DOI: 10.1214/EJP.v16-900

Electronic Journal of Probability

Vol. 16 (2011), Paper no. 34, pages 1001–1019.

Journal URL
http://www.math.washington.edu/~ejpecp/

Three Kinds of Geometric Convergence for Markov Chains and the Spectral Gap Property

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Abstract

In this paper we investigate three types of convergence for geometrically ergodic Markov chains (MCs) with countable state space, which in general lead to different 'rates of convergence'. For reversible Markov chains it is shown that these rates coincide. For general MCs we show some connections between their rates and those of the associated reversed MCs. Moreover, we study the relations between these rates and a certain family of isoperimetric constants. This sheds new light on the connection of geometric ergodicity and the so-called spectral gap property, in particular for non-reversible MCs, and makes it possible to derive sharp upper and lower bounds for the spectral radius of certain non-reversible chains.

Key words: Markov chain, countable state space; geometric ergodicity; spectral gap property; isoperimetric constant; reversibility; bounds for the spectral radius.

AMS 2010 Subject Classification: Primary 60J10.

Submitted to EJP on October 9, 2010, final version accepted April 20, 2011.

*Research has been supported by the DFG

1 Introduction

For positive recurrent Markov chains (MCs) one of the central questions is the convergence of their transition kernels to the invariant distribution. The ‘geometrically ergodic’ case when this convergence takes place at a geometric rate is of particular importance. A profound analysis of this subject can be found in the monographs by Meyn and Tweedie [7] and by Nummelin [8].

In this paper we are concerned with three different kinds of rates of geometric convergence. In Section 2 we present an example to illustrate the differences between the definitions; in Section 3 several connections between these rates for a MC and the corresponding rates for the reversed chain are proved. In Section 4 we show that for reversible Markov chains (under a mild condition) the different types of rates of convergence actually coincide. In Section 5 we analyze geometrically ergodic MCs by applying the concept of isoperimetric constants, which has been used in [14] to establish necessary and sufficient conditions for the spectral gap property. We show that this property and geometric ergodicity are equivalent for normal Markov chains, generalizing the results of Roberts and Tweedie [11] and Roberts and Rosenthal [12]. Moreover, it is shown how a certain sequence of isoperimetric constants can be used to obtain bounds for the rates of geometric convergence, and prove that these bounds are sharp in some cases. In Section 6 we present an example which shows that geometric ergodicity (GE) does not imply the spectral gap property (SGP) and calculate exact rates of geometric convergence applying the method of isoperimetric constants.

Throughout this paper let $\xi_1, \xi_2, \ldots$ be a positive recurrent MC with countable state space Ω, transition kernel $p(\cdot,\cdot)$ and invariant probability measure π. Let

$$p^*(i,j) = \frac{\pi(j)p(j,i)}{\pi(i)}, \quad i,j \in \Omega \tag{1}$$

be the transition probabilities of the reversed MC (a realization of which we denote by $\xi_1^*, \xi_2^*, \ldots$). We need the standard MC operators P, P^* and Π defined by

$$Pf(i) = \sum_{j\in\Omega} f(j)p(i,j), \tag{2}$$

$$P^*f(i) = \sum_{j\in\Omega} f(j)p^*(i,j), \tag{3}$$

$$\Pi f(i) = \sum_{j\in\Omega} f(j)\pi(j). \tag{4}$$

for all real-valued functions f on Ω for which the corresponding series converge. In particular, for all $f \in L^2(\pi)$ it easily follows from Jensen's inequality and the stationarity of π that the sums in (2), (3) and (4) converge and that Pf, P^*f and Πf are in $L^2(\pi)$. Note that we consider Π as the operator that maps every $f \in L^2(\pi)$ to the function constantly equal to the π-expected value of f. The scalar product on $L^2(\pi)$ is of course

$$\langle f, g\rangle_\pi = \sum_{j\in\Omega} f(j)g(j)\pi(j).$$

It is easy to show that

$$\langle Pf, g\rangle_\pi = \langle f, P^*g\rangle_\pi,$$

so P^* is the adjoint operator of P on $L^2(\pi)$. We say that P has the *spectral gap property* (SGP) on $L^2(\pi)$ if

$$\rho = \lim_{n\to\infty} \sup_{f\in L^2_{0,1}(\pi)} ||P^n f||_{L^2(\pi)}^{\frac{1}{n}} < 1, \tag{5}$$

where

$$L^2_{0,1}(\pi) = \{f \in L^2(\pi) : ||f||_{L^1(\pi)} = 0, ||f||_{L^2(\pi)} = 1\},$$

and

$$||f||_{L^1(\pi)} = \sum_{j\in\Omega} f(j)\pi(j), \quad ||f||_{L^2(\pi)} = \Big(\sum_{j\in\Omega} f(j)^2\pi(j)\Big)^{1/2}.$$

Note that the limit in (5) always exists (see e.g. [10]). The total variation distance of two probability measures μ and ν on Ω is defined by

$$d(\mu,\nu) = ||\mu-\nu||_{TV} = \sup_{\phi:||\phi||_\infty=1} \sum_{j\in\Omega} (\mu(j)-\nu(j))\phi(j).$$

If we set $A = \{j \in \Omega : \mu(j) \geq \nu(j)\}$, then clearly

$$d(\mu,\nu) = 2|\mu(A)-\nu(A)|.$$

A Markov chain $\xi_1, \xi_2, \ldots$ is called *geometrically ergodic* (GE) if for some $\delta < 1$

$$K_\delta(i) = \sup_{n\in\mathbb{N}} \frac{||p^n(i,\cdot)-\pi||_{TV}}{\delta^n} < \infty \quad \forall i \in \Omega. \tag{6}$$

From [7] (Chapter 15) and [8] (Theorem 6.14 (iii)) it follows that the GE property is equivalent to the seemingly more restrictive condition

$$||K_\delta||_{L^1(\pi)} = \sum_{i=1}^{\infty} K_\delta(i)\pi(i) < \infty \tag{7}$$

for some $\delta < 1$, where K_δ is defined as in (6). Note that the δ in (7) may differ from the δ in (6).

