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Support theorem for a stochastic Cahn-Hilliard equation *

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Abstract

In this paper, we establish a Stroock-Varadhan support theorem for the global mild solution to a d ($d \leq 3$)-dimensional stochastic Cahn-Hilliard partial differential equation driven by a space-time white noise.

Key words: Stochastic Cahn-Hilliard equation, Space-time white noise, Stroock-Varadhan support theorem.

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1 Introduction and main result

In this paper, we consider the following stochastic Cahn-Hilliard equation:

$$\begin{cases} \partial u / \partial t = -\Delta [\Delta u + f(u)] + \sigma(u)\dot{W}, & \text{in } [0, T] \times D, \\ u(0) = \psi, \\ \partial u / \partial \mathbf{n} = \partial [\Delta u] / \partial \mathbf{n} = 0, & \text{on } [0, T] \times \partial D, \end{cases} \quad (1.1)$$

where Δ denotes the Laplace operator, the domain $D = [0, \pi]^d$ ($d = 1, 2, 3$), and $f : \mathbf{R} \rightarrow \mathbf{R}$ is a polynomial of degree 3 with positive dominant coefficient (which is due to the background of the equation from material science). Assume that $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is a bounded and Lipschitzian function and \dot{W} is a Gaussian space-time white noise on some complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ satisfying

$$\mathbf{E} [\dot{W}(x, t)\dot{W}(y, s)] = \delta(|t - s|)\delta(|x - y|), \quad (t, x), (s, y) \in [0, T] \times D.$$

Here $\delta(\cdot)$ is the Dirac delta function concentrated at the point zero.

The (deterministic) Cahn-Hilliard equation (i.e., $\sigma \equiv 0$ in (1.1)) has been extensively studied (see, e.g., [2; 3; 4; 5; 10; 15; 18]) as a well-known model of the macro-phase separation that occurs in an isothermal binary fluid, when a spatially uniform mixture is quenched below a critical temperature at which it becomes unstable. A stochastic version of the Cahn-Hilliard equation (when $\sigma \equiv 1$ in (1.1)) was first proposed by Da Prato and Debussche [8], and the existence, uniqueness and regularity of the global mild solution were explored. In Cardon-Weber [6], the authors considered this type of stochastic equation in a general case on σ , which is equivalent to the following form:

$$\begin{aligned} u(t, x) = & \int_D G_t(x, y)\psi(y)dy + \int_0^t \int_D \Delta G_{t-s}(x, y)f(u(s, y))dyds \\ & + \int_0^t \int_D G_{t-s}(x, y)\sigma(u(s, y))W(dy, ds), \end{aligned} \quad (1.2)$$

where $G_t(\cdot, *)$ denotes the Green kernel corresponding to the operator $\partial / \partial t + \Delta^2$ with the homogeneous Neumann's boundary condition as in (1.1). Since Stroock and Varadhan [17] established their famous support theorem for diffusion processes, there have been many research works on this issue, for example, a variety of support theorems for 1-dimensional second-order parabolic and hyperbolic stochastic partial differential equations (abbr. SPDEs) have been discussed in the literature (see, e.g., [1; 7; 13; 14]). Millet and Sanz-Solé [13] characterized the support of the law of the solution to a class of hyperbolic SPDEs, which simplified the proof in [17]. In Bally et al. [1], the authors proved a support theorem for a semi-linear parabolic SPDE. Moreover, a support result for a generalized Burgers SPDE (containing a quadratic term) was established in Cardon-Weber and Millet [7]. Herein, we are attempting to establish a support theorem of the law corresponding to the solution to Equation (1.1) in $C([0, T], L^p([D]))$ for $p \geq 4$. The main strategy used in this paper is an approximation procedure by using a space-time polygonal interpolation for the white noise, and we particularly adopt a localization argument, which was used in [7] for studying a support theorem of a Burgers-type equation. However here we need more technical estimates concerning the high-order Green kernel $G_t(\cdot, *)$, which sharp the estimates in [6] (see Appendix).

