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Critical constants for recurrence on groups of polynomial growth

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Abstract

The critical constant for recurrence, c_{rt}, is an invariant of the quotient space $H\backslash G$ of a finitely generated group. The constant is determined by the largest moment a probability measure on G can have without the induced random walk on $H\backslash G$ being recurrent. We present a description of which subgroups of groups of polynomial volume growth are recurrent. Using this we show that for such recurrent subgroups c_{rt} corresponds to the relative growth rate of H in G, and in particular $c_{rt} \in \{0, 1, 2\}$.

Key words: nilpotent group, Schreier graph, random walk, recurrence, volume growth.

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1 Introduction

Whether or not a Markov chain is recurrent is one of the most basic probabilistic questions we can ask. On a countably infinite state space the problem was first treated by Polya, who showed that the integer lattice carries a recurrent random walk only in dimensions one and two [Pól21]. Later this question was generalized to groups in the form of Kesten's conjecture: that a recurrent group has volume growth of degree at most two. This result was finally proven by Varopoulos [Var85], utilizing Gromov's theorem that polynomial volume growth is equivalent to containing a nilpotent subgroup of finite index [Gro81]. More recently, others have examined the question of recurrence for subgroups or quotients of finitely generated groups [Ers04; Ers05; Rev02].

Gallardo and Schott showed that a homogeneous space of a simply connected nilpotent Lie group is recurrent if and only if it has polynomial growth of degree at most two [GS80]. Theorem 1.2 can be viewed as an extension of this result to finitely generated nilpotent groups, though we approach this question without reference to the theory of Lie groups. Furthermore, by focusing solely on finitely generated groups we gain additional structural information about the subgroups and quotient spaces in question.

To study quotients we generalize the notion of a Cayley graph to that of a Schreier graph. The Schreier graph $\Gamma(G, H\backslash G, S)$ is obtained by considering the right action of G on $H\backslash G$ with respect to a generating set S of G. The vertices correspond to cosets in $H\backslash G$ and are connected by an edge when the action of an element of S sends one coset to another. The set of Schreier graphs is surprisingly large: all even-regular graphs are Schreier [Lub95].

In general the recurrence of quotients is hard to determine as we lack good heat kernel estimates and structural theorems. However, progress has been made for certain classes of groups. Nilpotent groups have a very strong algebraic structure which is inherited by their subgroups, and using this fact the first author showed that an analog of Varoupoulos's theorem holds for quotients of groups of polynomial growth[Rev02]. Another important class of examples comes from self-similar groups where many of the quotients are fractal[Nek05].

One way to study recurrence is to ask what conditions we can place on the law of a random walk to ensure recurrence. For example there are classical results that a finite first moment guarantees recurrence on $\mathbb{Z}$ and a finite second moment guarantees recurrence on $\mathbb{Z}^2$ (see Chapter 2, section 8 in [Spi76]). To generalize this to the quotient case we begin with a measure μ on G and use it to drive a random walk on $H\backslash G$. We say such a walk on $H\backslash G$ is induced by μ. To study this question we will use the critical constant for recurrence, c_{rt}, introduced by Erschler in [Ers05].

Definition 1.1. *For a finitely generated group G with subgroup H the critical constant for recurrence, c_{rt}, is defined as the supremum of the $\beta \geq 0$ such that there exists a measure μ on G with*

$$\sum_{g \in G} |g|^{\beta} \mu(g) < \infty \tag{1}$$

and whose induced random walk on $H\backslash G$ is transient.

The moment condition (1) is equivalent to the following tail condition:

$$\mu(G - B_G(R)) \leq CR^{-\beta} \tag{2}$$

for $R \geq 1$ and a constant $C > 0$. Generally it is more natural to check that the tail condition holds rather than the moment condition.

From the definition it is clear that $c_{rt}(G,H) = 0$ if H is of finite index in G and that $c_{rt}(G,H) = \infty$ if any simple random walk on the quotient is transient. On $\mathbb{Z}$ and $\mathbb{Z}^2$ one can construct measures with moments $1 - \epsilon$ and $2 - \epsilon$, respectively, for any $\epsilon > 0$ such that the random walks are transient (see Example 8.2 in [Spi76]). Thus $c_{rt}(\mathbb{Z}, \{e\}) = 1$ and $c_{rt}(\mathbb{Z}^2, \{e\}) = 2$. The goal of this paper is to show that this essentially exhausts the behavior of c_{rt} for quotients of groups of polynomial volume growth, in the form of the following theorem.

