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A new probability measure-valued stochastic process with Ferguson-Dirichlet process as reversible measure*

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Abstract

A new diffusion process taking values in the space of all probability measures over $[0, 1]$ is constructed through Dirichlet form theory in this paper. This process is reversible with respect to the Ferguson-Dirichlet process (also called Poisson Dirichlet process), which is the reversible measure of the Fleming-Viot process with parent independent mutation. The intrinsic distance of this process is in the class of Wasserstein distances, so it's also a kind of Wasserstein diffusion. Moreover, this process satisfies the Log-Sobolev inequality.

Key words: Wasserstein diffusion; Logarithmic Sobolev inequalities; Ferguson-Dirichlet process; Fleming-Viot process.

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1 Introduction

This work is motivated by Von Renesse, M-K. and Sturm, K.T. [24] about Wasserstein diffusion on one dimensional space. There they constructed a probability measure-valued stochastic process, which is reversible with respect to (for short, w.r.t.) an “entropy measure”. Let $\mathcal{P}([0, 1])$ denote the collection of all probability measures on $[0, 1]$. Then under the entropy measure, almost surely the measure $\mu \in \mathcal{P}([0, 1])$ has no absolutely continuous part and no discrete part. The topological support of μ is negligible w.r.t. Lebesgue measure.

In this paper we are interested in the Ferguson-Dirichlet process, which is also called Poisson Dirichlet process. It is a probability measure on the space of all probability measures over a measurable space. The concrete definition will be given in next section. Ferguson-Dirichlet process is an important probability measure on the space of probability measures. It plays important roles in many research fields such as population genetics and Bayesian statistics. As proved in [7], it is the reversible measure of the Fleming-Viot processes with parent independent mutation. It was shown in [7] and [8] that under the Ferguson-Dirichlet process, almost surely the measure $\mu \in \mathcal{P}([0, 1])$ is discrete and has full topological support. This is very different to the “entropy measure”. However, [24, section 3] has pointed out some connection between the “entropy measure” and Ferguson-Dirichlet process. Moreover, Stannat, W. [18] showed that the Fleming-Viot process doesn’t satisfies the Log-Sobolev inequality unless the cardinality of the support of ν_0 in the Ferguson-Dirichlet process is finite (see the definition of Ferguson-Dirichlet process in (2.4)). Döring, M. and Stannat, W. [3] showed that the Wasserstein diffusion satisfies the Log-Sobolev inequality. So based on the works [24] and [3], it’s of great interest to ask whether there exists a probability measure-valued stochastic process, which is reversible w.r.t. Ferguson-Dirichlet process and satisfies the Log-Sobolev inequality at the same time. If it does, then it provides us a process which not only converges exponentially in L^2 -norm to the Ferguson-Dirichlet process, but also converges exponentially in entropy to the Ferguson-Dirichlet process. To be more precise, due to general results of functional inequalities, for the semigroup $(P_t)_{t \geq 0}$ determined by a Dirichlet form, which admits invariant measure π , it’s known that the Poincaré inequality is equivalent to the L^2 -exponential convergence:

$$\|P_t f - \pi(f)\|_{L^2(\pi)}^2 \leq \text{Var}(f) \exp[-\lambda t]$$

for some positive constant λ , where $\text{Var}(f) = \pi(f^2) - \pi(f)^2$. However, the Log-Sobolev inequality can yield the Poincaré inequality and

$$\text{Ent}(P_t f) \leq \text{Ent}(f) \exp[-\sigma t]$$

for some positive constant σ , where $\text{Ent}(f) = \pi(f \log f) - \pi(f) \log \pi(f)$ for positive function f . The well known exponential ergodicity for Markov process means that

$$\|P_t(x, \cdot) - \pi\|_{\text{var}} \leq C(x) e^{-\alpha t} \text{ for some } \alpha > 0,$$

where $\|\cdot\|_{\text{var}}$ is the total variation norm. According to Chen [2, Theorem 1.9], for reversible Markov process the L^2 -exponential convergence is equivalent to π -a.e. exponential ergodicity. Refer to Chen [2, Chapter 8] for a survey of the relation between functional inequalities and ergodic theory. Especially, Chen’s [2, Theorem 1.9] presents a diagram of nine types of ergodicity. If we are restricted to the Fleming-Viot process with parent independent mutation, Ethier and Griffiths [4] estimated

the rate of convergence in exponential ergodicity based on the explicit formula of the transition semigroup.

In this paper, we consider the space of all probability measures on the interval $[0, 1]$. On it, we will construct a new probability measure-valued process, whose reversible measure is the Ferguson-Dirichlet process. We shall show our new process satisfies the Log-Sobolev inequality, so, by the theory of functional inequalities, its associated semigroup will converge in entropy to its equilibrium with exponential rate. Our process is constructed through classical Dirichlet form theory. Moreover, we show that the intrinsic metric of this Dirichlet form is in the class of Wasserstein distances. This means that our new probability measure-valued process is also a kind of Wasserstein diffusion.

The structure of this paper is as follows: in next section we recall some facts about the Fleming-Viot process, and via comparing with the Fleming-Viot process we sketch the idea of our construction of the new process. In the third section we are concerned with constructing the Dirichlet form. In order to prove the pre-defined bilinear form to be closable, we consider the quasi-invariance property of Ferguson-Dirichlet measure (see Theorem 3.4) and establish the integration by parts formula (see Theorem 3.6). Then we constructed a regular Dirichlet form with Ferguson-Dirichlet process to be the reversible measure in Theorem 3.8. Furthermore, we discuss the intrinsic metric of this Dirichlet form (see Theorem 3.11). The last section is devoted to establish the Log-Sobolev inequality for our new probability measure-valued process. This is proved by the method of finite dimensional approximation based on our construction of Dirichlet form and Döring, M. and Stannat, W.'s work [3].

2 Comparison with Fleming-Viot process

Let's first introduce the Fleming-Viot process. Let E be a Polish space, i.e. a complete separable metric space. $\mathcal{P}(E)$ denotes the set of all probability measures on E , and $\mathcal{P}(\mathcal{P}(E))$ denotes the set of all probability measures on $\mathcal{P}(E)$. The Fleming-Viot process is initially established as the scaling limit of the measure-valued Moran model. The Fleming-Viot operator is defined as

$$\begin{aligned} (\mathcal{L}\phi)(\mu) &= \frac{1}{2} \int_E \int_E \mu(dx) (\delta_x(dy) - \mu(dy)) \frac{\delta^2 \phi(\mu)}{\delta \mu(x) \delta \mu(y)} \\ &\quad + \int_E \mu(dx) A \left(\frac{\delta \phi(\mu)}{\delta \mu(\cdot)} \right) (x), \quad \phi \in \text{Cyl}(\mathcal{D}(A)), \end{aligned} \quad (2.1)$$

where $\frac{\delta \phi(\mu)}{\delta \mu(x)} = \frac{d}{dt} \Big|_{t=0} \phi(\mu + t\delta_x)$, δ_x denotes the Dirac measure at x and A is the generator of a Feller semigroup on $C(E)$ with domain $\mathcal{D}(A)$, which represents the random mutation. Here

$$\text{Cyl}(\mathcal{D}(A)) = \{ \phi : \phi(\mu) = F(\langle f_1, \mu \rangle, \dots, \langle f_n, \mu \rangle), F \in C^\infty(\mathbb{R}^n), f_i \in \mathcal{D}(A), n \in \mathbb{N} \}, \quad (2.2)$$

where $\langle f, \mu \rangle = \int_E f d\mu$ for $f \in \mathcal{B}(E)$. When A is the parent independent mutation, i.e.

$$(Af)(x) = \frac{\theta}{2} \int_E (f(y) - f(x)) \nu_0(dy), \quad (2.3)$$

where $\theta > 0$ and $\nu_0 \in \mathcal{P}(E)$, Shiga, T. [17] showed that there is a unique reversible measure $\Pi_{\theta, \nu_0} \in \mathcal{P}(\mathcal{P}(E))$ which is characterized by the property that whenever $k \geq 2$ and $\{\Lambda_1, \dots, \Lambda_k\}$ is a

partition of E into disjoint sets, the mapping $\mu \mapsto (\mu(\Lambda_1), \dots, \mu(\Lambda_k))$ under Π_{θ, ν_0} has the Dirichlet distribution with parameter $\theta \nu_0(\Lambda_1), \dots, \theta \nu_0(\Lambda_k)$, i.e.

$$\begin{aligned} \Pi_{\theta, \nu_0}(\mu(\Lambda_1) \in dx_1, \dots, \mu(\Lambda_k) \in dx_k) \\ = \frac{\Gamma(\theta)}{\prod_{i=1}^k \Gamma(\theta \nu_0(\Lambda_i))} x_1^{\theta \nu_0(\Lambda_1) - 1} \dots x_k^{\theta \nu_0(\Lambda_k) - 1} \delta_{(1 - \sum_{i=1}^{k-1} x_i)}(dx_k) dx_1 \dots dx_{k-1} \end{aligned} \quad (2.4)$$

on the $(k-1)$ -dimensional simplex $\{(x_1, \dots, x_k); x_i \geq 0, \sum_{i=1}^k x_i = 1\}$. The measure Π_{θ, ν_0} is called the Ferguson-Dirichlet process or Poisson Dirichlet process. Ferguson [8] and Ethier and Kurtz [5, 7] showed that

$$\Pi_{\theta, \nu_0}(\cdot) = \mathbb{P}\left(\sum_{i=1}^{\infty} \rho_i \delta_{\xi_i} \in \cdot\right), \quad (2.5)$$

where (ρ_1, ρ_2, \dots) has the Poisson-Dirichlet distribution with parameter θ , and ξ_1, ξ_2, \dots are i.i.d. random variables distributed as ν_0 , independent of (ρ_1, ρ_2, \dots) . From this, we obtain that Π_{θ, ν_0} -almost everywhere $\mu \in \mathcal{P}(E)$ is a purely atomic probability measure.