In what follows, we introduce the main result of this paper. To do it, we define the following Cameron-Martin space \mathcal{H} by

$$\mathcal{H} = \left\{ h(t, x) = \int_0^t \int_{\prod_{i=1}^d [0, x_i]} \mathbf{h}(s, y) dy ds; (t, x) = (t, (x_1, \dots, x_d)) \in [0, T] \times D, \right. \\ \left. \mathbf{h} \in L^2([0, T] \times D) \right\},$$

and the corresponding norm by

$$\|h\|_{\mathcal{H}} = \sqrt{\int_0^T \int_D |\mathbf{h}(s, y)|^2 dy ds}, \quad \text{for all } h \in \mathcal{H}.$$

Let \mathcal{H}_b represent the subset of \mathcal{H} , in which the first-order derivative \mathbf{h} of $h \in \mathcal{H}$ is bounded. For $h \in \mathcal{H}$, consider the following skeleton equation:

$$S(h)(t, x) = \int_D G_t(x, y) \psi(y) dy + \int_0^t \int_D \Delta G_{t-s}(x, y) f(S(h)(s, y)) dy ds \\ + \int_0^t \int_D G_{t-s}(x, y) \sigma(S(h)(s, y)) \mathbf{h}(s, y) dy ds. \quad (1.3)$$

Recall Equations (1.1) and (1.2). We make the following assumptions throughout the paper:

(H1). Assume that $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is bounded and belongs to $C^3(\mathbf{R})$ with bounded first to third-order partial derivatives, and

(H2). The initial function $\psi \in L^p(D)$ for $p \geq 4$, and ψ is $\varrho \in]0, 1]$ -Hölder continuous.

Now we are at the position to state the main result of this paper.

Theorem 1.1. *Under the assumptions (H1) and (H2), let $u = (u(t, x))_{(t,x) \in [0,T] \times D}$ be the unique solution to Equation (1.2) in $C([0, T], L^p(D))$ with $p \geq 4$ and $\mathbf{P} \circ u^{-1}$ denote the law (a probability measure) of the solution u . Recall the skeleton equation (1.3), and set $\mathbb{S}_{\mathcal{H}} = \{S(h); h \in \mathcal{H}\}$. Then we have*

(a) *Let $p > 6$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$ and $\bar{\beta} \in]0, \min\{2 - \frac{d}{2}, \varrho\}[$, the topological support $\text{supp}(\mathbf{P} \circ u^{-1})$ in $C^{\bar{\alpha}, \bar{\beta}}([0, T] \times D)$ of the law $\mathbf{P} \circ u^{-1}$ is the closure of $\mathbb{S}_{\mathcal{H}}$.*

(b) *Let $p \geq 4$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$, the topological support $\text{supp}(\mathbf{P} \circ u^{-1})$ in $C^{\bar{\alpha}}([0, T], L^p(D))$ of the law $\mathbf{P} \circ u^{-1}$ is the closure of $\mathbb{S}_{\mathcal{H}}$.*

The rest of this paper is organized as follows: In the coming section, we give a difference approximation to the $(d + 1)$ -dimensional space-time white noise $\dot{W}(x, t)$ and study some concrete properties of the approximating noises. In Section 3, we introduce a localization framework as in [7], and then switch to prove the support theorem by checking the conditions (C1) and (C2) below (see Section 3). Sections 4 and 5 are devoted to checking the validity of the conditions (C1) and (C2), respectively. In Section 6, we prove the continuity of the solution $S(h)$ to the skeleton equation (1.3) in \mathcal{H}_b and finally we complete the proof of Theorem 1.1.

2 Difference approximation to white noise

In this section, we give a difference approximation to the $(d+1)$ -dimensional space-time white noise \dot{W} , which is a space-time polygonal interpolation for \dot{W} .

