DOI: 10.1214/EJP.v16-853

Electronic Journal of Probability

Vol. 16 (2011), Paper no. 7, pages 216–229.

Journal URL
http://www.math.washington.edu/~ejpecp/

Integrability of Seminorms

Andreas Basse-O'Connor
Department of Mathematical Sciences
University of Aarhus,
Ny Munkegade, DK-8000 Århus C, Denmark.
E-mail: basse@imf.au.dk

Abstract

We study integrability and equivalence of L^p-norms of polynomial chaos elements. Relying on known results for Banach space valued polynomials, we extend and unify integrability for seminorms results to random elements that are not necessarily limits of Banach space valued polynomials. This enables us to prove integrability results for a large class of seminorms of stochastic processes and to answer, partially, a question raised by C. Borell (1979, Séminaire de Probabilités, XIII, 1–3).

Key words: integrability; chaos processes; seminorms; regularly varying distributions.

AMS 2000 Subject Classification: Primary 60G17; 60B11; 60B12; 60E15.

Submitted to EJP on September 25, 2009, final version accepted December 30, 2010.

1 Introduction

The purpose of the present paper is to unify and extend results on integrability of seminorms of polynomial chaos elements taking values in a topological vector space. The chaos are understood in the weak sense, in the spirit of Ledoux and Talagrand (1991). The motivation for this comes from stochastic processes. For example, in order to study $U := \sup_{t\in T}|X_t|$, where T is a countable set, we may think of X as a map from Ω into $l^\infty(T)$. However, the non-separability of $l^\infty(T)$ causes many problems, e.g. with measurability of X. The approach in this paper is instead to view X as a random element in the separable topological space $\mathbb{R}^T$. Then $U = N(X)$, where $N(f) = \sup_{t\in T}|f(t)|$ is a lower semicontinuous seminorm on $\mathbb{R}^T$ (taking values in $[0,\infty]$). When X is a weak chaos process, Theorem 2.2 provides conditions under which U is integrable.

Weak chaos processes appear in the context of multiple integral processes; see e.g. Krakowiak and Szulga (1988) for the α-stable case. Rademacher chaos processes are applied repeatedly when studying U-statistics; see de la Peña and Giné (1999). They are also used to study infinitely divisible chaos processes; see Basse and Pedersen (2009), Marcus and Rosiński (2003) and Rosiński and Samorodnitsky (1996). Using the results of the present paper, Basse-O'Connor and Graversen (2010) extend some results on Gaussian semimartingales (e.g. Jain and Monrad (1982) and Stricker (1983)) to a large class of chaos processes.

Let N be a measurable seminorm on $\mathbb{R}^T$. For X Gaussian, Fernique (1970) shows that $e^{\varepsilon N(X)^2}$ is integrable for some $\varepsilon > 0$. This result is extended to Gaussian chaos processes by Borell (1978), Theorem 4.1. Moreover, if X is α-stable for some $\alpha \in (0,2)$, de Acosta (1975), Theorem 3.2, shows that $N(X)^p$ is integrable for all $p < \alpha$. When X is infinitely divisible, Rosiński and Samorodnitsky (1993) provide conditions on the Lévy measure ensuring integrability of $N(X)$. See also Hoffmann-Jørgensen (1977) for further results.

Given a sequence $(Z_n)_{n\in\mathbb{N}}$ of independent random variables, Borell (1984) studies, under the condition

$$\sup_{n\geq 1}\frac{\|Z_n-\mathbb{E}Z_n\|_q}{\|Z_n-\mathbb{E}Z_n\|_2}<\infty,\qquad q\in(2,\infty], \tag{1.1}$$

integrability of Banach space valued random elements which are limits in probability of tetrahedral polynomials associated with $(Z_n)_{n\in\mathbb{N}}$. As shown in Borell (1984), (1.1) implies equivalence of L^p-norms for Hilbert space valued tetrahedral polynomials for $p \leq q$, but not for Banach space valued tetrahedral polynomials except in the case $q = \infty$. We impose the stronger condition C_q on $(Z_n)_{n\in\mathbb{N}}$, see (1.2)–(1.3), which in the case $q = \infty$ equals (1.1). Under C_q with $q < \infty$, Kwapień and Woyczyński (1992), Theorem 6.6.2, show equivalence of L^p-norms of Banach space valued tetrahedral polynomials. We extend and unify Borell (1984), Kwapień and Woyczyński (1992) and and others, by considering random elements which are not necessarily limits of tetrahedral polynomials. Moreover, for lower semicontinuous seminorms Borell (1978), de Acosta (1975) and Fernique (1970) are special cases of Theorem 2.1.

1.1 Chaos Processes and Condition C_q

Let $(\Omega,\mathscr{F},\mathbb{P})$ denote a probability space. When F is a topological space, a Borel measurable map $X\colon\Omega\to F$ is called an F-valued random element, however when $F = \mathbb{R}$, X is, as usual, called a random variable. For each $p > 0$ and random variable X we let $\|X\|_p := \mathbb{E}[|X|^p]^{1/p}$, which defines a norm when $p \geq 1$; moreover, let $\|X\|_\infty := \inf\{t \geq 0 : \mathbb{P}(|X| \leq t) = 1\}$. When F is a Banach space, $L^p(\mathbb{P}; F)$ denotes the space of all F-valued random elements, X, satisfying $\|X\|_{L^p(\mathbb{P};F)} = \mathbb{E}[\|X\|^p]^{1/p} < \infty$. Throughout the paper I denotes a set and for all $\xi \in I$, $\mathscr{H}_\xi$ is a family of independent random variables. Set $\mathscr{H} = \{\mathscr{H}_\xi : \xi \in I\}$. Furthermore, $d \geq 1$ is a natural number and F is a locally convex Hausdorff topological vector space (l.c.TVS) with dual space F^*, see Rudin (1991). Following Fernique (1997), a map N from F into $[0, \infty]$ is called a pseudo-seminorm if for all $x, y \in F$ and $\lambda \in \mathbb{R}$, we have

$$N(\lambda x) = |\lambda| N(x) \qquad \text{and} \qquad N(x + y) \leq N(x) + N(y).$$

For $\xi \in I$ let $\mathscr{P}_d(\mathscr{H}_\xi; F)$ denote the set of $p(Z_1, \dots, Z_n)$ where $n \in \mathbb{N}$, $Z_1, \dots, Z_n$ are different elements in $\mathscr{H}_\xi$ and p is an F-valued tetrahedral polynomial of order d. Recall that $p\colon \mathbb{R}^n \to F$ is called an F-valued tetrahedral polynomial of order d if there exist $x_0, x_{i_1,\dots,i_k} \in F$ and $l \geq 1$ such that

$$p(z_1, \dots, z_n) = x_0 + \sum_{k=1}^{d} \sum_{1 \leq i_1 < \dots < i_k \leq l} x_{i_1,\dots,i_k} \prod_{j=1}^{k} z_{i_j}.$$