With the Fleming-Viot operator \mathcal{L} and Poisson Dirichlet process Π_{θ, ν_0} , one can associate a symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as the closure of

$$\begin{aligned} \mathcal{E}(u, v) &= \int_{\mathcal{P}(E)} u(\mu) (\mathcal{L}v)(\mu) \Pi_{\theta, \nu_0}(d\mu) \\ &= \int_{\mathcal{P}(E)} \text{cor}_{\mu} \left(\frac{\delta u(\mu)}{\delta \mu(\cdot)}, \frac{\delta v(\mu)}{\delta \mu(\cdot)} \right) \Pi_{\theta, \nu_0}(d\mu), \quad u, v \in \text{Cyl}(\mathcal{D}(A)), \end{aligned}$$

where $\text{cor}_{\mu}(f, g) = \langle fg, \mu \rangle - \langle f, \mu \rangle \langle g, \mu \rangle$ for $f, g \in C(E)$. When A is the parent independent mutation, then $\mathcal{D}(A) = \mathcal{B}_b(E)$. Refer to Overbeck, L. et al. [14] for Dirichlet form theory about more general Fleming-Viot processes.

In the sequel of this paper, we only consider the Fleming-Viot process with parent independent mutation A as in (2.3). Then the Fleming-Viot operator acting on the cylindrical function $u(\mu) = F(\langle f_1, \mu \rangle, \dots, \langle f_n, \mu \rangle)$ has the representation

$$\begin{aligned} (\mathcal{L}u)(\mu) &= \frac{1}{2} \int_{E^2} \mu(dx) (\delta_x(dy) - \mu(dy)) \frac{\delta^2 u(\mu)}{\delta \mu(x) \delta \mu(y)} + \frac{\theta}{2} \int_E \mu(dx) \int_E \left(\frac{\delta u(\mu)}{\delta \mu(y)} - \frac{\delta u(\mu)}{\delta \mu(x)} \right) \nu_0(dy) \\ &= \frac{1}{2} \int_{E^2} \mu(dx) (\delta_x(dy) - \mu(dy)) \sum_{i,j=1}^n \partial_i \partial_j F(\langle \vec{f}, \mu \rangle) f_i(x) f_j(y) \\ &\quad + \frac{\theta}{2} \int_E \mu(dx) \int_E \left(\sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) f_i(y) - \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) f_i(x) \right) \nu_0(dy), \end{aligned}$$

where $\langle \vec{f}, \mu \rangle = (\langle f_1, \mu \rangle, \dots, \langle f_n, \mu \rangle)$. However, in the definition of $\frac{\delta u(\mu)}{\delta \mu(x)}$, $\mu + t \delta_x$ is not in $\mathcal{P}(E)$ for $t \neq 0$ any longer. When we look on $\mathcal{P}(E)$ as a space and consider to define the gradient for functionals depending only on its values on $\mathcal{P}(E)$, it's more appropriate to define

$$\nabla_x u(\mu) := \left. \frac{d}{dt} \right|_{t=0} u((1-t)\mu + t \delta_x). \quad (2.6)$$

It's easy to check through direct calculation that the Fleming-Viot operator \mathcal{L} can also be represented by

$$(\mathcal{L}u)(\mu) = \frac{1}{2} \int_E \nabla_x^2 u(\mu) \mu(dx) + \frac{\theta}{2} \int_E \mu(dx) \int_E (\nabla_y u(\mu) - \nabla_x u(\mu)) \nu_0(dy)$$

for $u \in \text{Cyl}$. Moreover, the associated Dirichlet form has the form

$$\mathcal{E}(u, v) = - \int_{\mathcal{P}(E)} u \mathcal{L}v \Pi_{\theta, \nu_0}(d\mu) = \int_{\mathcal{P}(E)} \langle \nabla \cdot u(\mu), \nabla \cdot v(\mu) \rangle_{\mu} \Pi_{\theta, \nu_0}(d\mu),$$

where $\langle f, g \rangle_{\mu} = \int_E f(x)g(x)\mu(dx)$ for $f, g \in C(E)$.

Recently there are many works looking on $\mathcal{P}(E)$ as an infinite dimensional Riemannian manifold. For instance, Jordan, R. et al. [10] defined a Riemannian manifold structure on the space of all probability measures on \mathbb{R}^d and constructed the solution of Fokker-Planck equation in \mathbb{R}^d through establishing gradient flow of relative entropy functional. Otto, F. and Villani, C. [13] used it as a guideline to find the interrelation between the transportation cost inequality and the Log-Sobolev inequality. This kind of viewpoint also stimulate them to find the so called "HWI" inequality, which includes three important quantities: "H" entropy, "W" Wasserstein distance, and "I" Fisher information into one inequality. Sturm, K.T. [20, 21] and Lott, J. and Villani, C. [11] further established the concept of lower bound of Ricci curvature on metric measure space E . Particularly, when E is a Riemannian manifold, their lower bound of Ricci curvature coincides with the geometric lower bound of Ricci curvature. Taking this viewpoint, as done in Schied [15], one can look on $L^2(\mu)$ as the tangent space at each point $\mu \in \mathcal{P}(E)$, i.e. $T_{\mu}\mathcal{P}(E) = L^2(\mu)$ and for any $f, g \in T_{\mu}\mathcal{P}(E)$, define its inner product by $\langle f, g \rangle_{\mu}$. Then the function $x \mapsto \nabla_x u(\mu)$ for some good functional u on $\mathcal{P}(E)$ is a tangent vector at μ and the Carré du champs operator $\Gamma(u, v) = \langle \nabla \cdot u(\mu), \nabla \cdot v(\mu) \rangle_{\mu}$ associated to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is just the inner product of tangent vectors. Define a mapping $S_f : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ for $f \in \mathcal{B}_b(E)$ by

$$d(S_f \mu) = \frac{e^f}{\langle e^f, \mu \rangle} d\mu. \quad (2.7)$$

This mapping is called the exponential map of the "Riemannian manifold" $\mathcal{P}(E)$ since the map $t \mapsto S_{tf} \mu$ from \mathbb{R} to $\mathcal{P}(E)$ generates a continuous curve in $\mathcal{P}(E)$ and

$$\left. \frac{d}{dt} \right|_{t=0} u(S_{tf} \mu) = \langle \nabla \cdot u(\mu), f \rangle_{\mu}. \quad (2.8)$$

Indeed, for $u \in \text{Cyl}$, for each $f \in T_{\mu}\mathcal{P}(E) = L^2(\mu)$ we have

$$\left. \frac{d}{dt} \right|_{t=0} u(S_{tf} \mu) = \langle \nabla \cdot u(\mu), f \rangle_{\mu} = \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) (\langle f_i f, \mu \rangle - \langle f_i, \mu \rangle \langle f, \mu \rangle).$$

About the exponential map S_f we should mention the work of Handa [9]. There he provided a characterization of a probability measure $\Pi \in \mathcal{P}(\mathcal{P}(E))$ to be reversible w.r.t. a general Fleming-Viot operator \mathcal{L} through the quasi-invariance property of the probability measure Π under the exponential map S_f for $f \in \mathcal{D}(A)$. Recall that A represents mutation in the Fleming-Viot operator. For more details on this topic refer to [9].

In this paper, we mainly consider a special case of E , that is, $E = [0, 1]$. Compared with the Fleming-Viot process on $\mathcal{P}([0, 1])$, we define the tangent space at μ as a subspace of $L^2((g_\mu)_*Leb)$, where g_μ denotes the cumulative distribution function of μ , Leb denotes the Lebesgue measure and $(g_\mu)_*Leb := Leb \circ g_\mu^{-1}$ denotes the push forward measure of Leb under the map $g_\mu : [0, 1] \rightarrow [0, 1]$. We will introduce a new exponential map $\tilde{S}_f : \mathcal{P}([0, 1]) \rightarrow \mathcal{P}([0, 1])$ for $f \in T_\mu\mathcal{P}([0, 1])$. Precisely, let \mathcal{G}_0 denote the space of all right continuous nondecreasing maps $g : [0, 1] \rightarrow [0, 1]$ with $g(0) = 0$, $g(1) = 1$. A C^2 -isomorphism $h \in \mathcal{G}_0$ means that $h : [0, 1] \rightarrow [0, 1]$ is increasing homeomorphism such that h and h^{-1} are bounded in $C^2([0, 1])$. For each C^2 -isomorphism $h \in \mathcal{G}_0$, define a mapping $\tilde{\tau}_h : \mathcal{P}([0, 1]) \rightarrow \mathcal{P}([0, 1])$ as: for each $\mu \in \mathcal{P}([0, 1])$ with cumulative distribution function g_μ , $\tilde{\tau}_h(\mu)$ is defined to be an absolutely continuous probability measure w.r.t. μ with density function \bar{h}_{g_μ} , namely,

$$\frac{d\tilde{\tau}_h(\mu)}{d\mu}(x) = \bar{h}_{g_\mu}(x), \quad (2.9)$$

where

$$\bar{h}_g(x) = \int_0^1 h'(r g(x^+) + (1-r)g(x^-)) dr, \quad g(x^+) = \lim_{y \rightarrow x^+} g(y), \quad g(x^-) = \lim_{y \rightarrow x^-} g(y).$$

When g is a continuous function, $\bar{h}_g(x) = h'(g(x))$. When g is of bounded variation, $dh(g(x)) = \bar{h}_g(x)dg(x)$ (refer to [1, Section 3.10] for chain rule of bounded variation functions).