Let $n \in \mathbf{N}$ and $t \in [0, T]$, set

$$\underline{t}_n = \max_{j \in \{0, 1, \dots, 2^n\}} \{jT2^{-n}; jT2^{-n} \leq t\}, \quad \text{and } t_n = [\underline{t}_n - T2^{-n}] \vee 0.$$

Let $\mathbf{k} := (k_1, \dots, k_d) \in \mathbf{I}_n^d := \{0, 1, \dots, n-1\}^d$. Define a partition $(\Delta_{j,\mathbf{k}})_{j=0,1,\dots,2^n-1, \mathbf{k} \in \mathbf{I}_n^d}$ of $\mathcal{O}_T := [0, T] \times D$ by

$$\Delta_{j,\mathbf{k}} = D_{\mathbf{k}} \times]jT2^{-n}, (j+1)T2^{-n}],$$

where $D_{\mathbf{k}} = \prod_{j=1}^d]k_j\pi n^{-1}, (k_j+1)\pi n^{-1}]$. For $x_j \in]k_j\pi n^{-1}, (k_j+1)\pi n^{-1}]$ with $j = 1, \dots, d$, we set $D_{\mathbf{k}}(x) = \prod_{j=1}^d]k_j\pi n^{-1}, (k_j+1)\pi n^{-1}]$. Further, for each $(t, x) \in \mathcal{O}_T$, we define the following difference approximation to \dot{W} by

$$\dot{W}_n(x, t) = \begin{cases} \frac{W(\Delta_{j-1,\mathbf{k}})}{|\Delta_{j-1,\mathbf{k}}|}, & (x, t) \in \Delta_{j,\mathbf{k}}, j = 1, \dots, 2^n - 1, \mathbf{k} \in \mathbf{I}_n^d, \\ 0, & (x, t) \in \Delta_{0,\mathbf{k}}, \mathbf{k} \in \mathbf{I}_n^d, \end{cases} \quad (2.1)$$

where $|\Delta_{j,\mathbf{k}}| = T\pi^d(n^d 2^n)^{-1}$ is the volume of the partition $\Delta_{j,\mathbf{k}}$ for each $j = 0, 1, \dots, 2^n - 1$ and $\mathbf{k} \in \mathbf{I}_n^d$.

Next we suppose that

(H3). the mappings $F, H, K : \mathbf{R} \rightarrow \mathbf{R}$ are bounded, globally Lipschitzian and $H \in C^3(\mathbf{R})$ with bounded first to third-order derivatives.

We now consider the following equations for $h \in \mathcal{H}_b$,

$$\begin{aligned} X_n(t, x) &= G_t * \psi(x) \\ &+ \int_0^t \int_D G_{t-s}(x, y) [F(X_n(s, y))W(dy, ds) + H(X_n(s, y))W_n(dy, ds)] \\ &+ \int_0^t \int_D G_{t-s}(x, y) [K(X_n(s, y))\mathbf{h}(s, y) \\ &- \dot{H}(X_n(s, y))[\alpha_n(s, y)F(X_n(s, y)) + \beta_n(s, y)H(X_n(s, y))]] dy ds \\ &+ \int_0^t \int_D \Delta_y G_{t-s}(x, y) f(X_n(s, y)) dy ds, \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} X(t, x) &= G_t * \psi(x) + \int_0^t \int_D G_{t-s}(x, y) [F + H](X(s, y))W(dy, ds) \\ &+ \int_0^t \int_D G_{t-s}(x, y) K(X(s, y))\mathbf{h}(s, y) dy ds \end{aligned}$$

$$+ \int_0^t \int_D \Delta_y G_{t-s}(x, y) f(X(s, y)) dy ds, \quad (2.3)$$

where

$$G_t * \psi(x) := \int_D G_t(x, y) \psi(y) dy,$$

and for each $n \in \mathbf{N}$,

$$\begin{aligned} \alpha_n(t, x) &:= n^d 2^n (T \pi^d)^{-1} \int_{t_n}^{\underline{t}_n} \int_{D_k(x)} G_{t-s}(x, y) dy ds, \\ \beta_n(t, x) &:= n^d 2^n (T \pi^d)^{-1} \int_{\underline{t}_n}^t \int_{D_k(x)} G_{t-s}(x, y) dy ds. \end{aligned}$$