Moreover, let $\overline{\mathscr{P}}_d(\mathscr{H}; F)$ denote the closure in distribution of $\cup_{\xi \in I} \mathscr{P}_d(\mathscr{H}_\xi; F)$, that is, $\overline{\mathscr{P}}_d(\mathscr{H}; F)$ is the set of all F-valued random elements X for which there exists a sequence $(X_k)_{k \in \mathbb{N}} \subseteq \cup_{\xi \in I} \mathscr{P}_d(\mathscr{H}_\xi; F)$ converging weakly to X. In the spirit of Ledoux and Talagrand (1991) we introduce the following:

Definition 1.1. An F-valued random element X is said to be a weak chaos element of order d associated with $\mathscr{H}$ if for all $n \in \mathbb{N}$ and $(x_i^*)_{i=1}^n \subseteq F^*$ we have $(x_1^*(X), \dots, x_n^*(X)) \in \overline{\mathscr{P}}_d(\mathscr{H}; \mathbb{R}^n)$, and in this case we write $X \in weak\text{-}\overline{\mathscr{P}}_d(\mathscr{H}; F)$. Similarly, a real-valued stochastic process $(X_t)_{t \in T}$ is said to be a weak chaos process of order d associated with $\mathscr{H}$ if for all $n \in \mathbb{N}$ and $(t_i)_{i=1}^n \subseteq T$ we have $(X_{t_1}, \dots, X_{t_n}) \in \overline{\mathscr{P}}_d(\mathscr{H}; \mathbb{R}^n)$.

In what follows we shall need the next conditions:

Condition C_q. For $q \in (0, \infty)$, $\mathscr{H}$ is said to satisfy C_q if there exists $\beta_1, \beta_2 > 0$ such that for all $Z \in \cup_{\xi \in I} \mathscr{H}_\xi$ there exists $c_Z > 0$ with $\mathbb{P}(|Z| \geq c_Z) \geq \beta_1$ and

$$\mathbb{E}[|Z|^q, |Z| > s] \leq \beta_2 s^q \mathbb{P}(|Z| > s), \qquad s \geq c_Z. \tag{1.2}$$

For $q = \infty$, $\mathscr{H}$ is said to satisfy C_∞ if $\cup_{\xi \in I} \mathscr{H}_\xi \subseteq L^1$ and

$$\sup_{\xi \in I} \sup_{Z \in \mathscr{H}_\xi} \left(\frac{\|Z - \mathbb{E}Z\|_\infty}{\|Z - \mathbb{E}Z\|_2} \right) = \beta_3 < \infty. \tag{1.3}$$

Let us start by noticing that C_q implies equivalence of moments, that is, if $\mathscr{H}$ satisfies C_q with $q \in (0, \infty)$ then for all $p \in (0, q)$ we have

$$\sup_{\xi \in I} \sup_{Z \in \mathscr{H}_\xi} \frac{\|Z\|_q}{\|Z\|_p} \leq (\beta_2 \vee 1)^{1/q} \beta_1^{-1/p} < \infty. \tag{1.4}$$

Equation (1.4) follows from the estimates (a)–(b):

$$\text{(a):}\qquad \begin{aligned}\mathbb{E}[|Z|^q] &= \mathbb{E}[|Z|^q, |Z| > c_Z] + \mathbb{E}[|Z|^q, |Z| \le c_Z] \\ &\le \beta_2 c_Z^q \mathbb{P}(|Z| > c_Z) + c_Z^q \mathbb{P}(|Z| \le c_Z) \le (\beta_2 \vee 1) c_Z^q,\end{aligned}$$

and

$$\text{(b):}\qquad c_Z^p \beta_1 \le c_Z^p \mathbb{P}(|Z| \ge c_Z) \le \mathbb{E}[|Z|^p].$$

2 Main results

Recall that an F-valued random element X is said to be a.s. separably valued if $P(X \in A) = 1$ for some separable closed subset A of F, and a map $f : F \to [-\infty, \infty]$ is said to be lower semicontinuous if $x_n \to x$ in F implies $f(x) \le \liminf_n f(x_n)$.

Theorem 2.1. *Let F denote a metrizable l.c.TVS, $X \in weak\text{-}\overline{\mathscr{P}}_d(\mathscr{H}; F)$ be an a.s. separably valued random element and N be a lower semicontinuous pseudo-seminorm on F such that $N(X) < \infty$ a.s. Assume that $\mathscr{H}$ satisfies C_q for some $q \in (0, \infty]$ and if $q < \infty$ and $d \ge 2$ that all elements in $\cup_{\xi \in I} \mathscr{H}_\xi$ are symmetric. Then for all finite $0 < p < r \le q$ we have*

$$\|N(X)\|_r \le k_{p,r,d,\beta} \|N(X)\|_p < \infty,$$

where $k_{p,r,d,\beta}$ depends only on p, q, d and the β's from C_q. Furthermore, in the case $q = \infty$ we have that $\mathbb{E}[e^{\varepsilon N(X)^{2/d}}] < \infty$ for all $\varepsilon < d/(e 2^{d+5} \beta_3^4 \|N(X)\|_2^{2/d})$, and $k_{2,r,d,\beta} = 2^{d^2/2+2d} \beta_3^{2d} r^{d/2}$.

For $q = \infty$, Theorem 2.1 answers in the case where the pseudo-seminorm is lower semicontinuous a question raised by Borell (1979) concerning integrability of pseudo-seminorms of Rademacher chaos elements. This additional assumption is satisfied in most examples, in particular the one considered in the Introduction. We prove Theorem 2.1 by representing N on the form $N(x) = \sup_{n \in \mathbb{N}} |x_n^*(x)|$ where $(x_n^*)_{n \in \mathbb{N}} \subseteq F^*$, which enables us to obtain the result by a suitable application of Kwapień and Woyczyński (1992) when $q < \infty$ and Borell (1984) when $q = \infty$.