Theorem 1.1. *Let G be an infinite group with recurrent subgroup H. If G has polynomial volume growth then $c_{rt}(G,H)$ is an integer at most 2.*

An immediate consequence of this result is the following corollary.

Corollary 1.1. *Let H be a recurrent subgroup of G. If $c_{rt}(G,H)$ is non-integral then G has super-polynomial volume growth.*

To obtain theorem (1.1) we need a classification of recurrent subgroups of groups of polynomial growth. This was done by Revelle in [Rev02]. We will prove this result in section 3. This result is obtained in part via a generalization of Wiener's recurrence test to groups of polynomial volume growth.

Theorem 1.2 (Recurrent Subgroups)**.** *Let G be a finitely generated group with polynomial volume growth. A subgroup $H \leq G$ is recurrent iff its relative degree of growth is within two the growth rate of G. In this case there exists H' and G' such that*

(i) $H < H'$ *and* $[H' : H] < \infty$*,*

(ii) $G' < G$ *and* $[G : G'] < \infty$*,*

(iii) $H' \cap G' \lhd G'$*,*

(iv) and $(H' \cap G') \backslash G'$ *is isomorphic to* $\mathbb{Z}^d$ *for some* $d \leq 2$*.*

Another important component of the proof of (1.1) is the following lemma which describes the behavior of c_{rt} over a series of subgroups. This lemma was originally shown in [Ers05], and will be proven here for the sake of completeness.

Lemma 1.1 (Ladder Lemma)**.** *If $H \leq K \leq G$ then*

(i) $c_{rt}(K,H) \leq c_{rt}(G,H)$*, where equality holds if* $[G : K] < \infty$*.*

(ii) $c_{rt}(G,K) \leq c_{rt}(G,H)$*, where equality holds if* $[K : H] < \infty$*.*

The converse to theorem 1.1 is not true. Groups of super-polynomial growth can have subgroups with integral values for c_{rt}, and in particular $c_{rt}(G \times \mathbb{Z}^d, G) = d$ when $d = 1, 2$ and for any group G regardless of its growth rate. Erschler showed in [Ers05] that c_{rt} can take on non-integral values, though little is understood about this behavior. Erschler's example is the first Grigorchuk group with the stabilizer of the group's action on $(0, 1]$. In this case the critical constant lies in the interval $(1/2, 1)$. Such stabilizer subgroups are important in the study of self-similar groups [Nek05], and it would be interesting to calculate c_{rt} for more examples of this kind. Another interesting question is whether or not the critical constant can take on values larger than two.

2 Notation

We will think about our random walks in terms of a right group action $X \cdot G \to X$ for finitely generated G and a space X. In this paper, X will be either G or a quotient $H\backslash G$. Denote the associated Markov chain with law μ by (X,μ), and fix a base point $o \in X$. Generally, o will be either the identity element of G or the identity coset of $H\backslash G$. The location of (X,μ) at time n is

$$Z_n = o \cdot X_1 X_2 \cdots X_n,$$

where the X_i are i.i.d. with distribution μ. The distribution of Z_n is given by the n-fold convolution of μ. We will assume throughout that μ is symmetric, however, we do not assume that the support of μ generates G.

The *volume growth* of a finitely generated group G with symmetric generating set S is

$$V_{G,S}(n) = \#\{g \in G : |g|_S \le n\},$$

where $|\cdot|_S$ is the word length corresponding to S. For our purposes we only care about the general form of this and similar functions. For monotone increasing functions f and g we write $f \preceq g$ if

$$f(n) \le C g(Cn)$$

for all $n \ge 1$ and some $C > 0$. If $g \preceq f$ as well, we write $f \simeq g$, and say that f and g are of the same *type*. It is well known and easy to see that the type of a group's volume growth is independent of the choice of generating set, and so we will use the notation V_G when the generating set is not important. Similarly, we will suppress mention of the generating set when referring to word length.

