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The center of mass for spatial branching processes and an application for self-interaction

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Abstract

Consider the center of mass of a d-dimensional supercritical branching-Brownian motion. We first show that it is a Brownian motion being slowed down such that it tends to a limiting position almost surely, and that this is also true for a model where branching Brownian motion is modified by *attraction/repulsion between particles*, where the strength of the attraction/repulsion is given by the parameter $\gamma \neq 0$.
We then put this observation together with the description of the interacting system as viewed from its center of mass, and get our main result in Theorem 16: If $\gamma > 0$ (attraction), then, as $n \to \infty$,

$$2^{-n} Z_n(\mathrm{d}y) \stackrel{w}{\Rightarrow} \left(\frac{\gamma}{\pi}\right)^{d/2} \exp\left(-\gamma|y-x|^2\right) \mathrm{d}y,\ P^x - \text{a.s.}$$

for almost all $x \in \mathbb{R}^d$, where $Z(\mathrm{d}y)$ denotes the discrete measure-valued process corresponding to the interacting branching particle system, $\stackrel{w}{\Rightarrow}$ denotes weak convergence, and P^x denotes the law of the particle system conditioned to have x as the limit for the center of mass.
A conjecture is stated regarding the behavior of the local mass in the repulsive case.
We also consider a supercritical super-Brownian motion, and show that, conditioned on survival, its center of mass is a continuous process having an a.s. limit as $t \to \infty$.

Key words: Branching Brownian motion, super-Brownian motion, center of mass, self-interaction, Curie-Weiss model, McKean-Vlasov limit, branching Ornstein-Uhlenbeck process, spatial branching processes, H-transform.

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1 Introduction

1.1 Notation

1. **Probability:** The symbol $E^{\complement}$ will denote the complement of the event E, and $X \oplus Y$ will denote the independent sum of the random variables X and Y.

2. **Topology and measures:** The boundary of the set B will be denoted by ∂B and the closure of B will be denoted by $\mathrm{cl}(B)$, that is $\mathrm{cl}(B) := B \cup \partial B$; the interior of B will be denoted by $\dot{B}$ and B^{ϵ} will denote the ϵ-neighborhood of B. We will also use the notation $\dot{B}^{\epsilon} := \{y \in B : B_{\epsilon} + y \subset B\}$, where $B + b := \{y : y - b \in B\}$ and $B_t := \{x \in \mathbb{R}^d : |x| < t\}$. By a *bounded rational rectangle* we will mean a set $B \subset \mathbb{R}^d$ of the form $B = I_1 \times I_2 \times \cdots \times I_d$, where I_i is a bounded interval with rational endpoints for each $1 \le i \le d$. The family of all bounded rational rectangles will be denoted by $\mathscr{R}$.

   $\mathscr{M}_f(\mathbb{R}^d)$ and $\mathscr{M}_1(\mathbb{R}^d)$ will denote the space of finite measures and the space of probability measures, respectively, on $\mathbb{R}^d$. For $\mu \in \mathscr{M}_f(\mathbb{R}^d)$, we define $\|\mu\| := \mu(\mathbb{R}^d)$. $|B|$ will denote the Lebesgue measure of B. The symbols "$\stackrel{w}{\Rightarrow}$" and "$\stackrel{v}{\Rightarrow}$" will denote convergence in the weak topology and in the vague topology, respectively.

3. **Functions:** For $f, g > 0$, the notation $f(x) = \mathscr{O}(g(x))$ will mean that $f(x) \le C g(x)$ if $x > x_0$ with some $x_0 \in \mathbb{R}, C > 0$; $f \approx g$ will mean that f/g tends to 1 given that the argument tends to an appropriate limit. For $\mathbb{N} \to \mathbb{R}$ functions the notation $f(n) = \Theta(g(n))$ will mean that $c \le f(n)/g(n) \le C \ \forall n$, with some $c, C > 0$.

4. **Matrices:** The symbol $\mathbf{I}_d$ will denote the d-dimensional unit matrix, and $\mathrm{r}(\mathbf{A})$ will denote the rank of a matrix $\mathbf{A}$.

5. **Labeling:** In this paper we will often talk about the 'i^{th} particle' of a branching particle system. By this we will mean that we label the particles randomly, but in a way that does not depend on their spatial position.

1.2 A model with self-interaction

Consider a dyadic (i.e. precisely two offspring replaces the parent) branching Brownian motion (BBM) in $\mathbb{R}^d$ with unit time branching and with the following interaction between particles: if Z denotes the process and Z_t^i is the i^{th} particle, then Z_t^i 'feels' the drift

$$\frac{1}{n_t} \sum_{1 \le j \le n_t} \gamma \cdot \left(Z_t^j - \cdot \right),$$

where $\gamma \ne 0$, that is the particle's infinitesimal generator is

$$\frac{1}{2}\Delta + \frac{1}{n_t} \sum_{1 \le j \le n_t} \gamma \cdot \left(Z_t^j - x \right) \cdot \nabla. \tag{1.1}$$

(Here and in the sequel, n_t is a shorthand for $2^{\lfloor t \rfloor}$, where $\lfloor t \rfloor$ is the integer part of t.) If $\gamma > 0$, then this means *attraction*, if $\gamma < 0$, then it means *repulsion*.

To be a bit more precise, we can define the process by induction as follows. Z_0 is a single particle at the origin. In the time interval $[m, m+1)$ we define a system of 2^m *interacting diffusions*, starting at the position of their parents at the end of the previous step (at time $m-0$) by the following system of SDE's:

$$\mathrm{d}Z_t^i = \mathrm{d}W_t^{m,i} + \frac{\gamma}{2^m} \sum_{1 \le j \le 2^m} (Z_t^j - Z_t^i)\,\mathrm{d}t;\ i = 1, 2, \ldots, 2^m, \tag{1.2}$$

where $W^{m,i}, i = 1, 2, \ldots, 2^m; m = 0, 1, \ldots$ are independent Brownian motions.

Remark 1 (Attractive interaction)**.** If there were no branching and the interval $[m, m+1)$ were extended to $[0, \infty)$, then for $\gamma > 0$ the interaction (1.2) would describe the *ferromagnetic Curie-Weiss model*, a model appearing in the microscopic statistical description of a spatially homogeneous gas in a granular medium. It is known that as $m \to \infty$, a Law of Large Numbers, the *McKean-Vlasov limit* holds and the normalized empirical measure

$$\rho_m(t) := 2^{-m} \sum_{i=1}^{2^m} \delta_{Z_t^i}$$

tends to a probability measure-valued solution of

$$\frac{\partial}{\partial t}\rho = \frac{1}{2}\Delta\rho + \frac{\gamma}{2}\nabla \cdot \left(\rho \nabla f^{\rho}\right),$$

where $f^{\rho}(x) := \int_{\mathbb{R}^d} |x - y|^2 \rho(dy)$. (See p. 24 in [8] and the references therein.) $\diamond$

Remark 2 (More general interaction)**.** It seems natural to replace the linearity of the interaction by a more general rule. That is, to define and analyze the system where (1.2) is replaced by

$$\mathrm{d}Z_t^i = \mathrm{d}W_t^{m,i} + 2^{-m} \sum_{1 \le j \le 2^m} g(|Z_t^j - Z_t^i|) \frac{Z_t^j - Z_t^i}{|Z_t^j - Z_t^i|}\,\mathrm{d}t;\ i = 1, 2, \ldots, 2^m,$$

where the function $g : \mathbb{R}_+ \to \mathbb{R}$ has some nice properties. (In this paper we treat the $g(x) = \gamma x$ case.) This is part of a future project with J. Feng. $\diamond$

1.3 Existence and uniqueness

Notice that the 2^m interacting diffusions on $[m, m+1)$ can be considered as a single $2^m d$-dimensional Brownian motion with linear (and therefore Lipschitz) drift $\mathbf{b} : \mathbb{R}^{2^m d} \to \mathbb{R}^{2^m d}$:

$$\mathbf{b}(x_1, x_2, \ldots, x_d, x_{1+d}, x_{2+d}, \ldots, x_{2d}, \ldots, x_{1+(2^m-1)d}, x_{2+(2^m-1)d}, \ldots, x_{2^m d}) \\ =: \gamma(\beta_1, \beta_2, \ldots, \beta_{2^m d})^T,$$

where

$$\beta_k = 2^{-m}(x_{\widehat{k}} + x_{\widehat{k}+d} + \ldots + x_{\widehat{k}+(2^m-1)d}) - x_k,\ \ 1 \le k \le 2^m d,$$

and $\widehat{k} \equiv k \pmod d$, $1 \le \widehat{k} \le d$. This yields existence and uniqueness for our model.

1.4 Results on the self-interacting model

We are interested in the long time behavior of Z, and also whether we can say something about the number of particles in a given compact set for n large. In the sequel we will use the standard notation $\langle g, Z_t\rangle = \langle Z_t, g\rangle := \sum_{i=1}^{n_t} g(Z_t^i)$.

In this paper we will first show that Z asymptotically becomes a branching Ornstein-Uhlenbeck process (inward for attraction and outward for repulsion), but

1. the origin is shifted to a random point which has d-dimensional normal distribution $\mathcal{N}(0, 2\mathbf{I}_d)$, and
2. the Ornstein-Uhlenbeck particles are not independent but constitute a system with a degree of freedom which is less than their number by precisely one.

The main result of this article concerns the local behavior of the system: we will prove a scaling limit theorem (Theorem 16) for the local mass when $\gamma > 0$ (attraction), and formulate and motivate a conjecture (Conjecture 18) when $\gamma < 0$ (repulsion).

1.5 The center of mass for supercritical super-Brownian motion

In Lemma 6 we will show that $\overline{Z}_t := \frac{1}{n_t}\sum_{i=1}^{n_t} Z_t^i$, the center of mass for Z satisfies $\lim_{t\to\infty} \overline{Z}_t = N$, where $N \sim \mathcal{N}(0, 2\mathbf{I}_d)$. In fact, the proof will reveal that $\overline{Z}$ moves like a Brownian motion, which is nevertheless slowed down tending to a final limiting location (see Lemma 6 and its proof).

Since this is also true for $\gamma = 0$ (BBM with unit time branching and no self-interaction), our first natural question is whether we can prove a similar result for the supercritical super-Brownian motion.

Let X be the $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$-superdiffusion with $\alpha, \beta > 0$ (supercritical super-Brownian motion). Here β is the 'mass creation parameter' or 'mass drift', while $\alpha > 0$ is the 'variance (or intensity) parameter' of the branching — see [6] for more elaboration and for a more general setting.

Let P_μ denote the corresponding probability when the initial finite measure is μ. (We will use the abbreviation $P := P_{\delta_0}$.) Let us restrict Ω to the survival set

$$S := \{\omega \in \Omega \mid X_t(\omega) > 0,\ \forall t > 0\}.$$

Since $\beta > 0$, $P_\mu(S) > 0$ for all $\mu \neq \mathbf{0}$.

It turns out that on the survival set the center of mass for X stabilizes:

Theorem 3. *Let $\alpha, \beta > 0$ and let $\overline{X}$ denote the center of mass process for the $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X, that is let*

$$\overline{X} := \frac{\langle \mathrm{id}, X\rangle}{\|X\|},$$

where $\langle f, X\rangle := \int_{\mathbb{R}^d} f(x)X(\mathrm{d}x)$ and $\mathrm{id}(x) = x$. Then, on S, $\overline{X}$ is continuous and converges P-almost surely.

Remark 4. A heuristic argument for the convergence is as follows. Obviously, the center of mass is invariant under H-transforms[1] whenever H is spatially (but not temporarily) constant. Let $H(t) := e^{-\beta t}$. Then X^H is a $(\frac{1}{2}\Delta, 0, e^{-\beta t}\alpha; \mathbb{R}^d)$-superdiffusion, that is, a critical super-Brownian motion with a clock that is slowing down. Therefore, heuristically it seems plausible that $\overline{X^H}$, the center of mass for the transformed process stabilizes, because after some large time T, if the process is still alive, it behaves more or less like the heat flow ($e^{-\beta t}\alpha$ is small), under which the center of mass does not move. $\diamond$

1.6 An interacting superprocess model

The next goal is to construct and investigate the properties of a measure-valued process with representative particles that are attracted to or repulsed from its center of mass.

There is one work in this direction we are aware of: motivated by the present paper, H. Gill [9] has constructed very recently a *superprocess with attraction to its center of mass*. More precisely, Gill constructs a supercritical interacting measure-valued process with representative particles that are attracted to or repulsed from its center of mass using Perkins's *historical stochastic calculus*.

In the attractive case, Gill proves the equivalent of our Theorem 16 (see later): on S, the mass normalized process converges almost surely to the stationary distribution of the Ornstein-Uhlenbeck process centered at the limiting value of its center of mass; in the repulsive case, he obtains substantial results concerning the equivalent of our Conjecture 18 (see later), using [7]. In addition, a version of Tribe's result [12] is presented in [9].

2 The mass center stabilizes

Returning to the discrete interacting branching system, notice that

$$\frac{1}{n_t} \sum_{1 \le j \le n_t} \left(Z_t^j - Z_t^i \right) = \overline{Z}_t - Z_t^i, \tag{2.1}$$

and so *the net attraction pulls the particle towards the center of mass* (net repulsion pushes it away from the center of mass).

Remark 5. The reader might be interested in the very recent work [11], where a one dimensional particle system is considered with interaction via the center of mass. There is a kind of attraction towards the center of mass in the following sense: each particle jumps to the right according to some common distribution F, but the rate at which the jump occurs is a monotone decreasing function of the signed distance between the particle and the mass center. Particles being far ahead slow down, while the laggards catch up. $\diamond$

Since the interaction is in fact through the center of mass, the following lemma is relevant:

Lemma 6 (Mass center stabilizes)**.** *There is a random variable $N \sim \mathcal{N}(\mathbf{0}, 2\mathbf{I}_d)$ such that* $\lim_{t\to\infty} \overline{Z}_t = N$ *a.s.*

[1] See Appendix B in [7] for H-transforms.