Obviously, (7) implies that for some $\delta < 1$

$$C(\delta) = \sup_{n\in\mathbb{N}} \frac{\sum_{i\in\Omega} ||p^n(i,\cdot)-\pi||_{TV}\,\pi(i)}{\delta^n} < \infty. \tag{8}$$

It is certainly of interest to find the best rate of 'geometric convergence'. However, considering (6)-(8) there are three possibilities to define an optimal lower bound for this rate: Let

$$\delta_0 = \inf\{\delta : 0 < \delta < 1 \text{ and (6) is satisfied}\} \tag{9}$$

$$\delta_1 = \inf\{\delta : 0 < \delta < 1 \text{ and (7) is satisfied}\} \tag{10}$$

$$\delta_2 = \inf\{\delta : 0 < \delta < 1 \text{ and (8) is satisfied}\}. \tag{11}$$

Definition 1. *Regarding the geometric rate of convergence we call δ_0 the optimal lower bound (OLB) in the weak sense, δ_1 the OLB in the strong sense and δ_2 the OLB in the $L^1(\pi)$ sense.*

It follows from the definitions that

$$\delta_1 \geq \delta_2 \geq \delta_0.$$

Are these inequalities in general strict, and under which conditions do they become equalities? Moreover, are these OLBs attained? We start with an example.

2 Introductory example: the reversed winning streak

Let us consider the MC with state space $\mathbb{N}$ and transition matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{12}$$

Its invariant measure π is given by

$$\pi(i) = \left(\frac{1}{2}\right)^i, \ \ i \in \mathbb{N}.$$

The crucial observation now is that

$$p(1,i) = \pi(i) \ \forall i \in \mathbb{N},$$

which immediately generalizes to

$$||p^i(i,\cdot) - \pi||_{TV} = 0 \ \forall i \in \mathbb{N}.$$

It follows that

$$||p^j(i,\cdot) - \pi||_{TV} = 0 \ \ \forall j \geq i, \ \ i \in \mathbb{N}.$$

For arbitrary $\delta > 0$ we conclude that

$$||p^n(i,\cdot) - \pi||_{TV} \leq 2(1/\delta)^{i-1}\delta^n \quad \forall n \in \mathbb{N}, \ i \in \mathbb{N}.$$

Since this holds true for all $\delta > 0$, we see that $K_\delta(i) \leq 2(1/\delta)^{i-1}$ and that the OLB in the weak sense is zero, i.e.,

$$\delta_0 = 0.$$

But of course the MC is not GE at rate zero (this rate of geometric ergodicity only occurs for MCs induced by a sequence of i.i.d. random variables); thus the infimum in (9) is not attained.

Next let us determine δ_1. Check that

$$|p^i(i+1,1) - \pi(1)| = \frac{1}{2} \quad \forall i \in \mathbb{N}.$$

Now consider an arbitrary $\delta < 1$ satisfying (10). Then

$$\frac{1}{2} \leq ||p^i(i+1,\cdot) - \pi||_{TV} \leq K_\delta(i+1)\delta^i,$$

so that

$$K_\delta(i) \geq \frac{\delta/2}{\delta^i} \quad \forall i \in \mathbb{N}.$$

So (7) holds for δ only if

$$\frac{\delta}{2}\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i\left(\frac{1}{\delta}\right)^i < \infty,$$

which is of course equivalent to $\delta > 1/2$. Hence,

$$\delta_1 \geq \frac{1}{2}. \tag{13}$$

On the other hand, if we choose $\delta = \frac{1}{2}+\epsilon$, we see that for any $\epsilon \in (0, \frac{1}{2})$ we have that $K_\delta(i) \leq (2-\epsilon)^i$. Moreover, a simple calculation shows that (7) is satisfied. This together with (13) implies that

$$\delta_1 = \frac{1}{2}.$$

The above reasoning implies that this MC is not GE with rate $\frac{1}{2}$ in the strong sense.

Regarding δ_2, so far we only know that $\delta_2 \leq \frac{1}{2}$. Its exact value will be derived in the next section, where we will also see how the different rates of convergence occur in a natural way when trying to bound δ_0^*, the OLB of the reversed chain in the weak sense, by the OLBs of the original MC.

3 The reversed chain

Assuming that a MC $\xi_1, \xi_2, \ldots$ is GE, what can we say about the reversed MC $\xi_1^*, \xi_2^*, \ldots$? We show that the GE property is preserved under time-reversion, but the behavior of the OLBs is more complicated.

Theorem 1. *If a MC is GE, then the reversed MC is also GE.*

Proof: Let $B_i^{(n)} = \{j \in \Omega : p^{*^n}(i,j) \geq \pi(j)\}$ and $\delta \in (\delta_2, 1)$. Then we have

$$\begin{aligned}
||p^{*^n}(i,\cdot) - \pi||_{TV} &= 2|p^{*^n}(i, B_i^{(n)}) - \pi(B_i^{(n)})| \\
&= 2\sum_{j\in B_i^{(n)}} \frac{\pi(j)}{\pi(i)}\left(p^n(j,i) - \pi(i)\right) \\
&\leq \frac{2}{\pi(i)}\sum_{j\in B_i^{(n)}} \pi(j)\frac{||p^n(j,\cdot)-\pi||_{TV}}{\delta^n}\delta^n \\
&\leq \frac{2}{\pi(i)}\frac{\sum_{j\in\Omega}\pi(j)||p^n(j,\cdot)-\pi||_{TV}}{\delta^n}\delta^n \\
&= \frac{2C(\delta)}{\pi(i)}\delta^n
\end{aligned} \tag{14}$$

and $C(\delta) < \infty$ since $\delta > \delta_2$.

□

Actually, we have just shown

Corollary 1. *If $\xi_1, \xi_2, \ldots$ is GE, then $\delta_0^* \leq \delta_2$.*

Theorem 2. *If $\xi_1, \xi_2, \ldots$ is GE, then*

$$\delta_2 = \delta_2^*, \tag{15}$$

where δ_2^* denotes the OLB of the reversed MC in the $L^1(\pi)$ sense.