For a subgroup $H \le G$ we define the *relative growth* of H in G as

$$V_G(n,H) = \#\{h \in H : |h| \le n\}.$$

The quotient space $H\backslash G$ inherits a metric structure from G: for $x \in H\backslash G$, $|x| = \inf\{|g| : g \in G,\ x \cdot g = H\}$. The volume growth of the quotient $H\backslash G$ is

$$V_{H\backslash G}(n) = \#\{x \in H\backslash G : |x| \le n\}. \tag{3}$$

The type of the relative volume growth and the volume growth of the quotient space are independent of the choice of generating set, so we will suppress mention of the generating set S unless otherwise necessary. Likewise, the type is invariant when moving to sub- or super-groups of finite index [Rev02].

We will be concerned with groups such that $V_G(n) \simeq n^d$, i.e. groups of polynomial volume growth, and we will refer to d as the degree of the volume growth. By the Bass-Guivarc'h formula [Bas72; Gui71], d can only take on integral values. For a history of this formula see Section VII.26 in [dlH00].

We will use hitting times throughout this paper. For a subset A of the state space of a Markov chain we will let τ_A (resp. τ_A^+) denote the first (resp. positive) hitting time of A.

3 Recurrent subgroups

3.1 Wiener recurrence test

Varopoulos's theorem tells us that all subsets of groups with growth of degree at most two are recurrent, however we need a tool to determine when transient random walks visit a set infinitely often. The following result of Revelle [Rev02] extends the classical version [DY69] of Wiener's recurrence test to groups of polynomial growth. This will be our principle tool for showing the recurrence of subgroups when the group itself is not recurrent. The *capacity* of a finite set A is defined as

$$\mathrm{Cap}(A) = \sum_{x \in A} P_x(\tau_A^+ = \infty).$$

Theorem 3.1 (Wiener's recurrence test). *If G is a group with polynomial growth of degree $d \geq 3$, then $A \subseteq G$ is recurrent for some symmetric, finitely supported random walk iff*

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(A_k)}{2^{k(d-2)}} = \infty, \tag{4}$$

where $A_k = \{x \in A \,:\, 2^{k-1} < |x| \leq 2^k\}$.

Proof. Let μ be a symmetric finitely supported measure on G. Gaussian estimates for μ^{*n} [HSC93] show that Green's function satisfies

$$c_1 |xy^{-1}|^{2-d} \leq G(x,y) \leq c_2 |xy^{-1}|^{2-d}, \tag{5}$$

for constants c_1, $c_2 > 0$. Thus we have,

$$\begin{aligned} P(\tau_{A_k} < \infty) &= \sum_{x \in A_k} G(e,x) P_x(\tau_{A_k}^+ = \infty) \\ &\leq \sum_{x \in A_k} c_2 |x|^{2-d} P_x(\tau_{A_k}^+ = \infty) \\ &\leq c_2 2^{(k-1)(2-d)} \sum_{x \in A_k} P_x(\tau_{A_k}^+ = \infty) \\ &\leq c_2 2^{d-2} \frac{\mathrm{Cap}(A_k)}{2^{k(d-2)}}. \end{aligned}$$

The set A is recurrent iff $\tau_{A_k} < \infty$ for infinitely many k. The Borel-Cantelli lemma then implies that the capacity series diverges if A is recurrent.

For sufficiency of the first criteria, fix l such that $(2^{l-2} - 1)^{d-2} > 2^{d-2}(c_2/c_1)$. If the capacity series diverges then the subseries

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(A_{kl})}{2^{kl}}, \tag{6}$$

also diverges. Set $B_k = \{x \,:\, |x| \leq 2^{kl-2}\}$. The set A_{kl} is thus contained in $B_{k+1} - B_k$. We define events $U_k = \{\tau_{A_{kl}} < \tau_{B_{k+1}^c}\}$ and $V_k = \{\tau_{B_{k+1}^c} < \tau_{A_{kl}} < \infty\}$ such that $U_k \cup V_k = \{\tau_{A_{kl}} < \infty\}$. We now obtain a lower bound for $P_y(U_k)$.