Proof. Let $m \in \mathbb{N}$. For $t \in [m, m+1)$ there are 2^m particles moving around and particle Z_t^i's motion is governed by

$$\mathrm{d}Z_t^i = \mathrm{d}W_t^{m,i} + \gamma(\overline{Z}_t - Z_t^i)\mathrm{d}t.$$

Since $2^m \overline{Z}_t = \sum_{i=1}^{2^m} Z_t^i$, we obtain that

$$\mathrm{d}\overline{Z}_t = 2^{-m} \sum_{i=1}^{2^m} \mathrm{d}Z_t^i = 2^{-m} \sum_{i=1}^{2^m} \mathrm{d}W_t^{m,i} + \frac{\gamma}{2^m} \left(2^m \overline{Z}_t - \sum_{i=1}^{2^m} Z_t^i \right) \mathrm{d}t = 2^{-m} \sum_{i=1}^{2^m} \mathrm{d}W_t^{m,i}.$$

So, for $0 \le \tau < 1$,

$$\overline{Z}_{m+\tau} = \overline{Z}_m + 2^{-m} \bigoplus_{i=1}^{2^m} W_\tau^{m,i} =: \overline{Z}_m + 2^{-m/2} B^{(m)}(\tau), \tag{2.2}$$

where we note that $B^{(m)}$ is a Brownian motion on $[m, m+1)$. Using induction, we obtain that[2]

$$\begin{aligned} \overline{Z}_t = B^{(0)}(1) \oplus \frac{1}{\sqrt{2}} B^{(1)}(1) \oplus \cdots \oplus \frac{1}{2^{k/2}} B^{(k)}(1) \oplus \\ \cdots \oplus \frac{1}{\sqrt{2^{\lfloor t \rfloor - 1}}} B^{(\lfloor t \rfloor - 1)}(1) \oplus \frac{1}{\sqrt{n_t}} B^{(\lfloor t \rfloor)}(\tau), \end{aligned} \tag{2.3}$$

where $\tau := t - \lfloor t \rfloor$.

By Brownian scaling $W^{(m)}(\cdot) := 2^{-m/2} B^{(m)}(2^m \cdot)$, $m \ge 1$ are (independent) Brownian motions. We have

$$\overline{Z}_t = W^{(0)}(1) \oplus W^{(1)}\left(\frac{1}{2}\right) \oplus \cdots \oplus W^{(\lfloor t \rfloor - 1)}\left(\frac{1}{2^{\lfloor t \rfloor - 1}}\right) \oplus W^{(\lfloor t \rfloor)}\left(\frac{\tau}{n_t}\right).$$

By the Markov property, in fact

$$\overline{Z}_t = \widehat{W}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{\lfloor t \rfloor - 1}} + \frac{\tau}{n_t}\right),$$

where $\widehat{W}$ is a Brownian motion (the concatenation of the $W^{(i)}$'s), and since $\widehat{W}$ has a.s. continuous paths, $\lim_{t\to\infty} \overline{Z}_t = \widehat{W}(2)$, a.s. □

For another proof see the remark after Lemma 9.

Remark 7. It is interesting to note that $\overline{Z}$ *is in fact a Markov process*. To see this, let $\{\mathscr{F}_t\}_{t\ge0}$ be the canonical filtration for Z and $\{\mathscr{G}_t\}_{t\ge0}$ the canonical filtration for $\overline{Z}$. Since $\mathscr{G}_s \subset \mathscr{F}_s$, it is enough to check the Markov property with $\mathscr{G}_s$ replaced by $\mathscr{F}_s$.

Assume first $0 \le s < t$, $\lfloor s \rfloor = \lfloor t \rfloor =: m$. Then the distribution of $\overline{Z}_t$ conditional on $\mathscr{F}_s$ is the same as conditional on Z_s, because Z itself is a Markov process. But the distribution of $\overline{Z}_t$ only depends on Z_s through $\overline{Z}_s$, as

$$\overline{Z}_t \stackrel{d}{=} \overline{Z}_s \oplus W^{(2^m)}\left(\frac{t-s}{2^m}\right), \tag{2.4}$$

whatever Z_s is. That is, $P(\overline{Z}_t \in \cdot \mid \mathscr{F}_s) = P(\overline{Z}_t \in \cdot \mid Z_s) = P(\overline{Z}_t \in \cdot \mid \overline{Z}_s)$. Note that this is even true when $s \in \mathbb{N}$ and $t = s+1$, because $\overline{Z}_t = \overline{Z}_{t-0}$.

Assume now that $s < \lfloor t \rfloor =: m$. Then the equation $P(\overline{Z}_t \in \cdot \mid \mathscr{F}_s) = P(\overline{Z}_t \in \cdot \mid \overline{Z}_s)$, is obtained by conditioning successively on $m, m-1, ..., \lfloor s \rfloor + 1, s$. ◇

[2] It is easy to check that, as the notation suggests, the summands are independent.

We will also need the following fact later.

Lemma 8. *The coordinate processes of Z are independent one dimensional interactive branching processes of the same type as Z.*

We leave the simple proof to the reader.

3 Normality via decomposition

Let again $m := \lfloor t \rfloor$. We will need the following result.

Lemma 9 (Decomposition)**.** *In the time interval $[m, m+1)$ the $d \cdot 2^m$-dimensional process $(Z_t^1, Z_t^2, ..., Z_t^{2^m})$ can be decomposed into two components: a d-dimensional Brownian motion and an independent $d(2^m - 1)$-dimensional Ornstein-Uhlenbeck process with parameter γ.*

More precisely, each coordinate process (as a 2^m-dimensional process) can be decomposed into two components: a one-dimensional Brownian motion in the direction $(1, 1, ..., 1)$ and an independent $(2^m - 1)$-dimensional Ornstein-Uhlenbeck process with parameter γ in the ortho-complement of the vector $(1, 1, ..., 1)$.

Furthermore, $(Z_t^1, Z_t^2, ..., Z_t^{n_t})$ is dn_t-dimensional joint normal for all $t \geq 0$.

Proof of lemma. By Lemma 8, we may assume that $d = 1$. We prove the statement by induction.

(i) For $m = 0$ it is trivial.

(ii) Suppose that the statement is true for $m - 1$. Consider the time m position of the 2^{m-1} particles $(Z_m^1, Z_m^2, ..., Z_m^{2^{m-1}})$ 'just before' the fission. At the instant of the fission we obtain the 2^m-dimensional vector

$$(Z_m^1, Z_m^1, Z_m^2, Z_m^2, ..., Z_m^{2^{m-1}}, Z_m^{2^{m-1}}),$$

which has the same distribution on the 2^{m-1} dimensional subspace

$$S := \{x \in \mathbb{R}^{2^m} \mid x_1 = x_2, x_3 = x_4, ..., x_{2^m - 1} = x_{2^m}\}$$

of $\mathbb{R}^{2^m}$ as the vector $\sqrt{2}(Z_m^1, Z_m^2, ..., Z_m^{2^{m-1}})$ on $\mathbb{R}^{2^{m-1}}$. Since, by the induction hypothesis, $(Z_m^1, Z_m^2, ..., Z_m^{2^{m-1}})$ is normal, the vector formed by the particle positions 'right after' the fission is a 2^m-dimensional degenerate normal. (The reader can easily visualize this for $m = 1$: the distribution of (Z_1^1, Z_1^1) is clearly $\sqrt{2}$ times the distribution of a Brownian particle at time 1, i.e. $\mathcal{N}(0, \sqrt{2})$ on the line $x_1 = x_2$.)

Since the convolution of normals is normal, therefore, by the Markov property, it is enough to prove the statement when the 2^m particles start at the origin and the clock is reset: $t \in [0, 1)$.

Define the 2^m-dimensional process Z^* on the time interval $t \in [0, 1)$ by

$$Z_t^* := (Z_t^1, Z_t^2, ..., Z_t^{2^m}),$$

starting at the origin. Because the interaction between the particles attracts the particles towards the center of mass, Z^* is a Brownian motion with drift

$$\gamma \left[(\overline{Z}_t, \overline{Z}_t, ..., \overline{Z}_t) - (Z_t^1, Z_t^2, ..., Z_t^{2^m}) \right].$$

Notice that this drift is orthogonal to the vector[3] $\mathbf{v} := (1, 1, ..., 1)$, that is, the vector $(\overline{Z}_t, \overline{Z}_t, ..., \overline{Z}_t)$ is nothing but the orthogonal projection of $(Z_t^1, Z_t^2, ..., Z_t^{2^m})$ to the line of $\mathbf{v}$. This observation immediately leads to the following decomposition. The process Z^* can be decomposed into two components:

- the component in the direction of $\mathbf{v}$ is a Brownian motion
- in S, the ortho-complement of $\mathbf{v}$, it is an independent Ornstein-Uhlenbeck process with parameter γ. □

Remark 10. (i) The proof also shows that $(Z_t^1, Z_t^2, ..., Z_t^{n_t})$ given Z_s is a.s. joint normal for all $t > s \geq 0$.

(ii) Consider the Brownian component in the decomposition appearing in the proof. Since, on the other hand, this coordinate is $2^{m/2}\overline{Z}_t$, using Brownian scaling, one obtains a slightly different way of seeing that $\overline{Z}_t$ stabilizes at a position which is distributed as the time $1+2^{-1}+2^{-2}+...+2^{-m}+... = 2$ value of a Brownian motion. (The decomposition shows this for $d = 1$ and then it is immediately upgraded to general d by independence.) ◇

Corollary 11 (Asymptotics for finite subsystem)**.** *Let $k \geq 1$ and consider the subsystem $(Z_t^1, Z_t^2, ..., Z_t^k)$, $t \geq m_0$ for $m_0 := \lfloor \log_2 k \rfloor + 1$. (This means that at time m_0 we pick k particles and at every fission replace the parent particle by randomly picking one of its two descendants.) Let the real numbers $c_1, ..., c_k$ satisfy*

$$\sum_{i=1}^{k} c_i = 0, \ \sum_{i=1}^{k} c_i^2 = 1. \tag{3.1}$$

Define $\Psi_t = \Psi_t^{(c_1,...,c_k)} := \sum_{i=1}^k c_i Z_t^i$ and note that Ψ_t is invariant under the translations of the coordinate system. Let $\mathscr{L}_t$ denote its law.

For every $k \geq 1$ and $c_1, ..., c_k$ satisfying (3.1), $\Psi^{(c_1,...,c_k)}$ is the same d-dimensional Ornstein-Uhlenbeck process corresponding to the operator $1/2\Delta - \gamma\nabla \cdot x$, and in particular, for $\gamma > 0$,

$$\lim_{t\to\infty} \mathscr{L}_t = \mathscr{N}\left(\mathbf{0}, \frac{1}{2\gamma}\mathbf{I}_d\right).$$

For example, taking $c_1 = 1/\sqrt{2}, c_2 = -1/\sqrt{2}$, we obtain that when viewed from a tagged particle's position, any given other particle moves as $\sqrt{2}$ times the above Ornstein-Uhlenbeck process.

Proof. By independence (Lemma 8) it is enough to consider $d = 1$. For m fixed, consider the decomposition appearing in the proof of Lemma 9 and recall the notation. By (3.1), whatever $m \geq m_0$ is, the 2^m dimensional unit vector

$$(c_1, c_2, ..., c_k, 0, 0, ..., 0)$$

is orthogonal to the 2^m dimensional vector $\mathbf{v}$. This means that $\Psi^{(c_1,...,c_k)}$ is a one dimensional projection of the Ornstein-Uhlenbeck component of Z^*, and thus it is itself a one dimensional Ornstein-Uhlenbeck process (with parameter γ) on the unit time interval.

Now, although as m grows, the Ornstein-Uhlenbeck components of Z^* are defined on larger and larger spaces ($S \subset \mathbb{R}^{2^m}$ is a 2^{m-1} dimensional linear subspace), the projection onto the direction of $(c_1, c_2, ..., c_k, 0, 0, ..., 0)$ is always the same one dimensional Ornstein-Uhlenbeck process, i.e. the different unit time 'pieces' of $\Psi^{(c_1,...,c_k)}$ obtained by those projections may be concatenated. □

[3] For simplicity, we use row vectors in this proof.

4 The interacting system as viewed from the center of mass

Recall that by (2.2) the interaction has no effect on the motion of $\overline{Z}$. Let us see now how the interacting system looks like when viewed from $\overline{Z}$.

4.1 The description of a single particle

Using our usual notation, assume that $t \in [m, m+1)$ and let $\tau := t - \lfloor t \rfloor$. When viewed from $\overline{Z}$, the relocation[4] of a particle is governed by

$$\mathrm{d}(Z_t^1 - \overline{Z}_t) = \mathrm{d}Z_t^1 - \mathrm{d}\overline{Z}_t = \mathrm{d}W_t^{m,1} - 2^{-m}\sum_{i=1}^{2^m}\mathrm{d}W_t^{m,i} - \gamma(Z_t^1 - \overline{Z}_t)\mathrm{d}t.$$

So if $Y^1 := Z^1 - \overline{Z}$, then

$$\mathrm{d}Y_t^1 = \mathrm{d}W_t^{m,1} - 2^{-m}\sum_{i=1}^{2^m}\mathrm{d}W_t^{m,i} - \gamma Y_t^1\mathrm{d}t.$$

Clearly,

$$W_\tau^{m,1} - 2^{-m}\bigoplus_{i=1}^{2^m} W_\tau^{m,i} = \bigoplus_{i=2}^{2^m} 2^{-m}W_\tau^{m,i} \oplus (1-2^{-m})W_\tau^{m,1};$$

and, by a trivial computation, the right hand side is a Brownian motion with mean zero and variance $(1-2^{-m})\tau\mathbf{I}_d := \sigma_m^2\tau\mathbf{I}_d$. That is,

$$\mathrm{d}Y_t^1 = \sigma_m\mathrm{d}\widetilde{W}_t^{m,1} - \gamma Y_t^1\mathrm{d}t,$$

where $\widetilde{W}^{m,1}$ is a standard Brownian motion.

