DOI: 10.1214/EJP.v16-859

Electronic Journal of Probability

Vol. 16 (2011), Paper no. 18, pages 504–530.

Journal URL
http://www.math.washington.edu/~ejpecp/

Mirror coupling of reflecting Brownian motion and an application to Chavel's conjecture*

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Abstract

In a series of papers, Burdzy et al. introduced the *mirror coupling* of reflecting Brownian motions in a smooth bounded domain $D \subset \mathbb{R}^d$, and used it to prove certain properties of eigenvalues and eigenfunctions of the Neumann Laplaceian on D.

In the present paper we show that the construction of the mirror coupling can be extended to the case when the two Brownian motions live in different domains $D_1, D_2 \subset \mathbb{R}^d$.

As applications of the construction, we derive a unifying proof of the two main results concerning the validity of Chavel's conjecture on the domain monotonicity of the Neumann heat kernel, due to I. Chavel ([12]), respectively W. S. Kendall ([16]), and a new proof of Chavel's conjecture for domains satisfying the ball condition, such that the inner domain is star-shaped with respect to the center of the ball.

Key words: couplings, mirror coupling, reflecting Brownian motion, Chavel's conjecture.

AMS 2000 Subject Classification: Primary 60J65, 60H20; Secondary: 35K05, 60H3.

Submitted to EJP on April 16, 2010, final version accepted January 26, 2011.

*The author kindly acknowledges the support from CNCSIS-UEFISCSU research grant PNII - IDEI 209/2007.

1 Introduction

The technique of coupling of reflecting Brownian motions is a useful tool, used by several authors in connection to the study of the Neumann heat kernel of the corresponding domain (see [2], [3], [6], [11], [16], [17], etc).

In a series of paper, Krzysztof Burdzy et al. ([1], [2], [3], [6], [10],) introduced the *mirror coupling* of reflecting Brownian motions in a smooth domain $D \subset \mathbb{R}^d$ and used it in order to derive properties of eigenvalues and eigenfunctions of the Neumann Laplaceian on D.

In the present paper, we show that the mirror coupling can be extended to the case when the two reflecting Brownian motions live in different domains $D_1, D_2 \subset \mathbb{R}^d$.

The main difficulty in the extending the construction of the mirror coupling comes from the fact that the stochastic differential equation(s) describing the mirror coupling has a singularity at the times when coupling occurs. In the case $D_1 = D_2 = D$ considered by Burdzy et al. this problem is not a major problem (although the technical details are quite involved, see [2]), since after the coupling time the processes move together. In the case $D_1 \neq D_2$ however, this is a major problem: after the processes have coupled, it is possible for them to decouple (for example in the case when the processes are coupled and they hit the boundary of one of the domains).

It is worth mentioning that the method used for proving the existence of the solution is new, and it relies on the additional hypothesis that the smaller domain D_2 (or more generally $D_1 \cap D_2$) is a convex domain. This hypothesis allows us to construct an explicit set of solutions in a sequence of approximating polygonal domains for D_2, which converge to the desired solution.

As applications of the construction, we derive a unifying proof of the two most important results on the challenging Chavel's conjecture on the domain monotonicity of the Neumann heat kernel ([12], [16]), and a new proof of Chavel's conjecture for domains satisfying the ball condition, such that the inner domain is star-shaped with respect to the center of the ball. This is also a possible new line of approach for Chavel's conjecture (note that by the results in [4], Chavel's conjecture does not hold in its full generality, but the additional hypotheses under which this conjecture holds are not known at the present moment).

The structure of the paper is as follows: in Section 2 we briefly describe the construction of Burdzy et al. of the mirror coupling in a smooth bounded domain $D \subset \mathbb{R}^d$.

In Section 3, in Theorem 3.1, we give the main result which shows that the mirror coupling can be extended to the case when $\overline{D_2} \subset D_1$ are smooth bounded domains in $\mathbb{R}^d$ and D_2 is convex (some extensions of the theorem are presented in Section 5).

Before proceeding with the proof of theorem, in Remark 3.4 we show that the proof can be reduced to the case when $D_1 = \mathbb{R}^d$. Next, in Section 3.1, we show that in the case $D_2 = (0, \infty) \subset D_1 = R$ the solution is essentially given by Tanaka's formula (Remark 3.5), and then we give the proof of the main theorem in the 1-dimensional case (Proposition 3.6).

In Section 3.2, we first prove the existence of the mirror coupling in the case when D_2 is a half-space in $\mathbb{R}^d$ and $D_1 = \mathbb{R}^d$ (Lemma 3.8), and then we use this result in order to prove the existence of the mirror coupling in the case when D_2 is a polygonal domain in $\mathbb{R}^d$ and $D_1 = \mathbb{R}^d$ (Theorem 3.9). In Proposition 3.10 we present some of the properties of the mirror coupling in the particular case when D_2 is a convex polygonal domain and $D_1 = \mathbb{R}^d$, which are essential for the construction of the general mirror coupling.

In Section 4 we give the proof of the main Theorem 3.1. The idea of the proof is to construct a sequence $\left(Y_t^n, X_t\right)$ of mirror couplings in $\left(D_n, \mathbb{R}^d\right)$, where $D_n \nearrow D_2$ is a sequence of convex polygonal domains in $\mathbb{R}^d$. Then, using the properties of the mirror coupling in convex polygonal domains (Proposition 3.10), we show that the sequence Y_t^n converges to a process Y_t, which gives the desired solution to the problem.

The last section of the paper (Section 5) is devoted to the applications and the extensions of the mirror coupling constructed in Theorem 3.1.

First, in Theorem 5.3 we use the mirror coupling in order to give a simple, unifying proof of the results of I. Chavel and W. S. Kendall on the domain monotonicity of the Neumann heat kernel (Chavel's Conjecture 5.1). The proof is probabilistic in spirit, relying on the geometric properties of the mirror coupling.

Next, in Theorem 5.4 we show that Chavel's conjecture also holds in the more general case when one can interpose a ball between the two domains, and the inner domain is star-shaped with respect to the center of the ball (instead of being convex). The analytic proof given here is parallel to the geometric proof of the previous theorem, and it can also serve as an alternate proof of it.

Without giving all the technical details, we discuss the extension of the mirror coupling to the case of smooth bounded domains $D_{1,2} \subset \mathbb{R}^d$ with non-tangential boundaries, such that $D_1 \cap D_2$ is a convex domain.

The paper concludes with a discussion of the non-uniqueness of the mirror coupling. The lack of uniqueness is due to the fact that after coupling the processes may decouple, not only on the boundary of the domain, but also when they are inside the domain.

The two basic solutions give rise to the *sticky*, respectively *non-sticky* mirror coupling, and there is a whole range of intermediate possibilities. The stickiness refers to the fact that after coupling the processes "stick" to each other as long as possible ("sticky" mirror coupling, constructed in Theorem 3.1), or they can immediately split apart after coupling ("non-sticky" mirror coupling), the general case (*weak/mild* mirror coupling) being a mixture of these two basic behaviors.

We developed the extension of the mirror coupling having in mind the application to Chavel's conjecture, for which the sticky mirror coupling is the "right" tool, but perhaps the other mirror couplings (the non-sticky and the mild mirror couplings) might prove useful in other applications.

2 Mirror couplings of reflecting Brownian motions

Reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ can be defined as a solution of the stochastic differential equation

$$X_t = x + B_t + \int_0^t \nu_D \left(X_s\right) dL_s^X, \tag{2.1}$$

where B_t is a d-dimensional Brownian motion, ν_D is the inward unit normal vector field on ∂D and L_t^X is the boundary local time of X_t (the continuous non-decreasing process which increases only when $X_t \in \partial D$).

In [1], the authors introduced the *mirror coupling* of reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ (piecewise C^2 domain in $\mathbb{R}^2$ with a finite number of convex corners or a C^2 domain in $\mathbb{R}^d$, $d \geq 3$).

They considered the following system of stochastic differential equations:

$$X_t = x + W_t + \int_0^t \nu_D(X_s)\, dL_s^X \tag{2.2}$$

$$Y_t = y + Z_t + \int_0^t \nu_D(X_s)\, dL_s^Y \tag{2.3}$$

$$Z_t = W_t - 2\int_0^t \frac{Y_s - X_s}{\left\|Y_s - X_s\right\|^2}(Y_s - X_s)\cdot dW_s \tag{2.4}$$

for $t < \xi$, where $\xi = \inf\{s > 0 : X_s = Y_s\}$ is the coupling time of the processes, after which the processes X and Y evolve together, i.e. $X_t = Y_t$ and $Z_t = W_t + Z_\xi - W_\xi$ for $t \geq \xi$.

In the notation of [1], considering the Skorokhod map

$$\Gamma : C\left([0,\infty) : \mathbb{R}^d\right) \to C\left([0,\infty) : \overline{D}\right),$$

we have $X = \Gamma(x + W)$, $Y = \Gamma(y + Z)$, and therefore the above system is equivalent to

$$Z_t = \int_0^{t\wedge\xi} G\left(\Gamma(y+Z)_s - \Gamma(x+W)_s\right) dW_s + 1_{t\geq\xi}\left(W_t - W_\xi\right), \tag{2.5}$$

where $\xi = \inf\left\{s > 0 : \Gamma(x+W)_s = \Gamma(y+Z)_s\right\}$. In [1] the authors proved the pathwise uniqueness and the strong existence of the process Z_t in (2.5) (given the Brownian motion W_t).

In the above $G : \mathbb{R}^d \to \mathscr{M}_{d\times d}$ denotes the function defined by

$$G(z) = \begin{cases} H\left(\frac{z}{\|z\|}\right), & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}, \tag{2.6}$$

where for a unitary vector $m \in \mathbb{R}^d$, $H(m)$ represents the linear transformation given by the $d \times d$ matrix

$$H(m) = I - 2m\, m', \tag{2.7}$$

that is

$$H(m)\nu = \nu - 2(m\cdot\nu)m \tag{2.8}$$

is the mirror image of $\nu \in \mathbb{R}^d$ with respect to the hyperplane through the origin perpendicular to m (m' denotes the transpose of the vector m, vectors being considered as column vectors).

The pair $\left(X_t, Y_t\right)_{t\geq 0}$ constructed above is called a *mirror coupling* of reflecting Brownian motions in D starting at $(x,y) \in \overline{D}\times\overline{D}$.

Remark 2.1. The relation (2.4) can be written in the equivalent form

$$dZ_t = G\left(X_t - Y_t\right) dW_t,$$

which shows that for $t < \xi$ the increments of Z_t are mirror images of the increments of W_t with respect to the hyperplane M_t of symmetry between X_t and Y_t, justifying the name of *mirror coupling*.

3 Extension of the mirror coupling

The main contribution of the author is the observation that the mirror coupling introduced above can be extended to the case when the two reflecting Brownian motion have different state spaces, that is when X_t is a reflecting Brownian motion in a domain D_1 and Y_t is a reflecting Brownian motion in a domain D_2. Although the construction can be carried out in a more general setup (see the concluding remarks in Section 5), in the present section we will consider the case when one of the domains is strictly contained in the other.