Let $y \in B_k$. Applying the strong Markov property gives

$$P_y(V_k) \leq \sup_{x \notin B_{k+1}} P_x(\tau_{A_{kl}} < \infty),$$

and thus

$$\begin{aligned} P_y(U_k) &\geq P_y(\tau_{A_{kl}} < \infty) - \sup_{x \notin B_{k+1}} P_x(\tau_{A_{kl}} < \infty) \\ &= \sum_{u \in A_{kl}} G(y,u) P_y(\tau^+_{A_{kl}} = \infty) - \sup_{x \notin B_{k+1}} \sum_{u \in A_{kl}} G(x,u) P_y(\tau^+_{A_{kl}} = \infty) \\ &\geq \left(\min_{y \in B_k,\ u \in A_{kl}} G(y,u) - \max_{x \in B^c_{k+1}} G(x,u) \right) \operatorname{Cap} A_{kl} \\ &\geq (c_1 r_k^{2-d} - c_2 R_k^{2-d}) \operatorname{Cap}(A_{kl}), \end{aligned}$$

where

$$R_k = \max_{x \in B_k,\ y \in A_{kl}} |x^{-1} y|,$$

and

$$r_k = \min_{x \in B^c_{k+1},\ y \in A_{kl}} |x^{-1} y|.$$

To explain our choice of l note that $r_k \leq 2^{kl+1}$ and $R_k \geq (2^{l-2}-1)2^{kl}$. Thus $R_k / r_k \geq (2^{l-2}-1)/2$, and so $c_1/r_k - c_2/R_k > 0$. Combining this with the above inequality yields $P_y(U_k) \geq c' \operatorname{Cap}(A_{kl})/2^{-kl} > 0$ for some $c' > 0$ independent of y.

This gives us that

$$P_e(U_i^c,\ i \geq n) \leq \prod_{in}^{\infty} \left(1 - c' \frac{\operatorname{Cap} A_{kl}}{2^{kl}}\right) \leq \exp\left(-c' \sum_{i=n}^{\infty} \frac{\operatorname{Cap} A_{kl}}{2^{kl}}\right). \tag{7}$$

As the capacity series diverges, this implies that U_i occurs for infinitely many i. Thus τ^+_A is finite almost surely. □

3.2 Growth of subgroups

The purpose of this section is to classify the relative volume growth of subgroups of nilpotent groups. The techniques used are elementary, but the notation required is cumbersome. Our aim is to prove a generalization of the Bass-Guivarc'h formula [Bas72; Gui71; dlH00] for the volume growth of a nilpotent group in terms of the lower central series.

Theorem 3.2. *Let G be a nilpotent group with lower central series $G = G_1 > G_2 > \ldots > G_{k+1} = \{e\}$, where $G_{i+1} = [G_i, G]$, and with subgroup H. Set $H_i = H \cap G_i$. Then $H_{i+1} \lhd H_i$, $H_{i+1} \backslash H_i$ is abelian, and the relative growth of H in G has degree*

$$r = \sum_{i=1}^{k} i \cdot rk(H_{i+1} \backslash H_i), \tag{8}$$

where rk denotes the free abelian rank. Furthermore the degree of growth of $H \backslash G$ is given by $d - r$ where d is the degree of growth of G.

We will prove the lower and upper bounds for $V_G(n,H)$ using lemmas (3.1) and (3.2) respectively. We start by showing the desired properties of the H_i and their quotients.

Proof. First we show that $H_{i+1} \lhd H_i$ and $H_{i+1}\backslash H_i$ is abelian. Take $x \in H_i$ and $y \in H_{i+1}$. It is easy to see that $x^{-1}yx = y[x,y]$, and by definition $y[x,y] \in H_{i+1}$. This implies that conjugation by any element in H_i preserves H_{i+1}, and thus $H_{i+1} \lhd H_i$. As $[H_i, H_i] \le H_{i+1}$ the quotient is abelian.

Now we select a generating set of G. Recall that k is the length of the lower central series of G. For $1 \le i \le k$ we let

$$A_i = \{h_{i,n} : 1 \le n \le \operatorname{rk} H_{i+1}\backslash H_i\}$$

denote a maximal set of elements in $H_i - H_{i+1}$ whose images in $H_{i+1}\backslash H_i$ are linearly independent. Analogously we define

$$B_i = \{g_{i,n} : 1 \le n \le \operatorname{rk} G_{i+1}\backslash G_i\}$$

such that $A_i \cup B_i$ is a maximal linearly independent set in $G_{i+1}\backslash G_i$. To complete the generating set we include the sets

$$C_i = \{x_{i,n} : 1 \le n \le \operatorname{rk} H_{i+1}\backslash H_i\}$$

where the $x_{i,n}$ are chosen to be a complete set of distinct coset representatives in the span of the image of A_i in $H_{i+1}\backslash H_i$. We choose a set D_i and label its elements $y_{i,n}$ in an analogous fashion for the image of B_i. We take