We have thus obtained that on the time interval $[m, m+1)$, Y^1 corresponds to the Ornstein-Uhlenbeck operator

$$\frac{1}{2}\sigma_m^2\Delta - \gamma x \cdot \nabla. \tag{4.1}$$

Since for m large σ_m is close to one, the relocation viewed from the center of mass is *asymptotically governed by an O-U process corresponding to* $\frac{1}{2}\Delta - \gamma x \cdot \nabla$.

Remark 12 (Asymptotically vanishing correlation between driving BM's)**.** Let $\widetilde{W}^{m,i,k}$ be the k^{th} coordinate of the i^{th} Brownian motion: $\widetilde{W}^{m,i} = (\widetilde{W}^{m,i,k}, k = 1, 2, ..., d)$ and $B^{m,i,k}$ be the k^{th} coordinate of $W^{m,i}$. For $1 \le i \ne j \le 2^m$, we have

$$\begin{aligned}
&E\left[\sigma_m\widetilde{W}_\tau^{m,i,k}\cdot\sigma_m\widetilde{W}_\tau^{m,j,k}\right] = \\
&E\left[\left(B_\tau^{m,i,k} - 2^{-m}\bigoplus_{r=1}^{2^m} B_\tau^{m,r,k}\right)\left(B_\tau^{m,j,k} - 2^{-m}\bigoplus_{r=1}^{2^m} B_\tau^{m,r,k}\right)\right] = \\
&-2^{-m}\left[\mathrm{Var}\left(B_\tau^{m,i,k}\right) + \mathrm{Var}\left(B_\tau^{m,j,k}\right)\right] + 2^{-2m}\cdot 2^m\tau = (2^{-m} - 2^{1-m})\tau = -2^{-m}\tau,
\end{aligned}$$

[4] I.e. the relocation between time m and time t.

that is, for $i \neq j$,

$$E\left[\widetilde{W}_\tau^{m,i,k}\,\widetilde{W}_\tau^{m,j,\ell}\right] = -\frac{\delta_{k\ell}}{2^m-1}\,\tau. \tag{4.2}$$

Hence *the pairwise correlation tends to zero* as $t \to \infty$ (recall that $m = \lfloor t \rfloor$ and $\tau = t - m \in [0,1)$).

And of course, for the variances we have

$$E\left[\widetilde{W}_\tau^{m,i,k}\,\widetilde{W}_\tau^{m,i,\ell}\right] = \delta_{k\ell}\cdot\tau, \text{ for } 1 \le i \le 2^m. \qquad \diamond \tag{4.3}$$

4.2 The description of the system; the 'degree of freedom'

Fix $m \ge 1$ and for $t \in [m, m+1)$ let $Y_t := (Y_t^1, ..., Y_t^{2^m})^T$, where $()^T$ denotes transposed. (This is a vector of length 2^m where each component itself is a d dimensional vector; one can actually view it as a $2^m \times d$ matrix too.) We then have

$$\mathrm{d}Y_t = \sigma_m \mathrm{d}\widetilde{W}_t^{(m)} - \gamma Y_t \mathrm{d}t,$$

where

$$\widetilde{W}^{(m)} = \left(\widetilde{W}^{m,1}, ..., \widetilde{W}^{m,2^m}\right)^T$$

and

$$\widetilde{W}_\tau^{m,i} = \sigma_m^{-1}\left(W_\tau^{m,i} - 2^{-m}\bigoplus_{j=1}^{2^m} W_\tau^{m,j}\right),\ i = 1,2,...,2^m$$

are mean zero Brownian motions with correlation structure given by (4.2)-(4.3).

Just like at the end of Subsection 1.2, we can consider Y as a single $2^m d$-dimensional diffusion. Each of its components is an Ornstein-Uhlenbeck process with asymptotically unit diffusion coefficient.

By independence, it is enough to consider the $d = 1$ case, and so from now on, in this subsection we assume that $d = 1$.

Let us first describe the distribution of $\widetilde{W}_t^{(m)}$ for $t \ge 0$ fixed. Recall that $\{W_s^{m,i},\ s \ge 0;\ i = 1,2,...,2^m\}$ are independent Brownian motions. By definition, $\widetilde{W}_t^{(m)}$ is a 2^m-dimensional multivariate normal:

$$\widetilde{W}_t^{(m)} = \sigma_m^{-1}\cdot\begin{pmatrix} 1-2^{-m} & -2^{-m} & \dots & -2^{-m} \\ -2^{-m} & 1-2^{-m} & \dots & -2^{-m} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ -2^{-m} & -2^{-m} & \dots & 1-2^{-m} \end{pmatrix} W_t^{(m)} =: \sigma_m^{-1}\mathbf{A}^{(m)} W_t^{(m)},$$

where $W_t^{(m)} = (W_t^{m,1}, ..., W_t^{m,2^m})^T$, yielding

$$\mathrm{d}Y_t = \mathbf{A}^{(m)}\mathrm{d}W_t^{(m)} - \gamma Y_t \mathrm{d}t.$$

Since we are viewing the system from the center of mass, $\widetilde{W}_t^{(m)}$ is a *singular* multivariate normal and thus Y is a degenerate diffusion. The 'true' dimension of $\widetilde{W}_t^{(m)}$ is $\mathrm{r}(\mathbf{A}^{(m)})$.

Lemma 13. $r(\mathbf{A}^{(m)}) = 2^m - 1$.

Proof. We will simply write $\mathbf{A}$ instead of $\mathbf{A}^{(m)}$. Since the columns of $\mathbf{A}$ add up to zero, the matrix $\mathbf{A}$ is not of full rank: $r(\mathbf{A}) \leq 2^m - 1$. On the other hand,

$$2^m\mathbf{A} + \begin{pmatrix} 1 & 1 & \dots 1 \\ 1 & 1 & \dots 1 \\ . & & \\ . & & \\ . & & \\ 1 & 1 & \dots 1 \end{pmatrix} = 2^m\mathbf{I},$$

where $\mathbf{I}$ is the 2^m-dimensional unit matrix, and so by subadditivity,

$$r(\mathbf{A}) + 1 = r(2^m\mathbf{A}) + 1 \geq 2^m. \qquad \square$$

By Lemma 13, $\widetilde{W}_t^{(m)}$ is concentrated on S, and there the vector $\widetilde{W}_t^{(m)}$ has *non-singular* multivariate normal distribution.[5] What this means is that even though $\widetilde{W}^{m,1}, \dots, \widetilde{W}^{m,2^m}$ are not independent, their 'degree of freedom' is $2^m - 1$, i. e. the 2^m-dimensional vector $\widetilde{W}_t^{(m)}$ is *determined by* $2^m - 1$ *independent components* (corresponding to $2^m - 1$ principal axes).

5 Asymptotic behavior

5.1 Conditioning

How can one put together that $\overline{Z}_t$ tends to a random final position with the description of the system 'as viewed from $\overline{Z}_t$?' The following lemma is the first step in this direction.

Lemma 14 (Independence). *Let $\mathscr{T}$ be the tail σ-algebra of $\overline{Z}$.*

1. *For $t \geq 0$, the random vector Y_t is independent of the path $\{\overline{Z}_s\}_{s \geq t}$.*
2. *The process $Y = (Y_t; t \geq 0)$ is independent of $\mathscr{T}$.*

Proof. In both parts we will refer to the following fact. Let $s \leq t$, $s \in [\widehat{m}, \widehat{m}+1)$; $t \in [m, m+1)$ with $\widehat{m} \leq m$. Since the random variables $Z_t^1, Z_t^2, \dots, Z_t^{2^m}$ are exchangeable, thus, denoting $\widehat{n} := 2^{\widehat{m}}, n := 2^m$, the vectors $\overline{Z}_t$ and $Z_s^1 - \overline{Z}_s$ are uncorrelated for $0 \leq s \leq t$. Indeed, by Lemma 8 we may assume that $d = 1$ and then

$$E[\overline{Z}_t \cdot \left(Z_s^1 - \overline{Z}_s\right)] = E\left[\frac{Z_t^1 + Z_t^2 + \dots + Z_t^n}{n} \cdot \left(Z_s^1 - \frac{Z_s^1 + Z_s^2 + \dots + Z_s^{\widehat{n}}}{\widehat{n}}\right)\right] =$$
$$\frac{1}{n}E\left(Z_t^1 \cdot Z_s^1\right) + \frac{n-1}{n}E\left(Z_s^1 \cdot Z_t^2\right) - \frac{\widehat{n}}{n\widehat{n}}E\left(Z_t^1 \cdot Z_s^1\right) - \frac{\widehat{n}(n-1)}{n\widehat{n}}E\left(Z_t^2 \cdot Z_s^1\right) = 0.$$

(Of course the index 1 can be replaced by i for any $1 \leq i \leq 2^m$.)

[5] Recall that S is the $(2^m - 1)$-dimensional linear subspace given by the orthogonal complement of the vector $(1, 1, \dots, 1)^T$.

Part (1): First, for any $t > 0$, the ($d \cdot 2^m$-dimensional) vector Y_t is independent of the (d-dimensional) vector $\overline{Z}_t$, because the $d(2^m + 1)$-dimensional vector

$$(\overline{Z}_t, Z_t^1 - \overline{Z}_t, Z_t^2 - \overline{Z}_t, \dots, Z_t^{2^m} - \overline{Z}_t)^T$$

is normal (since it is a linear transformation of the $d \cdot 2^m$ dimensional vector $(Z_t^1, Z_t^2, \dots, Z_t^{2^m})^T$, which is normal by Lemma 9), and so it is sufficient to recall that $\overline{Z}_t$ and $Z_t^i - \overline{Z}_t$ are uncorrelated for $1 \le i \le 2^m$.

To complete the proof of (a), recall (2.2) and (2.3) and notice that the conditional distribution of $\{\overline{Z}_s\}_{s \ge t}$ given $\mathscr{F}_t$ only depends on its starting point $\overline{Z}_t$, as it is that of a Brownian path appropriately slowed down, whatever Y_t (or, equivalently, whatever $Z_t = Y_t + \overline{Z}_t$) is. Since, as we have seen, Y_t is independent of $\overline{Z}_t$, we are done.

Part (2): Let $A \in \mathscr{T}$. By Dynkin's Lemma, it is enough to show that $(Y_{t_1}, ..., Y_{t_k})$ is independent of A for $0 \le t_1 < ... < t_k$ and $k \ge 1$. Since $A \in \mathscr{T} \subset \sigma(\overline{Z}_s; s \ge t_k + 1)$, it is sufficient to show that $(Y_{t_1}, ..., Y_{t_k})$ is independent of $\{\overline{Z}_s\}_{s \ge t_k+1}$.

To see this, similarly as in Part (1), notice that the conditional distribution of $\{\overline{Z}_s\}_{s \ge t_k+1}$ given $\mathscr{F}_{t_k+1}$ only depends on its starting point $\overline{Z}_{t_k+1}$, as it is that of a Brownian path appropriately slowed down, whatever the vector $(Y_{t_1}, ..., Y_{t_k})$ is. If we show that $(Y_{t_1}, ..., Y_{t_k})$ is independent of $\overline{Z}_{t_k+1}$, we are done.

To see why this latter one is true, one just have to repeat the argument in (a), using again normality[6] and recalling that the vectors $\overline{Z}_t$ and $Z_s^i - \overline{Z}_s$ are uncorrelated. □

Remark 15 (Conditioning on the final position of $\overline{Z}$)**.** Let $N := \lim_{t \to \infty} \overline{Z}_t$ (recall that $N \sim \mathscr{N}(0, 2\mathbf{I}_d)$) and

$$P^x(\cdot) := P(\cdot \mid N = x).$$

By Lemma 14, $P^x(Y_t \in \cdot) = P(Y_t \in \cdot)$ for almost all $x \in \mathbb{R}^d$. It then follows that the decomposition $Z_t = \overline{Z}_t \oplus Y_t$ as well as the result obtained for the distribution of Y in subsections 4.1 and 4.2 are true under P^x too, for almost all $x \in \mathbb{R}^d$. ◇

5.2 Main result and a conjecture

So far we have obtained that on the time interval $[m, m+1)$, Y^1 corresponds to the Ornstein-Uhlenbeck operator

$$\frac{1}{2}\sigma_m^2 \Delta - \gamma x \cdot \nabla,$$

where $\sigma_m \to 1$ as $m \to \infty$, with asymptotically vanishing correlation between the driving Brownian motions; that

$$\mathrm{d}Y_t = \mathbf{A}^{(m)} \mathrm{d}W_t^{(m)} - \gamma Y_t \mathrm{d}t,$$

where $\{W_s^{m,i},\ s \ge 0;\ i = 1, 2, ..., 2^m\}$ are independent Brownian motions and $\mathrm{r}(\mathbf{A}^{(m)}) = 2^m - 1$, and finally, the independence of the center of mass and the relative motions as in Lemma 14.

[6]We now need normality for finite dimensional distributions and not just for one dimensional marginals, but this is still true by Lemma 9.

