^+(t)$ with $C_K^+ = \max_k C_k^+$, and $C > 0$ some constant. We mentioned already that $E(H_\alpha^{\zeta/C_K})$ is finite. By Theorem D.3 (iii) of Piterbarg [11], we know that for any given ζ

$$
\lim_{u\to\infty} \frac{P\{\max_{t\in[0,\delta]} X_0^+(t)(1-\zeta t^\beta) > u \,|\, \zeta\}}{\Psi(u)} = 1
$$

The same result holds for the lower approximation with $X_0^-(t)$. The interval $(\delta, T]$ does not play a role in the asymptotic result, since using Theorem D.2 of Piterbarg [11] and the argument in the domination part as in proof of part a) above, we have

$$\begin{aligned}
&\sum_{k=1}^{[T/\delta]} P\{\max_{[0,\delta]} X_k^+(t)(1-\zeta(t+k\delta)^\beta) > u \,|\, \zeta\} \\
&\leq \sum_{k=1}^{[T/\delta]} P\{\max_{[0,\delta]} X_k^+(t)(1-\zeta(k\delta)^\beta) > u \,|\, \zeta\} \\
&\leq [T/\delta] \cdot P\{\max_{[0,\delta]} X_K^+(t) > u/(1-\zeta\delta^\beta) \,|\, \zeta\} \\
&\leq [T/\delta]\delta C H_\alpha u^{2/\alpha} \Psi(u) \exp(-\zeta\delta^\beta u^2)
\end{aligned}$$

for some constant $C > 0$. We note that $E(\exp(-\zeta\delta^\beta u^2)) = O(u^{-2/\beta}) = o(u^{-2/\alpha})$, since $E(\zeta^{-1/\beta}) < \infty$ and $\alpha > \beta$.
For $E(\exp(-\zeta\delta^\beta u^2)) = (\delta^\beta u^2)^{-1/\beta} \int_0^{\sigma_\zeta} e^{-z\delta^\beta u^2}(z\delta^\beta u^2)^{1/\beta} z^{-1/\beta} f_\zeta(z)dz \leq C(\delta^\beta u^2)^{-1/\beta}$ since $e^{-\nu}\nu$ is bounded. Then by using the dominated convergence, the third statement follows. □

This lemma is now applied in combination with Proposition 2.2 to prove the first main theorem.

Proof of Theorem 1.1:

(a) If $\alpha < \beta \in (0, \infty)$. Then by Lemma 3.1 (a) and Proposition 2.2, we get for $u \to \infty$

$$\begin{aligned}
P_{u,\beta} &= EP\{\max_{t\in[0,T]} \xi(t)(1-\frac{\zeta}{\eta}t^\beta) > \frac{u}{\eta}|\eta\} \\
&\sim \frac{H_\alpha C^{1/\alpha}(0)\Gamma(1/\beta)}{\beta} E\big[(\frac{u}{\eta})^{2/\alpha-2/\beta}\Psi(\frac{u}{\eta})E[(\frac{\zeta}{\eta})^{-1/\beta}|\eta]\big] \\
&= \frac{H_\alpha C^{1/\alpha}(0)\Gamma(1/\beta)}{\beta} u^{2/\alpha-2/\beta} \int_{s_0}^{\sigma(\eta)} y^{3/\beta-2/\alpha} E^{(\zeta)}(y) f_\eta(y)\Psi(u/y)\,\mathrm{d}y \\
&\sim (-1)^k \frac{H_\alpha C^{1/\alpha}(0)\Gamma(1/\beta)}{\beta} \sigma^{3/\beta-2/\alpha+3k+3} E^{(\zeta)}(\sigma) f_\eta^{(k)}(\sigma) u^{2/\alpha-2/\beta-2-2k}\Psi(u/\sigma)
\end{aligned}$$

where in Proposition 2.2 we use $g_1(y) = y^{3/\beta-2/\alpha}E^{(\zeta)}(y)$ and $g_2(y) = f_\eta(y)$.

(b) If $\alpha = \beta \in (0, 2]$, we apply Lemma 3.1 (b) and Proposition 2.2, to get

$$\begin{aligned}
P_{u,\beta} &= EP\{\max_{t\in[0,T]} \xi(t)(1-\frac{\zeta}{\eta}t^\beta) > \frac{u}{\eta}|\eta\} \\
&\sim E\Big[E\big[\exp\big(\max_{t\in[0,\infty)}(\chi(t) - \frac{\zeta}{C(0)\eta}t^\alpha)\big)\,|\,\eta\big]\Psi(\frac{u}{\eta})\Big] \\
&= \int_{s_0}^{\sigma(\eta)} E\big[\exp\big(\max_{t\in[0,\infty)}(\chi(t) - \frac{\zeta}{C(0)\eta}t^\alpha)\big)\,|\,\eta = y\big]\Psi(u/y) f_\eta(y)\,\mathrm{d}y \\
&\sim (-1)^k \sigma^{3k+3} f_\eta^{(k)}(\sigma)\widetilde{H}_\alpha(\sigma) u^{-2-2k}\Psi(u/\sigma) \qquad \text{as } u \to \infty.
\end{aligned}$$

(c) If $2 \geq \alpha > \beta > 0$. Then by Lemma 3.1 (c) and Proposition 2.2, we obtain

$$P_{u,\beta} = EP\{\max_{t\in[0,T]} \xi(t)(1-\frac{\zeta}{\eta}t^{\beta}) > \frac{u}{\eta} \mid \eta\} \sim E\left[\Psi(u/\eta)\right]$$
$$= \int_{s_0}^{\sigma(\eta)} \Psi(u/y) f_{\eta}(y)\,\mathrm{d}y \sim (-1)^k \sigma^{3k+3} f_{\eta}^{(k)}(\sigma) u^{-2-2k} \Psi(u/\sigma)$$

as $u \to \infty$. □

4 Proof of Theorem 1.2

In the following lemmas, we always assume that ζ is a non-negative bounded random variable independent of $\xi(\cdot)$ and its density function f_ζ is bounded. Denote $\widetilde{f}_\zeta := \sup_{z\in[0,\sigma(\zeta)]} f_\zeta(z) < \infty$.

From the stated theorem, we have to discuss different cases, depending on $\beta <,=$ or $> \alpha$ and whether $\alpha <,=$ or > 1 as shown in Figure 1. The following lemmas deal with the different cases in the given order of a) - f), by applying similar ideas. We begin with case (a).

Lemma 4.1. *Let $\alpha, \beta \in (0,1)$. Suppose that the density $f_\zeta(x)$, $x \geq 0$, is positive and continuous at 0. Then for any $T \in (0, (\sigma(\zeta))^{-1/\beta})$, as $u \to \infty$*

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^{\beta}) > u\} \sim H_\alpha \int_0^T C^{1/\alpha}(t)\, t^{-\beta} \mathrm{d}t \cdot f_\zeta(0) u^{2/\alpha-2} \Psi(u)$$

Proof: Split the interval $[0,T]$ into subintervals with length $u^{-2/\alpha}\log u$ and denote these subintervals by $\Delta_k = [ku^{-2/\alpha}\log u, (k+1)u^{-2/\alpha}\log u]$. Without loss of generality assume again $n = T/(u^{-2/\alpha}\log u)$ is a integer.

In each interval Δ_k we approximate the locally stationary Gaussian process $\xi(t)$ by the stationary Gaussian processes $X_k^+(t)$ (with $C(t_k)+\varepsilon_1 = C^+(t_k)$) and $X_k^-(t)$ (with $C(t_k)-\varepsilon_1 = C^-(t_k)$) where $C(t_k)-\varepsilon_1 \leq C(t) \leq C(t_k)+\varepsilon_1$ for $t \in \Delta_k$, for some $\varepsilon_1 \to 0$ as $u \to \infty$. We apply Slepian's lemma again as in Lemma 3.1, to get the following approximations.