$$S = \bigcup_{i=1}^{k} \left(A_i \cup B_i \cup C_i \cup D_i\right)$$

to be the generating set for G. For the sake of notational convenience we will let

$$a_i^{\alpha_i} = h_{i,1}^{\alpha_{i,1}} \cdots h_{i,\operatorname{rk} H_{i+1}\backslash H_i}^{\alpha_{i,\operatorname{rk} H_{i+1}\backslash H_i}}$$

and

$$b_i^{\beta_i} = g_{i,\operatorname{rk} G_{i+1}\backslash G_i}^{\beta_{i,\operatorname{rk} G_{i+1}\backslash G_i}} \cdots g_{i,1}^{\beta_{i,1}}.$$

We will commonly refer to generators with index i as generators of level i, and will exclude the second subscript when its value is unimportant. We next show that words in G have a normal form whose length we can estimate.

Lemma 3.1. *If $|g| = n$ then there exists $C > 0$ such that g can be uniquely written in the form*

$$x_1 a_1^{\alpha_1} x_2 a_2^{\alpha_2} \cdots a_{rk\ H_{i+1}\backslash H_i}^{\alpha_{rk\ H_{i+1}\backslash H_i}} b_{rk\ G_{i+1}\backslash G_i}^{\beta_{rk\ G_{i+1}\backslash G_i}} y_{rk\ G_{i+1}\backslash G_i} \cdots b_1^{\beta_1} y_1, \tag{9}$$

where for each i, $\max_m |\alpha_{i,m}|, \max_m |\beta_{i,m}| < Cn^i$

Proof. We will prove this inductively for each level of generators. We will use the following relations to manipulate the generators

(a) $g_i g_j = g_j g_i g_{i+j}$,

(b) $g_j h_i = h_i g_j g_{i+j}$,

(c) $x_i x_i' = x_i'' g$, $g \in G_{i+1}$

(d) $y_i y_i' = g y_i''$, $g \in G_{i+1}$.

The first two follow because $[H_i, H_j] \leq [G_i, G_j] \leq G_{i+j}$ and $[H_i, H_j] \leq H_{i+j}$, and the last relations follow because the x_i and y_i are coset representatives.

Suppose that $g = s_1 s_2 \cdots s_r$, where each $s_1, \ldots, s_r \in G_i$ and such that each generator of level i occurs no more than cn^i times for some constant $c > 0$. We use the following algorithm to write g in normal form

(I) Move the rightmost x_i to the left using (a). If two x_i are adjacent apply (c). If the resulting g is not a generator rewrite it in terms of the generators contained in G_{i+1}.

(II) Move each $h_{i,j}$ to the left starting with $h_{i,1}$ using (b)

(III) Move each g_i to the right starting with $g_{i,\mathrm{rk}\ G_{i+1} \backslash G_i}$ using (a).

(IV) Move the leftmost y_i to the right using (d) and (a)

This process results in a word of the form

$$x h_1^{\alpha_i}(i) \cdots h_{\mathrm{rk}\ H_{i+1} \backslash H_i}^{\alpha_{\mathrm{rk}\ H_{i+1} \backslash H_i}(i)} t_1 \cdots t_o g_{\mathrm{rk}\ G_{i+1} \backslash G_i}^{\beta_{\mathrm{rk}\ G_{i+1} \backslash G_i}(i)} \cdots g_1^{\beta_i}(i),$$

where $t_1, \ldots, t_o$ are generators in G_{i+1}. Iterating this process over i gives the desired form. Now we need to estimate $\max_m |\alpha_{i,m}|$ and $\max_m |\beta_{i,m}|$.

Let M denote the maximal length of any one of the g obtained using (c). Executing the first step of the algorithm yields a word of the form $x_i s_1' \cdots s_{(M+1)r}'$, and each generator of level $j \leq i$ in this word will appear at most $c(M+1)n^j$ times. If $j \geq i$ the new generators we create at level $i + j$ must also be transposed with an h_i with a generator of level at most j. Thus the maximum number of generators at level $i + j$ after we have run the algorithm satisfies

$$\max_m |\alpha_{i+j,m}| \leq c(M+1)n^i \mathrm{rk}\ (H_{i+1} \backslash H_i) \sum_{l=1}^{j} \max_m |\alpha_{l,m}|$$

where $\max_m |\alpha_{j,m}|$ is the maximum number of generators at level j after moving the h_i. The induction hypothesis implies that this is bounded by Cn^{i+j}. Analogously, we get the same estimate for the y_i and the g_i. This provides an upper bound on the relative growth rate of H. □

We next obtain a lower bound on the relative growth rate of H using the following lemma.