We now go beyond these preliminary results and state a theorem (the main result of this article) and a conjecture on the local behavior of the system. Recall that one can consider Z_n as an element of $\mathscr{M}_f(\mathbb{R}^d)$ by putting unit point mass at the site of each particle; with a slight abuse of notation we will write $Z_n(\mathrm{d}y)$. Let $\{P^x, x \in \mathbb{R}^d\}$ be as in Remark 15. Our main result is as follows.

Theorem 16 (Scaling limit for the attractive case)**.** *If $\gamma > 0$, then, as $n \to \infty$,*

$$2^{-n} Z_n(\mathrm{d}y) \overset{w}{\Rightarrow} \left(\frac{\gamma}{\pi}\right)^{d/2} \exp\left(-\gamma |y-x|^2\right) \mathrm{d}y,\ P^x - \text{a.s.} \tag{5.1}$$

for almost all $x \in \mathbb{R}^d$. Consequently,

$$2^{-n} E Z_n(\mathrm{d}y) \overset{w}{\Rightarrow} f^{\gamma}(y)\mathrm{d}y, \tag{5.2}$$

where

$$f^{\gamma}(\cdot) = \left(\pi(4+\gamma^{-1})\right)^{-d/2} \exp\left[\frac{-|\cdot|^2}{4+\gamma^{-1}}\right].$$

Remark 17.

(i) The proof of Theorem 16 will reveal that actually

$$2^{-t_n} Z_{t_n}(\mathrm{d}y) \overset{w}{\Rightarrow} \left(\frac{\gamma}{\pi}\right)^{d/2} \exp\left(-\gamma |y-x|^2\right) \mathrm{d}y,\ P^x - \text{a.s.}$$

holds for any given sequence $\{t_n\}$ with $t_n \uparrow \infty$ as $n \to \infty$. This, of course, is still weaker than P-a.s. convergence as $t \to \infty$, but one can probably argue, using the method of Asmussen and Hering, as in Subsection 4.3 of [4] to upgrade it to continuous time convergence. Nevertheless, since our model is defined with unit time branching anyway, we felt satisfied with (5.1).

(ii) Notice that f^{γ}, which is the limiting density of the intensity measure, is the density for $\mathscr{N}\left(\mathbf{0}, \left(2+\frac{1}{2\gamma}\right)\mathbf{I}_d\right)$. This is the convolution of $\mathscr{N}\left(\mathbf{0}, 2\mathbf{I}_d\right)$, representing the randomness of the final position of the center of mass (c.f. Lemma 6) and $\mathscr{N}\left(\mathbf{0}, \left(\frac{1}{2\gamma}\right)\mathbf{I}_d\right)$, representing the final distribution of the mass scaled Ornstein-Uhlenbeck branching particle system around its center of mass (c.f. (5.1)). For strong attraction, the contribution of the second term is negligible. ◇

Conjecture 18 (Dichotomy for the repulsive case)**.** *Let $\gamma < 0$.*

1. *If $|\gamma| \geq \frac{\log 2}{d}$, then Z suffers local extinction:*

$$Z_n(\mathrm{d}y) \overset{v}{\Rightarrow} \mathbf{0},\ \ P - \text{a.s.}$$

2. *If $|\gamma| < \frac{\log 2}{d}$, then*

$$2^{-n} e^{d|\gamma| n} Z_n(\mathrm{d}y) \overset{v}{\Rightarrow} \mathrm{d}y,\ \ P - \text{a.s.}$$

5.3 Discussion of the conjecture

The intuition behind the phase transition at $\log 2/d$ is as follows. For a branching diffusion on $\mathbb{R}^d$ with motion generator L, smooth nonzero spatially dependent exponential branching rate $\beta(\cdot) \geq 0$ and dyadic branching, it is known (see Theorem 3 in [5]) that either local extinction or local exponential growth takes place according to whether $\lambda_c \leq 0$ or $\lambda_c > 0$, where $\lambda_c = \lambda_c(L+\beta)$ is the *generalized principle eigenvalue*[7] of $L+\beta$ on $\mathbb{R}^d$. In particular, for $\beta \equiv B > 0$, the criterion for local exponential growth becomes $B > |\lambda_c(L)|$, where $\lambda_c(L) \leq 0$ is the generalized principle eigenvalue of L, which is also the 'exponential rate of escape from compacts' for the diffusion corresponding to L. The interpretation of the criterion in this case is that a large enough mass creation can compensate the fact that individual particles drift away from a given bounded set. (Note that if L correspond to a recurrent diffusion then $\lambda_c(L) = 0$.)

In our case, the situation is similar, with $\lambda_c = d\gamma$ for the outward Ornstein-Uhlenbeck process, taking into account that for unit time branching, the role of B is played by $\log 2$. The condition for local exponential growth should therefore be $\log 2 > d|\gamma|$.

The scaling $2^{-n}e^{d|\gamma|n}$ comes from a similar consideration, noting that in our unit time branching setting, 2^n replaces the term $e^{\beta t}$ appearing in the exponential branching case, while $e^{\lambda_c(L)t}$ becomes $e^{\lambda_c(L)n} = e^{d\gamma n}$.

For a continuous time result analogous to (2) see Example 11 in [4]. Note that since the rescaled (vague) limit of $Z_n(\mathrm{d}y)$ is translation invariant (i.e. Lebesgue), the final position of the center of mass plays no role.

Although we will not prove Conjecture 18, we will discuss some of the technicalities in section 8.

6 Proof of Theorem 3

Since α, β are constant, the branching is independent of the motion, and therefore N defined by

$$N_t := e^{-\beta t}\|X_t\|$$

is a nonnegative martingale (positive on S) tending to a limit almost surely. It is straightforward to check that it is uniformly bounded in L^2 and is therefore uniformly integrable (UI). Write

$$\overline{X}_t = \frac{e^{-\beta t}\langle \mathrm{id}, X_t\rangle}{e^{-\beta t}\|X_t\|} = \frac{e^{-\beta t}\langle \mathrm{id}, X_t\rangle}{N_t}.$$

We now claim that $N_\infty > 0$ a.s. on S. Let $A := \{N_\infty = 0\}$. Clearly $S^{\complement} \subset A$, and so if we show that $P(A) = P(S^{\complement})$, then we are done. As is well known, $P(S^{\complement}) = e^{-\beta/\alpha}$. On the other hand, a standard martingale argument (see the argument after formula (20) in [3]) shows that $0 \leq u(x) := -\log P_{\delta_x}(A)$ must solve the equation

$$\frac{1}{2}\Delta u + \beta u - \alpha u^2 = 0,$$

but since $P_{\delta_x}(A) = P(A)$ constant, therefore $-\log P_{\delta_x}(A)$ solves $\beta u - \alpha u^2 = 0$. Since N is UI, no mass is lost in the limit, giving $P(A) < 1$. So $u > 0$, which in turn implies that $-\log P_{\delta_x}(A) = \beta/\alpha$.

[7] If the operator is symmetric, the world 'generalized' can be omitted. For more on the subject, see Chapter 4 in [10].

Once we know that $N_\infty > 0$ a.s. on S , it is enough to focus on the term $e^{-\beta t}\langle \mathrm{id}, X_t\rangle$: we are going to show that it converges almost surely. Clearly, it is enough to prove this coordinate-wise.

Recall a particular case of the H-transform for the $(L,\beta,\alpha;\mathbb{R}^d)$-superdiffusion X (see Appendix B in [7]):

Lemma 19. *Define X^H by*

$$X^H_t := H(\cdot,t)X_t \quad \left(\textit{that is, } \frac{\mathrm{d}X^H_t}{\mathrm{d}X_t} = H(\cdot,t)\right), \quad t \ge 0. \tag{6.1}$$

If X is an $(L,\beta,\alpha;\mathbb{R}^d)$-superdiffusion, and $H(x,t) := e^{-\lambda t}h(x)$, where h is a positive solution of $(L+\beta)h = \lambda h$, then X^H is a $(L + a\frac{\nabla h}{h}\cdot\nabla, 0, e^{-\lambda t}\alpha h;\mathbb{R}^d)$-superdiffusion.

In our case $\beta(\cdot) \equiv \beta$. So choosing $h(\cdot) \equiv 1$ and $\lambda = \beta$, we have $H(t) = e^{-\beta t}$ and X^H is a $(\frac{1}{2}\Delta, 0, e^{-\beta t}\alpha;\mathbb{R}^d)$-superdiffusion, that is, a critical super-Brownian motion with a clock that is slowing down. Since, as noted above, it is enough to prove the convergence coordinate-wise, we can assume that $d = 1$. One can write

$$e^{-\beta t}\langle \mathrm{id}, X_t\rangle = \langle \mathrm{id}, X^H_t\rangle.$$

Let $\{\mathscr{S}_s\}_{s\ge 0}$ be the 'expectation semigroup' for X, that is, the semigroup corresponding to the operator $\frac{1}{2}\Delta + \beta$. The expectation semigroup $\{\mathscr{S}^H_s\}_{s\ge 0}$ for X^H satisfies $T_s := \mathscr{S}^H_s = e^{-\beta s}\mathscr{S}_s$ and thus it corresponds to Brownian motion. In particular then

$$T_s[\mathrm{id}] = \mathrm{id}. \tag{6.2}$$

(One can pass from bounded continuous functions to $f := \mathrm{id}$ by defining $f_1 := f\,\mathbf{1}_{x>0}$ and $f_2 := f\,\mathbf{1}_{x\le 0}$, then noting that by monotone convergence, $E_{\delta_x}\langle f_i, X^H_t\rangle = \mathbb{E}_x f_i(W_t) \in (-\infty,\infty),\ i = 1,2$, where W is a Brownian motion with expectation $\mathbb{E}$, and finally taking the sum of the two equations.) Therefore $M := \langle \mathrm{id}, X^H\rangle$ is a martingale:

$$\begin{aligned} &E_{\delta_x}\left(M_t \mid \mathscr{F}_s\right) = E_{\delta_x}\left(\langle \mathrm{id}, X^H_t\rangle \mid \mathscr{F}_s\right) = \\ &\quad E_{X_s}\langle \mathrm{id}, X^H_t\rangle = \int_{\mathbb{R}} E_{\delta_y}\langle \mathrm{id}, X^H_t\rangle\, X^H_s(\mathrm{d}y) = \int_{\mathbb{R}} y\, X^H_s(\mathrm{d}y) = M_s. \end{aligned}$$

We now show that M is UI and even uniformly bounded in L^2, verifying its a.s. convergence, and that of the center of mass. To achieve this, define g_n by $g_n(x) = |x|\cdot\mathbf{1}_{\{|x|<n\}}$. Then we have

$$E\langle \mathrm{id}, X^H_t\rangle^2 = E|\langle \mathrm{id}, X^H_t\rangle|^2 \le E\langle |\mathrm{id}|, X^H_t\rangle^2,$$

and by the monotone convergence theorem we can continue with

$$= \lim_{n\to\infty} E\langle g_n, X^H_t\rangle^2.$$

Since g_n is compactly supported, there is no problem to use the moment formula and continue with

$$= \lim_{n\to\infty}\int_0^t \mathrm{d}s\, e^{-\beta s}\langle \delta_0, T_s[\alpha g_n^2]\rangle = \alpha \lim_{n\to\infty}\int_0^t \mathrm{d}s\, e^{-\beta s} T_s[g_n^2](0).$$

Recall that $\{T_s; s \geq 0\}$ is the Brownian semigroup, that is, $T_s[f](x) = \mathbb{E}_x f(W_s)$, where W is Brownian motion. Since $g_n(x) \leq |x|$, therefore we can trivially upper estimate the last expression by

$$\alpha \int_0^t \mathrm{d}s\, e^{-\beta s}\mathbb{E}_0(W_s^2) = \alpha \int_0^t \mathrm{d}s\, s e^{-\beta s} = \alpha \left(\frac{1 - e^{-\beta t}}{\beta^2} - \frac{t e^{-\beta t}}{\beta} \right) < \frac{\alpha}{\beta^2}.$$

Since this upper estimate is independent of t, we are done:

$$\sup_{t \geq 0} E\langle \mathrm{id}, X_t^H \rangle^2 \leq \frac{\alpha}{\beta^2}.$$

Finally, we show that $\overline{X}$ has continuous paths. To this end we first note that we can (and will) consider a version of X where all the paths are continuous in the weak topology of measures. We now need a simple lemma.

Lemma 20. *Let $\{\mu_t,\ t \geq 0\}$ be a family in $\mathscr{M}_f(\mathbb{R}^d)$ and assume that $t_0 > 0$ and $\mu_t \overset{w}{\Rightarrow} \mu_{t_0}$ as $t \to t_0$. Assume furthermore that*

$$C = C_{t_0,\epsilon} := \mathrm{cl} \left(\bigcup_{t = t_0 - \epsilon}^{t_0 + \epsilon} \mathrm{supp}(\mu_t) \right)$$

is compact with some $\epsilon > 0$. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. Then $\lim_{t \to t_0} \langle f, \mu_t \rangle = \langle f, \mu_{t_0} \rangle$.