Proof of Theorem 2.1. Let B denote a Banach space and let $Y \in \mathscr{P}_d(\mathscr{H}; B)$. For all $0 < p < r \le q$ with $r < \infty$ we have

$$\|Y\|_{L^r(\mathbb{P};B)} \le k_{p,r,d,\beta} \|Y\|_{L^p(\mathbb{P};B)} < \infty, \tag{2.1}$$

where $k_{p,r,d,\beta}$ depends only on p, q, d and the β's from C_q. If $q = \infty$ and $p \ge 2$ we may choose $k_{p,r,d,\beta} = A_d \beta_3^{2d} r^{d/2}$ with $A_d = 2^{d^2/2+2d}$. For $q < \infty$ and $d = 1$, (2.1) is a consequence of Kwapień and Woyczyński (1992), Equation (2.2.4). Furthermore, for $q \in (1, \infty)$ and $d \ge 2$ it is taken from the proof of Kwapień and Woyczyński (1992), Theorem 6.6.2, and using Kwapień and Woyczyński (1992), Remark 6.9, the result is seen to hold also for $q \in (0, 1]$. For $q = \infty$, (2.1) is a consequence of Borell (1984), Theorem 4.1. In Borell (1984) the result is only stated for $2 \le p < r$, however, a standard application of Hölder's inequality shows that it is valid for all $0 < p < r$; see e.g. Pisier (1978), Lemme 1.1. Finally, in Borell (1984) there is no explicit expression for A_d; this can, however, be obtained by applying Lemma A.1 from the Appendix, in the proof of Borell (1984), Theorem 4.1, top of page 198.

Let l_∞^n be $\mathbb{R}^n$ equipped with the sup norm. Fix finite p, r with $0 < p < r \le r$ and let $C := k_{p,r,d,\beta}$. Let us show that for all $n \in \mathbb{N}$ and $Y \in \overline{\mathscr{P}}_d(\mathscr{H}; \mathbb{R}^n)$ we have

$$\|Y\|_{L^q(\mathbb{P}; l_\infty^n)} \le C \|Y\|_{L^p(\mathbb{P}; l_\infty^n)}. \tag{2.2}$$

Using (2.1) on $B = l_\infty^n$ we have

$$\|Y\|_{L^q(\mathbb{P}; l_\infty^n)} \le C \|Y\|_{L^p(\mathbb{P}; l_\infty^n)} < \infty, \qquad Y \in \mathscr{P}_d(\mathscr{H}_\xi; \mathbb{R}^n),\ \xi \in I. \tag{2.3}$$

Choose $(\xi_k)_{k\in\mathbb{N}} \subseteq I$ and $Y_k \in \mathscr{P}_d(\mathscr{H}_{\xi_k}; \mathbb{R}^n)$ for $k \in \mathbb{N}$ such that $Y_k \to_d Y$ ($\to_d$ denotes convergence in distribution). Moreover, let $U_k = \|Y_k\|_{l_\infty^n}$ and $U = \|Y\|_{l_\infty^n}$. Then, $U_k \to_d U$ showing that $(U_k)_{k\in\mathbb{N}}$ is bounded in L^0, and by (2.3) and Krakowiak and Szulga (1986), Corollary 1.4, $\{U_k^p : k \in \mathbb{N}\}$ is uniformly integrable. Hence,

$$\|U\|_q \le \liminf_{k\to\infty} \|U_k\|_q \le C \liminf_{k\to\infty} \|U_k\|_p = C\|U\|_p < \infty,$$

which shows (2.2).

Arguing as in Fernique (1997), Lemme 1.2.2, we will show that there exists $(x_n^*)_{n\in\mathbb{N}} \subseteq F^*$ such that

$$N(x) = \sup_{n\in\mathbb{N}} |x_n^*(x)| \qquad \text{for all } x \in F. \tag{2.4}$$

To show (2.4) let $A := \{x \in F : N(x) \le 1\}$. Then A is convex and balanced since N is a pseudo-seminorm and closed since N is lower semicontinuous. Thus by the Hahn-Banach theorem, see Rudin (1991), Theorem 3.7, for all $x \notin A$ there exists $x^* \in F^*$ such that $|x^*(y)| \le 1$ for all $y \in A$ and $x^*(x) > 1$, showing that

$$A^c = \bigcup_{x\in A^c} \{y \in F : |x^*(y)| > 1\}. \tag{2.5}$$

Since X is a.s. separably valued we may and will assume that F is separable and hence strongly Lindelöf since it is metrizable by assumption, see Gemignani (1990). Thus, since (2.5) is an open cover of A^c there exists $(x_n)_{n\in\mathbb{N}} \subseteq A^c$ such that

$$A^c = \bigcup_{n=1}^\infty \{y \in F : |x_n^*(y)| > 1\},$$

implying that $A = \{y \in F : \sup_{n\in\mathbb{N}} |x_n^*(y)| \le 1\}$. Thus by homogeneity we have $N(y) = \sup_{n\in\mathbb{N}} |x_n^*(y)|$ for all $y \in F$.

For $n \in \mathbb{N}$, let $X_n := x_n^*(X)$ and $U_n = \sup_{1\le k\le n} |X_k|$. Then $(U_n)_{n\in\mathbb{N}}$ converge almost surely to $N(X)$. For all finite $0 < p < r \le q$, (2.2) shows that $\|U_n\|_q \le C\|U_n\|_p < \infty$ for all $n \in \mathbb{N}$. This implies that $\{U_n^p : n \in \mathbb{N}\}$ is uniformly integrable and hence

$$\|N(X)\|_r \le \liminf_{n\to\infty} \|U_n\|_r \le C \liminf_{n\to\infty} \|U_n\|_p = C\|N(X)\|_p < \infty.$$

To prove the last statement of the theorem let $\varepsilon < d/(e2^{d+5}\beta_3^4 \|N(X)\|_2^{2/d})$. Since $k_{2,r,d,\beta} = 2^{d^2/2+2d}\beta_3^{2d} r^{d/2}$ we have

$$\mathbb{E}[e^{\varepsilon N(X)^{2/d}}] \le 1 + \sum_{k=1}^d \|N(X)\|_{2k/d}^{2k/d} + \sum_{k=d+1}^\infty \left(\varepsilon 2^{d+5}\beta_3^4 \|N(X)\|_2^{2/d}/d\right)^k \frac{k^k}{k!} < \infty,$$

which completes the proof. □

Let T denote a countable set and $F := \mathbb{R}^T$ be equipped with the product topology. Then F is a separable and locally convex Fréchet space and all $x^* \in F^*$ are of the form $x \mapsto \sum_{i=1}^n \alpha_i x(t_i)$, for some $n \in \mathbb{N}$, $t_1, \ldots, t_n \in T$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Thus for $X = (X_t)_{t\in T}$ we have that $X \in weak\text{-}\overline{\mathscr{P}}_d(\mathscr{H}; F)$ if and only if X is a weak chaos process of order d. Rewriting Theorem 2.1 in the case $F = \mathbb{R}^T$ we obtain the following result:

Theorem 2.2. *Assume $\mathscr{H}$ satisfies C_q for some $q \in (0, \infty]$ and if $q < \infty$ and $d \geq 2$ that all elements in $\cup_{\xi\in I}\mathscr{H}_\xi$ are symmetric. Let T denote a countable set, $(X_t)_{t\in T}$ be a weak chaos process of order d and N be a lower semicontinuous pseudo-seminorm on $\mathbb{R}^T$ such that $N(X) < \infty$ a.s. Then for all finite $0 < p < r \leq q$ we have*

$$\|N(X)\|_r \leq k_{p,r,d,\beta}\|N(X)\|_p < \infty,$$

and in the case $q = \infty$ that $\mathbb{E}[e^{\varepsilon N(X)^{2/d}}] < \infty$ for all $\varepsilon < d/(e2^{d+5}\beta_3^4\|N(X)\|_2^{2/d})$.