Lemma 3.2. *Let k be the length of the lower central series of G. Then there exists a $c > 0$ such that for $g \in G_k$, $|g^n| \geq cn^{1/k}$.*

Proof. We will show this inductively. For $k = 1$ the lemma is trivial as G_k is abelian. Let $m = \lceil n^{1/k} \rceil$, and choose a_1, a_2 such that $n = a_1 m^{k-1} + a_2$, $a_1 < m$, and $a_2 \le m^{k-1}$. Set $g = [x, y]$ for $x \in G$ and $y \in G_{k-1}$. We now apply the induction hypothesis to $G_k \backslash G$ and $\bar{y} \in G_k \backslash G$ to see that $\bar{y}^{m^{k-1}}$ and $\bar{y}^{a_2}$ have length at least $c'm$ in G/G_k. As $g^{ab} = [x^a, y^b]$, we have $g^n = [x^{a_1}, y^{m^{k-1}}][x, y^{a_2}]$, which has length $2a_1 + 4c'm + 2$. Since m grows like $n^{1/k}$, the length is bounded below by $cn^{1/k}$. □

Now we induct on the length of the central series to get the lower bound of the theorem. For $k = 1$, G is abelian and the bound is clear. Now suppose G has nilpotency class $k + 1$. Set $d = \sum_{i=1}^{k+1} i\mathrm{rk}\ G_{i+1} \backslash G_1$, and let $d' = d - (k+1)\mathrm{rk}\ G_{k+1}$. By the induction hypothesis $G_{k+1} \backslash G$ contains at least $Cn^{d'}$ elements of length n. We now lift this estimate to G, and consider the normal form with $H = G$ to see that there are $Cn^{d'}$ elements of the form gg_{k+1} in G. There are $C'n^{k\mathrm{rk}\ G_{k+1}}$ elements in G_{k+1}, and so there are $CC'n^d$ elements in the ball of radius $2n$ in G. As $d \ge r$, this shows the desired lower bound. □

3.3 Recurrent subgroups

We now combine the results of the two prior sections to determine when a subgroup is recurrent.

Lemma 3.3. *Let G be a group of polynomial growth of degree d, and let H be a subgroup of relative degree of growth r. Then H is recurrent iff $d - r \le 2$.*

Proof. We apply the Wiener recurrence test to H. Set $H_k = \{h \in H :\ 2^{k-1} \le |h| \le 2^k\}$. By translation invariance

$$P_e(\tau_H^+ = \infty) \le P_{h \in H_k}(\tau_{H_k}^+ = \infty) \le P_e(\tau_e^+ = \infty).$$

We multiply by $\#H_k 2^{-k(d-2)}$ and sum over k to get

$$\sum_{k=1}^{\infty} P_e(\tau_H^+ = \infty) \frac{\#H_k}{2^{k(d-2)}} \le \sum_{k=1}^{\infty} \frac{\mathrm{Cap}\ H_k}{2^{k(d-2)}} \le \sum_{k=1}^{\infty} P_e(\tau_e^+ = \infty) \frac{\#H_k}{2^{k(d-2)}}.$$

If $r \le d - 2$ this implies that H is transient.

For the converse suppose the subgroup is transient. Then the Wiener recurrence test tells us that the capacity series converges, so by the above we must have $P_e(\tau_h^+ = \infty) > 0$. Kronecker's lemma implies

$$2^{-n(d-2)} \sum_{k=1}^{n} \#H_k \to 0, \tag{10}$$

and so $r \le d - 2$. □

Next we seek to determine the conditions under which a subgroup has a relative growth rate close to that of the group.