First, we estimate the upper bound of the probability, by using the stationarity of $X_k^+(t)$ and Theorem D.2 in Piterbarg [11], with the time transformation since $C^+(t_k)$ is in general not equal to 1. For u sufficiently large, we have

$$P\{\max_{[0,T]} \xi(t)(1-\zeta t^{\beta}) > u\} \leq E\Big[\sum_{k=0}^{n-1} P\{\max_{\Delta_k} \xi(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^{\beta}} \mid \zeta\}\Big]$$
$$\leq \quad E\Big[\sum_{k=0}^{n-1} P\{\max_{\Delta_k} X_k^+(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^{\beta}} \mid \zeta\}\Big]$$
$$\leq \quad (1+\gamma(u))E\Big[H_\alpha \sum_{k=0}^{n-1} \frac{(C^+(ku^{-2/\alpha}\log u))^{1/\alpha}\log u}{\left(1-\zeta(ku^{-2/\alpha}\log u)^{\beta}\right)^{2/\alpha}} \Psi\Big(\frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^{\beta}}\Big)\Big]$$
$$\leq \quad (1+\gamma(u))H_\alpha \log u\, \Psi(u) E\Big[\sum_{k=0}^{n-1} \frac{(C^+(ku^{-2/\alpha}\log u))^{1/\alpha} e^{-\zeta k^{\beta} u^{2-2\beta/\alpha}(\log u)^{\beta}}}{\left(1-\zeta(ku^{-2/\alpha}\log u)^{\beta}\right)^{2/\alpha-1}}\Big]$$

$$\leq \quad (1+\gamma(u))H_\alpha \log u\, \Psi(u)\Big[C^+(0) + E\Big(\int_0^{\frac{Tu^{2/\alpha}}{\log u}} \frac{(C^+(xu^{-2/\alpha}\log u)(1+o(1)))^{1/\alpha} e^{-\zeta x^\beta u^{2-2\beta/\alpha}(\log u)^\beta}}{\big(1-\zeta(xu^{-2/\alpha}\log u)^\beta\big)^{2/\alpha-1}}\,\mathrm{d}x\Big)\Big]$$

$$= \quad (1+\gamma(u))H_\alpha C^+(0)\log u\, \Psi(u) + (1+\gamma'(u))\frac{H_\alpha u^{2/\alpha-2/\beta}\Psi(u)}{\beta} \cdot E\Big[\zeta^{-1/\beta}\int_0^{T^\beta u^2\zeta}\Big(\frac{1}{1-vu^{-2}}\Big)^{2/\alpha-1}(C^+((\frac{v}{\zeta u^2})^{1/\beta}))^{1/\alpha} e^{-v} v^{1/\beta-1}\,\mathrm{d}v\Big], \tag{5}$$

where $\gamma(u) \leq \gamma'(u) \downarrow 0$ as $u \to \infty$ and does not depend on ζ and k, because $C(t)$ is continuous and bounded.

By the assumptions of $f_\zeta(x)$, for any arbitrarily small $\epsilon > 0$, there exists some $\delta > 0$ satisfying that for all $0 \leq x \leq \delta$, $|f_\zeta(x) - f_\zeta(0)| \leq \epsilon$. Hence the expectation in (5) is bounded by $(f_\zeta(0)+\epsilon)$ times

$$\int_0^\delta z^{-1/\beta}\Big[\int_0^{T^\beta u^2 z} e^{-v} v^{1/\beta-1}\Big(\frac{1}{1-vu^{-2}}\Big)^{2/\alpha-1}(C^+((\frac{v}{zu^2})^{1/\beta}))^{1/\alpha}\,\mathrm{d}v\Big]\mathrm{d}z + C$$

$$= \frac{u^{2/\beta-2}}{T^{\beta-1}}\int_0^{T^\beta u^2\delta} y^{-1/\beta}\Big[\int_0^y e^{-v} v^{1/\beta-1}\Big(\frac{1}{1-vu^{-2}}\Big)^{2/\alpha-1}(C^+((\frac{v}{y})^{1/\beta}\,T))^{1/\alpha}\,\mathrm{d}v\Big]\mathrm{d}y + C$$

using the transformation $y = T^\beta u^2 z$, where C is some constant for the remaining integral on $[\delta, \sigma(\zeta)]$. By Fubini's theorem and dominated convergence, the above double integral equals

$$\int_0^{T^\beta u^2\delta}\Big(\int_v^{T^\beta u^2\delta} y^{-1/\beta}(C^+((\frac{v}{y})^{1/\beta}\,T))^{1/\alpha}\,\mathrm{d}y\Big) e^{-v} v^{1/\beta-1}\Big(\frac{1}{1-vu^{-2}}\Big)^{2/\alpha-1}\mathrm{d}v. \tag{6}$$

Let us consider the inner integral; we use the variable transformation $s = (v/y)^{1/\beta}T$:

$$\int_v^{T^\beta u^2\delta} y^{-1/\beta}(C^+((\frac{v}{y})^{1/\beta}\,T))^{1/\alpha}\,\mathrm{d}y = \beta T^{\beta-1} v^{1-1/\beta}\int_{(v/u^2\delta)^{1/\beta}}^T s^{-\beta}(C^+(s))^{1/\alpha}\,\mathrm{d}s.$$

For $v = o(u^2)$ the last integral is tending to the constant $\int_0^T s^{-\beta}(C^+(s))^{1/\alpha}\mathrm{d}s = J^+(T)$. Thus we split the outer integral in (6) into two parts for $v \leq g(u)u^2$ and $v > g(u)u^2$ with $g(u) \to 0$ such that $g(u)u^2 \to \infty$. The integral on $v > g(u)u^2$ is of smaller order than the first integral part because of the exponential function. Thus we approximate the first part which is equal to

$$(1+o(1))\,\beta\, J^+(T)\, T^{\beta-1}\int_0^{g(u)u^2} v^{1-1/\beta} e^{-v} v^{1/\beta-1}\Big(\frac{1}{1-vu^{-2}}\Big)^{2/\alpha-1}\mathrm{d}v$$

$$= (1+o(1))\,\beta\, J^+(T)\, T^{\beta-1}\int_0^\infty e^{-v}\,\mathrm{d}v = (1+o(1))\,\beta\, J^+(T)\, T^{\beta-1} \qquad \text{as } u \to \infty.$$

Since the first term of (5) equals $o(u^{2/\alpha-2}\Psi(u))$ as $u \to \infty$, we obtain by combining the approximations

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \leq (1+\gamma''(u))H_\alpha J^+(T)(f_\zeta(0)+\epsilon)u^{2/\alpha-2}\Psi(u),$$

where $\gamma''(u) \downarrow 0$ as $u \to \infty$. Note also that $J^+(T) \to J(T) = \int_0^T s^{-\beta} C^{1/\alpha}(s) \mathrm{d}s$ as $u \to \infty$, by letting $\varepsilon_1 \to 0$.

For the lower bound, by with the same intervals Δ_k and Bonferroni's inequality we have

$$
\begin{aligned}
P\{\max_{[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \geq E\Big[\sum_{k=0}^{n-1} P\{\max_{\Delta_k} \xi(t) > \frac{u}{1-\zeta((k+1)u^{-2/\alpha}\log u)^\beta} | \zeta\} \\
- \sum_{k=0}^{n-2} \sum_{l=k+1}^{n-1} P\{\max_{\Delta_k} \xi(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}, \max_{\Delta_l} \xi(t) > \frac{u}{1-\zeta(lu^{-2/\alpha}\log u)^\beta} | \zeta\} \Big].
\end{aligned} \tag{7}
$$

With the approximating stationary Gaussian process $X_k^-(t)$ and Theorem D.2 of Piterbarg [11], the expectation of the first sum in (7) is bounded below in a similar way by

$$
\begin{aligned}
& E\Big[H_\alpha \sum_{k=0}^{n-1} (1-\gamma(u)) \frac{(C^-(ku^{-2/\alpha}\log u))^{1/\alpha} \log u}{(1-\zeta((k+1)u^{-2/\alpha}\log u)^\beta)^{2/\alpha}} \Psi\Big(\frac{u}{1-\zeta((k+1)u^{-2/\alpha}\log u)^\beta} \Big) \Big] \\
\geq\ & (1-\gamma_1(u)) H_\alpha \Psi(u) u^{2/\alpha} \log u \\
& \times E\Big[\int_{u^{-2/\alpha}}^{T(\log u)^{-1}} (C^-(x\log u))^{1/\alpha} \exp\Big[-\frac{u^2}{2}\Big(\frac{2\zeta(x\log u)^\beta}{1-\zeta(x\log u)^\beta} + \big(\frac{\zeta(x\log u)^\beta}{1-\zeta(x\log u)^\beta}\big)^2 \Big) \Big] \mathrm{d}x \Big] \\
\geq\ & (1-\gamma_1(u))(f_\zeta(0)-\epsilon) H_\alpha \beta^{-1} \Psi(u) u^{2/\alpha - 2/\beta} \int_0^\delta z^{-1/\beta} \\
& \times \Big(\int_{z(\log u)^\beta u^{2-2\beta/\alpha}}^{T^\beta u^2 z} (C^-((\frac{v}{zu^2})^{1/\beta}))^{1/\alpha} \exp\big(-\frac{v}{1-vu^{-2}} - \frac{v^2 u^{-2}}{2(1-vu^{-2})^2} \big) v^{1/\beta - 1} \mathrm{d}v \Big) \mathrm{d}z, \\
=\ & (1-\gamma_1(u))(f_\zeta(0)-\epsilon) H_\alpha \beta^{-1} \Psi(u) u^{2/\alpha - 2/\beta} \\
& \times \int_0^{T^\beta u^2 \delta} J_u^-(T) \exp\Big(-\frac{v}{1-vu^{-2}} - \frac{v^2u^{-2}}{2(1-vu^{-2})^2} \Big) v^{1/\beta-1} \mathrm{d}v
\end{aligned} \tag{8}
$$

by interchanging the integration, and using again Fubini's theorem and dominated convergence, where $\gamma_1(u) \downarrow 0$ as $u \to \infty$ not depending on ζ, and

$$
J_u^-(T) = \int_{T^{-\beta}u^{-2}v}^{\delta^*} (C^-((\frac{v}{zu^2})^{1/\beta}))^{1/\alpha} z^{-1/\beta} \mathrm{d}z
$$

where $\delta^* = \min\{\delta, vu^{2\beta/\alpha - 2}/(\log u)^\beta\}$. By transforming the variable z to $s = (v/zu^2)^{1/\beta}$ we get

$$
J_u^-(T)/(\beta u^{2/\beta - 2} v^{1-1/\beta}) = \int_{s_0}^T (C^-(s))^{1/\alpha} s^{-\beta} \mathrm{d}s = J(T) + o(1)
$$

by letting $u \to \infty$ and $\varepsilon_1 \to 0$, since $s_0 := \max\{(v/\delta u^2)^{1/\beta}, u^{-2/\alpha}\log u\} \to 0$ as $u \to \infty$ for $v = o(u^2)$.