The main result is the following:

Theorem 3.1. *Let $D_{1,2} \subset \mathbb{R}^d$ be smooth bounded domains (piecewise C^2-smooth boundary with convex corners in $\mathbb{R}^2$, or C^2-smooth boundary in $\mathbb{R}^d$, $d \geq 3$ will suffice) with $\overline{D_2} \subset D_1$ and D_2 convex domain, and let $x \in \overline{D}_1$ and $y \in \overline{D}_2$ be arbitrarily fixed points.*

Given a d-dimensional Brownian motion $\left(W_t\right)_{t\geq 0}$ starting at 0 on a probability space $(\Omega, \mathscr{F}, P)$, there exists a strong solution of the following system of stochastic differential equations

$$X_t = x + W_t + \int_0^t \nu_{D_1}\left(X_s\right) dL_s^X \tag{3.1}$$

$$Y_t = y + Z_t + \int_0^t \nu_{D_2}\left(Y_s\right) dL_s^Y \tag{3.2}$$

$$Z_t = \int_0^t G\left(Y_s - X_s\right) dW_s \tag{3.3}$$

or equivalent

$$Z_t = \int_0^t G\left(\widetilde{\Gamma}\left(y+Z\right)_s - \Gamma\left(x+W\right)_s\right) dW_s, \tag{3.4}$$

where Γ and $\widetilde{\Gamma}$ denote the corresponding Skorokhod maps which define the reflecting Brownian motion $X = \Gamma(x+W)$ in D_1, respectively $Y = \widetilde{\Gamma}\left(y+Z\right)$ in D_2, and $G : \mathbb{R}^d \to \mathscr{M}_{d\times d}$ denotes the following modification of the function G defined in the previous section:

$$G(z) = \begin{cases} H\left(\frac{z}{\|z\|}\right), & \text{if } z \neq 0 \\ I, & \text{if } z = 0 \end{cases}. \tag{3.5}$$

Remark 3.2. As it will follow from the proof of the theorem, with the choice of G above, the solution of the equation (3.4) in the case $D_1 = D_2 = D$ is the same as the solution of the equation (2.5) considered by the authors in [1] (as also pointed out by the authors, the choice of $G(0)$ is irrelevant in this case).

Therefore, the above theorem is a natural generalization of the mirror coupling to the case when the two processes live in different spaces. We will refer to a solution (X_t, Y_t) given by the above theorem as a *mirror coupling* of reflecting Brownian motions in (D_1, D_2) starting from $(x, y) \in \overline{D_1} \times \overline{D_2}$, with driving Brownian motion W_t.

As indicated in Section 5, the solution of (3.4) is not pathwise unique, due to the fact that the stochastic differential equation has a singularity at the times when coupling occurs. The general mirror coupling can be thought as depending on a parameter which is a measure of the stickiness of the coupling: once the processes X_t and Y_t have coupled, they can either move together until one of them hits the boundary (*sticky mirror coupling* - this is in fact the solution constructed in the above theorem), or they can immediately split apart after coupling (*non-sticky mirror coupling*), and there is a whole range of intermediate possibilities (see the discussion at the end of Section 5).

As an application, in Section 5 we will use the former mirror coupling to give a unifying proof of Chavel's conjecture on the domain monotonicity of the Neumann heat kernel for domains $D_{1,2}$ satisfying the ball condition, although the other possible choices for the mirror coupling might prove useful in other contexts.

Before carrying out the proof, we begin with some preliminary remarks which will allow us to reduce the proof of the above theorem to the case $D_1 = \mathbb{R}^d$.

Remark 3.3. The main difference from the case when $D_1 = D_2 = D$ considered by the authors in [1] is that after the coupling time ξ the processes X_t and Y_t may decouple. For example, if $t \geq \xi$ is a time when $X_t = Y_t \in \partial D_2$, the process Y_t (reflecting Brownian motion in D_2) receives a push in the direction of the inward unit normal to the boundary of D_2, while the process X_t behaves like a free Brownian motion near this point (we assumed that D_2 is strictly contained in D_1), and therefore the processes X and Y will drift apart, that is they will *decouple*. Also, as shown in Section 5, because the function G has a discontinuity at the origin, it is possible that the solutions decouple even when they are inside the domain D_2. This shows that without additional assumptions, the mirror coupling is not uniquely determined (there is no pathwise uniqueness of (3.4)).

Remark 3.4. To fix ideas, for an arbitrarily fixed $\varepsilon > 0$ chosen small enough such that $\varepsilon < \operatorname{dist}\left(\partial D_1, \partial D_2\right)$, we consider the sequence $\left(\xi_n\right)_{n\geq 1}$ of *coupling times* and the sequence $\left(\tau_n\right)_{n\geq 0}$ of times when the processes are ε-decoupled (*ε-decoupling times*, or simply *decoupling times* by an abuse of language) defined inductively by

$$\begin{aligned}
\xi_n &= \inf\left\{t > \tau_{n-1} : X_t = Y_t\right\}, \qquad && n \geq 1,\\
\tau_n &= \inf\left\{t > \xi_n : \|X_t - Y_t\| > \varepsilon\right\}, \qquad && n \geq 1,
\end{aligned}$$

where $\tau_0 = 0$ and $\xi_1 = \xi$ is the first coupling time.

To construct the general mirror coupling (that is, to prove the existence of a solution to (3.1) – (3.3) above, or equivalent to (3.4)), we proceed as follows.

First note that on the time interval $[0,\xi]$, the arguments used in the proof of Theorem 2 in [1] (pathwise uniqueness and the existence of a strong solution Z of (3.4)) do not rely on the fact that $D_1 = D_2$, hence the same arguments can be used to prove the existence of a strong solution of (3.4) on the time interval $[0,\xi_1] = [0,\xi]$. Indeed, given W_t, (3.1) has a strong solution which is pathwise unique (the reflecting Brownian motion X_t in D_1), and therefore the proof of pathwise uniqueness and the existence of a strong solution of (3.4) is the same as in [1] considering $D = D_2$. Also note that as also pointed out by the authors, the value $G(0)$ is irrelevant in their proof, since the problem is constructing the processes until they meet, that is for $Y_t - X_t \neq 0$, for which their definition of G is the same as in (3.5).

We obtain therefore the existence of a strong solution Z_t to (3.4) on the time interval $[0,\xi_1]$. By this we understand that the process Z verifies (3.4) for all $t \leq \xi_1$ and Z_t is $\mathscr{F}_t$ measurable for $t \leq \xi_1$, where $(\mathscr{F}_t)_{t\geq 0}$ denotes the corresponding filtration of the driving Brownian motion W_t.

For an arbitrarily fixed $T > 0$, if $\xi_1 < T$, we can extend Z to a solution of (3.4) on the time interval

$[0,T]$ as follows. Consider $\xi_1^T = \xi_1 \wedge T$, and note that if Z solves (3.4), then

$$
\begin{aligned}
Z_{\xi_1^T+t} - Z_{\xi_1^T} &= \int_{\xi_1^T}^{\xi_1^T+t} G\left(\widetilde{\Gamma}\left(y+Z\right)_s - \Gamma\left(x+W\right)_s\right) dW_s \\
&= \int_0^t G\left(\widetilde{\Gamma}\left(y+Z\right)_{\xi_1^T+s} - \Gamma\left(x+W\right)_{\xi_1^T+s}\right) dW_{\xi_1^T+s}.
\end{aligned}
$$

By the uniqueness results on the Skorokhod map (in the deterministic sense), we have

$$
\widetilde{\Gamma}\left(y+Z\right)_{\xi_1^T+s} = \widetilde{\Gamma}\left(\widetilde{\Gamma}(y+Z)_{\xi_1^T} - Z_{\xi_1^T} + Z_{\xi_1^T+\cdot}\right)_s
$$

and

$$
\Gamma\left(x+W\right)_{\xi_1^T+s} = \Gamma\left(\Gamma(x+W)_{\xi_1^T} - W_{\xi_1^T} + W_{\xi_1^T+\cdot}\right)_s
$$

for $s \geq 0$.

It is known that $\widetilde{W_s} = W_{\xi_1^T+s} - W_{\xi_1^T}$ is a Brownian motion starting at the origin, with corresponding filtration $\widetilde{\mathscr{F}_s} = \sigma\left(B_{\xi_1^T+u} - B_{\xi_1^T} : 0 \leq u \leq s\right)$ independent of $\mathscr{F}_{\xi_1^T}$.

Setting $\widetilde{Z}_t = Z_{\xi_1^T+t} - Z_{\xi_1^T}$ and combining the above equations we obtain

$$
\widetilde{Z}_t = \int_0^t G\left(\widetilde{\Gamma}\left(\widetilde{\Gamma}\left(y+Z\right)_{\xi_1^T} + \widetilde{Z}\right)_s - \Gamma\left(\Gamma\left(x+W\right)_{\xi_1^T} + \widetilde{W}\right)_s\right) d\widetilde{W_s}, \tag{3.6}
$$

which is the same as the equation (3.4) for $\widetilde{Z}$, with the initial points x, y of the coupling replaced by $Y_{\xi_1^T} = \widetilde{\Gamma}(y+Z)_{\xi_1^T}$, respectively $X_{\xi_1^T} = \Gamma(x+W)_{\xi_1^T}$, and the Brownian motion W replaced by $\widetilde{W}$.

If we assume the existence of a strong solution $\widetilde{Z}_t$ of (3.6) until the first ε-decoupling time, by patching Z and $\widetilde{Z}$ we obtain that

$$
Z_t 1_{t \leq \xi_1^T} + \widetilde{Z}_{t-\xi_1^T} 1_{\xi_1^T \leq t \leq \tau_1^T}
$$

is a strong solution to (3.4) on the time interval $[0, \tau_1^T]$, where $\tau_1^T = \tau_1 \wedge T$.

If $\tau_1^T = T$, we are done. Otherwise, since at time τ_1^T the processes X and Y are ε units apart, we can apply again the results in [1] (with the Brownian motion $W_{\tau_1^T+t} - W_{\tau_1^T}$ instead of W_t, and the starting points of the coupling $X_{\tau_1^T}$ and $Y_{\tau_1^T}$ instead of x and y) in order to obtain a strong solution of (3.4) until the first coupling time. By patching we obtain the existence of a strong solution of (3.4) on the time interval $\left[0, \xi_2^T\right]$.

Proceeding inductively as indicated above, since only a finite number of coupling / decoupling times ξ_n and τ_n can occur in the time interval $[0,T]$, we can construct a strong solution Z to (3.4) on the time interval $[0,T]$ for any $T > 0$ (and therefore on $[0,\infty)$), provided we show the existence of strong solutions of equations of type (3.6) until the first ε-decoupling time.

In order to prove this claim, since $\widetilde{\Gamma}(y+Z)_{\xi_1^T}$ and $\Gamma\left(x+W\right)_{\xi_1^T}$ are $\mathscr{F}_{\xi_1^T}$ measurable and the σ-algebra $\mathscr{F}_{\xi_1^T}$ is independent of the filtration $\widetilde{\mathscr{F}} = (\widetilde{\mathscr{F}}_t)_{t\geq 0}$ of the driving Brownian motion $\widetilde{W}_t$, it suffices to show that for any starting points $x = y \in \overline{D_2}$ of the mirror coupling, there exists a strong solution of (3.4) until the first ε-decoupling time τ_1. Since $\varepsilon < \operatorname{dist}\left(\partial D_1, \partial D_2\right)$, it follows that the process X_t cannot reach the boundary ∂D_1 before the first ε-decoupling time τ_1, and therefore we can consider that X_t is a free Brownian motion in $\mathbb{R}^d$, that is, we can reduce the proof of Theorem 3.1 to the case when $D_1 = \mathbb{R}^d$.