Proof of Lemma 20. First, if $f = (f_1, ..., f_d)$ then all f_i are $\mathbb{R}^d \to \mathbb{R}$ continuous functions and $\lim_{t \to t_0} \langle f, \mu_t \rangle = \langle f, \mu_{t_0} \rangle$ simply means that $\lim_{t \to t_0} \langle f_i, \mu_t \rangle = \langle f_i, \mu_{t_0} \rangle$. Therefore, it is enough to prove the lemma for an $\mathbb{R}^d \to \mathbb{R}$ continuous function. Let k be so large that $C \subset I_k := [-k,k]^d$. Using a mollified version of $\mathbf{1}_{[-k,k]}$, it is trivial to construct a continuous function $\widehat{f} =: \mathbb{R}^d \to \mathbb{R}$ such that $\widehat{f} = f$ on I_k and $\widehat{f} = 0$ on $\mathbb{R}^d \setminus I_{2k}$. Then,

$$\lim_{t \to t_0} \langle f, \mu_t \rangle = \lim_{t \to t_0} \langle \widehat{f}, \mu_t \rangle = \langle \widehat{f}, \mu_{t_0} \rangle = \langle f, \mu_{t_0} \rangle,$$

since $\widehat{f}$ is a bounded continuous function. $\square$

Returning to the proof of the theorem, let us invoke the fact that for

$$C_s(\omega) := \mathrm{cl} \left(\bigcup_{z \leq s} \mathrm{supp}(X_z(\omega)) \right),$$

we have $P(C_s$ is compact$) = 1$ for all fixed $s \geq 0$ (compact support property; see [6]). By the monotonicity in s, there exists an $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ such that for $\omega \in \Omega_1$,

$$C_s(\omega) \text{ is compact } \forall s \geq 0.$$

Let $\omega \in \Omega_1$ and recall that we are working with a continuous path version of X. Then letting $f := \mathrm{id}$ and $\mu_t = X_t(\omega)$, Lemma 20 implies that for $t_0 > 0$, $\lim_{t \to t_0} \langle \mathrm{id}, X_t(\omega) \rangle = \langle \mathrm{id}, X_{t_0}(\omega) \rangle$. The right continuity at $t_0 = 0$ is similar. $\square$

7 Proof of Theorem 16

Before we prove (5.1), we note the following. Fix $x \in \mathbb{R}^d$. Abbreviate

$$\nu^{(x)}(\mathrm{d}y) := \left(\frac{\gamma}{\pi}\right)^{d/2} \exp\left(-\gamma|y-x|^2\right)\mathrm{d}y.$$

Since $2^{-n}Z_n(\mathrm{d}x), \nu^{(x)} \in \mathscr{M}_1(\mathbb{R}^d)$, therefore, as is well known[8], $2^{-n}Z_n(\mathrm{d}x) \overset{w}{\Rightarrow} \nu^{(x)}$ is in fact equivalent to

$$\forall g \in \mathscr{E}:\ 2^{-n}\langle g, Z_n\rangle \to \langle g, \nu^{(x)}\rangle,$$

where $\mathscr{E}$ is any given family of bounded measurable functions with $\nu^{(x)}$-zero (Lebesgue-zero) sets of discontinuity that is separating for $\mathscr{M}_1(\mathbb{R}^d)$.

In fact, one can pick a *countable* $\mathscr{E}$, which, furthermore, consists of compactly supported functions. Such an $\mathscr{E}$ is given by the indicators of sets in $\mathscr{R}$.

Fix such a family $\mathscr{E}$. Since $\mathscr{E}$ is countable, in order to show (5.1), it is sufficient to prove that for almost all $x \in \mathbb{R}^d$,

$$P^x(2^{-n}\langle g, Z_n\rangle \to \langle g, \nu^{(x)}\rangle) = 1,\ g \in \mathscr{E}. \tag{7.1}$$

We will carry out the proof of (7.1) in several subsections.

7.1 Putting Y and $\overline{Z}$ together

The following remark is for the interested reader familiar with [4] only. *It can be skipped without any problem.*

Remark 21. Once we have the description of Y as in Subsections 4.1 and 4.2 and Remark 15, we can try to put them together with the Strong Law of Large Numbers for the local mass from [4] for the process Y.

If the components of Y were independent and the branching rate were exponential, Theorem 6 of [4] would be readily applicable. However, since the 2^m components of Y are not independent (as we have seen, their degree of freedom is $2^m - 1$) and since, unlike in [4], we now have unit time branching, the method of [4] must be adapted to our setting. The reader will see that this adaptation requires quite a bit of extra work. $\diamond$

Let $\widetilde{f}(\cdot) = \widetilde{f}^{\gamma}(\cdot) := (\frac{\gamma}{\pi})^{d/2}\exp\left(-\gamma|\cdot|^2\right)$, and note that $\widetilde{f}$ is the density for $\mathscr{N}(\mathbf{0}, (2\gamma)^{-1}\mathbf{I}_d)$.
We now claim that in order to show (7.1), it is enough to prove that for almost all x,

$$P^x(2^{-n}\langle g, Y_n\rangle \to \langle g, \widetilde{f}\rangle) = 1,\ g \in \mathscr{E}. \tag{7.2}$$

This is because

$$\lim_{n\to\infty} 2^{-n}\langle g, Z_n\rangle = \lim_{n\to\infty} 2^{-n}\langle g, Y_n + \overline{Z}_n\rangle = \lim_{n\to\infty} 2^{-n}\langle g(\cdot + \overline{Z}_n), Y_n\rangle = I + II,$$

where

$$I := \lim_{n\to\infty} 2^{-n}\langle g(\cdot + x), Y_n\rangle$$

[8] See Proposition 4.8.12 and the proof of Propositions 4.8.15 in [1].

and

$$II := \lim_{n\to\infty} 2^{-n}\langle g(\cdot + \overline{Z}_n) - g(\cdot + x), Y_n\rangle.$$

Now, (7.2) implies that for almost all x, $I = \langle g(\cdot + x), \widetilde{f}(\cdot)\rangle$ P^x-a.s., while the compact support of g, and Heine's Theorem yields that $II = 0,\ P^x - a.s.$ Hence, $\lim_{n\to\infty} 2^{-n}\langle g, Z_n\rangle = \langle g(\cdot + x), \widetilde{f}(\cdot)\rangle = \langle g(\cdot), \widetilde{f}(\cdot - x)\rangle,\ P^x$-a.s., giving (7.1).

Next, let us see how (5.1) implies (5.2). Let g be continuous and bounded. Since $2^{-n}\langle Z_n, g\rangle \leq \|g\|_\infty$, it follows by bounded convergence that

$$\lim_{n\to\infty} E2^{-n}\langle Z_n, g\rangle = \int_{\mathbb{R}^d} E^x\left(\lim_{n\to\infty} 2^{-n}\langle Z_n, g\rangle\right) Q(\mathrm{d}x) = \int_{\mathbb{R}^d} \langle g(\cdot), \widetilde{f}(\cdot - x)\rangle\, Q(\mathrm{d}x),$$

where $Q \sim \mathscr{N}(\mathbf{0}, 2\mathbf{I}_d)$. Now, if $\widehat{f} \sim \mathscr{N}(\mathbf{0}, 2\mathbf{I}_d)$ then, since $f^\gamma \sim \mathscr{N}\left(\mathbf{0}, \left(2 + \frac{1}{2\gamma}\right)\mathbf{I}_d\right)$, it follows that $f^\gamma = \widehat{f} * \widetilde{f}$ and

$$\int_{\mathbb{R}^d} \langle g(\cdot), \widetilde{f}(\cdot - x)\rangle\, Q(\mathrm{d}x) = \langle g(\cdot), f^\gamma\rangle,$$

yielding (5.2).

Next, notice that it is in fact sufficient to prove (7.2) *under P instead of P^x*. Indeed, by Lemma 14,

$$\begin{aligned} P^x\left(\lim_{n\to\infty} 2^{-n}\langle g, Y_n\rangle = \langle g, \widetilde{f}\rangle\right) &= P\left(\lim_{n\to\infty} 2^{-n}\langle g, Y_n\rangle = \langle g, \widetilde{f}\rangle \mid N = x\right) \\ &= P\left(\lim_{n\to\infty} 2^{-n}\langle g, Y_n\rangle = \langle g, \widetilde{f}\rangle\right). \end{aligned}$$

Let us use the shorthand $U_n(\mathrm{d}y) := 2^{-n}Y_t(\mathrm{d}y)$; in general $U_t(\mathrm{d}y) := \frac{1}{n_t}Y_t(\mathrm{d}y)$. With this notation, our goal is to show that

$$P(\langle g, U_n\rangle \to \langle g, \widetilde{f}\rangle) = 1,\ g \in \mathscr{E}. \tag{7.3}$$

Now, as mentioned earlier, we may (and will) take $\mathscr{E} := \mathscr{I}$, where $\mathscr{I}$ is the family of indicators of sets in $\mathscr{R}$. Then, it remains to show that

$$P\left(U_n(B) \to \int_B \widetilde{f}(x)\mathrm{d}x\right) = 1,\ B \in \mathscr{R}. \tag{7.4}$$

7.2 Outline of the further steps

Notation 22. In the sequel $\{\mathscr{F}_t\}_{t\geq 0}$ will denote the canonical filtration for Y, rather than the canonical filtration for Z.

The following key lemma (Lemma 23) will play an important role. It will be derived using Lemma 25 and (7.12), where the latter one will be derived with the help of Lemma 25 too. Then, Lemma 23 together with (7.14) will be used to complete the proof of (7.4) and hence, that of Theorem 16.

Lemma 23. *Let $B \subset \mathbb{R}^d$ be a bounded measurable set. Then,*

$$\lim_{n\to\infty}\left[U_{n+m_n}(B) - E(U_{n+m_n}(B) \mid \mathscr{F}_n)\right] = 0,\ P - a.s. \tag{7.5}$$

7.3 Establishing the crucial estimate (7.12) and the key Lemma 23

Let Y_n^i denote the "ith" particle at time n, $i = 1, 2, ..., 2^n$. Since B is a fixed set, in the sequel we will simply write U_n instead of $U_n(B)$. Recall the time inhomogeneity of the underlying diffusion process and note that by the branching property, we have the *clumping decomposition*: for $n, m \geq 1$,

$$U_{n+m} = \sum_{i=1}^{2^n} 2^{-n} U_m^{(i)}, \tag{7.6}$$

where given $\mathscr{F}_n$, each member in the collection $\{U_m^{(i)} : i = 1, ..., 2^n\}$ is defined similarly to U_m but with Y_m replaced by the time m configuration of the particles starting at Y_n^i, $i = 1, ..., 2^n$, respectively, and with motion component $\frac{1}{2}\sigma_{n+k}\Delta - \gamma x \cdot \Delta$ in the time interval $[k, k+1)$.

7.3.1 The functions a and ζ

Next, we define two positive functions, a and ζ on $(1, \infty)$. Here is a rough motivation.

(i) The function $a_\cdot$ will be related (via (7.9) below) to the *radial speed* of the particle system Y.

(ii) The function $\zeta(\cdot)$, will be related (via (7.10) below) to the *speed of ergodicity* of the underlying Ornstein-Uhlenbeck process.

For $t > 1$, define

$$a_t := C_0 \cdot \sqrt{t}, \tag{7.7}$$

$$\zeta(t) := C_1 \log t, \tag{7.8}$$

where C_0 and C_1 are positive (non-random) constants to be determined later. Note that

$$m_t := \zeta(a_t) = C_3 + C_4 \log t$$

with $C_3 = C_1 \log C_0 \in \mathbb{R}$ and $C_4 = C_1/2 > 0$. We will use the shorthand

$$\ell_n := 2^{\lfloor m_n \rfloor}.$$

Recall that $\widetilde{f}^\gamma$ is the density for $\mathscr{N}(0, (2\gamma)^{-1}\mathbf{I}_d)$ and let $q(x, y, t) = q^{(\gamma)}(x, y, t)$ and $q(x, \mathrm{d}y, t) = q^{(\gamma)}(x, \mathrm{d}y, t)$ denote the transition density and the transition kernel, respectively, corresponding to the operator $\frac{1}{2}\Delta - \gamma x \cdot \nabla$. We are going to show below that for sufficiently large C_0 and C_1, the following holds. For each given $x \in \mathbb{R}^d$ and $B \subset \mathbb{R}^d$ nonempty bounded measurable set,

$$P\left(\exists n_0, \forall n_0 < n \in \mathbb{N} \,:\, \mathrm{supp}(Y_n) \subset B_{a_n}\right) = 1, \text{ and} \tag{7.9}$$

$$\lim_{t\to\infty} \sup_{z \in B_t, y \in B} \left| \frac{q(z, y, \zeta(t))}{\widetilde{f}^\gamma(y)} - 1 \right| = 0. \tag{7.10}$$

For (7.9), note that in Example 10 of [4] similar calculations were carried out for the case when the underlying diffusion is an Ornstein-Uhlenbeck process and the breeding is quadratic. It is important to point out that in that example the estimates followed from *expectation* calculations (see also

Remark 9 in [4]), and thus they can be mimicked in our case for the Ornstein-Uhlenbeck process performed by the particles in Y (which corresponds to the operator $\frac{1}{2}\sigma_m\Delta - \gamma x\cdot\nabla$ on $[m, m+1)$, $m \geq 1$), despite the fact that the particle motions are now correlated. These expectation calculations lead to the estimate that the growth rate of the support of Y satisfies (7.9) with a sufficiently large $C_0 = C_0(\gamma)$. The same example shows that (7.10) holds with a sufficiently large $C_1 = C_1(\gamma)$.