Proof: We have

$$\begin{aligned}
\sum_{i\in\Omega} \|p^{*^n}(i,\cdot) - \pi\|_{TV}\pi(i) &= 2\sum_{i\in\Omega} |p^{*^n}(i, B_i^{(n)}) - \pi(B_i^{(n)})|\pi(i) \\
&= 2\sum_{i\in\Omega} \sum_{j\in B_i^{*(n)}} \frac{\pi(j)}{\pi(i)} \left(p^n(j,i) - \pi(i)\right) \pi(i) \\
&= 2\sum_{i\in\Omega} \sum_{j\in B_i^{*(n)}} \pi(j) \left(p^n(j,i) - \pi(i)\right) \\
&\leq 2\sum_{j\in\Omega} \sum_{i\in\Omega} \pi(j) \left|p^n(j,i) - \pi(i)\right| \\
&= 2\sum_{i\in\Omega} \|p^n(i,\cdot) - \pi\|_{TV}\pi(i). \qquad (16)
\end{aligned}$$

For every $\delta > \delta_2$ there is a constant C such that the right-hand side of (16) is at most $C\delta^n$ for all n. It follows that $\delta_2 \geq \delta_2^*$. Using the fact that $p^{**}(\cdot,\cdot) = p(\cdot,\cdot)$ and carrying out the same calculations as in (16) with $p^{**}(\cdot,\cdot)$ instead of $p^*(\cdot,\cdot)$, we obtain $\delta_2 \leq \delta_2^*$.

□

Let us apply Theorem 2 to the example in Section 2. The transition matrix of the reversed MC is given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \ldots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots \end{pmatrix}. \tag{17}$$

This MC has a remarkable feature: there is a central state in the sense that this state can be reached from any other one in a single step with probability 1/2. This property immediately implies that

$$\sup_{i,j\in\Omega, A\subset\Omega} |p^*(i,A) - p^*(j,A)| \leq \frac{1}{2}. \tag{18}$$

It is interesting that (18) implies the classical condition which was used by Döblin [2] in order to establish uniform geometric convergence to the invariant measure (with respect to total-variation) for certain Markov chains, i.e.,

$$\exists \delta < 1: \ \sup_{n\geq 1} \sup_{i\in\Omega} \frac{\|p^n(i,\cdot) - \pi\|_{TV}}{\delta^n} = \sup_{i\in\Omega} K_\delta(i) < \infty.$$

Note that this is a stronger property than (6).

In [6] it is shown that (18) implies that

$$||p^{*^n}(i,\cdot)-\pi||_{TV} \leq 2\left(\frac{1}{2}\right)^n \tag{19}$$

(the constant 2 does not appear in [6] due to a different definition of the total variation norm). The proof is based on a coupling argument in which (18) is used to bound the expected coupling time, which in turn leads to the estimate for the total variation (see [6]). The factor $\frac{1}{2}$ in (19) is optimal in the sense that it is as small as possible. In fact,

$$\sup_{i\in\Omega} ||p^{*^n}(i,\cdot)-\pi||_{TV} \geq |p^{*^n}(1,n)-\pi(n)| = 2^{-n},$$

so $\delta_0^* = \frac{1}{2}$. From (19) it now follows immediately that

$$\delta_0^* = \delta_1^* = \delta_2^* = \frac{1}{2}. \tag{20}$$

The situation is completely different from what we have seen for the original chain, for which it has been shown that

$$0 = \delta_0 \leq \delta_2 \leq \delta_1 = \frac{1}{2}.$$

Let us determine δ_2, which had been left open at the end of Section 2. From Theorem 2 and (20) it follows that

$$\delta_2 = \delta_2^* = \frac{1}{2}.$$

A closer look at the proof of Theorem 2 yields even more. We obtain

$$\sum_{i\in\Omega} ||p^n(i,\cdot)-\pi||_{TV}\pi(i) \leq 4\left(\frac{1}{2}\right)^n,$$

so the OLB in the $L^1(\pi)$ sense, δ_2, is in fact attained. Recall that this was not the case for δ_0 and δ_1.

4 Reversible Markov chains

In this section we show that for reversible MCs δ_0, δ_1 and δ_2 coincide under the (rather weak) condition that the invariant distribution π has a finite $(1+\epsilon)$-moment ($\epsilon > 0$), i.e., if $M = \sum_{i=1}^{\infty} i^{1+\epsilon}\pi(i) < \infty$.

Theorem 3. *If a MC is reversible, GE and its invariant distribution π has a finite $(1+\epsilon)$-moment for some $\epsilon > 0$, then*

$$\delta_0 = \delta_1 = \delta_2, \tag{21}$$

and all these OLBs are attained.

Proof: Without loss of generalization we can assume that $\Omega = \mathbb{N}$ and

$$\pi(i) \geq \pi(i+1)\ \forall i \in \mathbb{N}.$$

Define $\delta_i : \mathbb{N} \to \mathbb{R}$ by

$$\delta_i(k) = \begin{cases} 1 & : \quad i = k \\ 0 & : \quad i \neq k \end{cases}$$

and

$$\rho_i = \limsup_{n\to\infty} ||(P^n - \Pi)\frac{\delta_i}{\pi}||_{L^2(\pi)}^{\frac{1}{n}}, \tag{22}$$

with $||f||_{L^2(\pi)} = [\sum_{j\in\Omega} f(j)^2 \pi(j)]]^{1/2}$. Now we apply the spectral representation theorem (see e.g. [10]) with spectral measure

$$\nu_i(\lambda) = \left\langle E_\lambda \frac{\delta_i/\pi}{||\delta_i/\pi||_{L^2(\pi)}}, \frac{\delta_i/\pi}{||\delta_i/\pi||_{L^2(\pi)}} \right\rangle$$

associated to $P - \Pi$ and $(\delta_i/\pi)/||\delta_i/\pi||_{L^2(\pi)}$, E_λ denoting the corresponding projection operator. We obtain