For $\alpha_n(t, x)$ and $\beta_n(t, x)$, by virtue of (A.4) in Lemma A.1, we claim that

$$\sup_{(t, x) \in \mathcal{O}_T} |\alpha_n(t, x)| \leq C n^d, \quad \text{and} \quad (2.4)$$

$$\sup_{(t, x) \in \mathcal{O}_T} |\beta_n(t, x)| \leq C n^d. \quad (2.5)$$

Indeed, using (A.4), we have for each $t \in [0, T]$,

$$\begin{aligned} \sup_{x \in D} |\alpha_n(t, x)| &\leq C n^d 2^n \max_{k \in \mathbf{I}_n^d} \left\{ \mathbf{1}_{D_k}(x) \left| \int_{t_n}^{\underline{t}_n} \int_{D_k} |G_{t-s}(x, y)| dy ds \right| \right\} \\ &\leq C n^d 2^n |\underline{t}_n - t_n| \\ &\leq C n^d, \end{aligned}$$

and

$$\sup_{x \in D} |\beta_n(t, x)| \leq C n^d 2^n |t - \underline{t}_n| \leq C n^d,$$

follows from the equality (A.19).

In the following, let $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be the natural filtration generated by W , i.e.,

$$\mathcal{F}_t = \sigma\{W(B \times [0, s]); s \in [0, t], B \in \mathcal{B}(D)\}.$$

Then for every $t \in [0, T]$ and $n \in \mathbf{N}$ fixed, $(\dot{W}_n(x, t))_{x \in D}$ given by (2.1) is \mathcal{F}_t -adapted. More precisely, it is $\mathcal{F}_{\underline{t}_n}$ -adapted and which is independent of the information \mathcal{F}_{t_n} .

Lemma 2.1. *For each fixed $n \in \mathbf{N}$ and $p \geq 1$, we have*

$$\sup_{(t, x) \in \mathcal{O}_T} \mathbf{E} \left[|\dot{W}_n(x, t)|^p \right] \leq C_p n^{\frac{dp}{2}} 2^{\frac{np}{2}}.$$

Proof. By virtue of the definition (2.1),

$$\begin{aligned}
& \sup_{(t,x) \in \mathcal{O}_T} \mathbf{E} \left[|\dot{W}_n(x, t)|^p \right] \\
&= \sup_{(t,x) \in \mathcal{O}_T} \mathbf{E} \left[\left| \sum_{j=1}^{2^n-1} \sum_{\mathbf{k} \in \mathbf{I}_n^d} \frac{W(\Delta_{j-1, \mathbf{k}})}{|\Delta_{j-1, \mathbf{k}}|} \mathbf{1}_{\Delta_{j-1, \mathbf{k}}}(x, t) \right|^p \right] \\
&\leq C_p \max \left\{ \mathbf{E} \left[\left| \frac{W(\Delta_{j-1, \mathbf{k}})}{|\Delta_{j-1, \mathbf{k}}|} \right|^p \right]; j = 1, \dots, 2^n - 1, \mathbf{k} \in \mathbf{I}_n^d \right\}.
\end{aligned}$$

Note that for each $j = 1, \dots, 2^n - 1$ and $\mathbf{k} \in \mathbf{I}_n^d$,

$$\frac{W(\Delta_{j-1, \mathbf{k}})}{|\Delta_{j-1, \mathbf{k}}|} \sim N(0, |\Delta_{j-1, \mathbf{k}}|^{-1}).$$

For any random variable $Z \sim N(0, \sigma^2)$, it holds that

$$\mathbf{E}|Z|^p = \frac{1}{\sqrt{\pi}} \sqrt{2^p \sigma^{2p}} \Gamma\left(\frac{p}{2} + \frac{1}{2}\right),$$

where Γ denotes the Gamma function. This yields that

$$\sup_{(t,x) \in \mathcal{O}_T} \mathbf{E} \left[|\dot{W}_n(x, t)|^p \right] \leq C_p \max \left\{ \sqrt{|\Delta_{j-1, \mathbf{k}}|^{-p}}; j = 1, \dots, 2^n - 1, \mathbf{k} \in \mathbf{I}_n^d \right\},$$

and which proves the lemma. \square

Let $n \in \mathbf{N}$ be fixed. For $\alpha > 0$ and $t \in]0, T]$, we now define an event $\bar{\Omega}_{n,t}^\alpha$ by

$$\bar{\Omega}_{n,t}^\alpha = \left\{ \omega \in \Omega; \sup_{(s,y) \in [0,t] \times D} |\dot{W}(y, s; \omega)| \leq \alpha n^d 2^{\frac{n}{2}} \right\}. \quad (2.6)$$