Now we consider the approximation of the integral in (8) from below, similar to the upper approximation with $g(u) \to 0$ such that $g(u)u^2 \to \infty$ as $u \to \infty$.

$$\begin{aligned}\int_0^{T^\beta u^2\delta} & J_u^-(T) \exp\Big(-\frac{v}{1-vu^{-2}} - \frac{v^2u^{-2}}{2(1-vu^{-2})^2}\Big) v^{1/\beta-1}\,\mathrm{d}v \\ &\geq \beta u^{2/\beta}(J(T)+o(1)) \int_0^{g(u)u^2} \exp\Big(-\frac{v}{1-vu^{-2}} - \frac{v^2u^{-2}}{2(1-vu^{-2})^2}\Big)\,\mathrm{d}v \\ &\geq \beta u^{2/\beta}(J(T)+o(1)) \int_0^{g(u)u^2} \exp\Big(-v(1+o(1))\Big)\,\mathrm{d}v = \beta u^{2/\beta}(J(T)+o(1)).\end{aligned} \tag{9}$$

Combining the bounds we note that the lower bound of the first sum in (7) converges to the same bound as the corresponding upper approximation by letting $u \to \infty$ and ε_1, $\varepsilon \to 0$.

It remains to approximate the double sum in (7) by deriving an upper bound. The double-sum in (7) is bounded by

$$\begin{aligned}&\sum_{k=0}^{n-2}\sum_{l=k+1}^{n-1} P\{\max_{\Delta_k}\xi(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}, \max_{\Delta_l}\xi(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta} \mid \zeta\} \\ &\leq \sum_{k=0}^{n-2} P\{\max_{\Delta_k}\xi(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}, \max_{\Delta_{k+1}}\xi(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta} \mid \zeta\} \\ &+\sum_{k=0}^{n-3}\sum_{l=k+2}^{N-1} P\{\max_{\Delta_k}\xi(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}, \max_{\Delta_l}\xi(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta} \mid \zeta\} \\ &\quad+\sum_{k=0}^{n-N}\sum_{l=k+N}^{n} P\{\max_{\Delta_k\times\Delta_l} \xi(t)+\xi(s) > 2u\}\end{aligned} \tag{10}$$

where $N := [\widetilde{\epsilon}/(4u^{-2/\alpha}\log u)]$, with $\widetilde{\epsilon} \in (0, T \wedge 1/2)$ which is chosen such that $1 - C_{\max}\frac{1}{2}|t|^\alpha \geq r(t) \geq 1 - 2C_{\max}|t|^\alpha$ for all $t \in [0, \widetilde{\epsilon}]$, with $C_{\max} = \max_{t\in[0,T]} C(t)$.
For the third sum of (10), note that for all $k + N \leq l \leq n$, the variance of the Gaussian field $\xi(t) + \xi(s)$, with $(t,s) \in \Delta_k \times \Delta_l$,

$$\mathrm{Var}\big(\xi(t)+\xi(s)\big) = 2+2r(t,s) \leq 4 - 2\min_{|t-s|\geq\widetilde{\epsilon}/4}(1-r(t,s)) < 4;$$

there exists a constant b satisfying

$$\begin{aligned}P\big\{\max_{\Delta_k\times\Delta_l}(\xi(t)+\xi(s)) > b\big\} &\leq P\big\{\max_{[0,T]\times[0,T]}(\xi(t)+\xi(s)) > b\big\} \\ &= P\big\{\max_{[0,T]}\xi(t) > \frac{b}{2}\big\} \leq \frac{1}{2}.\end{aligned}$$

Therefore by the Borel theorem (cf. Theorem D.1 in Piterbarg [11]), we get

$$\sum_{k=0}^{n-N}\sum_{l=k+N}^{n} P\big\{\max_{\Delta_k\times\Delta_l}(\xi(t)+\xi(s)) > 2u\big\} \leq 2\Big(\frac{Tu^{2/\alpha}}{\log u}\Big)^2\Psi\Big(\frac{2u-b}{\sqrt{4-\widetilde{\delta}}}\Big) = o(u^{2/\alpha-2}\Psi(u)),$$

as $u \to \infty$, where $\widetilde{\delta} := 2\min_{|t-s|\geq\widetilde{\epsilon}/4}(1-r(t,s)) > 0$.

In the conditional probabilities with respect to the neighboring intervals Δ_k and Δ_l with $l-k \leq N$, we approximate $\xi(t)$ by $X^+(t)$ with $C_{\max}+\varepsilon_1 = \tilde{C}$ instead of $C^+(t_k)$ and $C^+(t_l)$, not depending on k. By stationarity and Theorem D.2 of Piterbarg [11], we estimate the first sum of (10):

$$\begin{aligned}
&\sum_{k=0}^{n-2} P\{\max_{\Delta_k} X^+(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}, \max_{\Delta_{k+1}} X^+(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta} \,|\, \zeta\} \\
&= \sum_{k=0}^{n-2} \Big[2P\{\max_{\Delta_k} X^+(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta} \,|\, \zeta\} \\
&\qquad\qquad - P\{\max_{\Delta_k\cup\Delta_{k+1}} X^+(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta} \,|\, \zeta\}\Big] \\
&\leq \sum_{k=0}^{n-2} \Big[2(1+\gamma(u)) \frac{H_\alpha \tilde{C}^{1/\alpha}\log u}{\big(1-\zeta(ku^{-2/\alpha}\log u)^\beta\big)^{2/\alpha}} \Psi\big(\frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}\big) \\
&\quad - (1-\widetilde{\gamma}(u)) \frac{H_\alpha \tilde{C}^{1/\alpha} 2\log u}{\big(1-\zeta(ku^{-2/\alpha}\log u)^\beta\big)^{2/\alpha}} \Psi\big(\frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}\big)\Big] \\
&= 2(\gamma(u)+\widetilde{\gamma}(u)) \Big[\sum_{k=0}^{n-2} \frac{H_\alpha \tilde{C}^{1/\alpha}\log u}{\big(1-\zeta(ku^{-2/\alpha}\log u)^\beta\big)^{2/\alpha}} \Psi\big(\frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}\big)\Big],
\end{aligned}$$

where $\gamma(u), \widetilde{\gamma}(u) \downarrow 0$ as $u \to \infty$, not depending on k and ζ. Then with the proof for the upper bound of the probability, we obtain that the expectation of the first term of (10) equals $o(u^{2/\alpha-2}\Psi(u))$ as $u \to \infty$.