$$\begin{aligned} ||(P^n - \Pi)\frac{\delta_i}{\pi}||_{L^2(\pi)}^{\frac{1}{n}} &= \left\langle (P^n - \Pi)^2 \frac{\delta_i}{\pi}, \frac{\delta_i}{\pi} \right\rangle_{L^2(\pi)}^{\frac{1}{2n}} \\ &= \left(\int_{-1}^{1} \lambda^{2n} \langle dE_\lambda \frac{\delta_i}{\pi}, \frac{\delta_i}{\pi} \rangle_{L^2(\pi)} \right)^{\frac{1}{2n}} \\ &= ||\frac{\delta_i}{\pi}||_{L^2(\pi)}^{\frac{1}{n}} \left(\int_{-1}^{1} \lambda^{2n} \nu_i(d\lambda) \right)^{\frac{1}{2n}} \\ &= \left(\frac{1}{\sqrt{\pi(i)}} \right)^{\frac{1}{n}} \left(\int_{-1}^{1} \lambda^{2n} \nu_i(d\lambda) \right)^{\frac{1}{2n}} \end{aligned} \tag{23}$$

From (22) and (23) it follows that

$$\rho_i = \max[-\inf \operatorname{supp}(\nu_i(\lambda)), \sup \operatorname{supp}(\nu_i(\lambda))]. \tag{24}$$

We have

$$\begin{aligned} ||p^n(i,\cdot) - \pi||_{TV} &= \sup_{\phi:||\phi||_\infty = 1} \sum_{j\in\Omega} \left(p^n(i,j) - \pi(j)\right) \phi(j) \\ &= \sup_{\phi:||\phi||_\infty = 1} \sum_{j\in\Omega} \sum_{k\in\Omega} \frac{\delta_i(k)}{\pi(k)} \left(p^n(k,j) - \pi(j)\right) \phi(j)\pi(k) \\ &= \sup_{\phi:||\phi||_\infty = 1} \left\langle \frac{\delta_i}{\pi}, (P^n - \Pi)\phi \right\rangle_{L^2(\pi)} \\ &= \sup_{\phi:||\phi||_\infty = 1} \left\langle (P^n - \Pi)\frac{\delta_i}{\pi}, \phi \right\rangle_{L^2(\pi)} \\ &\leq ||(P^n - \Pi)\frac{\delta_i}{\pi}||_{L^2(\pi)} \\ &\leq \rho_i^n \frac{1}{\sqrt{\pi(i)}} \end{aligned} \tag{25}$$

$$
\begin{aligned}
&\leq \sup_{j\in\Omega} \rho_j^n \frac{1}{\sqrt{\pi(i)}} && (26)\\
&\leq \frac{1}{\sqrt{\pi(i)}}\rho^n && (27)
\end{aligned}
$$

where the first two inequalities follow from Cauchy-Schwarz and the identities (23)-(24), respectively. The last inequality follows from the definition of ρ. From the equivalence of (i) and (iii) in Theorem 2.1 of [12] it follows that the upper bound ρ_i for the rate in (25) is optimal in the sense that

$$\sup_{n\geq 1} \frac{||p^n(i,\cdot)-\pi||_{TV}}{\delta^n} = \infty \ \forall \delta < \rho_i \ \forall i \in \Omega.$$

This implies that

$$\delta_0 = \sup_{j\in\Omega} \rho_j.$$

From (26) it follows that δ_0 is attained, i.e., that (6) holds for $\delta = \delta_0$.

Now let us prove (21). By (26), it is enough to show that

$$||\frac{1}{\sqrt{\pi(\cdot)}}||_{L^1(\pi)} = \sum_{i=1}^{\infty} \sqrt{\pi(i)} < \infty. \tag{28}$$

Let $K = \sum_{i=1}^{\infty} i^{-(1+\epsilon)}$. We obtain

$$
\begin{aligned}
\sum_{i=1}^{\infty} \sqrt{\pi(i)} &= \sum_{i=1}^{\infty} i^{\frac{1+\epsilon}{2}} \sqrt{\pi(i)} \frac{1}{i^{\frac{1+\epsilon}{2}}}\\
&\leq \sqrt{KM} < \infty. && (29)
\end{aligned}
$$

□

From the last proof we immediately obtain

Corollary 2. *For a reversible MC the following two statements are equivalent:*

1. $\rho = \sup_{j\in\Omega} \rho_j$.
2. $\delta_0 = \rho$.

The estimate in (27) is the well-known $\frac{1}{\sqrt{\pi(i)}}$-bound for the total variation in terms of the spectral radius. For Markov chains with finite state space this can be found in [13].

5 Geometric ergodicity and spectral theory

The following theorem due to [11] and [12] shows the close connection between geometric ergodicity and the spectral gap property.

Theorem 4. *For a reversible MC $\xi_1, \xi_2, \ldots$ the following two statements are equivalent:*

1. $\xi_1, \xi_2, \ldots$ is GE.

2. P satisfies the SGP.

Moreover,

$$\rho = \delta_0.$$

The original proof of this result can be found in [12]. A very short derivation of the first part was given in [14]. The key observation there was that the spectral radius of a MC can be expressed by a rescaled function of a sequence of isoperimetric constants (see Theorem 5 below). It turns out that these rescaled constants are a suitable tool for studying geometric ergodicity in the sense that they can be related to the different notions of geometric speed of convergence.

The isoperimetric constants in question are

$$k_n = \inf_{A \subset \Omega} k_n(A), \quad k_{P^{*n}P^n} = \inf_{A \subset \Omega} k_{P^{*n}P^n}(A), \; n \in \mathbb{N}$$

where

$$\begin{aligned} k_n(A) &= \frac{1}{\pi(A)\pi(A^c)} \sum_{i \in A} p^n(i, A^c)\pi(i) \\ k_{P^{*n}P^n}(A) &= \frac{1}{\pi(A)\pi(A^c)} \sum_{i \in A} \sum_{j \in \Omega} p^{*^n}(i,j) p^n(j, A^c)\pi(i). \end{aligned}$$

The following theorem from [14] relates spectral properties to the rescaled limits of isoperimetric constants.