Let $\mathscr{G}$ denote a vector space of Gaussian random variables and $\overline{\Pi}_d(\mathscr{G}; \mathbb{R})$ be the closure in probability of the random variables $p(Z_1, \ldots, Z_n)$, where $n \in \mathbb{N}$, $Z_1, \ldots, Z_n \in \mathscr{G}$ and $p\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree at most d (not necessary tetrahedral). Recall that a sequence of independent, identically distributed random variables $(Z_n)_{n\in\mathbb{N}}$ such that $P(Z_1 = \pm 1) = 1/2$ is called a Rademacher sequence.

Proposition 2.3. *Suppose F is a l.c.TVS and X is an F-valued random element such that $x^*(X) \in \overline{\Pi}_d(\mathscr{G}; \mathbb{R})$ for all $x^* \in F^*$. Then $X \in weak\text{-}\overline{\mathscr{P}}_d(\mathscr{H}; F)$ where $\mathscr{H} = \{\mathscr{H}_0\}$ and $\mathscr{H}_0$ is a Rademacher sequence. Thus, if X is a.s. separably valued and N is a lower semicontinuous pseudo-seminorm on F such that $N(X) < \infty$ a.s. then for all $r > 2$,*

$$\|N(X)\|_r \leq 2^{d^2/2+2d} r^{d/2}\|N(X)\|_2 < \infty,$$

and $\mathbb{E}[e^{\varepsilon N(X)^{2/d}}] < \infty$ for all $\varepsilon < d/(e2^{d+5}\|N(X)\|_2^{2/d})$.

Proof. Let $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in F^*$ and $W = (x_1^*(X), \ldots, x_n^*(X))$. We need to show that $W \in \overline{\mathscr{P}}_d(\mathscr{H}; \mathbb{R}^n)$. For all $k \geq 1$ we may choose polynomials $p_k\colon \mathbb{R}^k \to \mathbb{R}^n$ of degree at most d and $Y_{1,k}, \ldots, Y_{k,k}$ independent standard normal random variables such that with $Y_k = (Y_{1,k}, \ldots, Y_{k,k})$ we have $\lim_k p_k(Y_k) = W$ in probability. Hence it suffices to show $p_k(Y_k) \in \overline{\mathscr{P}}_d(\mathscr{H}; \mathbb{R}^n)$ for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and let us write p and Y for p_k and Y_k. Reenumerate $\mathscr{H}_0$ as k independent Rademacher sequences $(Z_{i,m})_{i\geq 1}$ with $m = 1, \ldots, k$ and set

$$U_j = \frac{1}{\sqrt{j}}\sum_{i=1}^{j}(Z_{1,i}, \ldots, Z_{k,i}), \qquad j \in \mathbb{N}.$$

Then, by the central limit theorem $U_j \to_d Y$ and hence $p(U_j) \to_d p(Y)$. Due to the fact that all $Z_{i,m}$ only takes on the values ± 1, $p(U_j) \in \mathscr{P}_d(\mathscr{H}_0; \mathbb{R}^n)$ for all $j \in \mathbb{N}$, showing that $p(Y) \in \overline{\mathscr{P}}_d(\mathscr{H}; \mathbb{R}^n)$. By applying Theorem 2.1, the conclusion follows since $\mathscr{H}$ satisfies C_∞ with $\beta_3 = 1$. □

The integrability of $e^{\varepsilon N(X)^{2/d}}$, in Proposition 2.3, is a consequence of the seminal work Borell (1978), Theorem 4.1. However, Proposition 2.3 provides a simple proof of this result and also provides equivalence of L^p-norms and explicit constants. When $F = \mathbb{R}^T$ for some countable set T, Proposition 2.3 covers processes $X = (X_t)_{t\in T}$, where all time variables, X_t, have the following representation in terms of multiple Wiener-Itô integrals with respect to a Brownian motion W,

$$X_t = \sum_{k=0}^{d} \int_{\mathbb{R}_+^k} f(t,k;s_1,\dots,s_k)\, dW_{s_1}\cdots dW_{s_k}, \qquad t\in T.$$

For basic fact about multiple integrals see Nualart (2006).

The next result is known from Arcones and Giné (1993), Theorem 3.1, for general Gaussian polynomials.

Proposition 2.4. *Assume that $\mathscr{H} = \{\mathscr{H}_0\}$ satisfies C_q for some $q \in [2,\infty]$ and $\mathscr{H}_0$ consists of symmetric random variables. Let F denote a Banach space and X an a.s. separably valued random element in F with $x^*(X) \in \overline{\mathscr{P}}_d(\mathscr{H};\mathbb{R})$ for all $x^* \in F^*$. Then there exist $x_0, x_{i_1,\dots,i_k} \in F$ and $\{Z_n : n \geq 1\} \subseteq \mathscr{H}_0$ such that for all finite $p \leq q$*

$$X = \lim_{n\to\infty} \Big(x_0 + \sum_{k=1}^{d} \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1,\dots,i_k} \prod_{j=1}^{k} Z_{i_j} \Big) \qquad \textit{a.s. and in } L^p(\mathbb{P};F).$$