We will give the proof of the Theorem 3.1 first in the 1-dimensional case, then we will extend it to the case of polygonal domains in $\mathbb{R}^d$, and we will conclude with the proof in the general case.

3.1 The 1-dimensional case

From Remark 3.4 it follows that in order to construct the mirror coupling in the 1-dimensional case, it suffices to consider $D_1 = \mathbb{R}$ and $D_2 = (0, a)$, and to show that for an arbitrary choice $x \in [0, a]$ of the starting point of the mirror coupling, $\varepsilon \in (0, a)$ sufficiently small and $\left(W_t\right)_{t\geq 0}$ a 1-dimensional Brownian motion starting at $W_0 = 0$, we can construct a strong solution on $[0, \tau_1]$ of the following system

$$\begin{aligned} X_t &= x + W_t && (3.7)\\ Y_t &= x + Z_t + L_t^Y && (3.8)\\ Z_t &= \int_0^t G\left(Y_s - X_s\right) dW_s && (3.9) \end{aligned}$$

where $\tau_1 = \inf\left\{s > 0 : |X_s - Y_s| > \varepsilon\right\}$ is the first ε-decoupling time and the function $G : \mathbb{R} \to \mathscr{M}_{1\times 1} \equiv \mathbb{R}$ is given in this case by

$$G(x) = \begin{cases} -1, & \text{if } x \neq 0 \\ +1, & \text{if } x = 0 \end{cases}. \tag{3.10}$$

Remark 3.5. Before proceeding with the proof, it is worth mentioning that the heart of the construction is Tanaka's formula. To see this, consider for the moment $a = \infty$, and note that Tanaka formula

$$\left|x + W_t\right| = x + \int_0^t \operatorname{sgn}\left(x + W_s\right) dW_s + L_t^0(x + W)$$

gives a representation of the reflecting Brownian motion $\left|x + W_t\right|$ in which the increments of the martingale part of $\left|x + W_t\right|$ are the increments of W_t when $x + W_t \in [0, \infty)$, respectively the opposite (minus) of the increments of W_t in the other case ($L_t^0(x + W)$ denotes here the local time at 0 of $x + W_t$).

Since $x + W_t \in [0, \infty)$ is the same as $\left|x + W_t\right| = x + W_t$, from the definition of the function G it follows that the above can be written in the form

$$\left|x + W_t\right| = x + \int_0^t G\left(\left|x + W_s\right| - \left(x + W_s\right)\right) dW_s + L_t^{x+W},$$

which shows that a strong solution to (3.7) – (3.9) above (in the case $a = \infty$) is given explicitly by $X_t = x + W_t$, $Y_t = \left|x + W_t\right|$ and $Z_t = \int_0^t \operatorname{sgn}\left(x + W_s\right) dW_s$.

We have the following:

Proposition 3.6. *Given a 1-dimensional Brownian motion $(W_t)_{t\geq 0}$ starting at $W_0 = 0$, a strong solution on $[0,\tau_1]$ of the system (3.7) – (3.9) is given by*

$$\begin{cases} X_t = x + W_t \\ Y_t = \left|a - \left|x + W_t - a\right|\right| \\ Z_t = \int_0^t \operatorname{sgn}\left(W_s\right)\operatorname{sgn}\left(a - W_s\right) dW_s \end{cases},$$

where $\tau_1 = \inf\left\{s > 0 : \left|X_s - Y_s\right| > \varepsilon\right\}$ and

$$\operatorname{sgn}(x) = \begin{cases} +1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

Proof. Since $\varepsilon < a$, it follows that for $t \leq \tau_1$ we have $X_t = x + W_t \in (-a, 2a)$, and therefore

$$Y_t = \left|a - \left|x + W_t - a\right|\right| = \begin{cases} -\left(x + W_t\right), & x + W_t \in (-a, 0) \\ x + W_t, & x + W_t \in [0, a] \\ 2a - x - W_t, & x + W_t \in (a, 2a) \end{cases}. \tag{3.11}$$

Applying the Tanaka-Itô formula to the function $f(z) = |a - |z - a||$ and to the Brownian motion $X_t = x + W_t$, for $t \leq \tau_1$ we obtain

$$\begin{aligned} Y_t &= x + \int_0^t \operatorname{sgn}\left(x + W_s\right)\operatorname{sgn}\left(a - x - W_s\right) d\left(x + W_s\right) + L_t^0 - L_t^a \\ &= x + \int_0^t \operatorname{sgn}\left(x + W_s\right)\operatorname{sgn}\left(a - x - W_s\right) dW_s + \int_0^t \nu_{D_2}\left(Y_s\right) d\left(L_s^0 + L_s^a\right), \end{aligned}$$

where $L_t^0 = \sup_{s\leq t}\left(x + W_s\right)^-$ and $L_t^a = \sup_{s\leq t}\left(x + W_s - a\right)^+$ are the local times of $x + W_t$ at 0, respectively at a, and $\nu_{D_2}(0) = +1$, $\nu_{D_2}(a) = -1$.
From (3.11) and the definition (3.10) of the function G we obtain

$$\begin{aligned} \operatorname{sgn}\left(x + W_s\right)\operatorname{sgn}\left(a - x - W_s\right) &= \begin{cases} -1, & x + W_s \in (-a, 0) \\ +1, & x + W_s \in [0, a] \\ -1, & x + W_s \in (a, 2a) \end{cases} \\ &= \begin{cases} +1, & X_s = Y_s \\ -1, & X_s \neq Y_s \end{cases} \\ &= G\left(Y_s - X_s\right), \end{aligned}$$

and therefore the previous formula can be written equivalently

$$Y_t = x + Z_t + \int_0^t \nu_{D_2}\left(Y_s\right) dL_s^Y,$$

where

$$Z_t = \int_0^t G\left(Y_s - X_s\right) dW_s$$

and $L_t^Y = L_t^0 + L_t^a$ is a continuous nondecreasing process which increases only when $x + W_t \in \{0, a\}$, that is only when $Y_t \in \partial D_2$. $\square$

3.2 The case of polygonal domains

In this section we will consider the case when $D_2 \subset D_1 \subset \mathbb{R}^d$ are polygonal domains (domains bounded by hyperplanes in $\mathbb{R}^d$). From Remark 3.4 it follows that we can consider $D_1 = \mathbb{R}^d$ and therefore it suffices to prove the existence of a strong solution of the following system

$$X_t = X_0 + W_t \tag{3.12}$$

$$Y_t = Y_0 + Z_t + \int_0^t \nu_{D_2}(Y_s)\, dL_s^Y \tag{3.13}$$

$$Z_t = \int_0^t G(Y_s - X_s)\, dW_s \tag{3.14}$$

or equivalently of the equation

$$Z_t = \int_0^t G\left(\widetilde{\Gamma}(Y_0 + Z)_s - X_0 - W_s\right) dW_s, \tag{3.15}$$

where W_t is a d-dimensional Brownian motion starting at $W_0 = 0$ and $X_0 = Y_0 \in \overline{D_2}$.

The construction relies on the following skew product representation of Brownian motion in spherical coordinates:

$$X_t = R_t \Theta_{\sigma_t}, \tag{3.16}$$

where $R_t = \|X_t\| \in \mathrm{BES}(d)$ is a Bessel process of order d and $\Theta_t \in \mathrm{BM}\left(S^{d-1}\right)$ is an independent Brownian motion on the unit sphere S^{d-1} in $\mathbb{R}^d$, run at speed

$$\sigma_t = \int_0^t \frac{1}{R_s^2} ds, \tag{3.17}$$

which depends only on R_t.

Remark 3.7. One way to construct the Brownian motion $\Theta_t = \Theta_t^{d-1}$ on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is to proceed inductively on $d \geq 2$, using the following skew product representation of Brownian motion on the sphere $\Theta_t^{d-1} \in S^{d-1}$ (see [15]):

$$\Theta_t^{d-1} = \left(\cos\theta_t^1, \sin\theta_t^1 \Theta_{\alpha_t}^{d-2}\right),$$

where $\theta^1 \in \mathrm{LEG}(d-1)$ is a Legendre process of order $d-1$ on $[0,\pi]$, and $\Theta_t^{d-2} \in S^{d-2}$ is an independent Brownian motion on S^{d-2}, run at speed

$$\alpha_t = \int_0^t \frac{1}{\sin^2\theta_s^1} ds.$$

Therefore, if $\theta_t^1, \ldots \theta_t^{d-1}$ are independent processes, with $\theta^i \in \mathrm{LEG}(d-i)$ on $[0,\pi]$ for $i = 1, \ldots, d-2$, and θ_t^{d-1} is a 1-dimensional Brownian (note that $\Theta_t^1 = \left(\cos\theta_t^1, \sin\theta_t^1\right) \in S^1$ is a Brownian motion on S^1), Brownian motion Θ_t^{d-1} on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is given by

$$\Theta_t^{d-1} = \left(\cos\theta_t^1, \sin\theta_t^1\cos\theta_t^2, \sin\theta_t^1\sin\theta_t^2\cos\theta_t^3, \ldots, \sin\theta_t^1 \cdot \ldots \cdot \sin\theta_t^{d-1}\sin\theta_t^{d-1}\right),$$

or by

$$\Theta_t^{d-1} = \left(\theta_t^1, \ldots, \theta_t^{d-2}, \theta_t^{d-1}\right) \tag{3.18}$$

in spherical coordinates.

To construct the solution of (3.12) – (3.14), we first consider the case when D_2 is a half-space $\mathscr{H}_d^+ = \left\{\left(z^1, \ldots, z^d\right) \in \mathbb{R}^d : z^d > 0\right\}$.

Given an angle $\varphi \in \mathbb{R}$, we introduce the rotation matrix $R(\varphi) \in \mathscr{M}_{d\times d}$ which leaves invariant the first $d-2$ coordinates and rotates clockwise by the angle α the remaining 2 coordinates, that is

$$R(\alpha) = \begin{pmatrix} 1 & & 0 & 0 & 0 \\ & \ddots & & \cdots & \cdots \\ 0 & & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cos\varphi & -\sin\varphi \\ 0 & \cdots & 0 & \sin\varphi & \cos\varphi \end{pmatrix}. \tag{3.19}$$

We have the following:

Lemma 3.8. *Let $D_2 = \mathscr{H}_d^+ = \left\{\left(z^1, \ldots, z^d\right) \in \mathbb{R}^d : z^d > 0\right\}$ and assume that*

$$Y_0 = R\left(\varphi_0\right) X_0 \tag{3.20}$$

for some $\varphi_0 \in \mathbb{R}$.