Theorem 5. *Assume that the operator P is normal. Then the spectral radius ρ is given by*

$$\rho = \lim_{n \to \infty} \left(\sqrt{1 - k_{P^{*n}P^n}} \right)^{\frac{1}{n}}. \tag{30}$$

In particular, for reversible Markov chains this yields

$$\rho = \lim_{n \to \infty} \left(\sqrt{1 - k_{2n}} \right)^{\frac{1}{n}}.$$

Moreover, if P is in addition positive, we have

$$\rho = \lim_{n \to \infty} \left(1 - k_n \right)^{\frac{1}{n}}.$$

Based on this result, we can show

Theorem 6. *If the underlying MC is GE, then*

$$\sup_{A \subset \Omega} \limsup_{n \to \infty} (1 - k_{P^{*n}P^n}(A))^{\frac{1}{2n}} \leq \sqrt{\delta_2}.$$

If P is in addition normal, then the MC satisfies SGP and the spectral radius ρ can be estimated by

$$\delta_0 \leq \rho \leq \sqrt{\delta_2}. \tag{31}$$

Proof: An easy calculation shows that

$$1-k_{P^{*n}P^n}(A)=\frac{1}{\pi(A)\pi(A^c)}\sum_{i\in\Omega}(p^n(i,A^c)-\pi(A^c))^2\pi(i). \tag{32}$$

Hence, for every $\epsilon\in(0,1-\delta_2)$,

$$\begin{aligned}
&\limsup_{n\to\infty}(1-k_{P^{*n}P^n}(A))^{\frac{1}{2n}}\\
&\quad=\limsup_{n\to\infty}\left(\frac{1}{\pi(A)\pi(A^c)}\sum_{i\in\Omega}(p^n(i,A^c)-\pi(A^c))^2\pi(i)\right)^{\frac{1}{2n}}\\
&\quad\le\sqrt{\epsilon+\delta_2}\limsup_{n\to\infty}\left(\frac{2}{\pi(A)\pi(A^c)}\right)^{\frac{1}{2n}}\left(\frac{\sum_{i\in\Omega}||p^n(i,\cdot)-\pi||_{TV}\pi(i)}{(\epsilon+\delta_2)^n}\right)^{\frac{1}{2n}}\\
&\quad\le\sqrt{\epsilon+\delta_2}.
\end{aligned}\tag{33}$$

This proves the first assertion of the theorem.

The first inequality in (31) follows from the second part of Theorem 4. Let us prove the second inequality. It was shown in [14] that for $l<n$ we have

$$(1-k_{P^{*l}P^l}(A))^{\frac{1}{2l}}\le(1-k_{P^{*n}P^n}(A))^{\frac{1}{2n}}. \tag{34}$$

Thus, by (34) and (32),

$$\begin{aligned}
(1-k_{P^{*l}P^l}(A))^{\frac{1}{2l}} &\le \left(\frac{1}{\pi(A)\pi(A^c)}\sum_{i\in\Omega}(p^n(i,A^c)-\pi(A^c))^2\pi(i)\right)^{\frac{1}{2n}}\\
&\le \left(\frac{2}{\pi(A)\pi(A^c)}\right)^{\frac{1}{2n}}\left(\sum_{i\in\Omega}||p^n(i,\cdot)-\pi||_{TV}\pi(i)\right)^{\frac{1}{2n}}.
\end{aligned}\tag{35}$$

Now first letting $n\to\infty$, then taking the supremum over all $A\subset\Omega$, thereafter letting $l\to\infty$ and applying Theorem 5 yields $\rho\le\sqrt{\delta_2}$.

□

From this theorem we immediately obtain

Corollary 3. *If P is normal, then the following statements are equivalent:*

1. *$\xi_1,\xi_2,\ldots$ is GE.*
2. *$\xi_1,\xi_2,\ldots$ satisfies SGP .*

Next we want to prove the equivalence in Corollary 3 for certain non-reversible MCs. Note that normality of the operator P is only needed to ensure that (34) holds. So it seems natural to start with a modified version of (34). Define

$$a(n,A)=(1-k_{P^{*n}P^n}(A))^{\frac{1}{2n}}. \tag{36}$$

Corollary 4. *Assume that for every $A \subset \Omega$ the sequence $(a(n,A))_{n\in\mathbb{N}}$ has a nondecreasing subsequence $(a(n_k,A))_{k\in\mathbb{N}}$ with $n_1 = 1$. Then the GE property and SGP are equivalent and*

$$\rho \le \sqrt{1 - \frac{\kappa}{8}\left(1 - \delta_2^2\right)^2}, \tag{37}$$

where $\kappa \ge 1$ is a constant which does not depend on the underlying MC.

Note that the subsequence $(n_k)_{k\ge 2}$ is allowed to depend on A. The fact that $\kappa \ge 1$ has been established in [5], from which the following definition of κ is taken: Let $\mathscr{D}$ denote the set of all possible distributions of pairs (X,Y) of i.i.d random variables each having variance 1. Then

$$\kappa = \inf_{\mathscr{D}} \sup_{c\in\mathbb{R}} \frac{E\left(|(X+c)^2 - (Y+c)^2|\right)}{E((X+c)^2)}. \tag{38}$$

Proof: The implication SGP $\Longrightarrow$ GE can be derived in a similar way as (25). More precisely, in the derivation of (25) we have to take the adjoint in the inner product, i.e. to replace $P^n - \Pi$ by $P^{*n} - \Pi$. The result follows by applying Cauchy-Schwarz in (25) and the fact that $||P^n - \Pi||_{L^2(\pi)} = ||P^{n*} - \Pi||_{L^2(\pi)}$.

GE $\Longrightarrow$ SGP follows immediately from (37), since $\delta_2 < 1$ implies $\rho < 1$. So let us show (37). Since $(a(n_k,A))_{k\in\mathbb{N}}$ is nondecreasing, we can carry out the same calculation as in the proof of Theorem 6 with n replaced by n_k. By assumption, we have $n_1 = 1$ for all $A \subset \Omega$. This yields

$$(1 - k_{P^*P}(A))^{\frac{1}{2}} \le \delta_2, \tag{39}$$

which implies that

$$(1 - k_{P^*P})^{\frac{1}{2}} \le \delta_2.$$

Now (37) follows from Proposition 1 of [16].