Proof. We follow Arcones and Giné (1993), Lemma 3.4. Since X is a.s. separably valued we may and do assume F that is separable, which implies that $F_1^* := \{x^* \in F^* : \|x^*\| \leq 1\}$ is metrizable and compact in the weak*-topology by the Banach-Alaoglu theorem; see Rudin (1991), Theorem 3.15+3.16. Moreover, the map $x^* \mapsto x^*(X)$ from F_1^* into $L^0(\mathbb{P})$ is trivially weak*-continuous and thus a weak*-continuous map into $L^2(\mathbb{P})$ by a combination of the equivalence of norms from Theorem 2.1 and Krakowiak and Szulga (1986), Corollary 1.4. This shows that $\{x^*(X) : x^* \in F_1^*\}$ is compact in $L^2(\mathbb{P})$ and hence separable. By definition of $\overline{\mathscr{P}}_d(\mathscr{H};\mathbb{R})$, this implies that there exists a countable set $\{Z_n : n \in \mathbb{N}\} \subseteq \mathscr{H}_0$ such that

$$x^*(X) = \sum_{A\in N_d} a(A,x^*) Z_A, \qquad \text{in } L^2(\mathbb{P}),$$

for some $a(A,x^*) \in \mathbb{R}$, where $N_d = \{A \subseteq \mathbb{N} : |A| \leq d\}$ and $Z_A = \prod_{i\in A} Z_i$ for $A \in N_d$. For $A \in N_d$, the map $x^* \mapsto a(A,x^*)$ from F^* into $\mathbb{R}$ is linear and weak*-continuous and hence there exists $x_A \in F$ such that $a(A,x^*) = x^*(x_A)$, showing that

$$x^*(X) = \lim_{n\to\infty} x^*\Big(\sum_{A\in N_d^n} x_A Z_A \Big), \qquad \text{in } L^2(\mathbb{P}), \tag{2.6}$$

where $N_d^n = \{A \in N_d : A \subseteq \{1,\dots,n\}\}$. Since F is separable, (2.6) and Kwapień and Woyczyński (1992), Theorem 6.6.1, show that

$$\lim_{n\to\infty} \sum_{A\in N_d^n} x_A Z_A = X \qquad \text{a.s.}$$

As above it follows that the convergence also takes place in $L^p(\mathbb{P};F)$ for all finite $p \leq q$, which completes the proof. □

The above proposition gives rise to the following corollary:

Corollary 2.5. *Assume that $\mathscr{H} = \{\mathscr{H}_0\}$ satisfies C_q for some $q \in [2,\infty]$ and $\mathscr{H}_0$ consists of symmetric random variables. Let T denote a set, $V(T) \subseteq \mathbb{R}^T$ a separable Banach space where the maps $f \mapsto f(t)$ from $V(T)$ into $\mathbb{R}$ is continuous for all $t \in T$, and $X = (X_t)_{t\in T}$ a stochastic process with sample paths in $V(T)$ satisfying $X_t \in \overline{\mathscr{P}}_d(\mathscr{H};\mathbb{R})$ for all $t \in T$. Then there exists $x_0, x_{i_1,\dots,i_k} \in V(T)$ and $\{Z_n : n \geq 1\} \subseteq \mathscr{H}_0$ such that*

$$X = \lim_{n\to\infty} \Big(x_0 + \sum_{k=1}^{d} \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1,\dots,i_k} \prod_{j=1}^{k} Z_{i_j} \Big)$$

a.s. in $V(T)$ and in $L^p(\mathbb{P};V(T))$ for all finite $p \leq q$.

Proof. For $t \in T$, let $\delta_t : V(T) \to \mathbb{R}$ denote the map $f \mapsto f(t)$. Since $V(T)$ is a separable Banach space and $\{\delta_t : t \in T\} \subseteq V(T)^*$ separate points in $V(T)$ we have

(i) the Borel σ-field on $V(T)$ equals the cylindrical σ-field $\sigma(\delta_t : t \in T)$,

(ii) $\{\sum_{i=1}^{n} \alpha_i \delta_{t_i} : \alpha_i \in \mathbb{R},\ t_i \in T,\ n \geq 1\}$ is sequentially weak*-dense in $V(T)^*$,

see e.g. Rosiński (1986), page 287. By (i) we may regard X as a random element in $V(T)$ and by (ii) it follows that $x^*(X) \in \overline{\mathscr{P}}_d(\mathscr{H};\mathbb{R})$ for all $x^* \in V(T)^*$. Hence the result is a consequence of Proposition 2.4. □

Borell (1984), Theorem 5.1, shows Corollary 2.5 assuming (1.1), T is a compact metric space, $V(T) = C(T)$ and $X \in L^q(P;V(T))$. By assuming C_q instead of the weaker condition (1.1) we can omit the assumption $X \in L^q(P;V(T))$. Note also that by Theorem 2.2 the last assumption is satisfied under C_q. When $\mathscr{H}_0$ consists of symmetric α-stable random variables and $d = 1$, Corollary 2.5 is known from Rosiński (1986), Corollary 5.2. The separability assumption on $V(T)$ in Corollary 2.5 is crucial. Indeed, for all $p > 1$, Jain and Monrad (1983), Proposition 4.5, construct a separable centered Gaussian process $X = (X_t)_{t\in[0,1]}$ with sample paths in the non-separable Banach space B_p of functions of finite p-variation on $[0,1]$ such that the range of X is a non-separable subset of B_p and hence the conclusion in Corollary 2.5 can not be true. However, for the non-separable Banach space B_1 a result similar to Corollary 2.5 is shown in Jain and Monrad (1982) for Gaussian processes, and extended to weak chaos processes in Basse-O'Connor and Graversen (2010).

3 A class of infinitely divisible processes

An important example of a weak chaos process of order one is $(X_t)_{t\in T}$ of the form

$$X_t = \int_S f(t,s)\,\Lambda(ds), \qquad t \in T, \tag{3.1}$$

where Λ is an independently scattered infinitely divisible random measure (or random measure for short) on some non-empty space S equipped with a δ-ring $\mathscr{S}$, and $s \mapsto f(t,s)$ are Λ-integrable deterministic functions in the sense of Rajput and Rosiński (1989). To obtain the associated $\mathscr{H}$ let I be the set of all ξ given by $\xi = \{A_1,\dots,A_n\}$ for some $n \in \mathbb{N}$ and disjoint sets $A_1,\dots,A_n$ in $\mathscr{S}$, and let

$$\mathscr{H}_\xi = \{\Lambda(A_1),\dots,\Lambda(A_n)\} \qquad \text{and} \qquad \mathscr{H} = \{\mathscr{H}_\xi\}_{\xi\in I}. \tag{3.2}$$

Then, by definition of the stochastic integral (3.1) as the limit of integrals of simple functions, $(X_t)_{t\in T}$ is a weak chaos process of order one associated with $\mathscr{H}$.