Consider the reflecting Brownian motion $\widetilde{\theta}_t^{d-1}$ on $[0, \pi]$ with driving Brownian motion θ_t^{d-1}, where θ_t^{d-1} is the $(d-1)$ spherical coordinate of $G\left(Y_0 - X_0\right) X_t$, given by (3.16) – (3.18) above, that is:

$$\widetilde{\theta}_t^{d-1} = \theta_t^{d-1} + L_t^0\left(\widetilde{\theta}^{d-1}\right) - L_t^\pi\left(\widetilde{\theta}^{d-1}\right), \qquad t \geq 0,$$

and $L_t^0\left(\widetilde{\theta}^{d-1}\right)$, $L_t^\pi\left(\widetilde{\theta}^{d-1}\right)$ represent the local times of $\widetilde{\theta}^{d-1}$ at 0, respectively at π.

A strong solution of the system (3.12) – (3.14) is explicitly given by

$$Y_t = \begin{cases} R\left(\varphi_t\right) G\left(Y_0 - X_0\right) X_t, & t < \xi \\ \left|X_t\right|_d, & t \geq \xi \end{cases} \tag{3.21}$$

where $\xi = \inf\left\{t > 0 : X_t = Y_t\right\}$ is the coupling time, the rotation angle φ_t is given by

$$\varphi_t = L_t^0\left(\widetilde{\theta}^{d-1}\right) - L_t^\pi\left(\widetilde{\theta}^{d-1}\right), \qquad t \geq 0,$$

and for $z = \left(z^1, z^2 \ldots, z^d\right) \in \mathbb{R}^d$ we denoted by $|z|_d = \left(z^1, z^2, \ldots, \left|z^d\right|\right)$.

Proof. Recall that for $m \in \mathbb{R}^d - \{0\}$, $G(m)v$ denotes the mirror image of $v \in \mathbb{R}^d$ with respect to the hyperplane through the origin perpendicular to m.

By Itô formula, we have

$$Y_{t\wedge\xi} = Y_0 + \int_0^{t\wedge\xi} R\left(\varphi_s\right) G\left(Y_0 - X_0\right) dX_s + \int_0^{t\wedge\xi} R\left(\varphi_s + \frac{\pi}{2}\right) G\left(Y_0 - X_0\right) dL_s. \tag{3.22}$$

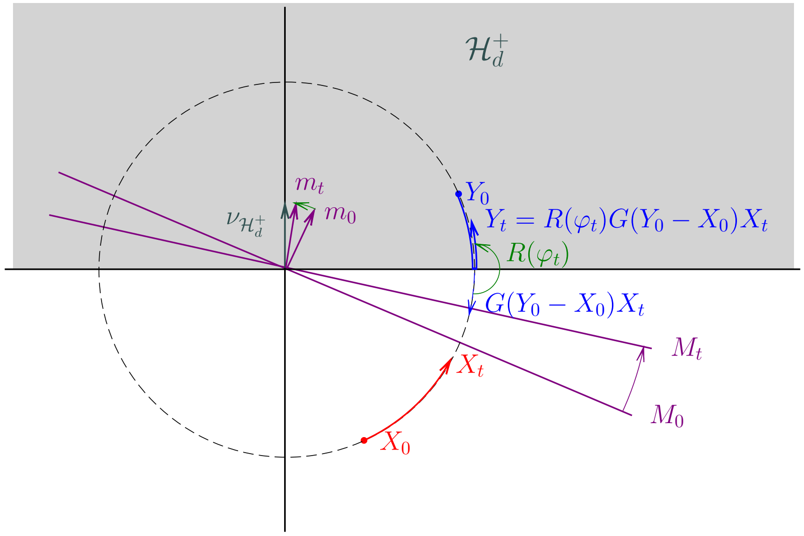

Figure 1: The mirror coupling of a free Brownian motion X_t and a reflecting Brownian motion Y_t in the half-space $\mathscr{H}_d^+$.

Note that the composition $R \circ G$ (a symmetry followed by a rotation) is a symmetry, and since $\|Y_t\| = \|X_t\|$ for all $t \geq 0$, it follows that X_t and Y_t are symmetric with respect to a hyperplane passing through the origin for all $t \leq \xi$. Therefore, from the definition (3.5) of the function G it follows that we have $Y_t = G\left(Y_t - X_t\right) X_t$ for all $t \leq \xi$.

Also note that when $L_s^0\left(\widetilde{\theta}^{d-1}\right)$ increases, $Y_s \in \partial D_2$ and we have

$$R\left(\varphi_s + \frac{\pi}{2}\right) G\left(Y_0 - X_0\right) X_s = R\left(\frac{\pi}{2}\right) Y_s = \nu_{D_2}\left(Y_s\right),$$

and if $L_s^\pi\left(\widetilde{\theta}^{d-1}\right)$ increases, $Y_s \in \partial D_2$ and we have

$$R\left(\varphi_s + \frac{\pi}{2}\right) G\left(Y_0 - X_0\right) X_s = R\left(\frac{\pi}{2}\right) Y_s = -\nu_{D_2}\left(Y_s\right).$$

It follows that the relation (3.22) can be written in the equivalent form

$$Y_{t\wedge\xi} = Y_0 + \int_0^{t\wedge\xi} G\left(Y_s - X_s\right) dX_s + \int_0^{t\wedge\xi} \nu_{D_2}\left(Y_s\right) dL_s^Y,$$

where $L_t^Y = L_t^0\left(\widetilde{\theta}^{d-1}\right) + L_t^\pi\left(\widetilde{\theta}^{d-1}\right)$ is a continuous non-decreasing process which increases only when $Y_t \in \partial D_2$, and therefore Y_t given by (3.21) is a strong solution of the system (3.12) – (3.14) for $t \leq \xi$.

For $t \geq \xi$, we have $Y_t = \left|X_t\right|_d = \left(X_t^1, X_t^2, \ldots, \left|X_t^d\right|\right)$, and proceeding similarly to the 1-dimensional case, by Tanaka formula we obtain:

$$\begin{aligned} Y_{t\vee\xi} &= Y_\xi + \int_\xi^{t\vee\xi} \left(1,\ldots,1,\operatorname{sgn}\left(X_s^d\right)\right) dX_s + \int_\xi^{t\vee\xi} (0,\ldots,0,1)\, L_t^0\left(X^d\right) \\ &= Y_\xi + \int_\xi^{t\vee\xi} G\left(Y_s - X_s\right) dX_s + \int_\xi^{t\vee\xi} \nu_{D_2}\left(Y_s\right) L_t^Y, \end{aligned} \tag{3.23}$$

since in this case

$$\begin{aligned} G\left(Y_s - X_s\right) &= \begin{cases} (1,\ldots,1,+1), & X_s = Y_s \\ (1,\ldots,1,-1), & X_s \neq Y_s \end{cases} \\ &= \begin{cases} (1,\ldots,1,+1), & X_s^d \geq 0 \\ (1,\ldots,1,-1), & X_s^d < 0 \end{cases} \\ &= \left(1,\ldots,1,\operatorname{sgn}\left(X_s^d\right)\right). \end{aligned}$$

The process $L_t^Y = L_t^0\left(X^d\right)$ in (3.23) is a continuous non-decreasing process which increases only when $Y_t \in \partial D_2$ ($L_t^0\left(X^d\right)$ represents the local time at 0 of the last cartesian coordinate X^d of X), which shows that Y_t also solves (3.12) – (3.14) for $t \geq \xi$, and therefore Y_t is a strong solution of (3.12) – (3.14) for $t \geq 0$, concluding the proof. □

Consider now the case of a general polygonal domain $D_2 \subset \mathbb{R}^d$. We will show that a strong solution of the system (3.12) – (3.14) can be constructed from the previous lemma by choosing the appropriate coordinate system.

Consider the times $\left(\sigma_n\right)_{n\geq 0}$ at which the solution Y_t hits different bounding hyperplanes of ∂D_2, that is $\sigma_0 = \inf\left\{s \geq 0 : Y_s \in \partial D_2\right\}$ and inductively

$$\sigma_{n+1} = \inf\left\{ t \geq \sigma_n : \begin{array}{l} Y_t \in \partial\mathscr{D}_2 \text{ and } Y_t, Y_{\sigma_n} \text{ belong to different}^1 \\ \text{bounding hyperplanes of } \partial D_2 \end{array} \right\},\ n \geq 0. \tag{3.24}$$

If $X_0 = Y_0 \in \partial D_2$ belong to a certain bounding hyperplane of D_2, we can chose the coordinate system so that this hyperplane is $\mathscr{H}_d = \left\{\left(z^1,\ldots,z^d\right) \in \mathbb{R}^d : z^d = 0\right\}$ and $D_2 \subset \mathscr{H}_d^+$, and we let $\mathscr{H}_d$ be any bounding hyperplane of D_2 otherwise.

By Lemma 3.8 it follows that on the time interval $[\sigma_0, \sigma_1)$, the strong solution of (3.12) – (3.14) is given explicitly by (3.21).

If $\sigma_1 < \infty$, we distinguish two cases: $X_{\sigma_1} = Y_{\sigma_1}$ and $X_{\sigma_1} \neq Y_{\sigma_1}$. Let $\mathscr{H}$ denote the bounding hyperplane of D which contains Y_{σ_1}, and let $\nu_{\mathscr{H}}$ denote the unit normal to $\mathscr{H}$ pointing inside D_2.

If $X_{\sigma_1} = Y_{\sigma_1} \in \mathscr{H}$, choosing again the coordinate system conveniently, we may assume that $\mathscr{H}$ is the hyperplane is $\mathscr{H}_d = \left\{\left(z^1,\ldots,z^d\right) \in \mathbb{R}^d : z^d = 0\right\}$, and on the time interval $[\sigma_1, \sigma_2)$ the coupling $\left(X_{\sigma_1+t}, Y_{\sigma_1+t}\right)_{t\in[0,\sigma_2-\sigma_1)}$ is given again by Lemma 3.8.

If $X_{\sigma_1} \neq Y_{\sigma_1} \in \mathscr{H}$, in order to apply Lemma 3.8 we have to show that we can choose the coordinate system so that the condition (3.20) holds. If $Y_{\sigma_1} - X_{\sigma_1}$ is a vector perpendicular to $\mathscr{H}$, by choosing

[1] Since 2-dimensional Brownian motion does not hit points a.s., the d-dimensional Brownian motion Y_t does not hit the edges of D_2 ($(d-2)$-dimensional hyperplanes in $\mathbb{R}^d$) a.s., thus there is no ambiguity in the definition.

the coordinate system so that $\mathscr{H} = \mathscr{H}_d = \left\{ \left(z^1, \dots, z^d \right) \in \mathbb{R}^d : z^d = 0 \right\}$, the problem reduces to the 1-dimensional case (the first $d-1$ coordinates of X and Y are the same), and it can be handled as in Proposition 3.6 by the Tanaka formula. The proof being similar, we omit it.