□

Because of its generality, the upper bound in (37) is not sharp in most cases. In order to improve this upper bound for certain MCs we show the following generalization of Theorem 5.
We need the Hilbert space $L_0^2(\pi) = \{f \in L^2(\pi) : \sum_{j\in\Omega} f(j)\pi(j) = 0\}$.

Theorem 7. *For a positive recurrent MC the spectral radius $\rho = \rho(P)$ of the associated Markov operator P on $L_0^2(\pi)$ is given by*

$$\rho = \lim_{n\to\infty} \lim_{l\to\infty} \left(1 - k_{(P^{*^n}P^n)^l}\right)^{\frac{1}{2nl}}. \tag{40}$$

Proof: Since $P^{*^n}P^n$ is positive and selfadjoint, Theorem 5 yields

$$\rho(P^{*^n}P^n) = \lim_{l\to\infty} \left(1 - k_{(P^{*^n}P^n)^l}\right)^{\frac{1}{l}}.$$

By the Rayleigh-Ritz principle (see e.g. [5]) it follows that

$$\sup_{f\in L_{0,1}^2(\pi)} \langle P^{*^n}P^n f, f\rangle_\pi = \lim_{l\to\infty} \left(1 - k_{(P^{*^n}P^n)^l}\right)^{\frac{1}{l}}. \tag{41}$$

Since the left-hand side in (41) equals $||P^n||^2_{L^2_0(\pi)}$, we obtain

$$||P^n||^{\frac{1}{n}}_{L^2_0(\pi)} = \lim_{l\to\infty} \left(1 - k_{(P^{*^n}P^n)^l}\right)^{\frac{1}{2nl}}.$$

Now $n \to \infty$ leads to the assertion.

□

Corollary 5. *Assume that there exists an $n_0 \in \mathbb{N}$ such that*

$$P^{*^n}P^n = (P^*P)^n \quad \forall n \geq n_0. \tag{42}$$

Then

$$\rho(P) = \sqrt{\rho(P^*P)} = \lim_{n\to\infty} \left(1 - k_{P^{*^n}P^n}\right)^{\frac{1}{2n}}$$

and

$$\delta_0 \leq \rho(P) \leq \sqrt{\delta_2}. \tag{43}$$

Proof: From Theorem 7 it follows that

$$\begin{aligned} \rho(P) &= \lim_{n\to\infty}\lim_{l\to\infty} \left(1 - k_{(P^{*^n}P^n)^l}\right)^{\frac{1}{2nl}} \\ &= \lim_{n\to\infty}\lim_{l\to\infty} \left(1 - k_{(P^*P)^{nl}}\right)^{\frac{1}{2nl}} \\ &= \sqrt{\rho(P^*P)} \\ &= \lim_{n\to\infty} \left(1 - k_{(P^*P)^n}\right)^{\frac{1}{2n}} \\ &= \lim_{n\to\infty} \left(1 - k_{(P^{*^n}P^n)}\right)^{\frac{1}{2n}}. \end{aligned} \tag{44}$$

The inequalities (43) can be shown in the same way as in the proof of Theorem 6.

□

The upper bound in (43) is better than that in (37). To show this, note that since we do not know the exact value of κ, the estimate (37) can only be applied with $\kappa = 1$. Therefore we have to prove that

$$\sqrt{\delta_2} \leq \sqrt{1 - \frac{1}{8}\left(1 - \delta_2^2\right)^2},$$

which is equivalent to

$$\frac{1}{8}(1 - \delta_2)^2(1 + \delta_2)^2 \leq 1 - \delta_2.$$

Actually, $\sqrt{\delta_2}$ is smaller than the right-hand side of (37) whenever $\max_{\delta\in[0,1]}(1-\delta)(1+\delta)^2 \leq 8/\kappa$. This is the case as long as $\kappa \leq 27/4$.

Observe that normality of a MC implies condition (42). Let us again consider the example of Section 2 to show that this implication cannot be reversed. Let P and P^* be given by (12) and (17), respectively. It can be readily seen that for $i \geq 2$ and $j \in \mathbb{N}$ we have

$$(P^*P)_{i,j} = \frac{1}{2}\pi_j + \frac{1}{2}\delta_{i,j}$$

and

$$(PP^*)_{i,j} = \frac{1}{2}\delta_{0,j} + \frac{1}{2}\delta_{i,j}.$$

This implies that $P^*P \neq PP^*$, so the MC is not normal. However, a short calculation shows that

$$((P^*P)^2)_{i,j} = \frac{3}{4}\pi_j + \frac{1}{4}\delta_{i,j} = (P^{*^2}P^2)_{i,j}. \tag{45}$$

By (45),

$$\begin{aligned} P^{*^3}P^3 &= P^*(P^{*^2}P^2)P = P^*(P^*P)^2P = P^{*^2}PP^*P^2 \\ &= P^{*^2}P^2(P^{-1}P^{*^{-1}})P^{*^2}P^2 = (P^*P)^2(P^*P)^{-1}(P^*P)^2 \\ &= (P^*P)^3. \end{aligned} \tag{46}$$

By complete induction, it is now seen that (42) is satisfied with $n_0 = 2$.

The spectral gap in this example has already been determined in [14]. We give a very short alternative derivation. From Corollary 5 it follows that

$$\rho(P) = \sqrt{\rho(P^*P)}.$$

But

$$P^*P = \frac{1}{2}I + \frac{1}{2}\Pi, \tag{47}$$

where I denotes the identity operator, i.e., $If = f$. Since P^*P is selfadjoint, we obtain

$$\begin{aligned} \rho(P) &= \sqrt{\rho(P^*P)} = \sqrt{||P^*P||_{L_0^2(\pi)}} \\ &= \sqrt{||\frac{1}{2}I + \frac{1}{2}\Pi||_{L_0^2(\pi)}} \\ &= \sqrt{\frac{1}{2}}. \end{aligned} \tag{48}$$

Note that the inequality $\rho \leq \sqrt{\delta_2} = \sqrt{\frac{1}{2}}$, which has been derived in Corollary 5, is in fact sharp! We can use this in order to obtain an estimate for κ. Insert $\rho = \sqrt{\frac{1}{2}}$ into (37) we obtain that

$$\kappa \leq \frac{64}{9}.$$

The computations in the proof of Theorem 3 lead to the following modification of Corollary 5:

Corollary 6. *If the operator P of a geometrically ergodic MC satisfies (42) and the invariant distribution π has a finite $(1+\epsilon)$-moment for some $\epsilon > 0$, then*

$$\delta_2 \leq \rho \leq \sqrt{\delta_2}.$$

The following result provides lower bounds for δ_0 and δ_2.