For the second term of (10), we apply Lemma 2.4, letting

$$s_1 = 0,\ t_1 = u^{-2/\alpha}\log u,\ s_2 = lu^{-2/\alpha}\log u,\ t_2 = (l+1)u^{-2/\alpha}\log u.$$

We have,

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{l=2}^{N-1} P\{\max_{\Delta_i} X^+(t) > \frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta},\ i=0,l \,|\, \zeta\} \\
&\quad \leq C_0 C_1 \sum_{k=0}^{n} \Psi\left(\frac{u}{1-\zeta(ku^{-2/\alpha}\log u)^\beta}\right) \sum_{l=2}^{N-1} \exp\left(-\frac{a\tilde{C}}{8}((l-1)\log u - 1)^\alpha\right) \\
&\quad \leq C_0 C_1 \Psi(u) \sum_{k=0}^{n} e^{-\zeta k^\beta (\log u)^\beta u^{2-2\beta/\alpha}} \sum_{l=2}^{N-1} \exp\left(-\frac{a\tilde{C}}{8}((l-1)\log u - 1)^\alpha\right) \\
&\quad \leq C_2 \Psi(u) \sum_{k=0}^{n} e^{-\zeta k^\beta (\log u)^\beta u^{2-2\beta/\alpha}},
\end{aligned}$$

for a suitable constant C_2. The expectation of the last sum is at most

$$\int_0^{\sigma(\zeta)} \Big[\int_0^{\frac{Tu^{2/\alpha}}{\log u}} \exp(-zx^\beta(\log u)^\beta u^{2-2\beta/\alpha})\,\mathrm{d}x\Big] f_\zeta(z)\,\mathrm{d}z + 1$$

$$\leq \int_0^{\frac{Tu^{2/\alpha}}{\log u}} \big[\widetilde{f}_\zeta \int_0^{\sigma(\zeta)} \exp(-zx^\beta (\log u)^\beta u^{2-2\beta/\alpha})\,\mathrm{d}z\big]\,\mathrm{d}x + 1$$

$$\leq \int_0^{\frac{Tu^{2/\alpha}}{\log u}} \frac{\widetilde{f}_\zeta}{(\log u)^\beta u^{2-2\beta/\alpha} x^\beta}\,\mathrm{d}x + 1 \leq \frac{T^{1-\beta}\widetilde{f}_\zeta}{1-\beta}(\log u)^{-1} u^{2/\alpha-2} + 1 = o(u^{2/\alpha-2})$$

as $u \to \infty$.

Therefore the expectation of the second sum of (10) is $o(\Psi(u)u^{2/\alpha-2})$, as $u \to \infty$.

Since ϵ and ε_1 are arbitrary, we conclude

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \sim H_\alpha J(T) f_\zeta(0) u^{2/\alpha-2}\Psi(u)$$

as $u \to \infty$. □

Now we deal with the case (b).

Lemma 4.2. *Let $\alpha \in (0,1]$ and $\beta = 1$. Suppose that the density $f_\zeta(x)$, $x \geq 0$, is positive and continuous at 0. Then for any $T \in (0, (\sigma(\zeta))^{-1})$,*

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t) > u\} \sim 2C^{1/\alpha}(0) H_\alpha f_\zeta(0) u^{2/\alpha-2}\Psi(u)\log u \qquad \text{as } u \to \infty.$$

Proof: For any $h > 1$ and u sufficiently large, we split the interval $[0,T]$ into subintervals with length $hu^{-2/\alpha}$, denote these subintervals by $\Delta_k = [khu^{-2/\alpha}, (k+1)hu^{-2/\alpha}]$ and assume again $n = T/(hu^{-2/\alpha}) \in \mathbb{N}$ without loss of generality. Let $t_k = khu^{-2/\alpha}$.

By using the approximation as above with the stationary Gaussian processes $X_k^+(t)$ with $C^+(t_k)$, its stationarity, Slepian's lemma, Lemma 2.1 and finally Lemma 2.3 in the last step, we estimate the upper bound of the probability in a similar way, by denoting $h_k = (C^+(t_k))^{1/\alpha}h$.

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t) > u\} \leq E\Big[\sum_{k=0}^{n-1} P\{\max_{\Delta_0} X_k^+(t)(1-\zeta t) > \frac{u}{1-k\zeta hu^{-2/\alpha}} \mid \zeta\}\Big]$$

$$\leq (1+\gamma(u)) E\Big[\sum_{k=0}^{n-1} \Psi(\frac{u}{1-k\zeta hu^{-2/\alpha}}) E\big[\exp\big(\max_{[0,h_k]} \chi(t)\big) \mid \zeta\big]\Big]$$

$$\leq (1+\gamma(u))\Psi(u) \int_0^{\sigma(\zeta)} \sum_{k=0}^{n-1} \exp(-kzhu^{2-2/\alpha}) E\exp\big(\max_{[0,h_k]} \chi(t)\big) f_\zeta(z)\,\mathrm{d}z$$

where $\gamma(u) \downarrow 0$ as $u \to \infty$, not depending on k and ζ. Now we split the sum into two parts for $k \leq \varepsilon n$ and $> \varepsilon n$ with some $\varepsilon > 0$. In the first partial sum we use the bound $E\exp\big(\max_{[0,h_k]} \chi(t)\big) \leq E\exp\big(\max_{[0,h_0]} \chi(t)\big) + \gamma'(\varepsilon)$ with $\gamma'(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence the first partial sum is bounded above by

$$(1+\gamma(u))\Psi(u)[E\exp\big(\max_{[0,h_0)} \chi(t)\big) + \gamma'(\varepsilon)] \int_0^{\sigma(\zeta)} \frac{1-\exp(-\varepsilon Tzu^2)}{1-\exp(-zhu^{2-2/\alpha})} f_\zeta(z)\,\mathrm{d}z$$

$$\leq \quad (1+\gamma(u)) \frac{2f_\zeta(0)}{h} [E\exp\big(\max_{[0,(C^+(0))^{1/\alpha}h]} \chi(t)\big) + \gamma'(\varepsilon)] u^{2/\alpha-2}\Psi(u)\log u.$$

The second partial sum is of smaller order since $\exp(-\varepsilon Tzu^2) \to 0$ and also $E(\exp(-\varepsilon Tzu^2))\to 0$, and we can use $\tilde{C} = C_{\max} + \varepsilon_1$ in $E\exp\big(\max_{[0,h_k]}\chi(t)\big)$ to get the upper bound $E\exp\big(\max_{[0,h\tilde{C}^{1/\alpha}]}\chi(t)\big)$ with $\varepsilon_1 > 0$.

For the lower bound, by Bonferroni's inequality, we have

$$E\Big[P\{\max_{t\in[0,T]}\xi(t)(1-\zeta t)>u\,|\,\zeta\}\Big] \geq E\Big[\sum_{k=0}^{n-1}P\{\max_{\Delta_k}\xi(t)(1-\zeta t)>u\,|\,\zeta\}\Big]$$
$$-E\Big[\sum_{0\leq k<l\leq n-1}P\{\max_{\Delta_k}\xi(t)(1-\zeta t)>u,\max_{\Delta_l}\xi(t)(1-\zeta t)>u\,|\,\zeta\}\Big]. \tag{11}$$

Choose $\epsilon > 0$ small and let u be large enough. Then with Lemma 2.1, the first sum in (11) is bounded below now with the use of $X^-(t)$ with $C(t_k) > C(0) - \varepsilon_1 = C^-(0)$ (not depending on k for $k \leq \varepsilon n$) by

$$E\Big[\sum_{k=0}^{[\epsilon n]}P\{\max_{\Delta_0}X^-(t)(1-\frac{\zeta t}{1-k\zeta hu^{-2/\alpha}})>\frac{u}{1-k\zeta hu^{-2/\alpha}}\,|\,\zeta\}\Big]$$
$$\geq E\Big[\sum_{k=0}^{[\epsilon n]}P\{\max_{\Delta_0}X^-(t)(1-\frac{\zeta t}{1-\epsilon T\sigma(\zeta)})>\frac{u}{1-k\zeta hu^{-2/\alpha}}\,|\,\zeta\}\Big]$$
$$\geq (1-\gamma_1(u))E\Big[\sum_{k=0}^{[\epsilon n]}\Psi(\frac{u}{1-k\zeta hu^{-2/\alpha}})$$
$$\times E\big[\exp\big(\max_{[0,hC^-(0)^{1/\alpha}]}\chi(t)-\frac{\zeta(C^-(0))^{-1/\alpha}}{1-\epsilon T\sigma(\zeta)}(\frac{u}{1-k\zeta hu^{-2/\alpha}})^{2-2/\alpha}t\big)\,|\,\zeta\big]\Big]$$
$$\geq (1-\gamma_1(u))\Psi(u)(1-\epsilon T\sigma(\zeta))E\Big[\sum_{k=0}^{[\epsilon n]}\exp\big(-\frac{k\zeta hu^{2-2/\alpha}}{(1-\epsilon T\sigma(\zeta))^2}\big) \tag{12}$$
$$\times E\big[\exp\big(\max_{[0,h(C^-(0))^{1/\alpha}]}\chi(t)-\frac{\zeta(C^-(0))^{-1/\alpha}}{1-\epsilon T\sigma(\zeta)}(\frac{u}{1-k\zeta hu^{-2/\alpha}})^{2-2/\alpha}t\big)\,|\,\zeta\big]\Big]$$
$$\geq (1-\gamma_1(u))\Psi(u)(1-\epsilon T\sigma(\zeta))E\Big[\frac{1-\exp(-\epsilon(1-\epsilon T\sigma(\zeta))^{-2}T\zeta u^2)}{1-\exp(-(1-\epsilon T\sigma(\zeta))^{-2}\zeta hu^{2-2/\alpha})}$$
$$\times E\big[\exp\big(\max_{[0,h(C^-(0))^{1/\alpha}]}\chi(t)-\frac{\zeta(C^-(0))^{-1/\alpha}t}{(1-\epsilon T\sigma(\zeta))^{3-2/\alpha}}\big)\,|\,\zeta\big]\Big],$$