Lemma 3.4 (Recurrent subgroups of nilpotent groups)**.** *Let G be a nilpotent group with growth of degree d, and a recurrent subgroup H with relative growth of degree $r \ge d - 2$. Then there exist H' and G' such that*

- *$H < H'$ and $[H' : H] < \infty$*
- *$G' < G$ and $[G : G'] < \infty$*
- *$H' \lhd G'$*
- *$H' \backslash G'$ is isomorphic to one of $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$.*

Proof. From the generalized Bass-Guivarc'h formula we know that $d - r \le 2$ only if $\operatorname{rk} H_{i+1} \backslash H_i = \operatorname{rk} G_{i+1} \backslash G_i$ for $i \ge 3$. If this is the case either $\operatorname{rk} H_3 \backslash H_2 = \operatorname{rk} G_3 \backslash G_2 - 1$ and $\operatorname{rk} H_2 \backslash H_1 = \operatorname{rk} G_2 \backslash G_1$ or $\operatorname{rk} H_3 \backslash H_2 = \operatorname{rk} G_3 \backslash G_2$. We will show that the first case is not possible.

The fact that $\operatorname{rk} H_{i+1} \backslash H_i = \operatorname{rk} G_{i+1} \backslash G_i$ for $i \ge 3$ implies that $[G_3 : H_3] < \infty$. We create a finite extension of H, H', by adding the y_i for $i \ge 3$, so that $H' \cap G_3 = H'$. Set $H'_i = H' \cap G_i$. Suppose $\operatorname{rk} H_3 \backslash H_2 = \operatorname{rk} H'_3 \backslash H'_2 \ne \operatorname{rk} G_3 \backslash G_2$. Our hypothesis on the relative growth rate holds only if $\operatorname{rk} H_2 \backslash H_1 = \operatorname{rk} G_2 \backslash G_1$. Similarly we exclude the y_i for $i \le 2$ from G to create a G' of finite index in G generated by H' and some elements of G_2 not in H'. We pick one of these elements from G_2 and call it g_2. The images of the elements in A_1 and C_1 generate $G'_2 \backslash G'_1$, and the image of their commutators generates $G'_3 \backslash G'_2$. Thus g_2 is a product of elements of A_i and some element of G'_3. However, this means that $g_2 \in H'$ which is a contradiction.

We now know that $\operatorname{rk} H_3 \backslash H_2 = \operatorname{rk} G_3 \backslash G_2$. This implies that $[G_2 : H_2] < \infty$. Pick H' containing H such that $H'_2 = G_2$. We take G' to be generated by all of the generators in S except those in D_1. This means that $H'_2 = G'_2$, and our assumptions on the growth rate implies that G' is generated by H' and at most two distinct generators in A_1. By definition the images of these generators are linearly independent in $G_2 \backslash G_1$. Thus they can only generate $\mathbb{Z}^d$ for $d \le 2$. □

Combining lemma 3.4 with lemma 3.3 suffices to classify when a subgroup of a virtually nilpotent group is recurrent, and so we have proved theorem 1.2.

4 The ladder lemma

It is evident from (1.2) that $c_{rt}(G', H') \in \{0, 1, 2\}$, and that this value is determined by the degree of growth of $H \backslash G$. Now we need a means for arriving at $c_{rt}(G, H)$ from $c_{rt}(G', H')$. This is done via lemma (1.1). The following proof is based on Erschler's.

Proof. (i) Let S be a generating set of G and $T \subset S$ a generating set of K. A measure μ on K is also a measure on G, so if

$$\mu(K - B_{K,T}(r)) \le C r^{-\alpha},$$

then

$$\mu(G - B_{G,S}(r)) \le C r^{-\alpha}.$$

It is also clear that if $(H \backslash K, \mu)$ is transient then so is $(H \backslash G, \mu)$. Thus $c_{rt}(K, H) \le c_{rt}(G, H)$.

Now suppose that $[G : K] \le \infty$, and fix a measure μ on G. Since K is of finite index it will be visited infinitely often, and thus the first entry distribution, ν, of K is well defined. In particular for any $k \in K$, we have $\nu(k) = P_e(Z_{\tau_K^+} = k)$.

The tail of ν decays exponentially when μ is finitely supported. Note that τ_K^+ is also the first positive hitting time of the identity coset for $(K\backslash G,\mu)$. For finitely supported μ we have

$$\{Z_{\tau_K^+} \in K - B_G(r)\} \subset \{\tau_K^+ > r\}. \tag{11}$$

The probability of the first event is $\nu(K - B_G(r))$, and the second is the tail of the hitting time for a finite reversible Markov Chain. This decays exponentially in r (see Chapter 2, section 4.3 in [AF02]). Thus ν has all polynomial moments.