For this event, we have

Lemma 2.2. *If choose $\alpha > 2\sqrt{\frac{\log 2}{T\pi^d}}$, then*

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\left[\bar{\Omega}_{n,T}^\alpha \right]^c \right) = 0.$$

Proof. Let $Z \sim N(0, 1)$ be a standard normal random variable. Then according to the definition (2.1) for $\dot{W}_n(x, t)$,

$$\begin{aligned}
\mathbf{P} \left(\left[\bar{\Omega}_{n,T}^\alpha \right]^c \right) &= \mathbf{P} \left(\max_{(j, \mathbf{k}) \in \{1, \dots, 2^n-1\} \times \mathbf{I}_n^d} \left\{ \frac{W(\Delta_{j-1, \mathbf{k}})}{|\Delta_{j-1, \mathbf{k}}|} \right\} \geq \alpha n^d 2^{\frac{n}{2}} \right) \\
&\leq n^d 2^n \mathbf{P} \left(|Z| \geq \alpha \sqrt{T\pi^d} n^{d/2} \right)
\end{aligned}$$

$$\begin{aligned}
&= n^d 2^n \mathbf{P} \left(\frac{|Z|^2}{4} \geq \frac{\alpha^2 T \pi^d}{4} n^d \right) \\
&\leq n^d 2^n \exp \left[-\frac{\alpha^2 T \pi^d}{4} n^d \right] \mathbf{E} \left[\exp \left(\frac{|Z|^2}{4} \right) \right].
\end{aligned} \tag{2.7}$$

Note that $\mathbf{E} \left[\exp \left(\frac{|Z|^2}{4} \right) \right] = \sqrt{2}$. Then (2.7) further yields that

$$\begin{aligned}
0 \leq \mathbf{P} \left(\left[\bar{\Omega}_{n,T}^\alpha \right]^c \right) &\leq \sqrt{2} n^d \exp \left[n \log 2 - \frac{\alpha^2 T \pi^d}{4} n^d \right] \\
&\leq \sqrt{2} n^d \exp \left[n^d \left(\log 2 - \frac{\alpha^2 T \pi^d}{4} \right) \right] \\
&\rightarrow 0, \text{ as } n \rightarrow \infty,
\end{aligned} \tag{2.8}$$

if $\alpha > 2\sqrt{\frac{\log 2}{T \pi^d}}$. Thus the proof of the lemma is complete. \square

3 Localization framework

In this section, we adopt a localization method used in [7] to deal with Equation (1.1). In addition, we will prove a key proposition, which is useful in the proof of Theorem 1.1.

Proposition 3.1. *Under the assumptions (H1) and (H2), let $X = (X(t, x))_{(t,x) \in [0,T] \times D}$ (resp. X_n) be the unique solution to Equation (2.3) (resp. (2.2)) in $C([0, T], L^p(D))$ with $p \geq 4$. Recall the skeleton equation (1.3), and set $\mathbb{S}_{\mathcal{H}} = \{S(h); h \in \mathcal{H}\}$. Then we have*

- (i) *Let $p > 6$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$ and $\bar{\beta} \in]0, \min\{2 - \frac{d}{2}, \varrho\}[$, the sequence X_n converges in probability to X in $C^{\bar{\alpha}, \bar{\beta}}([0, T] \times D)$.*
- (ii) *Let $p \geq 4$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$, the sequence X_n converges in probability to X in $C^{\bar{\alpha}}([0, T], L^p(D))$.*