If $X_{\sigma_1} \neq Y_{\sigma_1} \in \mathscr{H}$ and $Y_{\sigma_1} - X_{\sigma_1}$ is not orthogonal to $\mathscr{H}$, consider $\widetilde{X}_{\sigma_1} = \mathrm{pr}_{\mathscr{H}} X_{\sigma_1}$ the projection of X_{σ_1} onto $\mathscr{H}$, and therefore $\widetilde{X}_{\sigma_1} \neq Y_{\sigma_1}$. The plane of symmetry of X_{σ_1} and Y_{σ_1} intersects the line determined by $\widetilde{X}_{\sigma_1}$ and Y_{σ_1} at a point, and we consider this point as the origin of the coordinate system (note that the intersection cannot be empty, for otherwise the vectors $Y_{\sigma_1} - X_{\sigma_1}$ and $Y_{\sigma_1} - \widetilde{X}_{\sigma_1}$ were parallel, which is impossible since then $Y_{\sigma_1} - X_{\sigma_1}, Y_{\sigma_1} - \widetilde{X}_{\sigma_1}$ and $Y_{\sigma_1} - \widetilde{X}_{\sigma_1}, X_{\sigma_1} - \widetilde{X}_{\sigma_1}$ were perpendicular pairs of vectors, contradicting $\widetilde{X}_{\sigma_1} \neq Y_{\sigma_1}$ – see Figure 2).

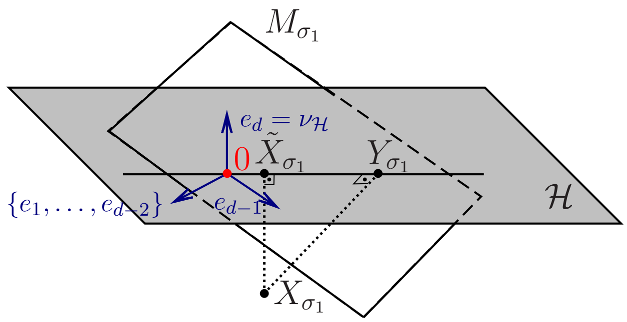

Figure 2: Construction of the appropriate coordinate system.

Choose an orthonormal basis $\{e_1, \dots, e_d\}$ in $\mathbb{R}^d$ such that $e_d = \nu_{\mathscr{H}}$ is the normal vector to $\mathscr{H}$ pointing inside D_2, $e_{d-1} = \frac{1}{\|Y_{\sigma_1} - X_{\sigma_1}\|} \left(Y_{\sigma_1} - X_{\sigma_1} \right)$ is a unit vector lying in the 2-dimensional plane determined by the origin and the vectors e_d and $Y_{\sigma_1} - X_{\sigma_1}$, and $\{e_1, \dots, e_{d-2}\}$ is a completion of $\{e_{d-1}, e_d\}$ to an orthonormal basis in $\mathbb{R}^d$ (see Figure 2).

Note that by the construction, the vectors $e_1, \dots, e_{d-2}$ are orthogonal to the 2-dimensional hyperplane containing the origin and the points X_{σ_1} and Y_{σ_1}, and therefore X_{σ_1} and Y_{σ_1} have the same (zero) first $d-2$ coordinates; also, since X_{σ_1} and Y_{σ_1} are at the same distance from the origin, it follows that Y_{σ_1} can be obtained from X_{σ_1} by a rotation which leaves invariant the first $d-2$ coordinates, which shows that the condition (3.20) of Lemma 3.8 is satisfied.

Since by construction the bounding hyperplane $\mathscr{H}$ of D_2 at Y_{σ_1} is given by $\mathscr{H}_d = \left\{ \left(z^1, \dots, z^d \right) \in \mathbb{R}^d : z^d = 0 \right\}$ and $D_2 \subset \mathscr{H}_d^+ = \left\{ \left(z^1, \dots, z^d \right) \in \mathbb{R}^d : z^d > 0 \right\}$, we can apply Lemma 3.8 to deduce that on the time interval $[\sigma_1, \sigma_2)$ a solution of (3.12) – (3.14) is given by $\left(X_{\sigma_1 + t}, Y_{\sigma_1 + t} \right)_{t \in [0, \sigma_2 - \sigma_1)}$.

Repeating the above argument we can construct inductively (in the appropriate coordinate systems) the solution of (3.12) – (3.14) on any time interval $[\sigma_n, \sigma_{n+1})$, $n \geq 1$, and therefore we obtain a strong solution of (3.12) – (3.14) defined for $t \geq 0$.

We summarize the above discussion in the following:

Theorem 3.9. *If $D_2 \subset \mathbb{R}^d$ is a polygonal domain, for any $X_0 = Y_0 \in \overline{D_2}$, there exists a strong solution of the system (3.12) - (3.14).*

Moreover, between successive hits of different bounding hyperplanes of D_2 (i.e. on each time interval $[\sigma_n, \sigma_{n+1})$ in the notation above), the solution is given by Lemma 3.8 in the appropriately chosen coordinate system.

We will refer to the solution $\left(X_t, Y_t\right)_{t\geq 0}$ constructed in the previous theorem as a *mirror coupling* of reflecting Brownian motions in $\left(\mathbb{R}^d, D_2\right)$ with starting point $X_0 = Y_0 \in \overline{D_2}$.

If $X_t \neq Y_t$, the hyperplane M_t of symmetry between X_t and Y_t (the hyperplane passing through $\frac{X_t+Y_t}{2}$ with normal $m_t = \frac{1}{\|Y_t - X_t\|}\left(Y_t - X_t\right))$ will be referred to as the *mirror of the coupling*. For definiteness, when $X_t = Y_t$ we let M_t denote any hyperplane passing through $X_t = Y_t$, for example we can choose M_t such that it is a left continuous function with respect to t.

In the particular case of a convex polygonal domain D_2, some of the properties of the mirror coupling are contained in the following:

Proposition 3.10. *If $D_2 \subset \mathbb{R}^d$ is a convex polygonal domain, for any $X_0 = Y_0 \in \overline{D_2}$, the mirror coupling given by the previous theorem has the following properties:*

i) If the reflection takes place in the bounding hyperplane $\mathscr{H}$ of D_2 with inward unitary normal $\nu_{\mathscr{H}}$, then the angle $\angle(m_t; \nu_{\mathscr{H}})$ decreases monotonically to zero.

ii) When processes are not coupled, the mirror M_t lies outside D_2.

iii) Coupling can take place precisely when $X_t \in \partial D_2$. Moreover, if $X_t \in D_2$, then $X_t = Y_t$.

iv) If $D_\alpha \subset D_\beta$ are two polygonal domains and $(Y_t^\alpha; X_t)$, $(Y_t^\beta; X_t)$ are the corresponding mirror coupling starting from $x \in \overline{D_\alpha}$, for any $t > 0$ we have

$$\sup_{s\leq t} \|Y_s^\alpha - Y_s^\beta\| \leq \mathrm{Dist}\left(D^\alpha, D^\beta\right) := \max_{\substack{x_\alpha \in \partial D_\alpha, x_\beta \in \partial D_\beta \\ \left(x_\beta - x_\alpha\right)\cdot \nu_{D_\alpha}\left(x_\alpha\right) \leq 0}} \|x_\alpha - x_\beta\|. \tag{3.25}$$

Proof. i) In the notation of Theorem 3.9, on the time interval $[\sigma_0, \sigma_1)$ we have $Y_t = X_t$, so $\angle\left(m_t, \nu_{\mathscr{H}}\right) = 0$ and therefore the claim is verified in this case.

On an arbitrary time interval $[\sigma_n, \sigma_{n+1})$, in the appropriately chosen coordinate system, Y_t is given by Lemma 3.8. For $t < \xi$, Y_t is given by the rotation $R\left(\varphi_t\right)$ of $G\left(Y_0 - X_0\right) X_t$ which leaves invariant the first $(d-2)$ coordinates, and therefore

$$\angle\left(m_t, \nu_{\mathscr{H}}\right) = \angle\left(m_0, \nu_{\mathscr{H}}\right) + \frac{L_t^0 - L_t^\pi}{2},$$

which proves the claim in this case (note that before the coupling time ξ only one of the non-decreasing processes L_t^0 and L_t^π is not identically zero).

Since for $t \geq \xi$ we have $Y_t = \left(X_t^1, \ldots, \left|X_t^d\right|\right)$, we have $\angle\left(m_t, \nu_{\mathscr{H}}\right) = 0$ which concludes the proof of the claim.

ii) On the time interval $[\sigma_0, \sigma_1)$ the processes are coupled, so there is nothing to prove in this case.

On the time interval $[\sigma_1, \sigma_2)$, in the appropriately chosen coordinate system we have $Y_t = \left(X_t^1, \ldots, \left|X_t^d\right|\right)$, thus the mirror M_t coincides with the boundary hyperplane $\mathscr{H}_d =$

$\left\{\left(z^1,\dots,z^d\right)\in\mathbb{R}^d : z^d=0\right\}$ of D_2 where the reflection takes place, and therefore $M_t\cap D_2=\varnothing$ in this case.

Inductively, assume the claim is true for $t<\sigma_n$. By continuity, $M_{\sigma_n}\cap D_2=\varnothing$, thus D_2 lies on one side of M_{σ_n}. By the previous proof, the angle $\angle\left(m_t,\nu_{\mathscr{H}}\right)$ between m_t and the inward unit normal $\nu_{\mathscr{H}}$ to bounding hyperplane $\mathscr{H}$ of D_2 where the reflection takes place decreases to zero. Since D_2 is a convex domain, it follows that on the time interval $[\sigma_n,\sigma_{n+1})$ we have $M_t\cap D_2=\varnothing$, concluding the proof.

iii) The first part of the claim follows from the previous proof (when the processes are not coupled, the mirror (hence X_t) lies outside D_2; by continuity, it follows that at the coupling time ξ we must have $X_\xi=Y_\xi\in\partial D_2$).

To prove the second part of the claim, consider an arbitrary time interval $[\sigma_n,\sigma_{n+1})$ between two successive hits of Y_t to different bounding hyperplanes of D_2. In the appropriately chosen coordinate system, Y_t is given by Lemma 3.8. After the coupling time ξ, Y_t is given by $Y_t=\left(X_t^1,\dots,\left|X_t^d\right|\right)$, and therefore if $X_t\in D_2$ (thus $X_t^d\geq 0$) we have $Y_t=\left(X_t^1,\dots,X_t^d\right)=X_t$, concluding the proof.

iv) Let M_t^α and M_t^β denote the mirrors of the coupling in D^α, respectively D^β, with the same driving Brownian motion X_t.

Since Y_t^α and X_t are symmetric with respect to M_t^α, and Y_t^β and X_t are symmetric with respect to M_t^β, it follows that Y_t^β is obtained from Y_t^β by a rotation which leaves invariant the hyperplane $M_t^\alpha\cap M_t^\beta$, or by a translation by a vector orthogonal to M_t^α (in the case when M_t^α and M_t^β are parallel).

The angle of rotation (respectively the vector of translation) is altered only when either Y_t^α or Y_t^β are on the boundary of D_α, respectively D_β. Since $D_\alpha\subset D_\beta$ are convex domains, the angle of rotation (respectively the vector of translation) decreases when $Y_t^\beta\in D_\beta$ or when $Y_t^\alpha\in\partial D_\alpha$ and $\left(Y_t^\beta-Y_t^\alpha\right)\cdot\nu_{D_\alpha}\left(Y_t^\alpha\right)>0$ (in these cases Y_t^β and Y_t^α receive a push such that the distance $\|Y_t^\alpha-Y_t^\beta\|$ is decreased), thus the maximum distance $\|Y_t^\alpha-Y_t^\beta\|$ is attained when $Y_t^\alpha\in\partial D_\alpha$ and $\left(Y_t^\beta-Y_t^\alpha\right)\cdot\nu_{D_\alpha}\left(Y_t^\alpha\right)\leq 0$, and the formula follows. □

4 The proof of Theorem 3.1

By Remark 3.4, it suffices to consider the case when $D_1=\mathbb{R}^d$ and $D_2\subset\mathbb{R}^d$ is a convex bounded domain with smooth boundary. To simplify the notation, we will drop the index and write D for D_2 in the sequel.