Given a function $\phi \in C^\infty([0, 1], \mathbb{R})$ satisfying $\phi(0) = \phi(1) = 0$, let $t \mapsto X(t, x)$ be the unique solution of the following ODE:

$$\frac{dX_t}{dt} = \phi(X_t), \quad X_0 = x. \quad (2.10)$$

Put $e_{t\phi}(x) = X(t, x)$, then $e_{t\phi}(x) = e_\phi(t, x) = e_{t\phi}(1, x)$ and $e_{(t+s)\phi} = e_{t\phi} \circ e_{s\phi}$ for $t, s \in \mathbb{R}$ and $x \in [0, 1]$. The assumption $\phi(0) = \phi(1) = 0$ yields that $e_{t\phi}(0) = 0$, $e_{t\phi}(1) = 1$ for all $t \in \mathbb{R}$. Hence, $e_{t\phi}$ is a C^2 -isomorphism in \mathcal{G}_0 . Set

$$\mathcal{H}_0 = \{\phi \in C^\infty([0, 1], \mathbb{R}); \phi(0) = \phi(1) = 0\}. \quad (2.11)$$

Now we define $\tilde{S}_\phi : \mathcal{P}([0, 1]) \rightarrow \mathcal{P}([0, 1])$ by

$$d\tilde{S}_\phi(\mu) = d\tilde{\tau}_{e_\phi}(\mu), \quad \text{where } \phi \in \mathcal{H}_0, \quad (2.12)$$

then it holds $\tilde{S}_{(t+s)\phi} = \tilde{S}_{t\phi} \circ \tilde{S}_{s\phi}$ for $t, s \in \mathbb{R}$. For a function $u : \mathcal{P}([0, 1]) \rightarrow \mathbb{R}$, define its *directional derivative* along $\phi \in \mathcal{H}_0$ by

$$D_\phi u(\mu) = \lim_{t \rightarrow 0} \frac{1}{t} [u(\tilde{\tau}_{e_{t\phi}}(\mu)) - u(\mu)] \quad (2.13)$$

provided the limit exists. The tangent space of $\mathcal{P}([0, 1])$ at a point μ is now defined to be the closure of \mathcal{H}_0 in the norm of $L^2((g_\mu)_*Leb)$, denoted by $T_\mu\mathcal{P}$. We say that a function u has a gradient at μ if there exists a function $x \mapsto \nabla_x u(\mu)$ such that $D_\phi u(\mu) = \langle \nabla \cdot u(\mu), \phi \rangle_{T_\mu\mathcal{P}} := \int_{[0,1]} \nabla_{g_\mu(x)} u(\mu) \phi(g_\mu(x)) dx$ for each $\phi \in \mathcal{H}_0$. Define a symmetric bilinear form by

$$\mathcal{E}(u, v) = \int_{\mathcal{P}([0,1])} \langle \nabla \cdot u(\mu), \nabla \cdot v(\mu) \rangle_{T_\mu\mathcal{P}} \Pi_{\theta, v_0}(d\mu), \quad (2.14)$$

for $u, \nu \in \text{Cyl}$, where Cyl is defined in (3.4) below. We shall prove that $(\mathcal{E}, \text{Cyl})$ is closable and by classical Dirichlet form theory, we can obtain a probability measure-valued process. This process is reversible w.r.t. the Ferguson-Dirichlet process Π_{θ, ν_0} . Moreover, we will show this process satisfies the Log-Sobolev inequality.

3 Construction of the Dirichlet form

3.1 Quasi-invariance property

In order to prove the symmetric bilinear form \mathcal{E} defined in (2.14) is closable, we need to consider first the quasi-invariance of Π_{θ, ν_0} under the map \tilde{S}_f for $f \in \mathcal{H}_0$. From now on for simplicity of notation, we set $\mathcal{P} = \mathcal{P}([0, 1])$ and $\Pi_\theta = \Pi_{\theta, \text{Leb}}$ (that is, the Ferguson-Dirichlet measure Π_{θ, ν_0} with $\nu_0 = \text{Lebesgue measure}$). It's known that \mathcal{P} is compact and complete under the weak topology, and its weak topology coincides with the topology determined by L^p -Wasserstein distance $d_{w,p}$, $p \geq 1$, where

$$d_{w,p}(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left(\int_{[0,1]^2} |x - y|^p \pi(dx, dy) \right)^{1/p}.$$

Here and in the sequel, $\mathcal{C}(\mu, \nu)$ stands for the collection of all probability measures on $[0, 1] \times [0, 1]$ with marginals μ and ν respectively. Set $d_w = d_{w,2}$. Refer to [23] for these fundamental results on probability measure space.

Recall that \mathcal{G}_0 denotes the space of all right continuous nondecreasing maps $g : [0, 1] \rightarrow [0, 1]$. Each $g \in \mathcal{G}_0$ can be extended to the full interval by setting $g(1) = 1$. \mathcal{G}_0 is equipped with L^2 -distance

$$\|g_1 - g_2\|_{L^2} = \left(\int_0^1 |g_1(t) - g_2(t)|^2 dt \right)^{1/2}.$$

Definition 3.1 For $\theta > 0$, there exists a unique probability measure Q_0^θ on \mathcal{G}_0 , called Dirichlet process, with the property that for each $n \in \mathbb{N}$, and each family $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1$,

$$Q_0^\theta(g_{t_1} \in dx_1, \dots, g_{t_n} \in dx_n) = \frac{\Gamma(\theta)}{\prod_{i=1}^n \Gamma(\theta(t_{i+1} - t_i))} \prod_{i=1}^n (x_{i+1} - x_i)^{\theta(t_{i+1} - t_i) - 1} dx_1 \cdots dx_n,$$

with $x_{n+1} = 1$.

The measure Q_0^θ is sometimes called entropy measure, but as in [24], in this paper we use the entropy measure to only denote the push forward measure of Q_0^θ under the map $g \mapsto g_* \text{Leb}$. Define the map $\zeta : \mathcal{G}_0 \rightarrow \mathcal{P}$, $g \mapsto dg$. It's easy to see that $(\zeta)_* Q_0^\theta := Q_0^\theta \circ \zeta^{-1} = \Pi_\theta$. Its inverse ζ^{-1} assigns to each probability measure its distribution function. In the following we will study the quasi-invariance property of Π_{θ, ν_0} through Q_0^θ and ζ . Von Renesse, M-K. and Sturm, K.T. [24] has studied the quasi-invariance property of Q_0^θ on \mathcal{G}_0 and under the map $\chi : \mathcal{G}_0 \rightarrow \mathcal{P}$, $g \mapsto g_* \text{Leb}$, Q_0^θ is pushed forward to a probability measure on \mathcal{P} , which is called entropy measure there. Then through Dirichlet form theory, a stochastic process is constructed on \mathcal{P} . Since its intrinsic metric of this Dirichlet form is just the L^2 -Wasserstein distance on \mathcal{P} , this process is usually called Wasserstein diffusion. Our present work also depends on the knowledge of Q_0^θ . Let's recall the quasi-invariance property of Q_0^θ .

Theorem 3.2 ([24] **Theorem 4.3**) *Each C^2 -isomorphism $h \in \mathcal{G}_0$ induces a bijection map $\tau_h : \mathcal{G}_0 \rightarrow \mathcal{G}_0$, $g \mapsto h \circ g$, which leaves Q_0^θ quasi-invariant:*

$$dQ_0^\theta(h \circ g) = Y_{h,0}^\theta(g) dQ_0^\theta(g),$$

and $Y_{h,0}^\theta$ is bounded from above and below. Here

$$Y_{h,0}^\theta(g) = X_h^\theta(g) Y_{h,0}(g), \quad (3.1)$$

where

$$Y_{h,0}(g) = \frac{1}{\sqrt{h'(0)h'(1)}} \prod_{a \in J_g} \frac{\sqrt{h'(g(a^-)) - h'(g(a^+))}}{\frac{\delta(h \circ g)}{\delta g}(a)},$$

$$X_h(g) = \exp \left(\theta \int_0^1 \log h'(g(s)) ds \right),$$

$$J_g = \{x \in [0, 1]; g(x^+) \neq g(x^-)\}, \quad \frac{\delta(h \circ g)}{\delta g}(a) = \frac{h(g(a^+)) - h(g(a^-))}{g(a^+) - g(a^-)}.$$

Due to the compactness of the interval $[0, 1]$ and \mathcal{P} , several well known topologies on \mathcal{G}_0 and \mathcal{P} coincide. More precisely, for each sequence $(g_n) \subset \mathcal{G}_0$, and each $g \in \mathcal{G}_0$, the following types of convergence are equivalent:

- $g_n(t) \rightarrow g(t)$ for each $t \in [0, 1]$ in which g is continuous;
- $g_n \rightarrow g$ in $L^p([0, 1])$ for each $p \geq 1$;
- $\mu_{g_n} \rightarrow \mu_g$ weakly;
- $\mu_{g_n} \rightarrow \mu_t$ in the L^p -Wasserstein distance for each $p \geq 1$.

Refer to [24] for a sketch of the idea of the argument.