Theorem 8. *If the MC is GE,*

$$\delta_2 \geq \sup_{A\subset\Omega} \limsup_{n\to\infty} |1-k_{2n}(A)|^{\frac{1}{2n}}. \tag{49}$$

$$\delta_0 \geq \sup_{A\subset\Omega:\min(|A|,|A^c|)<\infty} \limsup_{n\to\infty} |1-k_{2n}(A)|^{\frac{1}{2n}}. \tag{50}$$

If for every $A \subset \Omega$ the sequence $(|1-k_{2n}(A)|^{\frac{1}{2n}})_{n\in\mathbb{N}}$ is nondecreasing, we even have

$$\delta_2 \geq \lim_{n\to\infty} |1-k_{2n}|^{\frac{1}{2n}}. \tag{51}$$

Moreover, for every sequence $(A_{2n})_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\left(\frac{1}{\pi(A_n)\pi(A_n^c)}\right)^{\frac{1}{2n}} = 1$ we have

$$\delta_2 \geq \limsup_{n\to\infty} |1-k_{2n}(A_{2n})|^{\frac{1}{2n}}. \tag{52}$$

Proof: We only show the third inequality of Theorem 8 because the proofs of the others are similar. We have by assumption that, for arbitrary $\delta > \delta_2$,

$$\begin{aligned}
|1-k_{2n_0}(A)|^{\frac{1}{2n_0}} &\leq \lim_{n\to\infty} |1-k_{2n}(A)|^{\frac{1}{2n}} \\
&= \lim_{n\to\infty} \left|1-\frac{1}{\pi(A)\pi(A^c)}\sum_{i\in A} p^{2n}(i,A^c)\pi(i)\right|^{\frac{1}{2n}} \\
&\leq \lim_{n\to\infty}\left(\frac{1}{\pi(A)\pi(A^c)}\right)^{\frac{1}{2n}} \limsup_{n\to\infty}\left(\sum_{i\in A} ||p^{2n}(i,\cdot)-\pi||_{TV}\pi(i)\right)^{\frac{1}{2n}} \\
&\leq \limsup_{n\to\infty}\left(\sum_{i\in\Omega} ||p^{2n}(i,\cdot)-\pi||_{TV}\pi(i)\right)^{\frac{1}{2n}} \\
&\leq \limsup_{n\to\infty} C(\delta)^{\frac{1}{2n}}\delta = \delta.
\end{aligned} \tag{53}$$

Now $\delta \to \delta_2$ and $n_0 \to \infty$ yields the result.

□

Let us apply this result to our example. A good choice of the set A is of key importance in order to obtain a non-trivial lower bound. We try $A = \{2,4,6,8,\ldots\}$. Then

$$\begin{aligned}
k_{2n}(A) &= \frac{1}{\pi(A)\pi(A^c)}\sum_{i\in A} p^{2n}(i,A^c)\pi(i) \\
&= \frac{1}{\pi(A)\pi(A^c)}\sum_{i=1}^{n} \pi(A^c)\pi(2i) = 3\frac{1/4-(1/4)^{n+1}}{3/4} = 1-\left(\frac{1}{4}\right)^n.
\end{aligned} \tag{54}$$

This implies that

$$(1-k_{2n}(A))^{\frac{1}{2n}} = \frac{1}{2}$$

for all n. Applying Theorem 8 yields

$$\delta_2 \geq \frac{1}{2}.$$

By what has been shown before, this bound is again sharp. One can prove that the above choice of A is optimal in the sense that

$$k_{2n}(A) = k_{2n}.$$

So we have just seen that in our example we have

$$(1 - k_{2n})^{\frac{1}{2n}} = \delta_2 \quad \forall n. \tag{55}$$

It would be nice to have this relation in general, at least asymptotically, but this result fails to be true. In the next section we consider an example (originally due to Häggström [3]) of a MC that is GE and satisfies $k_{2n} = 0$ for all $n \in \mathbb{N}$. In this example the left-hand side in (55) is equal to one for every n, but by geometric ergodicity the right-hand side in (55) is less than one.

6 Example [GE $\not\Longrightarrow$ SGP]

Consider the MC with state space

$$\Omega = \{0\} \cup \{(a, b) : a \geq 1, b \in \{1, 2, \ldots, a\}\}$$

and transition kernel

$$p((a, b), (a, b-1)) = 1, \text{ for } b \geq 2, \; p((a, 1), 0) = 1,$$

$p(0, 0) = \frac{1}{2}$ and

$$p(0, (a, b)) = \begin{cases} 2^{-(a+1)} & : \quad a = b \\ 0 & : \quad \text{otherwise} \end{cases}.$$

The invariant distribution π can be calculated to be

$$\pi(0) = \frac{1}{2} \text{ and } \pi((a, b)) = 2^{-(a+2)} \text{ for } b \in \{1, 2, \ldots, a\}. \tag{56}$$

Häggström [3] has shown that this MC is GE with $\delta_0 = \frac{1}{2}$. In order to prove that $k_n = 0$ for all $n \in \mathbb{N}$, it suffices to show that $k_1 = 0$ (see [15]). This can be seen as follows: Define

$$A_{n,n} = \{(n, n), (n, n-1), \ldots, (n, 1)\} \text{ and } A_{n,1} = \{(n, 1)\}.$$

Then we have

$$\begin{aligned} k_1 &\leq k(A_{n,n}) = \frac{1}{n \cdot 2^{-(n+2)}} \frac{1}{1 - n \cdot 2^{-(n+2)}} \sum_{i \in A_{n,n}} p(i, A_{n,n}^c) \pi(i) \\ &\leq \frac{2}{n \cdot 2^{-(n+2)}} \sum_{i \in A_{n,1}} \pi(i) = \frac{2}{n}. \end{aligned} \tag{57}$$

Letting $n \to \infty$ yields $k_1 = 0$.