Let $\left(D_n\right)_{n\geq 1}$ be an increasing sequence of convex polygonal domains in $\mathbb{R}^d$ with $\overline{D_n}\subset D_{n+1}$ and $\cup_{n\geq 1}D_n=D$.

Consider $\left(Y_t^n,X_t\right)_{t\geq 0}$ a sequence of mirror couplings in $\left(D_n,\mathbb{R}^d\right)$ with starting point $x\in D_1$ and driving Brownian motion $\left(W_t\right)_{t\geq 0}$ with $W_0=0$, given by Theorem 3.9.

By Proposition 3.10, for any $t>0$ we have

$$\sup_{s\leq t}\left|Y_s^m-Y_s^n\right|\leq \operatorname{Dist}\left(D_n,D_m\right)=\max_{\substack{x_n\in\partial D_n,x_m\in\partial D_m\\ \left(x_m-x_n\right)\cdot\nu_{D_n}\left(x_n\right)\leq 0}}\left|x_n-x_m\right|\underset{n,m\to\infty}{\longrightarrow}0,$$

hence Y_t^n converges a.s. in the uniform topology to a continuous process Y_t.

Since $(Y^n)_{n\geq 1}$ are reflecting Brownian motions in $(D_n)_{n\geq 1}$ and $D_n \nearrow D$, the law of Y_t is that of a reflecting Brownian motion in D, that is Y_t is a reflecting Brownian motion in D starting at $x \in D$ (see [8]). Also note that since Y_t^n are adapted to the filtration $\mathscr{F}^W = (\mathscr{F}_t)_{t\geq 0}$ generated by the Brownian motion W_t, so is Y_t.

By construction, the driving Brownian motion Z_t^n of Y_t^n satisfies

$$Z_t^n = \int_0^t G\left(Y_t^n - X_t\right) dW_t, \qquad t \geq 0.$$

Consider the process

$$Z_t = \int_0^t G\left(Y_s - X_s\right) dW_s,$$

and note that since Y is $\mathscr{F}^W$-adapted and $||G|| = 1$, by Lévy's characterization of Brownian motion, Z_t is a free d-dimensional Brownian motion starting at $Z_0 = 0$, also adapted to the filtration $\mathscr{F}^W$.

We will show that Z is the driving process of the reflecting Brownian motion Y_t, that is, we will show that

$$Y_t = x + Z_t + L_t^Y = x + \int_0^t G\left(Y_s - X_s\right) dW_s + L_t^Y, \qquad t \geq 0.$$

Note that the mapping $z \longmapsto G(z)$ is continuous with respect to the norm $||A|| = \left|\left|\left(a_{ij}\right)\right|\right| = \sum_{i,j=1}^d a_{ij}^2$ of $d \times d$ matrices at all points $z \in \mathbb{R}^d - \{0\}$, hence $G\left(Y_s^n - X_s\right) \underset{n\to\infty}{\rightarrow} G\left(Y_s - X_s\right)$ if $Y_s - X_s \neq 0$. If $Y_s - X_s = 0$, then either $Y_s = X_s \in D$ or $Y_s = X_s \in \partial D$.

If $Y_s = X_s \in D$, since $D_n \nearrow D$, there exists $N \geq 1$ such that $X_s \in D_N$, hence $X_s \in D_n$ for all $n \geq N$. By Proposition 3.10, it follows that $Y_s^n = X_s$ for all $n \geq N$, hence in this case we also have $G\left(Y_s^n - X_s\right) = G(0) \underset{n\to\infty}{\rightarrow} G(0) = G\left(Y_s - X_s\right)$.

If $Y_s = X_s \in \partial D$, since $\overline{D_n} \subset D$ we have $Y_s^n - X_s \neq 0$, and therefore by the definition (3.5) of G we have:

$$\begin{aligned}
&\int_0^t \left\|G\left(Y_s^n - X_s\right) - G\left(Y_s - X_s\right)\right\|^2 1_{Y_s = X_s \in \partial D} ds \\
= \;& \int_0^t \left\|H\left(\frac{Y_s^n - X_s}{\left\|Y_s^n - X_s\right\|}\right) - I\right\|^2 1_{Y_s = X_s \in \partial D} ds \\
= \;& \int_0^t \left\|I - 2\frac{Y_s^n - X_s}{\left\|Y_s^n - X_s\right\|}\left(\frac{Y_s^n - X_s}{\left\|Y_s^n - X_s\right\|}\right)' - I\right\|^2 1_{Y_s = X_s \in \partial D} ds \\
= \;& \int_0^t \left\|2\frac{Y_s^n - X_s}{\left\|Y_s^n - X_s\right\|}\left(\frac{Y_s^n - X_s}{\left\|Y_s^n - X_s\right\|}\right)'\right\|^2 1_{Y_s = X_s \in \partial D} ds \\
= \;& 4\int_0^t 1_{Y_s = X_s \in \partial D} ds \\
\leq \;& 4\int_0^t 1_{\partial D}\left(Y_s\right) ds \\
= \;& 0,
\end{aligned}$$

since Y_t is a reflecting Brownian motion in D, and therefore it spends zero Lebesgue time on the boundary of D.

Since $||G|| = 1$, using the above and the bounded convergence theorem we obtain

$$\lim_{n\to\infty} \int_0^t \left\|G\left(Y_s^n - X_s\right) - G\left(Y_s - X_s\right)\right\|^2 ds = 0,$$

and therefore by Doob's inequality it follows that

$$E\sup_{s\leq t} \|Z_s^n - Z_s\|^2 \leq cE\|Z_t^n - Z_t\|^2 \leq cE\int_0^t \|G\left(Y_s^n - X_s\right) - G\left(Y_s - X_s\right)\|^2 ds \underset{n\to\infty}{\to} 0,$$

for any $t \geq 0$, which shows that Z_t^n converges uniformly on compact sets to $Z_t = \int_0^t G\left(Y_s - X_s\right) dW_s$.

By construction, Z_t^n is the driving Brownian motion for Y_t^n, that is

$$Y_t^n = x + Z_t^n + \int_0^t \nu_{D_n}\left(Y_s^n\right) dL_s^{Y_n},$$

and passing to the limit with $n \to \infty$ we obtain

$$Y_t = x + Z_t + A_t = x + \int_0^t G\left(Y_s - X_s\right) dW_s + A_t, \qquad t \geq 0,$$

where $A_t = \lim_{n\to\infty} \int_0^t \nu_{D_n}\left(Y_s^n\right) dL_s^{Y_n}$.

It remains to show that A_t is a process of bounded variation. For an arbitrary partition $0 = t_0 < t_1 < \ldots t_l = t$ of $[0, t]$ we have

$$\begin{aligned}
E\sum_{i=1}^{l}\|A_{t_i} - A_{t_{i-1}}\| &= \lim_{n\to\infty} E\sum_{i=1}^{l}\left\|\int_{t_{i-1}}^{t_i} \nu_{D_n}\left(Y_s^n\right) dL_s^{Y_n}\right\| \\
&\leq \limsup E\, L_t^{Y_n} \\
&= \limsup \int_0^t \int_{\partial D_n} p_{D_n}\left(s, x, y\right)\sigma_n\left(dy\right) ds \\
&\leq c\sqrt{t},
\end{aligned}$$

where σ_n is the surface measure on ∂D_n, and the last inequality above follows from the estimates in [5] on the Neumann heat kernels $p_{D_n}\left(t, x, y\right)$ (see the remarks preceding Theorem 2.1 and the proof of Theorem 2.4 in [7]).

From the above it follows that $A_t = Y_t - x - Z_t$ is a continuous, $\mathscr{F}^W$-adapted process (since Y_t, Z_t are continuous, $\mathscr{F}^W$-adapted processes) of bounded variation.

By the uniqueness in the Doob-Meyer semimartingale decomposition of the reflecting Brownian motion Y_t in D, it follows that

$$A_t = \int_0^t \nu_D\left(Y_s\right) dL_s^Y, \qquad t \geq 0,$$

where L^Y is the local time of Y on the boundary ∂D. It follows that the reflecting Brownian motion Y_t in D constructed above is a strong solution to

$$Y_t = x + \int_0^t G\left(Y_s - X_s\right) dW_s + \int_0^t \nu_D\left(Y_s\right) dL_s^Y, \qquad t \geq 0,$$

or equivalent, the driving Brownian motion $Z_t = \int_0^t G\left(Y_s - X_s\right) dW_s$ of Y_t is a strong solution to

$$Z_t = \int_0^t G\left(\widetilde{\Gamma}\left(y + Z\right)_s - X_s\right) dW_s, \qquad t \geq 0,$$

concluding the proof of Theorem 3.1.

5 Extensions and applications

As an application of the construction of mirror coupling, we will present a unifying proof of the two most important results on Chavel's conjecture.

It is not difficult to prove that the Dirichlet heat kernel is an increasing function with respect to the domain. Since for the Neumann heat kernel $p_D\left(t, x, y\right)$ of a smooth bounded domain $D \subset \mathbb{R}^d$ we have

$$\lim_{t\to\infty} p_D\left(t, x, y\right) = \frac{1}{\mathrm{vol}\left(D\right)},$$

the monotonicity in the case of the Neumann heat kernel should be reversed.

The above observation was conjectured by Isaac Chavel ([12]), as follows:

Conjecture 5.1 (Chavel's conjecture, [12]). *Let $D_{1,2} \subset \mathbb{R}^d$ be smooth bounded convex domains in $\mathbb{R}^d$, $d \geq 1$, and let $p_{D_1}(t,x,y)$, $p_{D_2}(t,x,y)$ denote the Neumann heat kernels in D_1, respectively D_2. If $D_2 \subset D_1$, then*

$$p_{D_1}(t,x,y) \leq p_{D_2}(t,x,y), \tag{5.1}$$

for any $t \geq 0$ and $x, y \in D_1$.

Remark 5.2. The smoothness assumption in the above conjecture is meant to insure the a.e. existence of the inward unit normal to the boundaries of D_1 and D_2, that is the boundaries should have a locally differentiable parametrization. Requiring that the boundary of the domain is of class $C^{1,\alpha}$ $(0 < \alpha < 1)$ is a convenient hypothesis on the smoothness of the domains $D_{1,2}$.

In order to simplify the proof, we will assume that $D_{1,2}$ are smooth C^2 domains (the proof can be extended to a more general setup, by approximating $D_{1,2}$ by less smooth domains).

Among the positive results on Chavel conjecture, the most general known results (and perhaps the easiest to use in practice) are due to I. Chavel ([12]) and W. Kendall ([16]), and they show that if there exists a ball B centered at either x or y such that $D_2 \subset B \subset D_1$, then the inequality (5.1) in Chavel's conjecture holds for any $t > 0$.