Lemma 3.3 *For each C^2 -isomorphism $h \in \mathcal{G}_0$, $\tilde{\tau}_h : \mathcal{P} \rightarrow \mathcal{P}$ defined by (2.9) is continuous.*

Proof. Let $\mu_n \in \mathcal{P}$, $n \geq 1$ and μ_n converges to μ as $n \rightarrow \infty$. g_{μ_n} and g_μ denote the corresponding cumulative distribution functions of μ_n and μ . Then g_{μ_n} converges to g_μ in $L^p([0, 1])$ for each $p \geq 1$. Set $\nu_n = \tilde{\tau}_h(\mu_n)$, $\nu = \tilde{\tau}_h(\mu)$. Then the cumulative function of ν_n and ν are $h \circ g_{\mu_n}$ and $h \circ g_\mu$ respectively. We denote by $d_{w,1}$ the L^1 -Wasserstein distance, that is, $d_{w,1}(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left\{ \int_{[0,1]^2} |x - y| \pi(dx, dy) \right\}$. Then according to [22, Theorem 2.18] about optimal transport on \mathbb{R} , we have

$$\begin{aligned} d_{w,1}(\nu_n, \nu) &= \int_0^1 |g_{\nu_n}^{-1}(t) - g_\nu^{-1}(t)| dt = \int_0^1 |g_{\nu_n}(t) - g_\nu(t)| dt \\ &= \int_0^1 |h \circ g_{\mu_n}(t) - h \circ g_\mu(t)| dt \leq \max_{s \in [0,1]} |h'(s)| \int_0^1 |g_{\mu_n}(t) - g_\mu(t)| dt, \end{aligned}$$

which yields that as μ_n weakly converges to μ , ν_n converges to ν as well. This is the desired result. In particular, $\tilde{\tau}_h$ is measurable from \mathcal{P} to \mathcal{P} for C^2 -isomorphism $h \in \mathcal{G}_0$. ■

Theorem 3.4 (Quasi-invariance) Let ν_0 be the Lebesgue measure on $[0, 1]$. For each C^2 -isomorphism $h \in \mathcal{G}_0$, the Ferguson-Dirichlet measure Π_{θ, ν_0} is quasi-invariant under the transformation $\tilde{\tau}_h$, and

$$d\Pi_{\theta, \nu_0}(\tilde{\tau}_h(\mu)) = Y_{h,0}^\theta(g_\mu) d\Pi_{\theta, \nu_0}(\mu), \quad (3.2)$$

where $Y_{h,0}^\theta(g_\mu)$ is defined as (3.1), and g_μ denotes the cumulative distribution function of μ .

Proof. For any bounded measurable function u on \mathcal{P} , it can induce a bounded measurable function \bar{u} on \mathcal{G}_0 by

$$\bar{u}(g) := u(\zeta(g)).$$

Note that for C^2 -isomorphism $h \in \mathcal{G}_0$, $\tau_h^{-1} = \tau_{h^{-1}}$ (see Theorem 3.2 for the definition), and

$$\zeta \circ \tau_h \circ \zeta^{-1} = \tilde{\tau}_h. \quad (3.3)$$

So $\tilde{\tau}_h$ is a bijection map and $\tilde{\tau}_h^{-1} = \tilde{\tau}_{h^{-1}}$. To see this, noting that $g_\mu = \zeta^{-1}(\mu)$, we have for any $f \in \mathcal{B}([0, 1])$, for any $\mu \in \mathcal{P}$,

$$\int_0^1 f(x) d(\zeta \circ \tau_h)(g_\mu) = \int_0^1 f(x) d\tau_h(g_\mu) = \int_0^1 \bar{h}(g_\mu(x)) dg_\mu(x) = \int_0^1 f(x) d\tilde{\tau}_h(\mu).$$

According to Theorem 3.2, we obtain

$$\begin{aligned} \int_{\mathcal{P}} u(\mu) d\Pi_{\theta, \nu_0}(\tilde{\tau}_h(\mu)) &= \int_{\mathcal{P}} u(\tilde{\tau}_h^{-1}(\mu)) d\Pi_{\theta, \nu_0}(\mu) = \int_{\mathcal{G}_0} \bar{u}(\tau_h^{-1}(g)) dQ_0^\theta(g) \\ &= \int_{\mathcal{G}_0} \bar{u}(g) dQ_0^\theta(\tau_h(g)) = \int_{\mathcal{G}_0} \bar{u}(g) Y_{h,0}^\theta(g) dQ_0^\theta(g) \\ &= \int_{\mathcal{P}} u(\mu) Y_{h,0}^\theta(g_\mu) d\Pi_{\theta, \nu_0}(\mu), \end{aligned}$$

which concludes the proof. ■

3.2 Integration by parts formula

In section 2, we have defined the map $e_{t\phi} : [0, 1] \rightarrow [0, 1]$ and the derivative of a function $u : \mathcal{P} \rightarrow \mathbb{R}$ by

$$D_\phi u(\mu) = \lim_{t \rightarrow 0} \frac{1}{t} [u(\tilde{\tau}_{e_{t\phi}})(\mu) - u(\mu)]$$

along $\phi \in \mathcal{H}_0$, provided the limit exists. Let Cyl be the set of all functions on \mathcal{P} in the form

$$u(\mu) = F(\langle f_1, \mu \rangle, \dots, \langle f_n, \mu \rangle), \quad (3.4)$$

where $F \in C^1(\mathbb{R}^n)$, $f_i \in C^1([0, 1])$ for $i = 1, \dots, n$ and $n \in \mathbb{N}$. Let $\text{Cyl}(\mathcal{G}_0)$ be the set of all functions $w : \mathcal{G}_0 \rightarrow \mathbb{R}$ in the form

$$w(g) = F\left(\int f_1 dg, \dots, \int f_n dg\right) \quad (3.5)$$

where $F \in C^1(\mathbb{R}^n)$, $f_i \in C^1([0, 1])$ and $n \in \mathbb{N}$. For a function $w : \mathcal{G}_0 \rightarrow \mathbb{R}$, define its directional derivative along $\phi \in \mathcal{H}_0$ by

$$D_\phi w(g) = \left. \frac{d}{dt} \right|_{t=0} w(\tau_{e_t\phi}(g)) = \left. \frac{d}{dt} \right|_{t=0} w(e_{t\phi} \circ g) \quad (3.6)$$

provided the limit exists.

Lemma 3.5 (i) For each $w \in \text{Cyl}(\mathcal{G}_0)$ in the form (3.5), $D_\phi w(g)$ exists for each $\phi \in \mathcal{H}_0$ at every point $g \in \mathcal{G}_0$, and

$$D_\phi w(g) = \sum_{i=1}^n \partial_i F \left(\int \vec{f} dg \right) \cdot \int_0^1 f_i \bar{\phi}_g dg, \quad (3.7)$$

where $\int \vec{f} dg = (\int f_1 dg, \dots, \int f_n dg)$ and $\bar{\phi}_g(x) = \int_0^1 \phi'(rg(x^+) + (1-r)g(x^-)) dr$ for $g \in \mathcal{G}_0$.

(ii) For each $u \in \text{Cyl}$ on \mathcal{P} in the form (3.4), $D_\phi u(\mu)$ exists for each direction $\phi \in \mathcal{H}_0$ at every $\mu \in \mathcal{P}$, and

$$D_\phi u(\mu) = \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) \cdot \int_0^1 f_i(x) \bar{\phi}_{g_\mu}(x) d\mu(x), \quad (3.8)$$

where $\langle \vec{f}, \mu \rangle = (\langle f_1, \mu \rangle, \dots, \langle f_n, \mu \rangle)$ and $\bar{\phi}_{g_\mu}$ defined as in (i).

Proof. (i) By virtue of integration by parts formula on $[0, 1]$, we have

$$\int_0^1 f_i(x) dg(x) = f_i(1)g(1) - f_i(0)g(0) - \int_0^1 f_i'(x)g(x)dx.$$

Using the chain rule for bounded variation function in Vol'pert average form (cf. [1, Thorem 3.96, Remark 3.98]), it holds

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \int_0^1 f_i(x) d(e_{t\phi} \circ g)(x) &= - \int_0^1 f_i'(x) \phi(g(x)) dx \\ &= \int_0^1 f_i(x) \bar{\phi}_g(x) dg(x) \end{aligned}$$

by noting that $\phi(0) = \phi(1) = 0$ for $\phi \in \mathcal{H}_0$. Therefore,

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} u(\tau_{e_t\phi} \circ g) &= \left. \frac{d}{dt} \right|_{t=0} u(e_{t\phi} \circ g) = \left. \frac{d}{dt} \right|_{t=0} F \left(\int \vec{f}(x) d(e_{t\phi} \circ g)(x) \right) \\ &= \sum_{i=1}^n \partial_i F \left(\int \vec{f}(x) dg(x) \right) \cdot \int_0^1 f_i(x) \bar{\phi}_g(x) dg(x). \end{aligned}$$

(ii) Define $\bar{u}(g) = u \circ \zeta(g)$, then \bar{u} is in the form

$$\bar{u}(g) = F \left(\int f_1 dg, \dots, \int f_n dg \right).$$

Invoking (3.3),

$$\begin{aligned}
\frac{d}{dt} \Big|_{t=0} u(\tilde{\tau}_{e_t\phi}(\mu)) &= \frac{d}{dt} \Big|_{t=0} u(\zeta(\tau_{e_t\phi}(g_\mu))) = \frac{d}{dt} \Big|_{t=0} \bar{u}(\tau_{e_t\phi}(g_\mu)) \\
&= \sum_{i=1}^n \partial_i F \left(\int \vec{f} d g_\mu \right) \cdot \int_0^1 f_i(x) \bar{\phi}_{g_\mu}(x) d g_\mu(x) \\
&= \sum_{i=1}^n \partial_i F(\vec{f}, \mu) \cdot \int_0^1 f_i(x) \bar{\phi}_{g_\mu}(x) d \mu(x).
\end{aligned}$$

which concludes the proof. \blacksquare

Before stating the integration by parts formula, we recall a result on the derivative of $g \mapsto Y_{h,0}^\theta(g)$ appeared in the quasi-invariance formula. According to [24, Lemma 5.7], for $\phi \in C^2([0,1])$ with $\phi(0) = \phi(1) = 0$ and $\theta \geq 0$,

$$\begin{aligned}
\frac{\partial}{\partial t} Y_{e_t\phi,0}^\theta(g) &= \sum_{a \in J_g} \left[\frac{\phi'(g(a^+)) + \phi'(g(a^-))}{2} - \frac{\phi(g(a^+)) - \phi(g(a^-))}{g(a^+) - g(a^-)} \right] \\
&\quad + \theta \int_0^1 \phi'(g(x)) dx - \frac{\phi'(0) + \phi'(1)}{2} \\
&=: V_\phi^\theta(g),
\end{aligned} \tag{3.9}$$

where $J_g = \{x \in [0,1]; g(x^+) \neq g(x^-)\}$.