where $\gamma_1(u)\downarrow 0$ as $u\to\infty$, not depending on k and ζ, and using $2-2/\alpha\leq 0$. Note that

$$g(z) := E\big[\exp\big(\max_{[0,hC^-(0)^{1/\alpha}]}\chi(t)-\frac{zC^-(0)^{-1/\alpha}t}{(1-\epsilon T\sigma(\zeta))^{3-2/\alpha}}\big)\big]$$

is positive and continuous at 0, hence so is $g(z)f_\zeta(z)$. Then with Lemma 2.3, we obtain that (12) is bounded below by

$$(1-\gamma_1(u))\Psi(u)(1-\epsilon T\sigma(\zeta))\int_0^{\sigma(\zeta)}\frac{1-\exp(-\epsilon(1-\epsilon T\sigma(\zeta))^{-2}Tzu^2)}{1-\exp(-(1-\epsilon T\sigma(\zeta))^{-2}zhu^{2-2/\alpha})}g(z)f_\zeta(z)\,\mathrm{d}z$$
$$\geq (1-\gamma_1'(u))(1-\epsilon T\sigma(\zeta))^3\frac{2f_\zeta(0)}{h}E\exp\big(\max_{[0,hC^-(0)^{1/\alpha}]}\chi(t)\big)u^{2/\alpha-2}\Psi(u)\log u,$$

where $\gamma_1'(u) \downarrow 0$ as $u \to \infty$, not depending on ζ.
Note that $E \exp\big(\max_{[0,h(C^-(0))^{1/\alpha}]} \chi(t)\big)/h \to (C^-(0))^{1/\alpha} H_\alpha$ as $h \to \infty$.
The double sum in (11) is split again into three parts,

$$\begin{aligned}
&\sum_{k=0}^{n-2}\sum_{l=k+1}^{n-1} P\{\max_{\Delta_k} \xi(t)(1-\zeta t) > u, \max_{\Delta_l} \xi(t)(1-\zeta t) > u \,|\, \zeta\} \\
&\leq \sum_{k=0}^{n-2} P\{ < \max_{\Delta_k} \xi(t) > \frac{u}{1-\zeta k T u^{-2/\alpha}}, \max_{\Delta_{k+1}} \xi(t) > \frac{u}{1-\zeta k T u^{-2/\alpha}} \,|\, \zeta\} \\
&\quad + \sum_{k=0}^{n-2}\sum_{l=2}^{N-1} P\{\max_{\Delta_0} \xi(t) > u + k\zeta T u^{1-2/\alpha}, \max_{\Delta_l} \xi(t) > u + k\zeta T u^{1-2/\alpha} \,|\, \zeta\} \\
&\quad + \sum_{k=0}^{n-2}\sum_{l=N}^{n-1} P\{\max_{\Delta_k \times \Delta_l} \xi(t) + \xi(s) > 2u\},
\end{aligned} \tag{13}$$

where $N := [\widetilde{\epsilon}/(4u^{-2/\alpha}T)]$, with $\widetilde{\epsilon} \in (0, T \wedge \frac{1}{2})$ which is chosen such that $1 - 2C_{\max}|t|^\alpha \leq r(t) \leq 1 - \frac{1}{2}C_{\max}|t|^\alpha$ for all $t \in [\,0, \widetilde{\epsilon}\,]$.

The third sum in (13) can be estimated similarly as in the proof of Lemma 4.1, i.e. with the Borel's lemma, we have

$$\sum_{k=0}^{n-2}\sum_{l=N}^{n-1} P\{\max_{\Delta_k \times \Delta_l} \xi(t) + \xi(s) > 2u\} \leq 2(au^{2/\alpha})^2 \Psi\Big(\frac{2u - b}{\sqrt{4 - \widetilde{\delta}}}\Big),$$

where $\widetilde{\delta} := 2\min_{|t-s| \geq \widetilde{\epsilon}/4}(1 - r(t,s)) > 0$, and b is a constant, as in the proof of Lemma 4.1. Therefore the third sum is $o(u^{2/\alpha - 2}\Psi(u)\log u)$ as $u \to \infty$.

By stationarity, Theorem D.2 of Piterbarg [11] and Lemma 2.3, we bound the expectation of the first sum in (13) as in the proof of Lemma 4.1 with the use of $X^+(t)$ and $\tilde{C} = C_{\max} + \varepsilon_1$ (not depending on k) by

$$
\begin{aligned}
&E\sum_{k=0}^{n-2}\Big[2P\big\{\max_{\Delta_0}X^+(t)>\frac{u}{1-\zeta khu^{-2/\alpha}}\,|\,\zeta\big\}-P\big\{\max_{\Delta_0\cup\Delta_1}X^+(t)>\frac{u}{1-\zeta khu^{-2/\alpha}}\,|\,\zeta\big\}\Big]\\
&\le 2(1+\gamma_2(u))E\Big[\sum_{k=0}^{n-2}\frac{H_\alpha\tilde{C}^{1/\alpha}h}{(1-k\zeta hu^{-2/\alpha})^{2/\alpha}}\Psi(\frac{u}{1-k\zeta hu^{-2/\alpha}})\Big]\\
&\qquad\qquad\qquad -(1-\gamma_2'(u))E\Big[\sum_{k=0}^{n-2}\frac{H_\alpha\tilde{C}^{1/\alpha}2h}{(1-k\zeta hu^{-2/\alpha})^{2/\alpha}}\Psi(\frac{u}{1-k\zeta hu^{-2/\alpha}})\Big]\\
&=2(\gamma_2(u)+\gamma_2'(u))E\Big[\sum_{k=0}^{n-2}\frac{H_\alpha\tilde{C}^{1/\alpha}h}{(1-k\zeta hu^{-2/\alpha})^{2/\alpha}}\Psi(\frac{u}{1-k\zeta hu^{-2/\alpha}})\Big]\\
&\le\tilde{\gamma}(u)H_\alpha\tilde{C}^{1/\alpha}h\cdot(1-T\sigma(\zeta))^{1-2/\alpha}\Psi(u)\int_0^{\sigma(\zeta)}\sum_{k=0}^{n-1}\exp\big(-kzhu^{2-2/\alpha}\big)f_\zeta(z)\,\mathrm{d}z\\
&\le\tilde{\gamma}(u)H_\alpha\tilde{C}^{1/\alpha}h\cdot(1-T\sigma(\zeta))^{1-2/\alpha}\Psi(u)\int_0^{\sigma(\zeta)}\frac{1-\exp(-Tzu^2)}{1-\exp(-zhu^{2-2/\alpha})}f_\zeta(z)\,\mathrm{d}z\\
&\le 2\tilde{\gamma}(u)(1+\gamma_2''(u))H_\alpha\tilde{C}^{1/\alpha}f_\zeta(0)(1-T\sigma(\zeta))^{1-2/\alpha}u^{2/\alpha-2}\Psi(u)\log u\\
&=o(u^{2/\alpha-2}\Psi(u)\log u),
\end{aligned}
$$

since $\gamma_2(u),\gamma_2'(u),\gamma_2''(u)\downarrow 0$ as $u\to\infty$, not depending on k and ζ with $\tilde{\gamma}(u)=2(\gamma_2(u)+\gamma_2'(u))$.
We estimate the expectation of the second sum in (13) by Lemma 6.3 of Piterbarg [11], since its conditions are satisfied by $X^+(t)$ with $\tilde{C}$.

$$
\begin{aligned}
&E\Big[\sum_{k=0}^{n-2}\sum_{l=2}^{N-1}P\big\{\max_{\Delta_0}X^+(t)>u+k\zeta hu^{1-2/\alpha},\max_{\Delta_l}X^+(t)>u+k\zeta hu^{1-2/\alpha}\,|\,\zeta\big\}\Big]\\
&\le E\Big[\sum_{k=0}^{n-2}\sum_{l=2}^{N-1}C_1h^2\Psi(u+k\zeta hu^{1-2/\alpha})\exp\big(-\frac{1}{8}(l-1)^\alpha h^\alpha\big)\Big]\\
&\le C_1h^2\Psi(u)\sum_{l=0}^{N-3}\exp\big(-2^{\alpha-4}h^\alpha(l^\alpha+1)\big)E\Big[\sum_{k=0}^{n-1}\exp\big(-k\zeta hu^{2-2/\alpha}\big)\Big]\\
&\le C_2h^2\exp(-2^{\alpha-4}h^\alpha)\Psi(u)\int_0^{\sigma(\zeta)}\frac{1-\exp(-Tzu^2)}{1-\exp(-zhu^{2-2/\alpha})}f_\zeta(z)\,\mathrm{d}z\\
&\le(1+\gamma_2''(u))2C_2h\exp(-2^{\alpha-4}h^\alpha)f_\zeta(0)u^{2/\alpha-2}\Psi(u)\log u\\
&=o(u^{2/\alpha-2}\Psi(u)\log u)
\end{aligned}
$$

as $h\to\infty$, where C_1 and C_2 are some constants, not depending on h, and by using the concavity of the function x^α. We applied also Lemma 2.3 in the second last step.