While there are also other positive results which suggest that Chavel's conjecture is true for certain classes of domains (see for example [11], [14]), in [4] R. Bass and K. Burdzy showed that Chavel's conjecture does not hold in its full generality (i.e. without additional hypotheses).

We note that both the proof of Chavel (the case when D_1 is a ball centered at either x or y) and Kendall (the case when D_2 is a ball centered at either x or y) relies in an essential way that one of the domains is a ball: the first uses an integration by parts technique, while the later uses a coupling argument of the radial parts of Brownian motion, and none of these proofs seem to be easily applicable to the other case.

Using the mirror coupling, we can derive a simple, unifying proof of these two important results, as follows:

Theorem 5.3. *Let $D_2 \subset D_1 \subset \mathbb{R}^d$ be smooth bounded domains and assume that D_2 is convex. If for $x, y \in D_2$ there exists a ball B centered at either x or y such that $D_2 \subset B \subset D_1$, then for all $t \geq 0$ we have*

$$p_{D_1}(t,x,y) \leq p_{D_2}(t,x,y). \tag{5.2}$$

Proof. Consider $x, y \in D_2$ arbitrarily fixed and assume that $D_2 \subset B = B(y,R) \subset D_1$ for some $R > 0$.

By eventually approximating the convex domain D_2 by convex polygonal domains, it suffices to prove the claim in the case when D_2 is a convex polygonal domain.

Let (X_t, Y_t) be a mirror coupling of reflecting Brownian motions in (D_1, D_2) starting at $x \in D_2$. The idea of the proof is to show that for all times $t \geq 0$, Y_t is at a distance from y is no greater than the distance from X_t to y.

Let $t_0 \geq 0$ be a time when the processes are at the same distance from y, and let $t_1 \geq t_0$ be the first time after t_0 when the process X_t hits the boundary of D_1.

Note that by the ball condition we have $\|X_t - y\| = R > \|Y_t - y\|$ for any $t \geq 0$, and in particular this holds for $t = t_1$. Since the processes X_t and Y_t are continuous, the distances from X_t and Y_t to y are continuous functions of t, and therefore in order to prove the claim it suffices to show that

$\|Y_t - y\| \le \|X_t - y\|$ for all $t \in [t_0, t_1]$. Also note that on the time interval $[t_0, t_1]$ the process X_t behaves like a free Brownian motion.

We distinguish the following cases:

i) The processes are coupled at time t_0 (i.e. $X_{t_0} = Y_{t_0}$);

In this case, the distances from X_t and Y_t to y will remain equal until the first time when the processes hit the boundary of D_2. Since on the time interval $[t_0, t_1]$ the process X_t behaves like a free Brownian motion, by Proposition 3.10 ii) it follows that when processes are not coupled, the mirror M_t of the coupling lies outside the domain D_2. Since the domain D_2 is assumed convex, this shows in particular that the mirror M_t of the coupling cannot separate the points Y_t and y, and therefore the distance from Y_t to y is smaller than or equal to the distance from X_t to y, for all $t \in [t_0, t_1]$.

ii) The processes are decoupled at time t_0;

In this case, since $\left|Y_{t_0} - y\right| = \left|X_{t_0} - y\right|$ and $X_{t_0} \ne Y_{t_0}$, the hyperplane M_{t_0} of symmetry between X_{t_0} and Y_{t_0} passes through the point y, so M_{t_0} does not separate the points Y_{t_0} and y.

The processes X_t and Y_t will remain at the same distance from y until the first time when $Y_t \in \partial D_2$. Since on the time interval $[t_0, t_1]$ the process X_t behaves like a free Brownian motion, by Theorem 3.9, it follows that between successive hits of different boundary hyperplanes of D_2, the mirror M_t of the coupling describes a rotation which leaves invariant $d-2$ coordinate axes. Moreover, by Proposition 3.10 the rotation is directed in such a way that the angle $\angle(m_t, \nu_{\mathscr{H}})$ between the normal $m_t = \frac{1}{\|Y_t - X_t\|}\left(Y_t - X_t\right)$ of M_t and the inner normal $\nu_{\mathscr{H}}$ of the bounding hyperplane $\mathscr{H}$ of D_2 where the reflection takes place decreases monotonically to zero (see Figure 1).

Since the hyperplane M_{t_0} does not separate the points Y_{t_0} and y, simple geometric considerations show that M_t will not separate the points Y_t and y for all $t \in [t_0, t_1]$, and therefore $\|Y_t - y\| \le \|X_t - y\|$ for all $t \in [t_0, t_1]$, concluding the proof of the claim.

We showed that for any $t \ge 0$ we have $\|Y_t - y\| \le \|X_t - y\|$, and therefore

$$P^x\left(\|X_t - y\| < \varepsilon\right) \le P^x\left(\|Y_t - y\| < \varepsilon\right),$$

for any $\varepsilon > 0$ and $t \ge 0$.

Dividing the above inequality by the volume of the ball $B\left(y, \varepsilon\right)$ and passing to the limit with $\varepsilon \searrow 0$, from the continuity of the transition density of the reflecting Brownian motion in the space variable we obtain

$$p_{D_1}\left(t, x, y\right) \le p_{D_2}\left(t, x, y\right), \qquad t \ge 0,$$

concluding the proof of the theorem. □

As also pointed out by Kendall in [16], we note that in the above theorem the convexity of the larger domain D_1 is not needed in order to derive the validity of condition (5.1) in Chavel's conjecture. We can also replace the hypothesis on the convexity of the smaller domain D_2 by the weaker hypothesis that D_2 is a star-shaped domain with respect to either x or y, as follows:

Theorem 5.4. *Let $D_2 \subset D_1 \subset \mathbb{R}^d$ be smooth bounded domains. If for $x, y \in D_2$ there exists a ball B centered at either x or y such that $D_2 \subset B \subset D_1$ and D_2 is star-shaped with respect to the center of the ball, then for all $t \ge 0$ we have*

$$p_{D_1}\left(t, x, y\right) \le p_{D_2}\left(t, x, y\right). \tag{5.3}$$

Proof. We will present an analytic proof which parallels the geometric proof of the previous theorem.

Consider $x, y \in D_2$ arbitrarily fixed and assume that $D_2 \subset B = B\left(y, R\right) \subset D_1$ for some $R > 0$ and D_2 is a star-shaped domain with respect to y.

By eventually approximating D_2 with star-shaped polygonal domains, it suffices to prove the claim in the case when D_2 is a polygonal star-shaped domain.

Let $\left(X_t, Y_t\right)$ be a mirror coupling of reflecting Brownian motions in $\left(D_1, D_2\right)$ starting at $x \in D_2$. The idea of the proof is to show that for all times $t \geq 0$, Y_t is at a distance from y is no greater than the distance from X_t to y.

We can reduce the proof to the case when $D_1 = \mathbb{R}^d$ as follows. Consider the sequences of stopping times $(\xi_n)_{n\geq 1}$ and $(\tau_n)_{n\geq 1}$ defined inductively by

$$\begin{aligned}
\tau_0 &= 0, \\
\xi_n &= \inf\left\{t > \tau_{n-1} : X_t \in \partial D_1\right\}, \qquad n \geq 1, \\
\tau_n &= \inf\left\{t > \xi_n : \|X_t - y\| = \|Y_t - y\|\right\}, \qquad n \geq 1.
\end{aligned}$$

Note that by the ball condition we have $\|X_{\xi_n} - y\| > \|Y_{\xi_n} - y\|$ for any $n \geq 1$, and therefore $\|X_t - y\| \geq \|Y_t - y\|$ for any $n \geq 1$ and any $t \in \left[\xi_n, \tau_n\right]$. In order to prove that the same inequality holds on the intervals $\left[\tau_n, \xi_{n+1}\right]$ for $n \geq 0$, we proceed as follows.

On the set $\{\tau_n < \infty\}$, the pair $(\widetilde{X}_t, \widetilde{Y}_t) = \left(X_{\tau_n+t}, Y_{\tau_n+t}\right)$ defined for $t \leq \xi_{n+1} - \tau_n$ is a mirror coupling in $(\mathbb{R}^d, D_2)$ with driving Brownian motion $\widetilde{W}_t = W_{\tau_n+t} - W_{\tau_n}$ (and $\widetilde{Z}_t = Z_{\tau_n+t} - Z_{\tau_n}$), and starting points $(\widetilde{X}_0, \widetilde{Y}_0) = (X_{\tau_n}, Y_{\tau_n})$ independent of the filtration of $\widetilde{B}_t$ (see Remark 3.4). In order to prove the claim it suffices therefore to show that for any points $u \in \mathbb{R}^d$ and $v \in \overline{D_2}$ with $\|u - y\| = \|v - y\|$, the mirror coupling (X_t, Y_t) in $(\mathbb{R}^d, D_2)$ with starting points $(X_0, Y_0) = (u, v)$ verifies

$$\|X_t - y\| \geq \|Y_t - y\|, \qquad t \geq 0. \tag{5.4}$$

Consider therefore a mirror coupling (X_t, Y_t) in $(\mathbb{R}^d, D_2)$ with starting points $(X_0, Y_0) = (u, v) \in \mathbb{R}^d \times \overline{D_2}$ satisfying $\|u - y\| = \|v - y\|$.

If $u = v$, from the construction of the mirror coupling it follows that $X_t = Y_t$ until the process Y_t hits the boundary of D_2, and therefore the inequality in (5.4) holds for these values of t. After the process Y_t hits a bounding hyperplane of D_2, by Lemma 3.8 it follows that in an appropriate coordinate system Y_t is given by $Y_t = (X_t^1, \ldots, X_t^{d-1}, |X_t^d|)$, until the time σ when the process Y hits a different bounding hyperplane of D_2, and therefore the inequality in (5.4) is again verified for the corresponding values of t (in the chosen coordinate system we must have $y = (y^1, \ldots, y^d)$ with $y^d > 0$, and therefore $\|X_t - y\|^2 - \|Y_t - y\|^2 = 2y^d\left(|X_t^d| - X_t^d\right) \geq 0$). If at time σ the processes are coupled (i.e. $X_\sigma = Y_\sigma \in \partial D_2$), we can apply the above argument inductively, and find a time σ_1 when the processes are decoupled and $\|X_t - y\| \geq \|Y_t - y\|$ for all $t \leq \sigma_1$.

The above discussion shows that without loss of generality we may further reduce the proof of the claim to the case when $(u, v) \in \mathbb{R}^d \times \overline{D_2}$ with $u \neq v$ and $\|u - y\| \geq \|v - y\|$. Also, the above discussion shows that it is enough to prove (5.4) for all values of $t \leq \zeta$, where $\zeta = \inf\{s > 0 : X_s = Y_s\}$ is the first coupling time.

The mirror coupling defined by (3.1) – (3.3) becomes in the case

$$X_t = u + W_t \tag{5.5}$$

$$Y_t = v + Z_t + \int_0^t \nu_{D_2}\left(Y_s\right) dL_s^Y \tag{5.6}$$

$$Z_t = \int_0^t G\left(Y_s - X_s\right) dW_s \tag{5.7}$$

where G is given by (3.5). In order to prove the claim we will show that

$$R_t = \|X_t - y\|^2 - \|Y_t - y\|^2 \geq 0, \qquad t \leq \zeta, \tag{5.8}$$

where ζ is the first coupling time.