Theorem 3.6 (Integration by parts formula) (i) For each $\phi \in \mathcal{H}_0$, $u \in \text{Cyl}(\mathcal{G}_0)$, it holds that

$$\int_{\mathcal{G}_0} v(g) D_\phi u(g) dQ_0^\theta(g) = \int_{\mathcal{G}_0} u(g) D_\phi^* v(g) dQ_0^\theta(g) \tag{3.10}$$

for any $v \in \text{Cyl}(\mathcal{G}_0)$, where

$$D_\phi^* v(g) = -D_\phi v(g) - V_\phi^\theta(g) v(g). \tag{3.11}$$

(ii) For each $\phi \in \mathcal{H}_0$, $u \in \text{Cyl}$, it holds that

$$\int_{\mathcal{P}} v(\mu) D_\phi u(\mu) d\Pi_\theta(\mu) = \int_{\mathcal{P}} u(\mu) D_\phi^* v(\mu) d\Pi_\theta(\mu) \tag{3.12}$$

for any $v \in \text{Cyl}$, where

$$D_\phi^* v(\mu) = -D_\phi v(\mu) - V_\phi^\theta(g_\mu) v(\mu). \tag{3.13}$$

Proof. We shall only prove (i), and (ii) can be proved by the similar method used in the previous lemma. By the quasi-invariance of Q_0^θ , one has

$$\begin{aligned}
\int_{\mathcal{G}_0} v(g) D_\phi u(g) dQ_0^\theta(g) &= \lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathcal{G}_0} v(g) (u(e_{t\phi} \circ g) - u(g)) dQ_0^\theta(g) \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \int_{\mathcal{G}_0} u(g) (v(e_{-t\phi} \circ g) Y_{e_t\phi}^\theta(g) - 1) dQ_0^\theta(g) \\
&= \int_{\mathcal{G}_0} u(g) (-D_\phi v(g) - V_\phi^\theta(g) v(g)) dQ_0^\theta(g),
\end{aligned}$$

which concludes (i). ■

3.3 Tangent space and Dirichlet form

The goal of this subsection is to obtain our stochastic process through establishing a Dirichlet form on \mathcal{P} . The key point is how to specify a suitable pre-Hilbert norm $\|\cdot\|_\mu$ on $T_\mu\mathcal{P}$ such that the direction derivative $D_\phi u(\mu)$ of a nice function u on \mathcal{P} could determine a bounded linear function $\phi \mapsto D_\phi u(\mu)$ on \mathcal{H}_0 . Then Riesz representation theorem yields that there exists a unique element $Du(\mu) \in T_\mu\mathcal{P}$ with

$$D_\phi u(\mu) = \langle Du(\mu), \phi \rangle_{T_\mu\mathcal{P}}, \quad \forall \phi \in T_\mu\mathcal{P}.$$

Here $T_\mu\mathcal{P}$ is the completion of \mathcal{H}_0 w.r.t. the pre-Hilbert norm $\|\cdot\|_\mu$.

Naturally, the nice functions are cylindrical functions. Let $u \in \text{Cyl}$ in the form (3.4). For $\phi \in \mathcal{H}_0$,

$$\begin{aligned} D_\phi u(\mu) &= \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) \cdot \int_0^1 f_i(x) \bar{\phi}_{g_\mu}(x) dg_\mu(x) \\ &= - \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) \cdot \int_0^1 f'_i(x) \phi(g_\mu(x)) dx \\ &= - \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) \cdot \int_0^1 f'_i(g_\mu^{-1}(x)) \phi(x) d(g_\mu)_* \text{Leb} \\ &= \int_0^1 \left(- \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) \cdot f'_i(g_\mu^{-1}(x)) \right) \phi(x) d(g_\mu)_* \text{Leb}, \end{aligned} \tag{3.14}$$

where $\langle \vec{f}, \mu \rangle = (\langle f_1, \mu \rangle, \dots, \langle f_n, \mu \rangle)$, $\int_0^1 f(x) d(g_\mu)_* \text{Leb} = \int_0^1 f(g_\mu(x)) dx$ for $f \in \mathcal{B}_b([0, 1])$. According to this expression, we choose the pre-Hilbert norm to be the norm of $L^2((g_\mu)_* \text{Leb})$ at $\mu \in \mathcal{P}$. Then $\phi \mapsto D_\phi u(\mu)$ is a bounded linear function on \mathcal{H}_0 . So it can be extended to be a bounded linear functional on the completion of \mathcal{H}_0 w.r.t. the norm of $L^2((g_\mu)_* \text{Leb})$. Therefore, the tangent space $T_\mu\mathcal{P}$ is defined to be the completion of \mathcal{H}_0 w.r.t. the norm of $L^2((g_\mu)_* \text{Leb})$.

Definition 3.7 The gradient of function $u : \mathcal{P} \rightarrow \mathbb{R}$ is said to exist at $\mu \in \mathcal{P}$, if there exists a function $\psi : [0, 1] \rightarrow \mathbb{R}$ such that for any $\phi \in \mathcal{H}_0$,

$$D_\phi u(\mu) = \int_0^1 \psi(x) \phi(x) d(g_\mu)_* \text{Leb} = \int_0^1 \psi(g_\mu(x)) \phi(g_\mu(x)) dx.$$

Then the gradient $\psi(\cdot)$ at μ is denoted by $\nabla \cdot u(\mu)$.

Similarly, for a function $u : \mathcal{G}_0 \rightarrow \mathbb{R}$, if there exists a function $\psi : [0, 1] \rightarrow \mathbb{R}$ such that

$$D_\phi u(g) = \int_0^1 \phi(x) \psi(x) dg_* \text{Leb}, \quad \forall \phi \in \mathcal{H}_0,$$

then its gradient is said to exist at $g \in \mathcal{G}_0$ and is denoted by $\nabla \cdot u(g)$.

Due to the calculation (3.14), we know that for a cylindrical function u in the form (3.4), its gradient exists at every $\mu \in \mathcal{D}$ and

$$\nabla_x u(\mu) = - \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) \cdot f'_i(g_\mu^{-1}(x)). \quad (3.15)$$

Next, we define a symmetric bilinear form by

$$\mathcal{E}(u, v) = \int_{\mathcal{D}} \langle \nabla u(\mu), \nabla v(\mu) \rangle_{T_\mu \mathcal{D}} \Pi_\theta(d\mu), \quad \text{for } u, v \in \text{Cyl}, \quad (3.16)$$

where $\Pi_\theta = \Pi_{\theta, Leb}$, $\langle f_1, f_2 \rangle_{T_\mu \mathcal{D}} = \int_0^1 f_1(x) f_2(x) d(g_\mu)_* Leb$ for any $f_1, f_2 \in \mathcal{B}_b([0, 1])$.

Let Cyl_0 be the set of all cylindrical functions in the form $u(\mu) = F(\langle f_1, \mu \rangle, \dots, \langle f_n, \mu \rangle)$ with $F \in C^\infty(\mathbb{R}^n)$, $f_i \in C^\infty([0, 1])$ satisfying $f'_i(0) = f'_i(1) = 0$, $i = 1, \dots, n$, and $n \in \mathbb{N}$.

Theorem 3.8 (i) $(\mathcal{E}, \text{Cyl})$ is closable. Its closure $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular recurrent Dirichlet form with reversible measure $\Pi_\theta = \text{Ferguson-Dirichlet measure}$.

(ii) The set Cyl_0 is a core for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

(iii) The generator $(\widetilde{\mathcal{L}}, \mathcal{D}(\widetilde{\mathcal{L}}))$ of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the Friedrichs extension of the operator $(\mathcal{L}, \text{Cyl})$ given by

$$\begin{aligned} (\widetilde{\mathcal{L}}u)(\mu) &= - \sum_{i=1}^n \left(D_{f'_i \circ g_\mu^{-1}} \partial_i F(\langle \vec{f}, \cdot \rangle) \right)(\mu) \\ &= - \sum_{i=1}^n \sum_{j=1}^n \partial_j \partial_i F(\langle \vec{f}, \mu \rangle) \cdot \int_0^1 f'_i(x) f'_j(x) dx + \sum_{i=1}^n V_{f'_i \circ g_\mu^{-1}}^\theta(g_\mu) \partial_i F(\langle \vec{f}, \mu \rangle), \end{aligned} \quad (3.17)$$

where $V_\phi^\theta(g)$ for $\phi \in \mathcal{H}_0$, $g \in \mathcal{G}_0$ is defined in (3.9).