As we saw in Section 2, C_q is crucial in order to obtain integrability results and equivalence of L^p-norms, so let us consider some cases where the important example (3.1) does or does not satisfy C_q. For this purpose let us introduce the following distributions: The inverse Gaussian distribution $\mathrm{IG}(\mu,\lambda)$ with $\mu,\lambda>0$ is the distribution on $\mathbb{R}_+$ with density

$$f(x;\mu,\lambda)=\left[\frac{\lambda}{2\pi x^3}\right]^{1/2}e^{-\lambda(x-\mu)^2/(2\mu^2x)},\qquad x>0. \tag{3.3}$$

Moreover, the normal inverse Gaussian distribution $\mathrm{NIG}(\alpha,\beta,\mu,\delta)$ with $\mu\in\mathbb{R}$, $\delta\geq 0$, and $0\leq\beta\leq\alpha$, is symmetric if and only if $\beta=\mu=0$, and in this case it has the following density

$$f(x;\alpha,\delta)=\frac{\alpha e^{\delta\alpha}}{\pi\sqrt{1+x^2\delta^{-2}}}K_1\left(\delta\alpha(1+x^2\delta^{-2})^{1/2}\right),\qquad x\in\mathbb{R},$$

where K_1 is the modified Bessel function of the third kind and index 1 given by $K_1(z)=\frac{1}{2}\int_0^\infty e^{-z(y+y^{-1})/2}\,dy$ for $z>0$.

For each finite number $t_0>0$, a random measure Λ is said to be induced by a Lévy process $Y=(Y_t)_{t\in[0,t_0]}$ if $S=[0,t_0]$, $\mathscr{S}=\mathscr{B}([0,t_0])$ and $\Lambda(A)=\int_A dY_s$ for all $A\in\mathscr{S}$. By the scaling property it is not difficult to show that if Λ is a symmetric α-stable random measure with $\alpha\in(0,2)$, then $\mathscr{H}$ satisfies C_q if and only if $q<\alpha$. The next result studies C_q in some non-trivial cases.

Proposition 3.1. *Let $t_0\geq 1$ be a finite number, Λ a random measure induced by a Lévy process $Y=(Y_t)_{t\in[0,t_0]}$ and $\mathscr{H}$ be given by* (3.2).

(i) If Y_1 has an IG-distribution, then $\mathscr{H}$ satisfies C_q if and only if $q\in(0,\frac{1}{2})$.

(ii) If Y_1 has a symmetric NIG-distribution, then $\mathscr{H}$ satisfies C_q if and only if $q\in(0,1)$.

(iii) If Y is non-deterministic and has no Gaussian component, then $\mathscr{H}$ does not satisfy C_q for any $q\geq 2$. In fact, all integrable non-deterministic Lévy processes Y satisfies $\lim_{t\to 0}(\|Y_t\|_2/\|Y_t\|_1)=\infty$.

Proof. Assume that Λ is a random measure induced by a Lévy process $Y=(Y_t)_{t\in[0,T]}$. For arbitrary $A\in\mathscr{S}$ let $Z=\Lambda(A)$.

To prove the *if*-implication of (i) let $q\in(0,\frac{1}{2})$ and assume that $Y_1=_d\mathrm{IG}(\mu,\lambda)$. Then $Z=_d\mathrm{IG}(m(A)\mu,m(A)^2\lambda)$, where m is the Lebesgue measure, and hence with $c_Z=m(A)^2\lambda$ we have that $Z/c_Z=_d\mathrm{IG}(\mu/(\lambda m(A)),1)$, which has a density which on $[1,\infty)$ is bounded from below and above by constants (not depending on x) times $g_Z(x)$, where

$$g_Z\colon\mathbb{R}_+\to\mathbb{R}_+,\qquad x\mapsto x^{-3/2}\exp[-x(\lambda m(A))^2/(2\mu^2)].$$

Thus there exists a constant $c>0$, not depending on A or s, such that

$$\frac{\mathbb{E}[|Z/c_Z|^q,|Z/c_Z|>s]}{s^q\mathbb{P}(|Z/c_Z|>s)}\leq c\sup_{u>0}\left(\frac{\int_u^\infty x^{q-3/2}e^{-x}\,dx}{u^q\int_u^\infty x^{-3/2}e^{-x}\,dx}\right)\qquad s\geq 1. \tag{3.4}$$

Using e.g. l'Hôpital's rule it is easily seen that (3.4) is finite, showing (1.2). Therefore C_q follows by the inequality

$$\mathbb{P}(Z/c_Z \geq 1) \geq \frac{e^{-1/2}}{\sqrt{2\pi}} \int_1^\infty x^{-3/2} \exp[-x(\lambda T)^2/(2\mu^2)]\, dx.$$

To show the *only if*-implication of (i) note that $n^2 Y_{1/n} \to_d X$ as $n \to \infty$, where X follows a $\frac{1}{2}$-stable distribution on $\mathbb{R}_+$. Assume that $\mathscr{H}$ satisfies C_q for some $q \geq 1/2$. Then, by (1.4) there exists $c > 0$ such that $\|Y_t\|_{1/2} \leq c\|Y_t\|_{1/4}$ for all $t \in [0,1]$, and since $\{n^2 Y_{1/n} : n \geq 1\}$ is bounded in L^0 it is also bounded in $L^{1/2}$. But this contradicts

$$\infty = \|X\|_{1/2} \leq \liminf_{n\to\infty} \|n^2 Y_{1/n}\|_{1/2},$$

and shows that $\mathscr{H}$ does not satisfy C_q.

To show the *if*-implication of (ii) assume that $Y_1 =_d \text{NIG}(\alpha, 0, 0, \delta)$. Then, $Z = \Lambda(A)$ follows a $\text{NIG}(\alpha, 0, 0, m(A)\delta)$ distribution and with $c_Z = m(A)\delta$ we have that $Z/c_Z =_d \varepsilon U_Z^{1/2}$, where U_Z and ε are independent, $U_Z =_d \text{IG}(1/(m(A)\delta\alpha), 1)$ and $\varepsilon =_d \text{N}(0,1)$. For $q \in (0,1)$,

$$\mathbb{E}[|Z/c_Z|^q, |Z/c_Z| > s] = \sqrt{2\pi^{-1}}\Big(\int_0^s \mathbb{E}[|xU_Z^{1/2}|^q, |xU_Z^{1/2}| > s] e^{-x^2/2}\, dx \\ + \int_s^\infty \mathbb{E}[|xU_Z^{1/2}|^q, |xU_Z^{1/2}| > s] e^{-x^2/2}\, dx\Big).$$

Using the above (i) on U_Z and $q/2$, there exists a constant $c_1 > 0$ such that

$$\int_0^s \mathbb{E}[|xU_Z^{1/2}|^q, |xU_Z^{1/2}| > s] e^{-x^2/2}\, dx \leq c_1 s^q \int_0^s \mathbb{P}\big(U_Z > (s/x)^2\big) e^{-x^2/2}\, dx \\ \leq c_1 s^q \int_0^\infty \mathbb{P}\big(xU_Z^{1/2} > s\big) e^{-x^2/2}\, dx = c_1 \sqrt{\pi 2^{-1}} s^q \mathbb{P}(|Z/c_Z| > s).$$