Kontoyiannis and Meyn [4] have proved that geometric ergodicity and SGP are not equivalent using the same example, but a different argument based on an Lyapunov function approach.

Häggström [3] originally used the example in order to present a sequence of random variables connected to a geometrically ergodic MC with finite second moments but not following the central limit theorem. In fact, this result implies that the MC cannot satisfy SGP, since by a theorem due to Cogburn [1] for every sequence of random variable connected to a Markov chain satisfying SGP and having finite second moments the central limit theorem holds.

We now show that

$$\delta_0 = \delta_1 = \delta_2 = \frac{1}{2}.$$

We start from the observation

$$p^n(0,0) = \frac{1}{2} \quad \forall n \in \mathbb{N}. \tag{58}$$

Define

$$d(0,(a,b)) = a - b + 1 \quad \forall (a,b) : a \geq 1, b \in \{1,2,\ldots,a\}$$

and

$$d((a,b),0) = b \quad \forall (a,b) : a \geq 1, b \in \{1,2,\ldots,a\}.$$

Using equality (58) it is not difficult to see that for all $n \geq d(0,(a,b))$ we have

$$p^n(0,(a,b)) = \pi((a,b)). \tag{59}$$

But this implies that for $n \geq d((a,b),0) = b$

$$\begin{aligned} ||p^n((a,b),\cdot) - \pi||_{TV} &\leq ||p^{n-b}(0,\cdot) - \pi||_{TV} \\ &\leq \pi(\{(a,b) : d(0,(a,b)) > n - b, a \geq 1, b \in \{1,2,\ldots,a\}\}) \\ &\leq C\, 2^b \left(\frac{1}{2}\right)^n \text{ for some } C > 0. \end{aligned} \tag{60}$$

This yields that $\frac{1}{2}$ is an upper bound for δ_0. To see that $\frac{1}{2}$ is also a lower bound, note that

$$||p^n(0,\cdot) - \pi||_{TV} \geq |p^n(0,(n+1,1)) - \pi((n+1,1))| = \pi((n+1,1)) = 2^{-4}\left(\frac{1}{2}\right)^n.$$

Next we show that $\delta_2 \leq \frac{1}{2}$. Similar calculations as in (59) yield for all $\epsilon \in (0,\frac{1}{2}]$ and $n \geq d((a,b),0) = b$

$$||p^n((a,b),\cdot) - \pi||_{TV} \leq C\,(2-\epsilon)^b \left(\frac{1}{2} + \epsilon\right)^n \text{ for some } C > 0.$$

Since f defined by $f((a,b)) = (2-\epsilon)^b$ is in $\in L^1(\pi)$, the desired inequality follows.

To see that $\frac{1}{2}$ is also a lower bound for δ_2, we calculate $1 - k_{2n}(A_{2n})$ for

$$A_{2n} = \{0\} \cup \{(a,a) : a \in \{1,2,\ldots,2n\}\},\ n \geq 2.$$

It is not difficult to show that

$$p^2(0,(j,j)) = \pi((j,j))\ \forall j \in \mathbb{N}.$$

This implies

$$p^k(0,(j,j)) = \pi((j,j)) \ \forall j \in \mathbb{N}, \forall k \geq 2. \tag{61}$$

Applying (58) and (61) we obtain

$$\begin{aligned}
k_{2n}(A_{2n}) &= \frac{1}{\pi(A_{2n}^c)} - \frac{1}{\pi(A_{2n})\pi(A_{2n}^c)} \sum_{i\in A_{2n}} p^{2n}(i,A_{2n})\pi(i) \\
&= \frac{1}{\pi(A_{2n}^c)} - \frac{1}{\pi(A_{2n}^c)} \frac{1}{\pi(A_{2n})} \sum_{i=0}^{2n} p^{2n-i}(0,A_{2n})\pi(i) \\
&= \frac{1}{\pi(A_{2n}^c)} - \frac{1}{\pi(A_{2n}^c)} \frac{1}{\pi(A_{2n})} \Big(\sum_{i=0}^{2n-2} \pi(A_{2n})\pi(i) \\
&\qquad + \pi(2n-1)p(0,A_{2n}) + \pi(2n) \Big) \\
&= \frac{1}{\pi(A_{2n}^c)} - \frac{1}{\pi(A_{2n}^c)\pi(A_{2n})} [\pi(A_{2n})^2 - \pi(A_{2n})(\pi(2n-1) + \pi(2n)) \\
&\qquad + \pi(2n-1)p(0,A_{2n}) + \pi(2n)] \\
&= 1 - \frac{-\pi(A_{2n})(\pi(2n-1)+\pi(2n)) + \pi(2n-1)p(0,A_{2n}) + \pi(2n)}{\pi(A_{2n}^c)\pi(A_{2n})} \\
&= 1 - \frac{1 + 2p(0,A_{2n}) - 3\pi(A_{2n})}{\pi(A_{2n}^c)\pi(A_{2n})} \pi(2n)
\end{aligned} \tag{62}$$

Now it can be easily deduced that

$$\lim_{n\to\infty} |1 - k_{2n}(A_{2n})|^{\frac{1}{2n}} = \frac{1}{2}.$$

Apply inequality (52) of Theorem 8 to conclude that $\delta_2 \geq \frac{1}{2}$. Altogether we have now shown that $\delta_0 = \delta_1 = \delta_2 = \frac{1}{2}$. Note that the infima δ_1 and δ_2 are not attained but the infimum δ_0 is.

Acknowledgement. This research is part of a project that is supported by the Deutsche Forschungsgemeinschaft.

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