Proof. a) Let $u(\mu) = F(\langle f_1, \mu \rangle, \dots, \langle f_n, \mu \rangle) \in \text{Cyl}$, $v(\mu) = G(\langle g_1, \mu \rangle, \dots, \langle g_m, \mu \rangle) \in \text{Cyl}$. It's not restrictive by assuming $n = m$. Since if not, for instance, $n > m$, one can set $g_{m+1} = \dots = g_n = 0$, and $\bar{G}(x_1, \dots, x_n) = G(x_1, \dots, x_m)$ to get $v(\mu) = \bar{G}(\langle g_1, \mu \rangle, \dots, \langle g_n, \mu \rangle)$. By (3.15),

$$\begin{aligned} \mathcal{E}(u, v) &= \int_{\mathcal{D}} \langle \nabla u(\mu), \nabla v(\mu) \rangle_{T_\mu \mathcal{D}} \Pi_\theta(d\mu) \\ &= \int_{\mathcal{D}} \left(\int_0^1 \sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) f'_i(g_\mu^{-1}(x)) \cdot \sum_{j=1}^n \partial_j G(\langle \vec{g}, \mu \rangle) g'_j(g_\mu^{-1}(x)) d(g_\mu)_* Leb \right) \Pi_\theta(d\mu) \\ &= \int_{\mathcal{D}} \left(\sum_{i=1}^n \partial_i F(\langle \vec{f}, \mu \rangle) \cdot \int_0^1 f'_i(x) \left(\sum_{j=1}^n \partial_j G(\langle \vec{g}, \mu \rangle) g'_j(x) \right) dx \right) \Pi_\theta(d\mu) \\ &= \int_{\mathcal{D}} \sum_{j=1}^n \partial_j G(\langle \vec{g}, \mu \rangle) D_{g'_j \circ g_\mu^{-1}} u(\mu) \Pi_\theta(d\mu). \end{aligned}$$

By the integration by parts formula (3.12), previous equality

$$\begin{aligned} &= \int_{\mathcal{D}} \sum_{j=1}^n D_{g_j' \circ g_\mu^{-1}}^* \partial_j G(\langle \vec{g}, \mu \rangle) u(\mu) \Pi_\theta(d\mu) \\ &= - \int_{\mathcal{D}} \widetilde{\mathcal{L}}v(\mu) \cdot u(\mu) \Pi_\theta(d\mu). \end{aligned}$$

This proves that $(\widetilde{\mathcal{L}}, \text{Cyl}_0)$ is a symmetric operator, and $(\mathcal{E}, \text{Cyl}_0)$ is closable, its generator coincides with the Friedrichs extension of $\widetilde{\mathcal{L}}$.

b) Now let's prove that Cyl_0 is dense in Cyl . For simplicity, assume that u is of form $u(\mu) = F(\langle f, \mu \rangle)$, $F \in C^1(\mathbb{R})$, and $f \in C^1([0, 1])$, that is, for simplicity, consider $n = 1$. Let $F_\varepsilon \in C^\infty(\mathbb{R})$, for $\varepsilon > 0$ be smooth approximation of F with $\|F - F_\varepsilon\|_\infty + \|F' - F'_\varepsilon\|_\infty \rightarrow 0$, as $\varepsilon \rightarrow 0$. Let $f_\varepsilon \in C^\infty$ with $f'_\varepsilon(0) = f'_\varepsilon(1) = 0$ be smooth approximation of f with $\|f - f_\varepsilon\|_\infty \rightarrow 0$ and $f'_\varepsilon(t) \rightarrow f'(t)$ for all $t \in (0, 1)$ as $\varepsilon \rightarrow 0$. Moreover, assume that $\sup_\varepsilon \|f'_\varepsilon\|_\infty < \infty$. Define $u_\varepsilon(\mu) = F_\varepsilon(\langle f_\varepsilon, \mu \rangle)$. Then $u_\varepsilon \in \text{Cyl}_0$ and $u_\varepsilon \rightarrow u$ in $L^2(\Pi_\theta)$ as $\varepsilon \rightarrow 0$ by dominated convergence theorem. Since

$$\begin{aligned} &\sup_\varepsilon \sup_{\mu \in \mathcal{D}} \int_0^1 F'_\varepsilon(\langle f_\varepsilon, \mu \rangle)^2 f'_\varepsilon(x)^2 dx \leq C, \\ &\int_0^1 F'_\varepsilon(\langle f_\varepsilon, \mu \rangle)^2 f'_\varepsilon(x)^2 dx \rightarrow \int_0^1 F'(\langle f, \mu \rangle)^2 f'(x)^2 dx, \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{E}(u_\varepsilon, u_\varepsilon) &= \int_{\mathcal{D}} \left(\int_0^1 F'_\varepsilon(\langle f_\varepsilon, \mu \rangle)^2 f'_\varepsilon(x)^2 dx \right) \Pi_\theta(d\mu) \\ &\rightarrow \int_{\mathcal{D}} \left(\int_0^1 F'(\langle f, \mu \rangle)^2 f'(x)^2 dx \right) \Pi_\theta(d\mu) \end{aligned}$$

as $\varepsilon \rightarrow 0$ by dominated convergence theorem in $L^2(\Pi_\theta)$. This implies that $(u_\varepsilon)_\varepsilon$ constitutes a Cauchy sequence relative to the norm

$$\|v\|_{\mathcal{E},1}^2 := \|v\|_{L^2}^2 + \mathcal{E}(v, v).$$

In fact, since $(u_\varepsilon)_\varepsilon$ is uniformly bounded w.r.t. $\|\cdot\|_{\mathcal{E},1}$, by weak compactness there exists a subsequence converging weakly in $(\mathcal{D}(\mathcal{E}), \|\cdot\|_{\mathcal{E},1})$. Since the associated norm converges,

$$\|u_\varepsilon - u\|_{\mathcal{E},1}^2 = \|u_\varepsilon\|_{\mathcal{E},1}^2 - 2\langle u_\varepsilon, u \rangle_{L^2} - 2\mathcal{E}(u_\varepsilon, u) + \|u\|_{\mathcal{E},1}^2 \rightarrow 0.$$

Moreover, as $u_\varepsilon \rightarrow u$ in $L^2(\Pi_\theta)$, the limit is unique. Hence the entire sequence converges to $u \in (\mathcal{D}(\mathcal{E}), \|\cdot\|_{\mathcal{E},1})$. Particularly, $\mathcal{E}(u_\varepsilon, u_\varepsilon) \rightarrow \mathcal{E}(u, u)$. This proves Cyl_0 is dense in Cyl . Combining with $(\mathcal{E}, \text{Cyl}_0)$ is closable proved in a), we get $(\mathcal{E}, \text{Cyl})$ is closable and that the closures of Cyl_0 and Cyl coincide.

c) Obviously $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ has the Markovian property. Hence it's a Dirichlet form. Since constant functions belong to $\mathcal{D}(\mathcal{E})$, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is recurrent. Furthermore, by Stone-Weierstrass theorem, the fact that Cyl separates the points in \mathcal{D} yields Cyl is dense in $(C(\mathcal{D}), \|\cdot\|_\infty)$. Hence, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular. ■

According to the theory of Dirichlet form (refer to [12]), any regular Dirichlet form possessing local property, i.e.

$$\mathcal{E}(u, v) = 0, \text{ for all } u, v \in \mathcal{D}(\mathcal{E}) \text{ with } \text{supp}[u] \cap \text{supp}[v] = \emptyset,$$

where $\text{supp}[u]$ denotes the support of u , admits a diffusion process, that is, a strong Markov process with continuous sample paths with probability 1. It's clear that Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ possesses the local property, so it admits a diffusion process. Note that here sample paths are continuous in the weak topology of \mathcal{P} . Is this process still continuous in the topology determined by the total variation norm? We tend to believe the negative answer, but we can't prove it now.

Remark 3.9 In the argument of Theorem 3.8, we know that operator $\widetilde{\mathcal{L}}$ is symmetric w.r.t. Π_θ , i.e.

$$\int_{\mathcal{P}} u(\mu)(\widetilde{\mathcal{L}}v)(\mu) d\Pi_\theta(\mu) = \int_{\mathcal{P}} v(\mu)(\widetilde{\mathcal{L}}u)(\mu) d\Pi_\theta(\mu), \quad u, v \in \mathcal{D}(\widetilde{\mathcal{L}}).$$

So the diffusion process associated $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is reversible w.r.t. Π_θ .

Our next goal is to give a description of the intrinsic metric associated to our Dirichlet form. In [24], they constructed a Dirichlet form on \mathcal{P} whose intrinsic metric is the L^2 -Wasserstein distance on \mathcal{P} . This result is obtained based on Rademacher theorem ([24, Theorem 7.9, Theorem 7.11]) on the space \mathcal{G}_0 . There they considered different kinds of cylindrical functions. Using their idea, we can establish Rademacher theorem in our setting on \mathcal{G}_0 .

Proposition 3.10 *It holds that*

$$\|g_1 - g_0\|_{L^2} = \sup \{u(g_1) - u(g_0); u \in \text{Cyl}(G_0), \|\nabla u(g)\|_{T_g \mathcal{G}_0} \leq 1, Q_0^\theta - a.e. \text{ on } \mathcal{G}_0\} \quad (3.18)$$

for all $g_0, g_1 \in \mathcal{G}_0$, where $\|g_1 - g_0\|_{L^2}^2 = \int_0^1 |g_1(x) - g_0(x)|^2 dx$.

Sketch of the proof. Note $\text{Cyl}(\mathcal{G}_0)$ can be proved to be a core of a Dirichlet form on \mathcal{G}_0 in the same method to construct $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on \mathcal{P} , so (3.18) equals to the intrinsic metric of this Dirichlet form on \mathcal{G}_0 . Almost following the same argument of [24] Theorems 7.9 and 7.11, we can establish Rademacher theorem, then this proposition corresponds to [24, Corollary 7.14]. ■

Theorem 3.11 *Let d_{ess} denote the intrinsic metric associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on \mathcal{P} , that is,*

$$d_{\text{ess}}(\mu_1, \mu_0) = \sup \{u(\mu_1) - u(\mu_0); u \in \text{Cyl}, \|\nabla u(\mu)\|_{T_\mu \mathcal{P}} \leq 1, \Pi_\theta\text{-a.e. on } \mathcal{P}\}$$

for all $\mu_0, \mu_1 \in \mathcal{P}$. Then $d_{\text{ess}}(\mu_1, \mu_0) = \|g_{\mu_1} - g_{\mu_0}\|_{L^2}$, and

$$d_{w,1}(\mu_1, \mu_0) \leq d_{\text{ess}}(\mu_1, \mu_0) \leq \sqrt{d_{w,1}(\mu_1, \mu_0)}, \quad (3.19)$$

where $d_{w,1}(\mu_1, \mu_0)$ denotes the L^1 -Wasserstein distance on \mathcal{P} .