Thus we have

$$\frac{2f_\zeta(0)}{h}E\exp\big(\max_{[0,(C^{1/\alpha}(0)+\varepsilon_1)h]}\chi(t)\big)\geq\lim_{u\to\infty}\frac{P\{\max_{t\in[0,T]}\xi(t)(1-\zeta t)>u\}}{u^{2/\alpha-2}\Psi(u)\log u}$$
$$\geq(1-\epsilon T\sigma(\zeta))^3\frac{2f_\zeta(0)}{h}E\exp\big(\max_{[0,h(C^{1/\alpha}(0)-\varepsilon_1)]}\chi(t)\big)-o(1).$$

We may select h arbitrarily large, thus using $E\exp\big(\max_{[0,h(C^{1/\alpha}(0)\pm\varepsilon_1)]}\chi(t)\big)/h\sim(C^{1/\alpha}(0)\pm\varepsilon_1)H_\alpha$. Since ϵ and ε_1 can tend to 0, we get the stated result. □

The next lemma considers the case (c) of Figure 1.

Lemma 4.3. *Let $1=\alpha>\beta$. Suppose that the density $f_\zeta(x)$, $x\geq 0$, is positive and continuous at 0. Then for any $T\in(0,(\sigma(\zeta))^{-1/\beta})$,*

$$P\{\max_{t\in[0,T]}\xi(t)(1-\zeta t^\beta)>u\}\sim\big(f_\zeta(0)J(T)+1\big)\Psi(u)\qquad \text{as } u\to\infty.$$

Proof: Note that

$$P\{\max_{t\in[0,T]}\xi(t)(1-\zeta t^\beta)>u\}=EP\{\max_{t\in[0,T]}\xi(t)(1-\zeta t^\beta)>u\,|\,\zeta\}$$
$$=\int_0^{u^{2\beta-2+\delta}}P\{\max_{t\in[0,T]}\xi(t)(1-\zeta t^\beta)>u\,|\,\zeta=z\}f_\zeta(z)\,\mathrm{d}z \tag{14}$$
$$+\int_{u^{2\beta-2+\delta}}^{\sigma(\zeta)}P\{\max_{t\in[0,T]}\xi(t)(1-\zeta t^\beta)>u\,|\,\zeta=z\}f_\zeta(z)\,\mathrm{d}z,$$

where δ is chosen in such a way that $0<\delta<2-2\beta$.

a) Considering the first term of (14), we split $[0,T]$ into subintervals with length $u^{-2}\log u$, denote the subintervals by $\Delta_k=[ku^{-2}\log u,(k+1)u^{-2}\log u]$ and without loss of generality assume again $n=T/(u^{-2}\log u)$. We use again the approximating Gaussian processes $X_k^+(t)$ with $C^+(t_k)$ on Δ_k, where $t_k=ku^{-2}\log u$. Then with Theorem D.2 of Piterbarg [11] and the fact that $H_1=1$, the first term of (14) has the upper bound

$$\int_0^{u^{2\beta-2+\delta}}\Big[\sum_{k=0}^{n-1}P\{\max_{\Delta_k}\xi(t)>u/(1-z(ku^{-2}\log u)^\beta)\}\Big]f_\zeta(z)\,\mathrm{d}z$$
$$\leq\int_0^{u^{2\beta-2+\delta}}\Big[\sum_{k=0}^{n-1}(1+\gamma(u))\frac{(C^+(t_k))^{1/\alpha}\log u}{\big(1-z(ku^{-2}\log u)^\beta\big)^2}\Psi\big(\frac{u}{1-z(ku^{-2}\log u)^\beta}\big)\Big]f_\zeta(z)\,\mathrm{d}z$$
$$\leq(1+\gamma(u))\log u\,\Psi(u)\int_0^{u^{2\beta-2+\delta}}\Big[\sum_{k=0}^{n-1}\frac{\exp(-zk^\beta u^{2-2\beta}(\log u)^\beta)}{1-z(ku^{-2}\log u)^\beta}\Big](C^+(t_k))^{1/\alpha}f_\zeta(z)\,\mathrm{d}z \tag{15}$$
$$\leq(1+\gamma(u))\log u\,\Psi(u)\big[u^{2\beta-2+\delta}+\int_0^{u^{2\beta-2+\delta}}\int_0^{\frac{Tu^2}{\log u}}\frac{e^{-zx^\beta u^{2-2\beta}(\log u)^\beta}C^*(\frac{t\log u}{u^2})}{1-z(xu^{-2}\log u)^\beta}\mathrm{d}x\,f_\zeta(z)\mathrm{d}z\big],$$

where $\gamma(u)\downarrow 0$ as $u\to\infty$, not depending on k and $\zeta=z$, and we write $C^*(t):=C^{1/\alpha}(t)+2\varepsilon_1$.

For any small $\epsilon > 0$, let u be sufficiently large so that for all $0 \le z \le u^{2\beta-2+\delta}$, $|f_\zeta(z) - f_\zeta(0)| \le \epsilon$; since the density f_ζ is positive and continuous at 0. Hence by Fubini's theorem and dominated convergence, we bound the integral in (15),

$$
\begin{aligned}
&\frac{u^{2-2/\beta}}{\beta \log u}\int_0^{u^{2\beta-2+\delta}} z^{-1/\beta}\Big(\int_0^{T^\beta u^2 z}\Big(\frac{C^*((v/z)^{1/\beta}u)}{1-vu^{-2}}\Big)e^{-v}v^{1/\beta-1}\,\mathrm{d}v\Big)f_\zeta(z)\,\mathrm{d}z\\
&\le (f_\zeta(0)+\epsilon)\frac{u^{2-2/\beta}}{\beta \log u}\int_0^{u^{2\beta-2+\delta}} z^{-1/\beta}\Big(\int_0^{T^\beta u^2 z}\Big(\frac{C^*((v/z)^{1/\beta}u)}{1-vu^{-2}}\Big)e^{-v}v^{1/\beta-1}\,\mathrm{d}v\Big)\,\mathrm{d}z\\
&= \frac{(f_\zeta(0)+\epsilon)T^{1-\beta}}{\beta\log u}\int_0^{T^\beta u^{2\beta+\delta}} y^{-1/\beta}\Big(\int_0^{y}\Big(\frac{C^*((v/y)^{1/\beta}T)}{1-vu^{-2}}\Big)e^{-v}v^{1/\beta-1}\,\mathrm{d}v\Big)\,\mathrm{d}y\\
&= \frac{(f_\zeta(0)+\epsilon)T^{1-\beta}}{\beta\log u}\int_0^{T^\beta u^{2\beta+\delta}}\int_v^{T^\beta u^{2\beta+\delta}} C^*((v/y)^{1/\beta}T)y^{-1/\beta}\,\mathrm{d}y\,\Big(\frac{1}{1-vu^{-2}}\Big)e^{-v}v^{1/\beta-1}\,\mathrm{d}v
\end{aligned}
$$

As in the proof of Lemma 4.1, we get for the inner integral that

$$
\begin{aligned}
\int_v^{T^\beta u^{2\beta+\delta}} C^*((v/y)^{1/\beta}T)y^{-1/\beta}\,\mathrm{d}y &= \beta(vT^\beta)^{1-1/\beta}\int_{v^{1/\beta}u^{-2-\delta/\beta}}^{T} C^*(s)s^{-\beta}\,\mathrm{d}s\\
&\to J^+(T)+O(\varepsilon_1) = J(T)+O(\varepsilon_1)
\end{aligned}
$$

as $u \to \infty$, since the lower boundary of the integral tends to 0 for $v \le g(u)u^{2\beta+\delta}$ with $g(u) \to 0$ such that $g(u)u^{2\beta+\delta} \to \infty$, and ε_1 is small. Therefore we obtain the upper bound for the first part ($v \le g(u)u^{2\beta+\delta}$) of the outer integral

$$
\begin{aligned}
&\frac{(f_\zeta(0)+\epsilon)(J^+(T)+o(1))}{\log u}\int_0^{g(u)u^{2\beta+\delta}}\Big(\frac{1}{1-vu^{-2}}\Big)e^{-v}\,\mathrm{d}v\\
&= \frac{(f_\zeta(0)+\epsilon)(J^+(T)+o(1))}{\log u}(1+o(1)) \quad \text{as } u \to \infty.
\end{aligned}
$$

The second part of the outer integral is of much smaller order because of the exponential term which implies that the first term of (14), is bounded by

$$
\begin{aligned}
&\Big((1+\gamma(u))u^{2\beta-2+\delta}\log u + (1+\gamma'(u))(f_\zeta(0)+\epsilon)(J^+(T)+o(1))\Big)\Psi(u)\\
&\sim \quad (f_\zeta(0)+\epsilon)J^+(T)\Psi(u)
\end{aligned}
$$

as $u \to \infty$, where $\gamma'(u) \downarrow 0$ as $u \to \infty$.