Remark 24. Denote by $\nu = \nu^\gamma \in \mathscr{M}_1(\mathbb{R}^d)$ the distribution of $\mathscr{N}(\mathbf{0}, (2\gamma)^{-1}\mathbf{I}_d)$. Let $B \subset \mathbb{R}^d$ be a nonempty bounded measurable set. Taking $t = a_n$ in (7.10) and recalling that $\zeta(a_n) = m_n$,

$$\lim_{n\to\infty} \sup_{z\in B_{a_n}, y\in B} \left| \frac{q(z, y, m_n)}{\widetilde{f}^\gamma(y)} - 1 \right| = 0.$$

Since $\widetilde{f}^\gamma$ is bounded, this implies that for any bounded measurable set $B \subset \mathbb{R}^d$,

$$\lim_{n\to\infty} \sup_{z\in B_{a_n}} \left[q(z, B, m_n) - \nu(B) \right] = 0. \tag{7.11}$$

We will use (7.11) in Subsection 7.4. $\diamond$

7.3.2 Covariance estimates

Let $\{Y^{i,j}_{m_n},\ j = 1, ..., \ell_n\}$ be the descendants of $Y_n{}^i$ at time $m_n + n$. So $Y^{1,j}_{m_n}$ and $Y^{2,k}_{m_n}$ are respectively the jth and kth particle at time $m_n + n$ of the trees emanating from the first and second particles at time n. It will be useful to control the covariance between $\mathbf{1}_B(Y^{1,j}_{m_n})$ and $\mathbf{1}_B(Y^{2,k}_{m_n})$, where B is a nonempty, bounded open set. To this end, we will need the following lemma, the proof of which is relegated to Section 9 in order to minimize the interruption in the main flow of the argument.

Lemma 25. *Let $B \subset \mathbb{R}^d$ be a bounded measurable set.*

(a) *There exists a non-random constant $K(B)$ such that if $C = C(B, \gamma) := \frac{3}{\gamma}|B|^2 K(B)$, then*

$$P\Bigg[\forall n \text{ large enough and } \forall \xi, \widetilde{\xi} \in \Pi_n,\ \xi \neq \widetilde{\xi} :$$
$$\left| P(\xi_{m_n}, \widetilde{\xi}_{m_n} \in B \mid \mathscr{F}_n) - P(\xi_{m_n} \in B \mid \mathscr{F}_n) P(\widetilde{\xi}_{m_n} \in B \mid \mathscr{F}_n) \right| \leq \frac{Cn}{2^n} \Bigg] = 1,$$

where Π_n denotes the collection of those ℓ_n particles, which, start at some time-n location of their parents and run for (an additional) time m_n.

(b) *Let $C = C(B) := \nu(B) - (\nu(B))^2$. Then*

$$P\left[\lim_{n\to\infty} \sup_{\xi\in\Pi_n} \left| \mathrm{Var}\left(\mathbf{1}_{\{\xi_{m_n}\in B\}} \mid \mathscr{F}_n \right) - C \right| = 0 \right] = 1.$$

Remark 26. In the sequel, instead of writing ξ_{m_n} and $\widetilde{\xi}_{m_n}$, we will use the notation $Y^{i_1,j}_{m_n}$ and $Y^{i_2,k}_{m_n}$ with $1 \leq i_1, i_2 \leq n; 1 \leq j, k \leq \ell_n$ satisfying that $i_1 \neq i_2$ or $j \neq k$. $\diamond$

7.3.3 The crucial estimate (7.12)

Let $B \subset \mathbb{R}^d$ be a bounded measurable set and $C = C(B,\gamma)$ as in Lemma 25. Define

$$\mathscr{Z}_i := \frac{1}{\ell_n} \sum_{j=1}^{\ell_n} \left[\mathbf{1}_B(Y_{m_n}^{i,j}) - P(Y_{m_n}^{i,j} \in B \mid Y_n^i) \right], \quad i = 1, 2, ..., 2^n.$$

With the help of Lemma 25, we will establish the following crucial estimate, the proof of which is provided in Section 9.

Claim 27. *There exists a non-random constant* $\widehat{C}(B,\gamma) > 0$ *such that*

$$P\left(\sum_{1 \le i \ne j \le 2^n} E\left[\mathscr{Z}_i \mathscr{Z}_j \mid \mathscr{F}_n \right] \le \widehat{C}(B,\gamma) n \ell_n \sum_{i=1}^{2^n} E\left[\mathscr{Z}_i^2 \mid \mathscr{F}_n \right], \text{ for large } n\text{'s} \right) = 1. \tag{7.12}$$

The significance of Claim 27 is as follows.

Claim 28. *Lemma 25 together with the estimate (7.12) implies Lemma 23.*

Proof of Claim 28. Assume that (7.12) holds. By the clumping decomposition under (7.6),

$$U_{n+m_n} - E(U_{n+m_n} \mid \mathscr{F}_n) = \sum_{i=1}^{2^n} 2^{-n} \left(U_{m_n}^{(i)} - E(U_{m_n}^{(i)} \mid \mathscr{F}_n) \right).$$

Since $U_{m_n}^{(i)} = \ell_n^{-1} \sum_{j=1}^{\ell_n} \mathbf{1}_B(Y_{m_n}^{i,j})$, therefore

$$U_{m_n}^{(i)} - E(U_{m_n}^{(i)} \mid \mathscr{F}_n) = U_{m_n}^{(i)} - E(U_{m_n}^{(i)} \mid Y_n^i) = \mathscr{Z}_i.$$

Hence,

$$E\left(\left[U_{n+m_n} - E(U_{n+m_n} \mid \mathscr{F}_n) \right]^2 \mid \mathscr{F}_n \right) = E\left(\left[\sum_{i=1}^{2^n} 2^{-n} \left(U_{m_n}^{(i)} - E(U_{m_n}^{(i)} \mid \mathscr{F}_n) \right) \right]^2 \mid \mathscr{F}_n \right)$$

$$= E\left(\left[\sum_{i=1}^{2^n} 2^{-n} \mathscr{Z}_i \right]^2 \mid \mathscr{F}_n \right) = 2^{-2n} \left[\sum_{i=1}^{2^n} E\left(\mathscr{Z}_i^2 \mid \mathscr{F}_n \right) + \sum_{1 \le i \ne j \le 2^n} E\left[\mathscr{Z}_i \mathscr{Z}_j \mid \mathscr{F}_n \right] \right]$$

By (7.12), P-almost surely, this can be upper estimated for large n's by

$$2^{-2n} \left[(Cn\ell_n + 1) \sum_{i=1}^{2^n} E\left(\mathscr{Z}_i^2 \mid \mathscr{F}_n \right) \right] \le 2^{-2n} \left[C' n \ell_n \sum_{i=1}^{2^n} E\left(\mathscr{Z}_i^2 \mid \mathscr{F}_n \right) \right],$$

where $\widehat{C}(B,\gamma) < C'$. Now note that by Lemma 25,

$$\begin{aligned}
&\ell_n^2 E[\mathscr{Z}_1^2 \mid \mathscr{F}_n] \\
&= \sum_{j,k=1}^{\ell_n} \left\{ P(Y_{m_n}^{1,j}, Y_{m_n}^{1,k} \in B \mid \mathscr{F}_n) - P(Y_{m_n}^{1,j} \in B \mid \mathscr{F}_n) P(Y_{m_n}^{1,k} \in B \mid \mathscr{F}_n) \right\} \\
&= (\ell_n^2 - \ell_n) \left\{ P(Y_{m_n}^{1,1}, Y_{m_n}^{1,2} \in B \mid \mathscr{F}_n) - P(Y_{m_n}^{1,1} \in B \mid \mathscr{F}_n) P(Y_{m_n}^{1,2} \in B \mid \mathscr{F}_n) \right\} \\
&+ \ell_n \mathrm{Var}\left(\mathbf{1}_{\{Y_{m_n}^{1,1} \in B\}} \mid \mathscr{F}_n \right) = \mathscr{O}(n 2^{-n} \ell_n^2) + \mathscr{O}(\ell_n).
\end{aligned}$$

(Here the first term corresponds to the $k \neq j$ case and the second term corresponds to the $k = j$ case.)

Since, by Lemma 25, this estimate remains *uniformly* valid when the index 1 is replaced by anything between 1 and 2^n, therefore,

$$\ell_n^2 \sum_{i=1}^{2^n} E[\mathscr{Z}_i^2 \mid \mathscr{F}_n] = \mathscr{O}(n\ell_n^2) + \mathscr{O}(2^n \ell_n) = \mathscr{O}(2^n \ell_n) \ a.s.$$

(Recall that $m_n = C_3 + C_4 \log n$.) Thus,

$$\sum_{i=1}^{2^n} E[\mathscr{Z}_i^2 \mid \mathscr{F}_n] = \mathscr{O}(2^n / \ell_n) \ a.s.$$

It then follows that, P-almost surely, for large n's,

$$E\left(\left[U_{n+m_n} - E(U_{n+m_n} \mid \mathscr{F}_n)\right]^2 \mid \mathscr{F}_n\right) \leq C'' \cdot n2^{-n}.$$

The summability immediately implies Lemma 23; nevertheless, since conditional probabilities are involved, we decided to write out the standard argument in details. First, we have that P-almost surely,

$$\sum_{n=1}^{\infty} E\left(\left[U_{n+m_n} - E(U_{n+m_n} \mid \mathscr{F}_n)\right]^2 \mid \mathscr{F}_n\right) < \infty.$$

Then, by the (conditional) Markov inequality, for any $\delta > 0$, P-almost surely,

$$\sum_{n=1}^{\infty} P\left(\left|U_{n+m_n} - E(U_{n+m_n} \mid \mathscr{F}_n)\right| > \delta \mid \mathscr{F}_n\right) < \infty.$$

Finally, by a well known conditional version of Borel-Cantelli (see e.g. Theorem 1 in [2]), it follows that

$$P\left(\left|U_{n+m_n} - E(U_{n+m_n} \mid \mathscr{F}_n)\right| > \delta \text{ occurs finitely often}\right) = 1,$$

which implies the result in Lemma 23. □

The following remark is intended to the interested reader familiar with [4] only, and *can be skipped without any trouble.*

Remark 29 (No spine argument needed). In [4], this part of the analysis was more complicated, because the upper estimate there involved the analogous term U_s, which, unlike here, was not upper bounded. Therefore, in [4] we proceeded with a *spine change of measure* along with some further calculations. That part of the work is saved now. The martingale by which the change of measure was defined in [4], now becomes identically one: $2^{-n}\langle 1, Y_n \rangle = 1$. (Because now 2^{-n} plays the role of $e^{-\lambda_c t}$ and the function 1 plays the role of the positive $(L + \beta - \lambda_c)$-harmonic function ϕ.) ◇

7.4 The rest of the proof

Recall the definition of ν and $\mathscr{R}$, and that our goal is to show that for any $B \in \mathscr{R}$,

$$P(\lim_{n\to\infty} U_n(B) = \nu(B)) = 1. \tag{7.13}$$

Let us fix $B \in \mathscr{R}$ for the rest of the subsection, and simply write U_t instead of $U_t(B)$.

Next, recall the limit in (7.11), but note that the underlying diffusion is only asymptotically Ornstein-Uhlenbeck[9], that is $\sigma_n^2 = 1 - 2^{-n}$, and so the transition kernels q_n defined by

$$q_n(x, \mathrm{d}y, k) := P(Y_k^1 \in \mathrm{d}y \mid Y_n^1 = x),\ k \geq n,$$

are slightly different from q. Note also the decomposition

$$E\left(U_{n+m_n} \mid \mathscr{F}_n\right) = \sum_{i=1}^{2^n} 2^{-n} E(U_{m_n}^{(i)} \mid \mathscr{F}_n) = 2^{-n} \sum_{i=1}^{2^n} q_n(Y_n^i, B, n + m_n).$$

In addition, recall the following facts.

1. If $A_n := \{\mathrm{supp}(Y_n) \not\subset B_{a_n}\}$, then $\lim_{n\to\infty} \mathbf{1}_{A_n} = 0$, P-a.s.;
2. $m_t = \zeta(a_t) = C_3 + C_4 \log t$;
3. Lemma 23.

From these it follows that the limit

$$\lim_{n\to\infty} \sup_{x_0 \in B_{a_n}} \left| q_n(x_0, B, n + m_n) - \nu(B) \right| = 0, \tag{7.14}$$

which we will verify below, implies (7.13) with U_n replaced by U_{n+m_n}.

Remark 30 (n and $N(n)$)**.** Notice that (7.13) must then also hold P-a.s. for U_n, and even for U_{t_n} with *any given* sequence $t_n \uparrow \infty$ replacing n. Indeed, define the sequence $N(n)$ by the equation

$$N(n) + m_{N(n)} = t_n.$$

Clearly, $N(n) = \Theta(t_n)$, and in particular $\lim_{n\to\infty} N(n) = \infty$. Now, it is easy to see that in the proof of Theorem 16, including the remainder of this paper, all the arguments go through when replacing n by $N(n)$, yielding thus (7.13) with U_n replaced by $U_{N(n)+m_{N(n)}} = U_{t_n}$. In those arguments it never plays any role that n is actually an integer. ◇

(We preferred to provide Remark 30 instead of presenting the proof with $N(n)$ replacing n everywhere, and to avoid notation even more difficult to follow[10].)

Motivated by Remark 30, we now show (7.14). To achieve this goal, first recall that on the time interval $[l, l+1)$, $Y = Y^1$ corresponds to the Ornstein-Uhlenbeck operator

$$\frac{1}{2}\sigma_l^2 \Delta - \gamma x \cdot \nabla,$$

[9] Unlike in [4], where $\sigma_n \equiv 1$.