Proof. Let $u \in \text{Cyl}$ such that $\|\nabla u(\mu)\|_{T_\mu \mathcal{P}} \leq 1, \Pi_\theta\text{-a.e. on } \mathcal{P}$. Set $\bar{u}(g) = u(\zeta(g))$, then $\bar{u} \in \text{Cyl}(\mathcal{G}_0)$. By (3.3), we get

$$D_\phi \bar{u}(g) = D_\phi u(\zeta(g)) \quad \text{for all } \phi \in \mathcal{H}_0,$$

which yields $\nabla \bar{u}(g) = \nabla u(\zeta(g))$ and $\|\nabla \bar{u}(g)\|_{T_g \mathcal{G}_0} = \|\nabla u(\zeta(g))\|_{T_{\zeta(g)} \mathcal{P}}$. It follows that

$$\|\nabla \bar{u}(g)\|_{T_g \mathcal{G}_0} \leq 1, \quad Q_0^\theta - a.e. \text{ in } \mathcal{P}.$$

As ζ is reversible, for each $\bar{u} \in \text{Cyl}(\mathcal{G}_0)$ with $\|\nabla \bar{u}(g)\|_{T_g \mathcal{G}_0} \leq 1$, Q_0^θ -a.e. in \mathcal{G}_0 , define $u(\mu) = \bar{u}(\zeta^{-1}(\mu))$. After similar deduction as above, we obtain $u \in \text{Cyl}$ and $\|\nabla u(\mu)\|_{T_\mu \mathcal{P}} \leq 1$, Π_θ -a.e. on \mathcal{P} . Due to Proposition 3.10, for $\mu_0, \mu_1 \in \mathcal{P}$,

$$\begin{aligned} & \|g_{\mu_1} - g_{\mu_0}\|_{L^2} \\ &= \sup\{\bar{u}(g_{\mu_1}) - \bar{u}(g_{\mu_0}); \bar{u} \in \text{Cyl}(\mathcal{G}_0), \|\nabla \bar{u}(g)\|_{T_g \mathcal{G}_0} \leq 1, Q_0^\theta\text{-a.e. on } \mathcal{P}\} \\ &= \sup\{u(\mu_1) - u(\mu_0); u \in \text{Cyl}, \|\nabla u(\mu)\|_{T_\mu \mathcal{P}} \leq 1, \Pi_\theta\text{-a.e. on } \mathcal{P}\}. \end{aligned}$$

Hence,

$$d_{\text{ess}}(\mu_1, \mu_0) = \|g_{\mu_1} - g_{\mu_0}\|_{L^2}. \quad (3.20)$$

It holds

$$\|g_{\mu_1} - g_{\mu_0}\|_{L^1} = \int_0^1 |g_{\mu_1}(x) - g_{\mu_0}(x)| dx = \int_0^1 |g_{\mu_1}^{-1}(x) - g_{\mu_0}^{-1}(x)| dx = d_{w,1}(\mu_1, \mu_0),$$

where the second equality is easily verified by seeing the area between graphs and the last inequality follows from [22, Theorem 2.18]. By Hölder's inequality,

$$d_{w,1}(\mu_1, \mu_0) = \|g_{\mu_1} - g_{\mu_0}\|_{L^1} \leq \|g_{\mu_1} - g_{\mu_0}\|_{L^2} = d_{\text{ess}}(\mu_1, \mu_0).$$

On the other hand, as $|g_{\mu_1}(x) - g_{\mu_0}(x)| \leq 1$ at each $x \in [0, 1]$,

$$\|g_{\mu_1} - g_{\mu_0}\|_{L^2} \leq \left(\int_0^1 |g_{\mu_1}(x) - g_{\mu_0}(x)| dx \right)^{1/2} = (\|g_{\mu_1} - g_{\mu_0}\|_{L^1})^{1/2}.$$

Therefore, $d_{\text{ess}}(\mu_1, \mu_0) \leq \sqrt{d_{w,1}(\mu_1, \mu_0)}$, and we get the desired results. \blacksquare

4 Log-Sobolev inequalities for the process

In this section, we shall establish the Log-Sobolev inequality for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Döring, M. and Stannat, W. [3] established the Log-Sobolev inequality for the Wasserstein diffusion constructed by [24] on \mathcal{P} . There they have done lots of calculation on finite dimensional approximation, especially they established the Log-Sobolev inequality for the Dirichlet distribution on finite simplex. We shall take advantage of their work and to establish the Log-Sobolev inequality for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on \mathcal{P} again by finite dimensional approximation. But to deal with our Dirichlet form, we should take different approximation sequence.

First let's recall the definition of Dirichlet process Q_0^θ on \mathcal{G}_0 for $\theta > 0$. Q_0^θ is the unique probability measure on \mathcal{G}_0 such that for any $n \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$,

$$\begin{aligned} & Q_0^\theta(g_{t_1} \in dx_1, \dots, g_{t_{n-1}} \in dx_{n-1}) \\ &= \nu_{\theta(t_1, t_2 - t_1, \dots, t_{n-1} - t_{n-2}, t_n - t_{n-1})}(dx_1, \dots, dx_{n-1}) \end{aligned}$$

where for $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{R}_+^n$

$$\begin{aligned} \nu_{\mathbf{q}}(d\mathbf{x}) &= \nu_{\mathbf{q}}(dx_1, \dots, dx_{n-1}) \\ &= \frac{\Gamma(|\mathbf{q}|)}{\prod_{i=1}^n \Gamma(q_i)} \prod_{i=1}^n (x_i - x_{i-1})^{q_i-1} dx_1 \dots dx_{n-1} \end{aligned}$$

on the space $\sum_n = \{(x_1, \dots, x_{n-1}, x_n) \in [0, 1]^n; 0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1\}$. Here $|\mathbf{q}| = q_1 + \dots + q_n$.

We now include a result proved in [3, Proposition 2.3] as follows:

Proposition 4.1 *Let $\mathbf{q} \in \mathbb{R}_+^n$ and $q^* = \min_{1 \leq i \leq n} q_i$. Then*

$$A_{\mathbf{q}}(f) := \sum_{i=1}^{n-1} \int_{\sum_n} x_i (\partial_i f)^2 d\nu_{\mathbf{q}}, \quad f \in C_b^1(\mathbb{R}_+^n)$$

satisfies the Log-Sobolev inequality with constant less than $4c_1/q^*$, that is,

$$\int_{\sum_n} f^2 \log f^2 d\nu_{\mathbf{q}} \leq 4 \frac{c_1}{q^*} A_{\mathbf{q}}(f), \quad f \in C_b^1(\mathbb{R}_+^n). \quad (4.1)$$

Here c_1 can be taken to be 160.

Theorem 4.2 (Log-Sobolev inequality) *There exists a universal constant $C > 0$ such that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on \mathcal{P} satisfies the Log-Sobolev inequality with constant less than C/θ , i.e.*

$$\int_{\mathcal{P}} u^2 \log u^2 d\Pi_{\theta} - \int_{\mathcal{P}} u^2 d\Pi_{\theta} \log \int_{\mathcal{P}} u^2 d\Pi_{\theta} \leq \frac{C}{\theta} \mathcal{E}(u, u), \quad u \in \mathcal{D}(\mathcal{E}). \quad (4.2)$$

Proof. We will use the idea of [3] to establish the Log-Sobolev inequality through finite dimensional approximation. But compared with [3], we use different type of approximation sequence. In order to use the calculation results of [3], we first make our calculations on \mathcal{G}_0 , then obtain the desired results under the help of map ζ .

Let $u \in \text{Cyl}(\mathcal{G}_0)$ in the form $u(g) = F(\int f_1 dg, \dots, \int f_m dg)$ for $F \in C^1(\mathbb{R}^m)$ and $f_i \in C^1([0, 1])$, $i = 1, \dots, m$. By Lemma 3.5 (i) and (3.15), its gradient in $L^2(g_*\text{Leb})$ equals to

$$\nabla \cdot u(g) = - \sum_{i=1}^m \partial_i F \left(\int \vec{f} dg \right) f_i'(g^{-1}(\cdot)).$$

Then

$$\begin{aligned} \bar{\mathcal{E}}(u, u) &:= \int_{\mathcal{G}_0} \|\nabla u(g)\|_{L^2(g_*\text{Leb})}^2 dQ_0^{\theta}(g) \\ &= \int_{\mathcal{G}_0} \left(\sum_{i,j=1}^m \partial_i F \left(\int \vec{f} dg \right) \partial_j F \left(\int \vec{f} dg \right) \int_0^1 f_i'(x) f_j'(x) dx \right) dQ_0^{\theta}(g). \end{aligned}$$

Hence we choose the approximation sequence by

$$\bar{\mathcal{E}}_n(u, u) := \int_{\mathcal{G}_0} \sum_{i,j=1}^m \partial_i F(s_n(\vec{f}, g)) \partial_j F(s_n(\vec{f}, g)) s_n(f'_i, f_j) dQ_0^\theta(g), \quad (4.3)$$

where $s_n(\vec{f}, g) := (s_n(f_1, g), \dots, s_n(f_m, g))$ and

$$s_n(f, g) := \sum_{k=1}^n f\left(\frac{k}{n}\right) \left(g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right)\right).$$

Note that this is different to Döring, M. and Stannat, W. [3]. There they used the approximation induced by

$$s'_n(f, g) := \frac{1}{n} \sum_{\ell=1}^{n-1} f\left(\frac{\ell}{n}\right) g\left(\frac{\ell}{n}\right).$$