Furthermore, it is well known that there exists a constant $c_2 > 0$ such that for all $s \geq 1$

$$\int_s^\infty \mathbb{E}[|xU_Z^{1/2}|^q, |xU_Z^{1/2}| > s] e^{-x^2/2}\, dx \\ \leq \mathbb{E}[U_Z^{q/2}] \int_s^\infty x^q e^{-x^2/2}\, dx \leq c_2 s^q \mathbb{E}[U_Z^{q/2}] \int_s^\infty e^{-x^2/2}\, dx.$$

Since U_Z has a density given by (3.3) it is easily seen that

$$\mathbb{E}[U_Z^{q/2}] \leq 1 + \frac{1}{\sqrt{2\pi}} \int_1^\infty x^{q/2-3/2}\, dx.$$

Moreover, using that $Z/c_Z =_d \text{NIG}(m(A)\alpha\delta, 0, 0, 1)$ and that $K_1(z) \geq e^{-z}/z$ for all $z > 0$, it is not difficult to show that there exists a constant c_3, not depending on s and A, such that

$$\int_s^\infty e^{-x^2/2}\, dx \leq c_3 \mathbb{P}(|Z/c_Z| > s), \qquad \text{for all } s \geq 1. \tag{3.5}$$

By combining the above we obtain (1.2) and by (3.5) applied on $s = 1$, C_q follows. The *only if*-implication of (ii) follows similar to the one of (i), now using that $(n^{-1}Y_{1/n})_{n\geq 1}$ converge weakly to a symmetric 1-stable distribution.

To show (iii) it is enough to prove that for all non-deterministic and square-integrable Lévy processes, Y, with no Gaussian component we have $\|Y_t\|_1 = o(t^{1/2})$ and $\|Y_t\|_2^2 \sim t\mathbb{E}[(Y_1 - \mathbb{E}[Y_1])^2]$ as $t \to 0$. The latter statement follows by the equality

$$\mathbb{E}[Y_t^2] = \mathrm{Var}(Y_t) + \mathbb{E}[Y_t]^2 = \mathrm{Var}(Y_1)t + \mathbb{E}[Y_1]^2 t^2.$$

To show that $\|Y_t\|_1 = o(t^{1/2})$ as $t \to 0$ we may assume that Y is symmetric. Indeed let $\mu = \mathbb{E}[Y_1]$, Y' an independent copy of Y and $\tilde{Y}_t = Y_t - Y_t'$. Then $\tilde{Y}$ is a symmetric square-integrable Lévy process and

$$\|Y_t\|_1 \leq \|Y_t - \mu t\|_1 + |\mu| \leq \|Y_t - \mu t - (Y_t' - \mu t)\|_1 + |\mu| t = \|\tilde{Y}_t\|_1 + |\mu| t.$$

Hence assume that Y is symmetric. Recall, e.g. from Hoffmann-Jørgensen (1994), Exercise 5.7, that for any random variable U we have

$$\|U\|_1 = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1 - \Re\varphi_U(s)}{s^2}\,ds,$$

where φ_U denotes the characteristic function of U. Using the inequalities $1 - e^{-x} \leq 1 \wedge x$ and $1 - \cos(x) \leq 4(1 \wedge x^2)$ for all $x \geq 0$ it follows that with $\psi(s) := 4\int (1 \wedge |sx|^2)\,\nu(dx)$ we have

$$\|Y_t\|_1 \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{1 - e^{-t\psi(s)}}{s^2}\,ds \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{|t\psi(s)| \wedge 1}{s^2}\,ds. \tag{3.6}$$

By substitution we get

$$\int_{\mathbb{R}} \frac{|t\psi(s)| \wedge 1}{s^2}\,ds \leq 2t^{1/2}\int_0^\infty \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2}\,ds. \tag{3.7}$$

From Lebesgue's dominated convergence theorem, it follows

$$\psi(s)s^{-2} = 4\int_{\mathbb{R}} (x^2 \wedge s^{-2})\,\nu(dx) \xrightarrow[s\to\infty]{} 0,$$

showing that for all $s > 0$, $t\psi(t^{-1/2}s) \to 0$ as $t \to 0$. With $c := 4\int_{\mathbb{R}}(1 \wedge |x|^2)\,\nu(dx)$ we have $\psi(s) \leq cs^2$ for all $s \geq 1$, and therefore for $t \in (0,1)$,

$$\frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2} \leq 1_{\{|s|\leq 1\}}c + 1_{\{|s|>1\}}s^{-2}, \qquad s \geq 0.$$

Thus,

$$\lim_{t\to 0}\int_0^\infty \frac{|t\psi(t^{-1/2}s)| \wedge 1}{s^2}\,ds = 0,$$

which by (3.6)–(3.7) completes the proof. □

A Appendix

To obtain explicit constant in Theorem 2.1 we need the following lemma:

Lemma A.1. *Let V denote a vector space, N a seminorm on V, $\varepsilon \in (0,1)$ and $x_0,\dots,x_d \in V$.*

$$\text{If } N\Big(\sum_{k=0}^{d} \lambda^k x_k\Big) \leq 1 \text{ for all } \lambda \in [-\varepsilon,\varepsilon] \quad \text{then} \quad N\Big(\sum_{k=0}^{d} x_k\Big) \leq 2^{d^2/2+d}\varepsilon^{-d}. \tag{A.1}$$

Proof. Assume first that $x_0,\dots,x_d \in \mathbb{R}$. By induction in d, let us show:

$$\text{If } \Big|\sum_{k=0}^{d} \lambda^k x_k\Big| \leq 1 \text{ for all } \lambda \in [-\varepsilon,\varepsilon] \quad \text{then} \quad \Big|\sum_{k=0}^{d} x_k\Big| \leq 2^{d^2/2+d}\varepsilon^{-d}. \tag{A.2}$$

For $d = 1,2$ (A.2) follows by a straightforward argument, so assume $d \geq 3$, (A.2) holds for $d-1$ and that the left-hand side of (A.2) holds for d. We have

$$\Big|\sum_{k=0}^{d} \lambda^k(\varepsilon^k x_k)\Big| \leq 1, \qquad \text{for all } \lambda \in [-1,1],$$

which by Pólya and Szegö (1954), Aufgabe 77, shows that $|x_d\varepsilon^d| \leq 2^d$ and hence $|x_d| \leq 2^d\varepsilon^{-d}$. For $\lambda \in [-\varepsilon,\varepsilon]$, the triangle inequality yields

$$\Big|\sum_{k=0}^{d-1} \lambda^k x_k\Big| \leq 1+2^d, \quad \text{and hence} \quad \Big|\sum_{k=0}^{d-1} \lambda^k \frac{x_k}{1+2^d}\Big| \leq 1.$$

The induction hypothesis implies

$$\Big|\sum_{k=0}^{d-1} x_k\Big| \leq \varepsilon^{-(d-1)}2^{(d-1)^2+(d-1)}(1+2^d),$$

and hence another application of the triangle inequality shows that

$$\begin{aligned}\Big|\sum_{k=0}^{d} x_k\Big| &\leq \varepsilon^{-d}2^d + \varepsilon^{-(d-1)}2^{(d-1)^2/2+(d-1)}(1+2^d) \\ &\leq \varepsilon^{-d}2^{d^2/2+d}\Big(2^{-d^2/2} + 2^{-1/2-d} + 2^{-1/2}\Big),\end{aligned}$$

which is less than or equal to $\varepsilon^{-d}2^{d^2/2+d}$ since $d \geq 3$. This completes the proof of (A.2).