[10] For example, one should replace 2^n with $2^{\lfloor N(n) \rfloor}$ or $n_{N(n)}$ everywhere.

where $\sigma_l^2 = 1 - 2^{-l}$, $l \in \mathbb{N}$. That is, if $\sigma^{(n)}(\cdot)$ is defined by $\sigma^{(n)}(s) := \sigma_{n+l}$ for $s \in [l, l+1)$, then, given $\mathscr{F}_n$ and with a Brownian motion W, one has that

$$\begin{aligned} Y_{m_n} - E(Y_{m_n} \mid \mathscr{F}_n) = Y_{m_n} - e^{-\gamma m_n} Y_0 &= \int_0^{m_n} \sigma^{(n)}(s) e^{\gamma(s-m_n)} \, \mathrm{d}W_s \\ &= \int_0^{m_n} e^{\gamma(s-m_n)} \, \mathrm{d}W_s - \int_0^{m_n} [1 - \sigma^{(n)}(s)] e^{\gamma(s-m_n)} \, \mathrm{d}W_s. \end{aligned}$$

Fix $\epsilon > 0$. By the Chebyshev inequality and the Itô-isometry,

$$\begin{aligned} & P\left(\left| \int_0^{m_n} [1 - \sigma^{(n)}(s)] e^{\gamma(s-m_n)} \, \mathrm{d}W_s \right| > \epsilon \right) \\ & \leq \epsilon^{-2} E\left[\left(\int_0^{m_n} [1 - \sigma^{(n)}(s)] e^{\gamma(s-m_n)} \, \mathrm{d}W_s \right)^2 \right] \\ & = \epsilon^{-2} \int_0^{m_n} [1 - \sigma^{(n)}(s)]^2 e^{2\gamma(s-m_n)} \, \mathrm{d}s. \end{aligned}$$

Now,

$$[1 - \sigma^{(n)}(s)]^2 \leq [1 - \sigma_n]^2 = (1 - \sqrt{1 - 2^{-n}})^2 = \left(\frac{2^{-n}}{1 + \sqrt{1 - 2^{-n}}} \right)^2 \leq 2^{-2n}.$$

Hence,

$$P\left(\left| \int_0^{m_n} [1 - \sigma^{(n)}(s)] e^{\gamma(s-m_n)} \, \mathrm{d}W_s \right| > \epsilon \right) \leq \epsilon^{-2} \int_0^{m_n} 2^{-2n} e^{2\gamma(s-m_n)} \, \mathrm{d}s.$$

Since $e^{-m_n} = e^{-C_3} n^{-C_4}$, we obtain that

$$\begin{aligned} & \epsilon^{-2} \int_0^{m_n} 2^{-2n} e^{2\gamma(s-m_n)} \, \mathrm{d}s = \epsilon^{-2} e^{-2\gamma C_3} 2^{-2n} n^{-2\gamma C_4} \int_0^{m_n} e^{2\gamma s} \, \mathrm{d}s \\ & = \epsilon^{-2} e^{-2\gamma C_3} 2^{-2n} n^{-2\gamma C_4} \cdot \frac{e^{2\gamma C_3} n^{2\gamma C_4} - 1}{2\gamma} \to 0, \text{ as } n \to \infty. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} P\left(\left| \int_0^{m_n} [1 - \sigma^{(n)}(s)] e^{\gamma(s-m_n)} \, \mathrm{d}W_s \right| > \epsilon \right) = 0. \tag{7.15}$$

We have

$$\begin{aligned} q_n(x_0, B, n + m_n) &= P(Y^1_{n+m_n} \in B \mid Y^1_n = x_0) \\ &= P\left(\int_0^{m_n} \sigma^{(n)}(s) e^{\gamma(s-m_n)} \, \mathrm{d}W_s \in B - x_0 e^{-\gamma m_n} \right), \end{aligned}$$

and

$$q(x_0, B, m_n) = P\left(\int_0^{m_n} e^{\gamma(s-m_n)} \, \mathrm{d}W_s \in B - x_0 e^{-\gamma m_n} \right).$$

For estimating $q_n(x_0, B, n+m_n)$ let us use the inequality

$$\dot{A}^\epsilon \subset A + b \subset A^\epsilon, \text{ for } A \subset \mathbb{R}^d, b \in \mathbb{R}^d, |b| < \epsilon, \epsilon > 0.$$

So, for any $\epsilon > 0$,

$$\begin{aligned}
&q_n(x_0, B, n+m_n) \\
&= P\left(\int_0^{m_n} e^{\gamma(s-m_n)}\,\mathrm{d}W_s - \int_0^{m_n}[1-\sigma^n(s)]e^{\gamma(s-m_n)}\,\mathrm{d}W_s \in B - x_0 e^{-\gamma m_n}\right) \\
&= P\left(\int_0^{m_n} e^{\gamma(s-m_n)}\,\mathrm{d}W_s \in B - x_0 e^{-\gamma m_n} + \int_0^{m_n}[1-\sigma^n(s)]e^{\gamma(s-m_n)}\,\mathrm{d}W_s\right) \\
&\le P\left(\int_0^{m_n} e^{\gamma(s-m_n)}\,\mathrm{d}W_s \in B^\epsilon - x_0 e^{-\gamma m_n}\right) \\
&\quad + P\left(\left|\int_0^{m_n}[1-\sigma^{(n)}(s)]e^{\gamma(s-m_n)}\,\mathrm{d}W_s\right| > \epsilon\right) \\
&= q(x_0, B^\epsilon, m_n) + P\left(\left|\int_0^{m}[1-\sigma^{(n)}(s)]e^{\gamma(s-m_n)}\,\mathrm{d}W_s\right| > \epsilon\right).
\end{aligned}$$

Taking $\limsup_{n\to\infty}\sup_{x_0 \in B_{a_n}}$, the second term vanishes by (7.15) and the first term becomes $\nu(B^\epsilon)$ by (7.11).

The lower estimate is similar:

$$\begin{aligned}
&q_n(x_0, B, n+m_n) \\
&\ge P\left(\int_0^{m_n} e^{\gamma(s-m_n)}\,\mathrm{d}W_s \in \dot{B}^\epsilon - x_0 e^{-\gamma m_n}\right) \\
&\quad - P\left(\left|\int_0^{m_n}[1-\sigma^{(n)}(s)]e^{\gamma(s-m_n)}\,\mathrm{d}W_s\right| > \epsilon\right) \\
&= q(x_0, \dot{B}^\epsilon, m_n) - P\left(\left|\int_0^{m_n}[1-\sigma^{(n)}(s)]e^{\gamma(s-m_n)}\,\mathrm{d}W_s\right| > \epsilon\right).
\end{aligned}$$

Taking $\liminf_{n\to\infty}\sup_{x_0 \in B_{a_n}}$, the second term vanishes by (7.15) and the first term becomes $\nu(\dot{B}^\epsilon)$ by (7.11).

Now (7.14) follows from these limits:

$$\lim_{\epsilon\downarrow 0} \nu(B^\epsilon) = \lim_{\epsilon\downarrow 0} \nu(\dot{B}^\epsilon) = \nu(B). \tag{7.16}$$

To verify (7.16) let $\epsilon \downarrow 0$ and use that, obviously, $\nu(\partial B) = 0$. Then $\nu(B^\epsilon) \downarrow \nu(\mathrm{cl}(B)) = \nu(B)$ because $B^\epsilon \downarrow \mathrm{cl}(B)$, and $\nu(\dot{B}^\epsilon) \uparrow \nu(\dot{B}) = \nu(B)$ because $\dot{B}^\epsilon \uparrow \dot{B}$.

The proof of (7.14) and that of Theorem 16 is now completed. □

8 On a possible proof of Conjecture 18

In this section we provide some discussion for the reader familiar with [4] and interested in a possible way of proving Conjecture 18.

The main difference relative to the attractive case is that, as we have mentioned earlier, in that case one does not need the spine change of measure from [4]. In the repulsive case however, one cannot bypass the spine change of measure. Essentially, an h-transform transforms the outward Ornstein-Uhlenbeck process into an inward Ornstein-Uhlenbeck process, and in the exponential branching clock setting (and with independent particles), this inward Ornstein-Uhlenbeck process becomes the 'spine.' A possible way of proving Conjecture 18 would be to try to adapt the spine change of measure to unit time branching and dependent particles.

9 The proof of Lemma 25 and that of (7.12)

9.1 Proof of Lemma 25

The proof of the first part is a bit tedious, the proof of the second part is very simple. We recall that $\{\mathscr{F}_t\}_{t\geq 0}$ denotes the canonical filtration for Y.

(a): Throughout the proof, we may (and will) assume that, the growth of the support of Y is bounded from above by the function a, because this happens with probability one. That is, we assume that

$$\exists n_0(\omega)\in\mathbb{N} \text{ such that } \forall n\geq n_0\ \forall \xi,\widetilde{\xi}\in\Pi_n:\ |\xi_0|,|\widetilde{\xi}_0|\leq C_0\sqrt{n}. \tag{9.1}$$

(Recall that C_0 is not random.)

<u>First assume $d=1$.</u>

Next, note that given $\mathscr{F}_n$ (or, what is the same[11], given Z_n), ξ_{m_n} and $\widetilde{\xi}_{m_n}$ have joint normal distribution. This is because by Remark 10, $(Z_t^1, Z_t^2, ..., Z_t^{n_t})$ given Z_n is a.s. joint normal for $t>n$, and $(\xi_{m_n},\widetilde{\xi}_{m_n})$ is a projection of $(Z_t^1, Z_t^2, ..., Z_t^{n_t})$. Therefore, denoting $\widehat{x}:=x-\xi_0$, $\widehat{y}:=y-\widetilde{\xi}_0$, the joint (conditional) density of ξ_{m_n} and $\widetilde{\xi}_{m_n}$ (given $\mathscr{F}_n$) on $\mathbb{R}^2$ is of the form

$$f^{(n)}(x,y)=f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{\widehat{x}^2}{\sigma_x^2}+\frac{\widehat{y}^2}{\sigma_y^2}-\frac{2\rho\widehat{x}\widehat{y}}{\sigma_x\sigma_y}\right]\right),$$

where σ_x^2,σ_y^2 and $\rho=\rho_n$ denote the (conditional) variances of the marginals and the (conditional) correlation[12] between the marginals, respectively, given $\mathscr{F}_n$. Abbreviating $\kappa:=\frac{1}{\sigma_x\sigma_y}$, one has

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{\widehat{x}^2}{\sigma_x^2}+\frac{\widehat{y}^2}{\sigma_y^2}\right]\right)\exp\left(\frac{\rho}{1-\rho^2}\kappa\widehat{x}\widehat{y}\right).$$

[11]Given $\mathscr{F}_n$, the distribution of $\left(\xi_{m_n},\widetilde{\xi}_{m_n}\right)$ will not change by specifying Z_n, that is, specifying $\overline{Z}_n$.

[12]Provided, of course, that $\rho_n\neq 1$, but we will see in (9.3) below that $\lim_{n\to\infty}\rho_n=0$.

Let $f_1^{(n)} = f_1$ and $f_2^{(n)} = f_2$ denote the (conditional) marginal densities of f, given $\mathscr{F}_n$. We now show that P-a.s., for all large enough n,

$$|f(x,y) - f_1(x)f_2(y)| \le K(B)n\rho, \text{ with some } K(B) > 0 \text{ on } B, \tag{9.2}$$

and that P-a.s.,

$$\rho = \rho_n = E\left[(\xi_{m_n} - E(\xi_{m_n} \mid \mathscr{F}_n))(\widetilde{\xi}_{m_n} - E(\widetilde{\xi}_{m_n} \mid \mathscr{F}_n)) \mid \mathscr{F}_n\right] \le \frac{3}{\gamma} \cdot 2^{-n}, \; n \ge 1. \tag{9.3}$$

Clearly, (9.2) and (9.3) imply the statement in (a):

$$\left|\int_{B\times B} f(x,y) - f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y\right| \le$$
$$\int_{B\times B} |f(x,y) - f_1(x)f_2(y)|\mathrm{d}x\mathrm{d}y \le |B|^2 K(B)n\rho_n = |B|^2 K(B)\frac{3}{\gamma} \cdot n2^{-n}.$$

To see (9.2), write

$$f(x,y) - f_1(x)f_2(y) =$$
$$\left\{ f(x,y) - \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{1}{2}\left[\frac{\widehat{x}^2}{\sigma_x^2} + \frac{\widehat{y}^2}{\sigma_y^2}\right]\right) \exp\left(\frac{\rho}{1-\rho^2}\kappa\widehat{x}\widehat{y}\right)\right\} +$$
$$\left\{\frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{1}{2}\left[\frac{\widehat{x}^2}{\sigma_x^2} + \frac{\widehat{y}^2}{\sigma_y^2}\right]\right) \exp\left(\frac{\rho}{1-\rho^2}\kappa\widehat{x}\widehat{y}\right) - f_1(x)f_2(y)\right\} =: I + II.$$

Now,

$$|I| = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{1}{2}\left[\frac{\widehat{x}^2}{\sigma_x^2} + \frac{\widehat{y}^2}{\sigma_y^2}\right]\right) \exp\left(\frac{\rho}{1-\rho^2}\kappa\widehat{x}\widehat{y}\right)$$
$$\cdot \left|\left(\frac{1}{\sqrt{1-\rho^2}} e^{\frac{1}{2}\left[\frac{\widehat{x}^2}{\sigma_x^2} + \frac{\widehat{y}^2}{\sigma_y^2}\right] - \frac{1}{2(1-\rho^2)}\left[\frac{\widehat{x}^2}{\sigma_x^2} + \frac{\widehat{y}^2}{\sigma_y^2}\right]} - 1\right)\right|$$
$$\le \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{\rho}{1-\rho^2}\kappa\widehat{x}\widehat{y}\right)$$
$$\cdot \left|\left(\frac{1}{\sqrt{1-\rho^2}} \exp\left\{\frac{1}{2}\left[\frac{\widehat{x}^2}{\sigma_x^2} + \frac{\widehat{y}^2}{\sigma_y^2}\right]\left(1 - \frac{1}{(1-\rho^2)}\right)\right\} - 1\right)\right|.$$

Since B is a fixed bounded measurable set, using (9.1) along with the approximations $1 - e^{-a} \approx a$ as $a \to 0$, and $1 - \sqrt{1-\rho^2} \approx \rho^2/2$ as $\rho \to 0$, one can see that *if (9.3) holds*, then there exists a $K(B) > 0$ such that P-a.s.,

$$|I| \le K(B)n\rho^2 \text{ for all large enough } n.$$

To see that the presence of the $\mathscr{F}_n$-dependent σ_x, σ_y do not change this fact, recall that ξ and $\widetilde{\xi}$ are both (time inhomogeneous) Ornstein-Uhlenbeck processes (see Section 4.1), and so σ_x and

σ_y are bounded between two positive (absolute) constants for $n \geq 1$. (Recall that the variance of an Ornstein-Uhlenbeck process is bounded between two positive constants, which depend on the parameters only, on the time interval (ϵ, ∞), for $\epsilon > 0$.)