To derive the lower bound of the first term of (14), we use Bonferroni's inequality,

$$\begin{aligned}
&\int_0^{u^{2\beta-2+\delta}} P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u \,|\, \zeta = z\} f_\zeta(z)\,\mathrm{d}z \\
&\geq \int_0^{u^{2\beta-2+\delta}} \Big[\sum_{k=0}^{n-1} P\{\max_{\Delta_k} \xi(t) > \frac{u}{1-\zeta\big((k+1)u^{-2}\log u\big)^\beta} \,|\, \zeta = z\}\Big] f_\zeta(z)\,\mathrm{d}z \\
&\quad - \int_0^{u^{2\beta-2+\delta}} \Big[\sum_{0\le k<l\le n-1} P\{\max_{\Delta_k} \xi(t) > \frac{u}{1-\zeta(ku^{-2}\log u)^\beta}, \\
&\qquad\qquad \max_{\Delta_l} \xi(t) > \frac{u}{1-\zeta(lu^{-2}\log u)^\beta} \,|\, \zeta = z\}\Big] f_\zeta(z)\,\mathrm{d}z.
\end{aligned} \tag{16}$$

From (8) and (9) in the proof of Lemma 4.1, we know the lower bound of the first term in (16) by setting the upper endpoint of the integration interval as $u^{2\beta-2+\delta}$, to derive the lower bound, similar to the upper bound,

$$\begin{aligned}
&\int_0^{u^{2\beta-2+\delta}} \Big[\sum_{k=0}^{n-1} P\{\max_{\Delta_k} X_k^-(t) > u/(1-\zeta\big((k+1)u^{-2}\log u\big)^\beta) \,|\, \zeta = z\}\Big] f_\zeta(z)\,\mathrm{d}z \\
&\geq (1-\gamma_1'(u))(f_\zeta(0)-\epsilon)(J^-(T)-o(1))\,\Psi(u),
\end{aligned}$$

where $\gamma_1'(u) \downarrow 0$ as $u \to \infty$, and $J^-(T) = \int_0^T C^-(s)s^{-\beta}\mathrm{d}s \to J(T)$ as $\varepsilon_1 \to 0$, where we set $C^-(s) = C(t) - 2\varepsilon_1$ instead of $C^*(t)$.

As in (10), we divide the double sum in the second term of (16) into three parts. Then from the proof of Lemma 4.1, we know that the integrand in the second term of (16) can be bounded by $C_1\Psi(u)$, where C_1 is some constant. Hence we have

$$\begin{aligned}
&\int_0^{u^{2\beta-2+\delta}} \Big[\sum_{0\le k<l\le n-1} P\{\max_{\Delta_k} \xi(t) > u/(1-\zeta(ku^{-2}\log u)^\beta), \\
&\qquad\qquad \max_{\Delta_l} \xi(t) > u/(1-\zeta(lu^{-2}\log u)^\beta) \,|\, \zeta = z\}\Big] f_\zeta(z)\,\mathrm{d}z \\
&\leq \int_0^{u^{2\beta-2+\delta}} C_1\Psi(u) f_\zeta(z)\,\mathrm{d}z \leq C_1\widetilde{f}_\zeta u^{2\beta-2+\delta}\Psi(u) = o(\Psi(u)) \qquad \text{as } u\to\infty.
\end{aligned}$$

b) For the second term of (14), we use the following derivation which is also needed in the proof of Lemma 4.5 dealing with the case (f). Therefore we formulate it for both cases together, assuming $\alpha \geq 1$ and $\alpha > \beta$ where $0 < \delta < 2 - 2\beta/\alpha$. We have

$$\begin{aligned}
&\int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u \,|\, \zeta = z\} f_\zeta(z)\,\mathrm{d}z \\
&\leq \int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} P\{\max_{[0,u^{-2/\alpha-\widetilde{\delta}/\beta}]} \xi(t)(1-\zeta t^\beta) > u \,|\, \zeta = z\} f_\zeta(z)\,\mathrm{d}z \\
&\quad + \int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} P\{\max_{[u^{-2/\alpha-\widetilde{\delta}/\beta},T]} \xi(t)(1-\zeta t^\beta) > u \,|\, \zeta = z\} f_\zeta(z)\,\mathrm{d}z,
\end{aligned} \tag{17}$$

where $\widetilde{\delta} \in (0, \delta)$.

With Theorem D.4 of Piterbarg [11], the second term of (17) is bounded by

$$
\begin{aligned}
&\int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} P\{\max_{[u^{-2/\alpha-\widetilde{\delta}/\beta},T]} \xi(t)(1-zu^{-2\beta/\alpha-\widetilde{\delta}}) > u \,|\, \zeta = z\} f_\zeta(z)\,\mathrm{d}z \\
&\le \int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} CTu^{2/\alpha}\Psi(\frac{u}{1-zu^{-2\beta/\alpha-\widetilde{\delta}}}) f_\zeta(z)\,\mathrm{d}z \qquad (18) \\
&\le C'\widetilde{f}_\zeta T u^{2/\alpha}\Psi(u)\int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} \exp(-zu^{2-2\beta/\alpha-\widetilde{\delta}})\,\mathrm{d}z \\
&\le C'\widetilde{f}_\zeta T\Psi(u)u^{2/\alpha+2\beta/\alpha+\widetilde{\delta}-2}\big[\exp(-u^{\delta-\widetilde{\delta}}) - \exp(-\sigma(\zeta)u^{2-2\beta/\alpha-\widetilde{\delta}})\big] = o(\Psi(u))
\end{aligned}
$$

as $u \to \infty$, where C and C' are some constants.

Since $u^{-2/\alpha-\widetilde{\delta}/\beta} = o(u^{-2/\alpha})$ as $u \to \infty$, we get for any $\epsilon > 0$, with Lemma D.1 of Piterbarg [11] using $X^+(t)$ and $C^+(0)$, that the first term of (17) is bounded by

$$
\begin{aligned}
&\int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} P\{\max_{[0,\epsilon u^{-2/\alpha}]} X^+(t) > u\} f_\zeta(z)\,\mathrm{d}z \\
&\le \int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} (1+\gamma_1(u))H_\alpha(\epsilon(C^+(0))^{1/\alpha})\Psi(u) f_\zeta(z)\,\mathrm{d}z \qquad (19) \\
&\le (1+\gamma_1(u))H_\alpha(\epsilon(C^+(0))^{1/\alpha})\Psi(u)
\end{aligned}
$$

where $\gamma_1(u) \downarrow 0$ as $u \to \infty$, not depending on ζ. Since $H_\alpha(\epsilon) \to 1$ as $\epsilon \to 0$, the estimate for the upper bound is obtained.

The lower bound of the probability is obvious, for any $\zeta > 0$

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u|\zeta\} \ge P\{\xi(0) > u\} \sim \Psi(u)$$

as $u \to \infty$, and thus

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \sim \Psi(u)$$

as $u \to \infty$.

c) Finally, putting the derived bounds together, using ϵ and $\varepsilon_1 \to 0$, we conclude

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \sim \big(f_\zeta(0)J(T)+1\big)\Psi(u)$$

as $u \to \infty$. □

In the next lemma we consider the two cases d) and e) of Figure 1 together.

Lemma 4.4. *For any $T \in (0, (\sigma(\zeta))^{-1/\beta})$,*

(a) if $\alpha < \beta \in (1, \infty)$, then as $u \to \infty$,

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u\} \sim \frac{C^{1/\alpha}(0)H_\alpha\Gamma(1/\beta)}{\beta} u^{2/\alpha-2/\beta}\Psi(u)E\zeta^{-1/\beta};$$

(b) *if* $\alpha = \beta \in (1,2]$, *then as* $u \to \infty$,

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^{\beta}) > u\} \sim EH_{\alpha}^{\zeta/C(0)}\Psi(u),$$

where $0 < H_{\alpha}^{\zeta} := E\exp\big(\max_{[0,\infty)}(\chi(t) - \zeta t^{\alpha})|\zeta\big) < \infty$.