When f, g are both continuous or f is continuous, g is of bounded variation, one has

$$\lim_{n \rightarrow \infty} s_n(f, g) = \int_0^1 f(x) dg(x).$$

Let $u_n(g) = F(s_n(\vec{f}, g))$. By the dominated convergence theorem, one has

$$\begin{aligned} \lim_{n \rightarrow \infty} u_n(g) &= u(g), \quad \lim_{n \rightarrow \infty} \bar{\mathcal{E}}_n(u, u) = \bar{\mathcal{E}}(u, u), \\ \lim_{n \rightarrow \infty} \int_{\mathcal{G}_0} |u_n(g) - u(g)|^2 dQ_0^\theta(g) &= 0. \end{aligned}$$

Formula (4.3) can be written more explicitly and take projection onto finite dimensional space. Put

$$\Sigma_n = \{(x_1, \dots, x_n) \in [0, 1]^n; 0 < x_1 < \dots < x_n < 1\}.$$

Then we have

$$\begin{aligned} \bar{\mathcal{E}}_n(u, u) &= \int_{\mathcal{G}_0} \sum_{i,j=1}^m \partial_i F\left(\sum_{k=1}^n f\left(\frac{k}{n}\right) \left(g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right)\right)\right) \\ &\quad \times \partial_j F\left(\sum_{k=1}^n f\left(\frac{k}{n}\right) \left(g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right)\right)\right) \sum_{k=1}^n f'_i\left(\frac{k}{n}\right) \left(f_j\left(\frac{k}{n}\right) - f_j\left(\frac{k-1}{n}\right)\right) dQ_0^\theta(g) \\ &= \int_{\Sigma_n} \sum_{i,j=1}^m \partial_i F\left(\sum_{k=1}^n f\left(\frac{k}{n}\right) (x_k - x_{k-1})\right) \partial_j F\left(\sum_{k=1}^n f\left(\frac{k}{n}\right) (x_k - x_{k-1})\right) \\ &\quad \times \sum_{k=1}^n f'_i\left(\frac{k}{n}\right) \left(f_j\left(\frac{k}{n}\right) - f_j\left(\frac{k-1}{n}\right)\right) dv_{\mathbf{q}}(x_1, \dots, x_{n-1}) \end{aligned} \quad (4.4)$$

Set $\tilde{s}_n(f, \mathbf{x}) = \sum_{k=1}^n f\left(\frac{k}{n}\right) (x_k - x_{k-1})$ for $f \in C([0, 1])$ on Σ_n . Define

$$\mathbb{E}_n(\phi, \phi) = n \sum_{i=1}^{n-1} \int_{\Sigma_n} (\partial_i \phi)^2 dv_{\mathbf{q}}, \quad \phi \in C(\mathbb{R}^{n-1}). \quad (4.5)$$

Put $\tilde{u}_n(\mathbf{x}) = F(\tilde{s}_n(\vec{f}, \mathbf{x}))$ on Σ_n , then

$$\begin{aligned}
\mathbb{E}_n(\tilde{u}_n, \tilde{u}_n) &= n \sum_{i=1}^{n-1} \int_{\Sigma_n} (\partial_i \tilde{u}_n(\mathbf{x}))^2 d\nu_{\mathbf{q}}(\mathbf{x}) \\
&= n \sum_{i=1}^{n-1} \int_{\Sigma_n} \left(\sum_{r=1}^m \partial_r F(\tilde{s}_n(\vec{f}, \mathbf{x})) (f_r(\frac{i}{n}) - f_r(\frac{i+1}{n})) \right)^2 d\nu_{\mathbf{q}}(\mathbf{x}) \\
&= \int_{\Sigma_n} \sum_{r,\ell=1}^m \partial_r F(\tilde{s}_n(\vec{f}, \mathbf{x})) \partial_\ell F(\tilde{s}_n(\vec{f}, \mathbf{x})) \\
&\quad \times n \sum_{i=1}^{n-1} \left(f_r(\frac{i}{n}) - f_r(\frac{i+1}{n}) \right) \left(f_\ell(\frac{i}{n}) - f_\ell(\frac{i-1}{n}) \right) d\nu_{\mathbf{q}}(\mathbf{x}).
\end{aligned}$$

Since

$$\begin{aligned}
&n \sum_{i=1}^{n-1} \left(f_r(\frac{i}{n}) - f_r(\frac{i+1}{n}) \right) \left(f_\ell(\frac{i}{n}) - f_\ell(\frac{i-1}{n}) \right) \\
&= \sum_{i=1}^{n-1} f_r'(\frac{i+1}{n}) \left(f_\ell(\frac{i}{n}) - f_\ell(\frac{i-1}{n}) \right) + o(1),
\end{aligned}$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$, combining this with (4.4), we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}_n(u_n, u_n) = \lim_{n \rightarrow \infty} \bar{\mathcal{E}}_n(u, u) = \bar{\mathcal{E}}(u, u). \quad (4.6)$$

By Proposition 4.1,

$$\begin{aligned}
&\int_{\Sigma_n} \tilde{u}_n^2 \log \tilde{u}_n^2 d\nu_{\mathbf{q}} - \int_{\Sigma_n} \tilde{u}_n^2 d\nu_{\mathbf{q}} \log \int_{\Sigma_n} \tilde{u}_n^2 d\nu_{\mathbf{q}} \\
&\leq 4 \frac{nc_1}{\theta} A_{\mathbf{q}}(\tilde{u}_n) = 4 \frac{nc_1}{\theta} \sum_{i=1}^{n-1} \int_{\Sigma_n} x_i (\partial_i \tilde{u}_n)^2 d\nu_{\mathbf{q}}(\mathbf{x}) \\
&\leq 4 \frac{c_1}{\theta} \mathbb{E}_n(\tilde{u}_n, \tilde{u}_n),
\end{aligned}$$

where the last inequality is due to $0 < x_i < 1$. Passing to the limit as $n \rightarrow \infty$ in the previous equation, we obtain

$$\int_{\mathcal{G}_0} u^2 \log u^2 dQ_0^\theta - \int_{\mathcal{G}_0} u^2 dQ_0^\theta \log \int_{\mathcal{G}_0} u^2 dQ_0^\theta \leq 4 \frac{c_1}{\theta} \bar{\mathcal{E}}(u, u), \quad u \in \text{Cyl}(\mathcal{G}_0). \quad (4.7)$$

Due to the 1-1 correspondence between functions in $\text{Cyl}(\mathcal{G}_0)$ and in Cyl , and $(\zeta)_* Q_0^\theta = \Pi_\theta$, the inequality (4.7) implies

$$\int_{\mathcal{P}} u^2 \log u^2 d\Pi_\theta - \int_{\mathcal{P}} u^2 d\Pi_\theta \log \int_{\mathcal{P}} u^2 d\Pi_\theta \leq 4 \frac{c_1}{\theta} \mathcal{E}(u, u), \quad u \in \text{Cyl}. \quad (4.8)$$

Since Cyl is a core of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, we get (4.8) holds for all $u \in \mathcal{D}(\mathcal{E})$, and complete the proof. \blacksquare

Remark 4.3 It's well known that inequality (4.2) implies that the semigroup $(e^{t\tilde{\mathcal{L}}})_{t \geq 0}$ is hypercontractive, i.e. $\|e^{t\tilde{\mathcal{L}}}\|_{2 \rightarrow 4} \leq 1$ for some $t > 0$, where $\|\cdot\|_{2 \rightarrow 4}$ denotes the operator norm from $L^2(\Pi_\theta)$ to $L^4(\Pi_\theta)$. The contraction properties of Markov semigroups including hypercontractivity, supercontractivity and ultracontractivity are closely related to various functional inequalities. To be more precise, given a complete, connected, noncompact Riemannian manifold M , let $(P_t)_{t \geq 0}$ be the semigroup of a diffusion process generated by $L = \Delta + Z$ for some C^1 -vector field Z satisfying

$$\text{Ric}(X, X) - \langle \nabla_X Z, Z \rangle \geq -K_Z |X|, \quad X \in TM$$

for some $K_Z \in \mathbb{R}$, where Ric denotes the Ricci curvature of M . Assume P_t admits an invariant measure μ which is positive and Radon. Then, according to [25, Theorem 5.7.1], if there exist $C, t > 0$, and $q > p > 1$ such that $\|P_t\|_{p \rightarrow q} \leq C$, then

$$\mu(f^2 \log f^2) \leq \beta_{K_Z, p, q} \mu(|\nabla f|^2) + \frac{pq}{q-p} \log C, \quad \forall f \in C_0^\infty(M), \mu(f^2) = 1,$$

for some positive constant $\beta_{K_Z, p, q}$. Conversely, if there exist $C_1, C_2 > 0$ such that

$$\mu(f^2 \log f^2) \leq C_1 \mu(|\nabla f|^2) + C_2, \quad f \in C_0^\infty(M), \mu(f^2) = 1,$$

then

$$\|P_t\|_{p \rightarrow q} \leq \exp \left[4C_2 \left(\frac{1}{p} - \frac{1}{q} \right) \right]$$

for $t > 0$ and $q > p > 1$ satisfying $\exp(4t/C_1) \geq (q-1)/(p-1)$. Refer to Wang's book [25] for more properties associated with Log-Sobolev inequalities and general discussion about functional inequalities including F -Sobolev inequalities, Harnack inequalities, and super and weak Poincaré inequalities.

Remark 4.4 In [19], the same Dirichlet form and Log-Sobolev inequality as this work have been obtained, but the generator and the intrinsic distance of the Dirichlet form haven't been given there. Moreover, the explicit form of the intrinsic metric enable us to establish the transportation cost inequalities for Wasserstein diffusions, which will be done in our forthcoming work [16].

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