Now let $x_0,\dots,x_d \in V$. Since N is a seminorm, the Hahn-Banach theorem (see Rudin (1991), Theorem 3.2) shows that there exists a family Λ of linear functionals on V such that

$$N(x) = \sup_{F\in\Lambda}|F(x)|, \qquad \text{for all } x \in V.$$

Assuming that the left-hand side of (A.1) is satisfied we have

$$\Big|\sum_{k=0}^{d} \lambda^k F(x_k)\Big| \leq 1, \qquad \text{for all } \lambda \in [-\varepsilon,\varepsilon] \text{ and all } F \in \Lambda,$$

which by (A.2) shows

$$\Big|F\Big(\sum_{k=0}^{d} x_k\Big)\Big| = \Big|\sum_{k=0}^{d} F(x_k)\Big| \leq 2^{d(d-1)}\varepsilon^{-d}, \qquad \text{for all } F \in \Lambda.$$

This completes the proof. □

References

Arcones, M. A. and E. Giné (1993). On decoupling, series expansions, and tail behavior of chaos processes. *J. Theoret. Probab. 6*(1), 101–122. MR1201060

Basse, A. and J. Pedersen (2009). Lévy driving moving averages and semimartingales. *Stochastic Process. Appl. 119*(9), 2970–2991. MR2554035

Basse-O'Connor, A. and S.-E. Graversen (2010). Path and semimartingale properties of chaos processes. *Stochastic Process. Appl. 120*(4), 522–540. MR2594369

Borell, C. (1978). Tail probabilities in Gauss space. In *Vector space measures and applications (Proc. Conf., Univ. Dublin, Dublin, 1977), II*, Volume 77 of *Lecture Notes in Phys.*, pp. 73–82. Berlin: Springer. MR0502400

Borell, C. (1979). On the integrability of Banach space valued Walsh polynomials. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, Volume 721 of *Lecture Notes in Math.*, pp. 1–3. Berlin: Springer. MR0544777

Borell, C. (1984). On polynomial chaos and integrability. *Probab. Math. Statist. 3*(2), 191–203. MR0764146

de Acosta, A. (1975). Stable measures and seminorms. *Ann. Probability 3*(5), 865–875. MR0391202

de la Peña, V. H. and E. Giné (1999). *Decoupling*. Probability and its Applications (New York). New York: Springer-Verlag. From dependence to independence, Randomly stopped processes. U-statistics and processes. Martingales and beyond. MR1666908

Fernique, X. (1970). Intégrabilité des vecteurs gaussiens. *C. R. Acad. Sci. Paris Sér. A-B 270*, A1698–A1699. MR0266263

Fernique, X. (1997). *Fonctions aléatoires gaussiennes, vecteurs aléatoires gaussiens*. Montreal, QC: Université de Montréal Centre de Recherches Mathématiques. MR1472975

Gemignani, M. C. (1990). *Elementary topology* (second ed.). New York: Dover Publications Inc. MR1088253

[=[

Hoffmann-Jørgensen, J. (1994). *Probability with a view toward statistics. Vol. I*. Chapman & Hall Probability Series. New York: Chapman & Hall.

Jain, N. C. and D. Monrad (1982). Gaussian quasimartingales. *Z. Wahrsch. Verw. Gebiete 59*(2), 139–159. MR0650607

Jain, N. C. and D. Monrad (1983). Gaussian measures in B_p. *Ann. Probab. 11*(1), 46–57. MR0682800

Krakowiak, W. and J. Szulga (1986). Random multilinear forms. *Ann. Probab. 14*(3), 955–973. MR0841596

Krakowiak, W. and J. Szulga (1988). A multiple stochastic integral with respect to a strictly p-stable random measure. *Ann. Probab. 16*(2), 764–777. MR0929077

Kwapień, S. and W. A. Woyczyński (1992). *Random series and stochastic integrals: single and multiple*. Probability and its Applications. Boston, MA: Birkhäuser Boston Inc. MR1167198

Ledoux, M. and M. Talagrand (1991). *Probability in Banach spaces*, Volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Berlin: Springer-Verlag. Isoperimetry and processes. MR1102015

Marcus, M. B. and J. Rosiński (2003). Sufficient conditions for boundedness of moving average processes. In *Stochastic inequalities and applications*, Volume 56 of *Progr. Probab.*, pp. 113–128. Basel: Birkhäuser. MR2073430

Nualart, D. (2006). *The Malliavin calculus and related topics* (Second ed.). Probability and its Applications (New York). Berlin: Springer-Verlag. MR2200233

Pisier, G. (1978). Les inégalités de Khintchine-Kahane, d'après C. Borell. In *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, pp. Exp. No. 7, 14. Palaiseau: École Polytech. MR0520209

Pólya, G. and G. Szegö (1954). *Aufgaben und Lehrsätze aus der Analysis. Zweiter Band. Funktionentheorie, Nullstellen, Polynome, Determinanten, Zahlentheorie*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Bd XX. Berlin: Springer-Verlag. 2te Aufl.

Rajput, B. S. and J. Rosiński (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Relat. Fields 82*(3), 451–487. MR1001524

Rosiński, J. (1986). On stochastic integral representation of stable processes with sample paths in Banach spaces. *J. Multivariate Anal. 20*(2), 277–302. MR0866076

Rosiński, J. and G. Samorodnitsky (1993). Distributions of subadditive functionals of sample paths of infinitely divisible processes. *Ann. Probab. 21*(2), 996–1014. MR1217577

Rosiński, J. and G. Samorodnitsky (1996). Symmetrization and concentration inequalities for multilinear forms with applications to zero-one laws for Lévy chaos. *Ann. Probab. 24*(1), 422–437. MR1387643

Rudin, W. (1991). *Functional analysis* (Second ed.). International Series in Pure and Applied Mathematics. New York: McGraw-Hill Inc. MR1157815