Proof: It is easy to see that if $\beta > 1$, then

$$E\zeta^{-1/\beta} = \int_0^{\sigma(\zeta)} z^{-1/\beta} f_{\zeta}(z)\,\mathrm{d}z \le \tilde{f}_{\zeta} \int_0^{\sigma(\zeta)} z^{-1/\beta}\,\mathrm{d}z = \frac{\tilde{f}_{\zeta}\beta}{\beta-1}(\sigma(\zeta))^{-1/\beta+1} < \infty.$$

Hence the conditions of Lemma 3.1 are fulfilled, and the results follow. □

It remains to consider the case (f) in Figure 1.

Lemma 4.5. *Let* $\alpha > \beta$, *where* $\alpha > 1$. *Then for any* $T \in (0, (\sigma(\zeta))^{-1/\beta})$,

$$P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^{\beta}) > u\} \sim \Psi(u) \qquad \text{as } u \to \infty.$$

Proof: For an upper bound of the probability, note that

$$\begin{aligned} P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^{\beta}) > u\} &= EP\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^{\beta}) > u \,|\, \zeta\} \\ = \int_0^{u^{2\beta/\alpha-2+\delta}} & P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^{\beta}) > u \,|\, \zeta = z\} f_{\zeta}(z)\,\mathrm{d}z \\ &+ \int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^{\beta}) > u \,|\, \zeta = z\} f_{\zeta}(z)\,\mathrm{d}z, \end{aligned} \tag{20}$$

where δ is chosen such that $0 < \delta < 2 - 2\beta/\alpha$.

For the first term in (20), we split $[0,T]$ into subintervals with length $u^{-2/\alpha}$ and assume again $n = Tu^{2/\alpha}$ without loss of generality. On the subintervals we use $X^{+}(t)$ with $\tilde{C} = C_{\max} + \varepsilon_1$ as approximating stationary Gaussian process and use the stationarity and Theorem D.2 of Piterbarg [11], to bound the first term of (20) for u large.

$$\begin{aligned} &\int_0^{u^{2\beta/\alpha-2+\delta}} \Big[\sum_{k=0}^{n-1} P\{\max_{t\in[0,u^{-2/\alpha}]} X^{+}(t) > u/(1-z(ku^{-2/\alpha})^{\beta})\}\Big] f_{\zeta}(z)\,\mathrm{d}z \\ &\le \int_0^{u^{2\beta/\alpha-2+\delta}} \Big[\sum_{k=0}^{n-1} (1+\gamma(u)) \frac{H_{\alpha}\tilde{C}^{1/\alpha}}{\big(1-z(ku^{-2/\alpha})^{\beta}\big)^{2/\alpha}} \Psi\Big(\frac{u}{1-z(ku^{-2/\alpha})^{\beta}}\Big)\Big] f_{\zeta}(z)\,\mathrm{d}z \\ &\le (1+\gamma(u))H_{\alpha}\tilde{C}^{1/\alpha}\Psi(u) \int_0^{u^{2\beta/\alpha-2+\delta}} \Big[\sum_{k=0}^{n-1} \frac{\exp(-zk^{\beta}u^{2-2\beta/\alpha})}{\big(1-T^{\beta}u^{2\beta/\alpha-2+\delta}\big)^{2/\alpha-1}}\Big] f_{\zeta}(z)\,\mathrm{d}z \\ &\le c\Psi(u) \int_0^{u^{2\beta/\alpha-2+\delta}} \Big[\sum_{k=0}^{n-1} \exp\big(-zk^{\beta}u^{2-2\beta/\alpha}\big)\Big]\,\mathrm{d}z \\ &\le c\Psi(u)\Big[2u^{2\beta/\alpha-2+\delta} + \int_0^{u^{2\beta/\alpha-2+\delta}} \int_1^{Tu^{2/\alpha}} \exp\big(-zx^{\beta}u^{2-2\beta/\alpha}\big)\,\mathrm{d}x\,\mathrm{d}z\Big] \\ &\le c\Psi(u)\Big[2u^{2\beta/\alpha-2+\delta} + u^{2\beta/\alpha-2} \int_1^{Tu^{2/\alpha}} \frac{1}{x^{\beta}}\,\mathrm{d}x\Big], \end{aligned} \tag{21}$$

by interchanging the integrals, where $\gamma(u) \downarrow 0$ as $u \to \infty$, not depending on k and ζ, c a suitable constant and using $f_\zeta(z) \leq \tilde{f}$. If $\beta = 1$, since $\int_1^{Tu^{2/\alpha}} x^{-1}\,\mathrm{d}x = \log(Tu^{2/\alpha})$,

$$(21) = c\Psi(u)\big[2u^{2\beta/\alpha-2+\delta} + u^{2\beta/\alpha-2}\log(Tu^{2/\alpha})\big] = o(\Psi(u));$$

if $\beta \neq 1$, since $\int_1^{Tu^{2/\alpha}} x^{-\beta}\,\mathrm{d}x = (1-\beta)^{-1}(T^{1-\beta}u^{2/\alpha-2\beta/\alpha} - 1)$,

$$(21) = c\Psi(u)\Big[2u^{2\beta/\alpha-2+\delta} + \frac{T^{1-\beta}}{1-\beta}u^{2/\alpha-2} - (1-\beta)^{-1}u^{2\beta/\alpha-2})\Big] = o(\Psi(u)).$$

Therefore we conclude that the first term of (20) is infinitely smaller than $\Psi(u)$ for any $\beta < \alpha \in (1,2]$ as $u \to \infty$.

The second term of (20) is approximated in the proof of Lemma 4.3, showing that

$$\int_{u^{2\beta/\alpha-2+\delta}}^{\sigma(\zeta)} P\{\max_{t\in[0,T]} \xi(t)(1-\zeta t^\beta) > u \,|\, \zeta = z\} f_\zeta(z)\,\mathrm{d}z \ \sim \Psi(u)$$

as $u \to \infty$. □

Proof of Theorem 1.2: We use the same ideas as in the proof of Theorem 1.1. Write

$$P_{u,\beta} = P\{\max_{[0,T]} \xi(t)(\eta - \zeta t^\beta) > u\} = EP\{\max_{[0,T]} \xi(t)(1-(\zeta/\eta)t^\beta) > u/\eta \,|\, \eta\}$$

and apply Lemma 4.1, 4.2, 4.3, 4.4 and 4.5 with fixed η. Then taking the expectation on η, we get the assertions of Theorem 1.2 from Proposition 2.2. □

Acknowledgement: We are very thankful to the reviewers of the first version of this paper, for their helpful comments on improving the presentation of our results.

References

[1] Adler, R.J., Samorodnitsky, G., and Gadrich, T. (1993) The expected number of level crossings for stationary, harmonizable, symmetric, stable processes. Annals Applied Probability, 3, 553-575. MR1221165

[2] Doucet, A., de Freitas, N., Murphy, K., and Russel, S. (2000) *Rao-Blackwellized Particle Filtering for Dynamic Bayesian Networks*. In 16th Conference on Uncertainty in AI, 176 – 183.

[3] Falk, M., Hüsler, H., and Reiss, R.-D. (2010) *Laws of Small Numbers: Extremes and Rare Events*. 3rd ed. Birkhäuser.

[4] Fedoruk, M. (1989) *Asymptotic methods in analysis. In: Analysis I: integral representations and asymptotic methods*. Encyclopaedia of mathematical sciences, vol. 13, Springer Verlag. MR1042759

[5] Hüsler, J. (1990) Extreme values and high boundary crossings of locally stationary Gaussian processes. Ann. Probab. **18**, 1141-1158. MR1062062

[6] Hüsler, J., Piterbarg, V., and Rumyantseva E. (2011) Extremes of Gaussian processes with a smooth random variance. Submitted for publication.

[7] Lototsky, S.V. (2004) *Optimal Filtering of Stochastic Parabolic Equations*. Recent Developments in Stochastic Analysis and Related Topics (Proceedings of the First Sino-German Conference on Stochastic Analysis), Albeverio, S., et al. Eds., World Scientific, 330-353. MR2200521

[8] Marks, T.K., Hershey, J., Roddey, J.C., and Movellan, J.R. (2005) Joint tracking of pose, expression, and texture using conditionally Gaussian filters. Advances in Neural Information Processing Systems 17, MIT Press.

[9] Nielsen, L.T. (1999) *Pricing and Hedging of Derivative Securities*. Textbook in continuous-time finance theory, Oxford University Press.

[10] Pickands III, J. (1969) Upcrossing probabilities for stationary Gaussian processes. Trans. Amer. Math. Soc. **145**, 75-86. MR0250368

[11] Piterbarg, V.I. (1996) *Asymptotic Methods in the Theory of Gaussian Processes and Fields*. AMS Translations of Mathematical Monographs 148, Providence, Rhode Island. MR1361884

[12] Piterbarg, V.I., and Prisyazhnyuk, V.P. (1979) Asymptotics of the probability of large excursions for a nonstationary Gaussian process. (English) Theory Probab. Math. Stat. **18**, 131-144.