A similar (but simpler) computation shows that *if (9.3) holds*, then there exists a $K(B) > 0$ (we can choose the two constants the same, so this one will be denoted by $K(B)$ too) such that P-a.s.,

$$|II| \leq K(B) n\rho, \ \forall x, y \in B \text{ for all large enough } n.$$

These estimates of I and II yield (9.2).

Thus, it remains to prove (9.3). Recall that we assume $d = 1$. Using similar notation as in Subsection 4.1, let $\widetilde{W}^{(i)}$ $(i = 1, 2)$ be Brownian motions, which, satisfy for $s \in [k, k+1)$, $0 \leq k < m_n$,

$$\begin{aligned} \sigma_{n+k} \widetilde{W}_s^{(1)} &= \bigoplus_{i \in I_{n+k}} 2^{-n-k} W_s^{k,i} \oplus (1 - 2^{-n-k}) W_s^{k,1}, \\ \sigma_{n+k} \widetilde{W}_s^{(2)} &= \bigoplus_{i \in J_{n+k}} 2^{-n-k} W_s^{k,i} \oplus (1 - 2^{-n-k}) W_s^{k,2}, \end{aligned} \tag{9.4}$$

where the $W^{k,i}$ are 2^{n+k} independent standard Brownian motions, and $I_{n+k} := \{i : \ 2 \leq i \leq 2^{n+k}\}$, $J_{n+k} := \{i : \ 1 \leq i \leq 2^{n+k}, i \neq 2\}$. Recall that, by (4.1), given $\mathscr{F}_n$, Y and $\widetilde{Y}$ are Ornstein-Uhlenbeck processes driven by $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$, respectively, and $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ are independent of $\mathscr{F}_n$.

Notation 31. We are going to use the following (slight abuse of) notation. For $r > 0$, the expression $\sum_{j=0}^{r-1} \int_j^{j+1} f(s)\, \mathrm{d}W_s$ will mean $\sum_{j=0}^{\lfloor r \rfloor -1} \int_j^{j+1} f(s)\, \mathrm{d}W_s + \int_{\lfloor r \rfloor}^{r} f(s)\, \mathrm{d}W_s$, where W is Brownian motion.

Using this notation with $r = m_n$ and recalling that $\sigma^{(n)}(s) := \sigma_{n+l}$ for $s \in [l, l+1)$, one has

$$\xi_{m_n} - E(\xi_{m_n} \mid \mathscr{F}_n) = \int_0^{m_n} \sigma^{(n)}(s) e^{\gamma(s - m_n)}\, \mathrm{d}\widetilde{W}_s^{(1)} = \sum_{j=0}^{m_n - 1} \sigma_{n+j} \int_j^{j+1} e^{\gamma(s-m_n)}\, \mathrm{d}\widetilde{W}_s^{(1)}$$

and

$$\widetilde{\xi}_{m_n} - E(\widetilde{\xi}_{m_n} \mid \mathscr{F}_n) = \int_0^{m_n} \sigma^{(n)}(s) e^{\gamma(s - m_n)}\, \mathrm{d}\widetilde{W}_s^{(2)} = \sum_{j=0}^{m_n - 1} \sigma_{n+j} \int_j^{j+1} e^{\gamma(s-m_n)}\, \mathrm{d}\widetilde{W}_s^{(2)},$$

where, of course, $E(\xi_{m_n} \mid \mathscr{F}_n) = e^{-\gamma m_n} \xi_0$ and $E(\widetilde{\xi}_{m_n} \mid \mathscr{F}_n) = e^{-\gamma m_n} \widetilde{\xi}_0$. Writing out $\sigma_{n+j}\, \mathrm{d}\widetilde{W}_s^{(1)}$ and $\sigma_{n+j}\, \mathrm{d}\widetilde{W}_s^{(2)}$ according to (9.4), one obtains, that given $\mathscr{F}_n$,

$$\begin{aligned} &I := \xi_{m_n} - E(\xi_{m_n} \mid \mathscr{F}_n) = \\ &\sum_{j=0}^{m_n-1} \left[\sum_{i \in I_{n+j}} 2^{-n-j} \int_j^{j+1} e^{\gamma(s-m_n)} \mathrm{d}W_s^{j,i} + (1 - 2^{-n-j}) \int_j^{j+1} e^{\gamma(s-m_n)} \mathrm{d}W_s^{j,1} \right], \end{aligned}$$

$$\begin{aligned} &II := \widetilde{\xi}_{m_n} - E(\widetilde{\xi}_{m_n} \mid \mathscr{F}_n) = \\ &\sum_{j=0}^{m_n-1} \left[\sum_{i \in J_{n+j}} 2^{-n-j} \int_j^{j+1} e^{\gamma(s-m_n)} \mathrm{d}W_s^{j,i} + (1 - 2^{-n-j}) \int_j^{j+1} e^{\gamma(s-m_n)} \mathrm{d}W_s^{j,2} \right]. \end{aligned}$$

Because I and II are jointly independent of $\mathscr{F}_n$, one has

$$E(I \cdot II \mid \mathscr{F}_n) = E(I \cdot II).$$

Since the Brownian motions $W^{j,i}$ are independent for fixed j and different i's, and the Brownian increments are also independent for different j's, therefore one has $E(I \cdot II) = E\sum_{j=0}^{m_n-1}(III + IV)$, where

$$III := (2^{n+j} - 2)2^{-2(n+j)} \left(\int_j^{j+1} e^{\gamma(s-m_n)}\,\mathrm{d}B_s \right)^2;$$
$$IV := 2^{1-n-j}(1 - 2^{-n-j}) \left(\int_j^{j+1} e^{\gamma(s-m_n)}\,\mathrm{d}B_s \right)^2,$$

and B is a generic Brownian motion. By Itô's isometry,

$$\begin{aligned}
&E(I \cdot II) = \\
&\sum_{j=0}^{m_n-1} \left[(2^{n+j} - 2)2^{-2(n+j)} + 2^{1-n-j}(1 - 2^{-n-j})\right] \int_j^{j+1} e^{2\gamma(s-m_n)}\mathrm{d}s = \\
&\frac{1}{2\gamma} \sum_{j=0}^{\lfloor m_n \rfloor -1} \left[3 \cdot 2^{-(n+j)} - 4 \cdot 2^{-2(n+j)}\right] [e^{2\gamma(j+1-m_n)} - e^{2\gamma(j-m_n)}] + R_n = \\
&\frac{1}{2\gamma} 2^{-n} \sum_{j=0}^{\lfloor m_n \rfloor -1} \left[3 \cdot 2^{-j} - 4 \cdot 2^{(-n-2j)}\right] [e^{2\gamma(j+1-m_n)} - e^{2\gamma(j-m_n)}] + R_n,
\end{aligned}$$

where

$$R_n := \frac{1}{2\gamma} 2^{-n} \cdot \left[3 \cdot 2^{-\lfloor m_n \rfloor} - 4 \cdot 2^{(-n-2\lfloor m_n \rfloor)}\right] [1 - e^{2\gamma(\lfloor m_n \rfloor - m_n)}] < \frac{3}{2\gamma} 2^{-n}.$$

(Note that $3 \cdot 2^{-j} > 4 \cdot 2^{(-n-2j)}$ and $\gamma > 0$.) Hence

$$\begin{aligned}
&0 < E(I \cdot II) \\
&< \frac{3}{2\gamma} 2^{-n} \sum_{j=0}^{\lfloor m_n \rfloor -1} [e^{2\gamma(j+1-m_n)} - e^{2\gamma(j-m_n)}] + R_n < \frac{3}{2\gamma} 2^{-n}(2 - e^{-2\gamma m_n}),
\end{aligned}$$

and so (9.3) follows, finishing the proof of part (a) for $d = 1$.

Assume that $d \geq 2$. It clear that (9.3) follows from the one dimensional case. As far as (9.2) is concerned, the computation is essentially the same as in the one dimensional case. Note, that although the formulæ are lengthier in higher dimension, the $2d$-dimensional covariance matrix is block-diagonal because of the independence of the d coordinates (Lemma 8), and this simplifies the computation significantly. We leave the simple details to the reader.

(b): Write

$$\mathrm{Var}\left(\mathbf{1}_{\{\xi_{m_n} \in B\}} \mid \mathscr{F}_n\right) = P(\xi_{m_n} \in B \mid \mathscr{F}_n) - P^2(\xi_{m_n} \in B \mid \mathscr{F}_n),$$

and note that $P(\xi_{m_n} \in B \mid \xi_0 = x) = q_n(x, B, n + m_n)$, and ξ_0 is the location of the parent particle at time n. Hence, (7.14) together with (7.9) implies the limit in (b). □

9.2 Proof of (7.12)

We will assume that $\nu(B) > 0$ (i.e. $C(B) = \nu(B) - (\nu(B))^2 > 0$), or equivalently, that B has positive Lebesgue measure. This does not cause any loss of generality, since otherwise the $\mathscr{X}_i$'s vanish a.s. and (7.12) is trivially true.

Now let us estimate $E\left[\mathscr{X}_i\mathscr{X}_j \mid \mathscr{F}_n\right]$ and $E\left[\mathscr{X}_i^2 \mid \mathscr{F}_n\right]$. The calculation is based on Lemma 25 as follows. First, by part (a) of Lemma 25, it holds P-a.s. that for all large enough n,

$$P(Y_{m_n}^{1,j} \in B, Y_{m_n}^{2,k} \in B \mid \mathscr{F}_n) - P(Y_{m_n}^{1,j} \in B \mid \mathscr{F}_n)P(Y_{m_n}^{2,k} \in B \mid \mathscr{F}_n) \le C(B,\gamma) \cdot n2^{-n}.$$

Therefore, recalling that $\ell_n = 2^{\lfloor m_n \rfloor}$, one has that P-a.s., for all large enough n,

$$\begin{aligned}
&\ell_n^2 E[\mathscr{X}_1\mathscr{X}_2 \mid \mathscr{F}_n] \\
&= \sum_{j,k=1}^{\ell_n} \left\{P(Y_{m_n}^{1,j} \in B, Y_{m_n}^{2,k} \in B \mid \mathscr{F}_n) - P(Y_{m_n}^{1,j} \in B \mid \mathscr{F}_n)P(Y_{m_n}^{2,k} \in B \mid \mathscr{F}_n)\right\} \\
&\le C(B,\gamma) \cdot n2^{-n}\ell_n^2.
\end{aligned}$$

This estimate holds when $Y_{m_n}^{1,j}$ and $Y_{m_n}^{2,k}$ are replaced by any $Y_{m_n}^{p,j}$ and $Y_{m_n}^{r,k}$, where $p \neq r$ and $1 \le p, r \le 2^n$; consequently, if

$$I_n := \sum_{1 \le i \neq j \le 2^n} E\left[\mathscr{X}_i\mathscr{X}_j \mid \mathscr{F}_n\right]$$

(which is the left hand side of the inequality in (7.12)) then one has that P-a.s., for all large enough n,

$$\ell_n^2 I_n \le 2^n \cdot (2^n - 1) C(B,\gamma) \cdot n2^{-n}\ell_n^2 < C(B,\gamma) \cdot n2^n\ell_n^2.$$

Hence, to finish the proof, it is sufficient to show that[13]

$$\ell_n^2 J_n = \Theta\left(n2^n\ell_n^2\right) \ a.s., \tag{9.5}$$

for

$$J_n = n\ell_n \sum_{i=1}^{2^n} E[\mathscr{X}_i^2 \mid \mathscr{F}_n]$$

(which is the right hand side of the inequality in (7.12) without the constant). To this end, we essentially repeat the argument in the proof of Claim 28. The only difference is that we now *also use the assumption* $C(B) > 0$, and obtain that

$$\ell_n^2 E[\mathscr{X}_1^2 \mid \mathscr{F}_n] = \mathscr{O}(n2^{-n}\ell_n^2) + \Theta(\ell_n),$$

as $n \to \infty$, a.s.

Just like in the proof of Claim 28, replacing 1 by i, the estimate holds uniformly for $1 \le i \le 2^n$, and so

$$\ell_n^2 \sum_{i=1}^{2^n} E[\mathscr{X}_i^2 \mid \mathscr{F}_n] = \mathscr{O}(n\ell_n^2) + \Theta(2^n\ell_n) = \Theta(2^n\ell_n) \ a.s.,$$

[13] What we mean here is that there exist $c, C > 0$ absolute constants such that for all $n \ge 1$, $c < J_n/n2^n < C$ a.s.

where in the last equality we used that $\ell_n = 2^{\lfloor m_n \rfloor}$ and $m_n = o(n)$. From here, (9.5) immediately follows:

$$\ell_n^2 J_n = n\ell_n^3 \sum_{i=1}^{2^n} E[\mathscr{Z}_i^2 \mid \mathscr{F}_n] = \Theta(n2^n\ell_n^2) \ a.s.,$$

and the proof of (7.12) is completed. □

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