If μ is not finitely supported, then the following holds

$$\{Z_{\tau_K} \in K - B_G(r)\} \subseteq \{X_i \in B_G(r/\tau_K),\ i \le \tau_K\}^c. \tag{12}$$

Thus

$$P(Z_{\tau_K^+} \in G - B_G(r)) \le P(\bigcup_{i=1}^{\tau_K^+}\{|X_i| \le r/\tau_K\}^c) \tag{13}$$

$$\le \tau_K^+ P(X_1 \in G - B_G(r/\tau_K^+)). \tag{14}$$

The left hand side is $\nu(G - B_G(r))$ which for some $c > 0$ bounds

$$\nu(K - B_K(ct)),$$

and the probability on the right hand side is bounded by $c_2\mu(G - B_G(c_1 r))$ for some constants $c_1, c_2 > 0$. We then take the expectation to see that the tail of ν decays at least as fast of that of μ. Thus

$$c_{rt}(K,H) \ge c_{rt}(G,H),$$

which suffices to show $c_{rt}(K,H) = c_{rt}(G,H)$.

(ii) For a given measure μ on G, if $(K\backslash G,\mu)$ is transient then $(H\backslash G,\mu)$ is transient. Hence $c_{rt}(G,K) \ge c_{rt}(G,H)$. Now suppose $[K : H] < \infty$. A random walk visiting K infinitely often will also visit H infinitely often, and so transience of H implies transience of K. Hence $c_{rt}(G,K) \le c_{rt}(G,H)$, and we get that $c_{rt}(G,K) = c_{rt}(G,H)$.

□

5 Proof of Theorem 1.1

We are now ready to prove the the main result.

Proof of Theorem 1.1. Let G be a group with recurrent subgroup H. Suppose G has polynomial volume growth of degree d. We apply (1.2) to obtain subgroups $G' \le G$ with $[G : G'] < \infty$ and $H' \le H$ with $[H : H'] < \infty$ such that $c_{rt}(G', H' \cap G') = s$ for some $s \in \{0,1,2\}$.

To apply the equality cases of lemma 1.1 we need to show that $[H : H \cap G'] < \infty$ and $[H' \cap G' : H \cap G'] < \infty$. For the first pair, if $h \in H$,

$$(H \cap G')h = H \cap Gh',$$

and thus a coset of $(H\cap G')\backslash H$ corresponds to a coset of $G'\backslash G$. As G' is of finite index in G there are only finitely many such cosets. For the second pair, take $h' \in H' \cap G'$. Then

$$(H\cap G')h' = Hh' \cap G'h',$$

so a coset in $(H\cap G')\backslash(H'\cap G')$ corresponds to the intersection of a coset of $H\backslash H'$ with a coset of $G'\backslash G$. As there are only finitely many cosets in both quotients we conclude that $(H\cap G')\backslash(H'\cap G')$ is finite.

We now apply lemma 1.1. By (ii) to $(H\cap G') \le H \le G$ we see that

$$c_{rt}(G,H) = c_{rt}(G,H\cap G'). \tag{15}$$

Next we apply (i) to $(H\cap G') \le G' \le G$ to get

$$c_{rt}(G,H\cap G') = c_{rt}(G',H\cap G'). \tag{16}$$

Finally, we apply (ii) to $(H\cap G') < (H'\cap G') < G'$ to see that

$$c_{rt}(G',H\cap G') = c_{rt}(G',H'\cap G') \tag{17}$$

Combining these identities gives $c_{rt}(G,H) = c_{rt}(G',H'\cap G')$ as desired. Note that if $c_{rt}(G,H) \notin \mathbb{Z}\cup\{\infty\}$ then (1.2) implies that G cannot have polynomial volume growth $\square$

The classification of c_{rt} can be made more precise using theorem 1.2, giving us the following corollary.

Corollary 5.1. *If G has polynomial growth and H is a subgroup such that $c_{rt}(G,H) < \infty$, then $c_{rt}(G,H)$ is the growth rate of the quotient $H\backslash G$. In particular, $c_{rt}(G,H) = d - r$ where d is the growth rate of G and r is the relative growth rate of H in G.*

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