Using the Itô formula it can be shown that the process R_t verifies the stochastic differential equation

$$R_t = R_0 - 2\int_0^t R_s \frac{Y_s - X_s}{\|Y_s - X_s\|^2} \cdot dW_s - 2\int_0^t (Y_s - y) \cdot \nu_{D_2}\left(Y_s\right) dL_s^Y, \quad t \leq \zeta. \tag{5.9}$$

The process $B_t = -2\int_0^t \frac{Y_s - X_s}{\|Y_s - X_s\|^2} \cdot dW_s$ is a continuous local martingale on $[0, \zeta)$, with quadratic variation

$$A_t = 4\sum_{i=1}^d \int_0^{t\wedge\zeta} \frac{(Y_s^i - X_s^i)^2}{\|Y_s - X_s\|^4} ds = \int_0^{t\wedge\zeta} \frac{4}{\|Y_s - X_s\|^2} ds, \qquad t \geq 0, \tag{5.10}$$

and therefore by Lévy's characterization of Brownian motion it follows that $\widetilde{B}_t = B_{\alpha_t \wedge \zeta}$ is a 1-dimensional Brownian motion (possibly stopped at time ζ, if $A_\zeta < \infty$), where the time change $\alpha_t = \inf\{s \geq 0 : A_s > t\}$ is the inverse of the nondecreasing process A_t.

Setting $\widetilde{X}_t = X_{\alpha_t\wedge\zeta}$, $\widetilde{Y}_t = Y_{\alpha_t\wedge\zeta}$, $\widetilde{R}_t = R_{\alpha_t\wedge\zeta}$ and $\widetilde{L}_t^Y = L_{\alpha_t\wedge\zeta}^Y$, from (5.9) we obtain

$$\widetilde{R}_t = \widetilde{R}_0 + \int_0^t \widetilde{R}_s d\widetilde{B}_s - \int_0^t (\widetilde{Y}_s - y) \cdot \nu_{D_2}\left(\widetilde{Y}_s\right) d\widetilde{L}_s^Y, \qquad t \geq 0. \tag{5.11}$$

Since the polygonal domain D_2 is assumed star-shaped with respect to the point y, geometric considerations show that

$$(z - y) \cdot \nu_{D_2}(z) \leq 0, \tag{5.12}$$

for all the points $z \in \partial D_2$ for which the inside pointing normal $\nu_{D_2}(z)$ at the boundary point z of D_2 is defined, that is for all points $z \in \partial D_2$ not lying on the intersection of two bounding hyperplanes of D_2. Since the reflecting Brownian motion Y_t does not hit the set of these exceptional points with positive probability, we may assume that the above condition is satisfied for all points, and therefore

$$(\widetilde{Y}_s - y) \cdot \nu_{D_2}(\widetilde{Y}_s) \leq 0 \qquad \text{a.s}, \tag{5.13}$$

for all times $s \geq 0$ when $\widetilde{Y}_s \in \partial D_2$.

Since $\widetilde{L}_t^Y$ is a nondecreasing process of $t \geq 0$, a standard comparison argument for solutions of stochastic differential equations shows that the solution $\widetilde{R}_t$ of (5.11) satisfies $\widetilde{R}_t \geq \rho_t$ for all $t \geq 0$, where ρ_t is the solution of the stochastic differential equation

$$\rho_t = \widetilde{R}_0 + \int_0^t \rho_s d\widetilde{B}_s, \qquad t \geq 0. \tag{5.14}$$

The last equation has the explicit solution $\rho_t = R_0 e^{\widetilde{B}_t - \frac{1}{2}t}$, and since by hypothesis $R_0 = \|u-y\|^2 - \|v-y\|^2 \geq 0$, we obtain

$$R_{\alpha_t \wedge \zeta} = \widetilde{R}_t \geq \rho_t = \widetilde{R}_0 e^{\widetilde{B}_t - \frac{1}{2}t} \geq 0, \qquad t \geq 0, \tag{5.15}$$

and therefore $R_t = \|X_t - y\|^2 - \|Y_t - y\|^2 \geq 0$ for all $t \leq \zeta$, concluding the proof of the claim.

By the initial remarks, it follows that if (X_t, Y_t) is a mirror coupling in (D_1, D_2) with starting point $X_0 = Y_0 = x$, then

$$\|X_t - y\| \geq \|Y_t - y\|, \qquad t \geq 0. \tag{5.16}$$

As in the proof of the last theorem, this shows that $p_{D_1}(t,x,y) \leq p_{D_2}(t,x,y)$ for all $t \geq 0$, concluding the proof. $\square$

We have chosen to carry out the construction of the mirror coupling in the case of smooth domains with $\overline{D_2} \subset D_1$ and D_2 convex, having in mind the application to Chavel's conjecture. However, although the technical details can be considerably longer, it is possible to construct the mirror coupling in a more general setup.

For example, in the case when D_1 and D_2 are disjoint domains, none of the difficulties encountered in the construction of the mirror coupling occur (the possibility of coupling/decoupling), so the constructions extends immediately to this case.

The two key ingredients in our construction of the mirror coupling were the hypothesis $\overline{D_2} \subset D_1$ (needed in order to reduce by a localization argument the construction to the case $D_1 = \mathbb{R}^d$) and the hypothesis on the convexity of the inner domain D_2 (which allowed us to construct a solution of the equation of the mirror coupling in the case $D_1 = \mathbb{R}^d$).

Replacing the first hypothesis by the condition that the boundaries ∂D_1 and ∂D_2 are not tangential (needed for the localization of the construction of the mirror coupling) and the second one by condition that $D_1 \cap D_2$ is a convex domain, the arguments in the present construction can be modified in order to give rise to a mirror coupling of reflecting Brownian motion in (D_1, D_2) (see Figure 3).

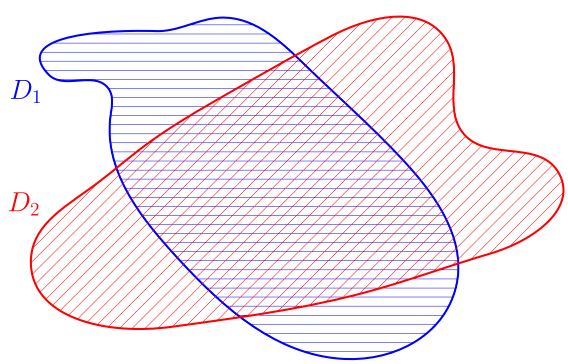

Figure 3: Generic smooth domains $D_{1,2} \subset \mathbb{R}^d$ for the mirror coupling: D_1, D_2 have non-tangential boundaries and $D_1 \cap D_2$ is a convex domain.

Remark 5.5. Even though the construction of the mirror coupling was carried out without the additional assumption on the convexity of the inner domain D_2 in the case when D_2 is a polygonal domain (see Theorem 3.9), we cannot extend the construction of the mirror coupling to the general case of smooth domains $D_2 \subset D_1$.

This is due to the fact that the stochastic differential equation which defines the mirror coupling has a singularity (discontinuity) when the processes couple, and we cannot prove the convergence of solutions in the approximating domains (as in the proof of Theorem 3.1). The convexity of the inner domain is an essential argument for this proof, which allowed us to handle the discontinuity of the stochastic differential equation which defines the mirror coupling: considering an increasing sequence of approximating domains $D_n \nearrow D_2$, the convexity of D_2 was used to show that if the coupling occurred in the case of the mirror coupling in $(\mathbb{R}^d, D_N)$, then coupling also occurred in the case of the mirror coupling in $(\mathbb{R}^d, D_n)$, for all $n \geq N$.

It is easy to construct an example of a non-convex domain D_2 and a sequence of approximating domains $D_n \nearrow D_2$ such that the mirror coupling (X_t, Y_t^n) in $(\mathbb{R}^d, D_n)$ does not have the above-mentioned property, and therefore we cannot prove the existence of the mirror coupling using the same ideas as in Theorem 3.1. However, this does not imply that the mirror coupling cannot be constructed by other methods in a more general setup.

We conclude with some remarks on the non-uniqueness of the mirror coupling in general domains. To simplify the ideas, we will restrict to the 1-dimensional case when $D_2 = (0, \infty) \subset D_1 = \mathbb{R}$.

Fixing $x \in (0, \infty)$ as starting point of the mirror coupling $\left(X_t, Y_t\right)$ in $\left(D_1, D_2\right)$, the equations of the mirror coupling are

$$\begin{aligned} X_t &= x + W_t && (5.17) \\ Y_t &= x + Z_t + L_t^Y && (5.18) \\ Z_t &= \int_0^t G\left(Y_s - X_s\right) dW_s && (5.19) \end{aligned}$$

where in this case

$$G(z) = \begin{cases} -1, & \text{if } z \neq 0 \\ +1, & \text{if } z = 0 \end{cases}.$$

Until the hitting time $\tau = \{s > 0 : Y_s \in \partial D_2\}$ of the boundary of ∂D_2 we have $L_t^Y \equiv 0$, and with the substitution $U_t = -\frac{1}{2}\left(Y_t - X_t\right)$, the stochastic differential for Y_t becomes

$$U_t = \int_0^t \frac{1 - G\left(Y_s - X_s\right)}{2} dW_s = \int_0^t \sigma\left(U_s\right) dW_s, \tag{5.20}$$

where

$$\sigma(z) = \frac{1 - G(z)}{2} = \begin{cases} 1, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}.$$

By a result of Engelbert and Schmidt ([13]) the solution of the above problem is not even weakly unique, for in this case the set of zeroes of the function σ is $N = \{0\}$ and σ^{-2} is locally integrable on $\mathbb{R}$.

In fact, more can be said about the solutions of (5.20) in this case. It is immediate that both $U_t \equiv 0$ and $U_t = W_t$ are solutions to 5.20, and it can be shown that an arbitrary solution can be obtained from W_t by delaying it when it reaches the origin (sticky Brownian motion with sticky point the origin).

Therefore, until the hitting time τ of the boundary, we obtain as solutions

$$Y_t = X_t = x + W_t \tag{5.21}$$

and

$$Y_t = X_t - 2W_t = x - W_t, \tag{5.22}$$

and an intermediate range of solutions, which agree with (5.21) for some time, then switch to (5.22) (see [18]).

Correspondingly, this gives rise to mirror couplings of reflecting Brownian motions for which the solutions stick to each other after they have coupled (as in (5.21)), or they immediately split apart after coupling (as in (5.22)), and there is a whole range of intermediate possibilities. The first case can be referred to as *sticky* mirror coupling, the second as *non-sticky* mirror coupling, and the intermediate possibilities as *weak/mild sticky* mirror coupling.

The same situation occurs in the general setup in $\mathbb{R}^d$, and it is the cause of lack uniqueness of the stochastic differential equations which defines the mirror coupling. In the present paper we detailed the construction of the sticky mirror coupling, which we considered to be the most interesting, both from the point of view of the construction and of the applications, although the other types of mirror coupling might prove useful in other applications.

Acknowledgements

I would like to thank Krzysztof Burdzy and Wilfried S. Kendall for the helpful discussions and the encouragement to undertake the task of the present project.

I would also like to thank the anonymous referee for carefully reading the manuscript and for the suggestion of extending the result in Theorem 5.3 to the more general case in Theorem 5.4.

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