Next we give a sketch for the proof of the conclusion (ii) in Proposition 3.1. The similar argument can also be used to prove the part (i). For $(t, x) \in \mathcal{O}_T$, set

$$Y_n(t, x) := X_n(t, x) - X(t, x).$$

From (2.2) and (2.3), it follows that

$$Y_n(t, x) = \sum_{i=1}^3 \Gamma_n^i(t, x) + \Lambda_n(t, x), \tag{3.1}$$

where

$$\begin{aligned}
\Gamma_n^1(t, x) &:= \int_0^t \int_D G_{t-s}(x, y) [(F + H)(X_n(s, y)) - (F + H)(X(s, y))] W(dy, ds), \\
\Gamma_n^2(t, x) &:= \int_0^t \int_D G_{t-s}(x, y) [K(X_n(s, y)) - K(X(s, y))] \mathbf{h}(s, y) dy ds,
\end{aligned}$$

$$\Gamma_n^3(t, x) := \int_0^t \int_D \Delta_y G_{t-s}(x, y) [f(X_n(s, y)) - f(X(s, y))] dy ds,$$

and

$$\begin{aligned} \Lambda_n(t, x) &:= \int_0^t \int_D G_{t-s}(x, y) H(X_n(s, y)) [W_n(dy, ds) - W(dy, ds)] \\ &\quad - \int_0^t \int_D G_{t-s}(x, y) \dot{H}(X_n(s, y)) \\ &\quad \times [\alpha_n(s, y) F(X_n(s, y)) + \beta_n(s, y) H(X_n(s, y))] dy ds. \end{aligned} \quad (3.2)$$

Introduce an auxiliary \mathcal{F}_{t_n} -adapted process

$$X_n^-(t, x) := G_{t-t_n}(x, X_n(t_n, \cdot)), \quad \text{for } (t, x) \in \mathcal{O}_T. \quad (3.3)$$

Recall the localization argument adopted in [7]. For $\gamma \in (0, 1)$ and $p \geq 4$, define

$$\Phi_n^{p, \gamma}(t) := \sup_{s \in [0, t]} \|Y_n(t, \cdot)\|_p + \sup_{s \neq s' \in [0, t]} \frac{\|Y_n(s, \cdot) - Y_n(s', \cdot)\|_p}{|s - s'|^\gamma},$$

where $\|\cdot\|_p$ corresponds to the norm of $L^p(D)$ and for $\delta > 0$,

$$\tau_n^\delta := \inf \{t > 0; \Phi_n^{p, \gamma}(t) \geq \delta\} \wedge T.$$

For $M > \delta$, define the following events

$$A_t(M - \delta) := \left\{ \omega \in \Omega; \sup_{s \in [0, t]} \|X(s, \cdot)\|_p \leq M - \delta \right\}, \quad (3.4)$$

$$A_n^M(t) := \left\{ \omega \in \Omega; \sup_{s \in [0, t]} \|X_n(s, \cdot)\|_p \vee \sup_{s \in [0, t]} \|X(s, \cdot)\|_p \leq M \right\}. \quad (3.5)$$

Then for $t \in]0, T]$,

$$A_t(M - \delta) \cap \{t \leq \tau_n^\delta\} \subseteq A_n^M(t). \quad (3.6)$$

In fact, from the inequality $|y| \leq |x - y| + |x|$, it follows that

$$\begin{aligned} &A_t(M - \delta) \cap \{t \leq \tau_n^\delta\} \\ &\subseteq \left\{ \sup_{s \in [0, t]} \|X(s, \cdot)\|_p \leq M - \delta \right\} \cap \left\{ \sup_{s \in [0, t]} \|X_n(s, \cdot) - X(s, \cdot)\|_p \leq \delta \right\} \\ &\subseteq \left\{ \sup_{s \in [0, t]} \|X(s, \cdot)\|_p \leq M \right\} \cap \left\{ \sup_{s \in [0, t]} \|X_n(s, \cdot)\|_p \leq M \right\} \\ &= A_n^M(t). \end{aligned}$$

Recall the event $\bar{\Omega}_{n,t}^\alpha$ defined by (2.6) in Section 2 and that $\alpha > 2\sqrt{\frac{\log 2}{T\pi^d}}$. For each fixed $\delta > 0$ and $V \in C^\gamma([0, T]; L^p(D))$, set

$$\|V\|_{\gamma, p, \tau_n^\delta} := \sup_{s \in [0, T \wedge \tau_n^\delta]} \|V(s, \cdot)\|_p + \sup_{s \neq s' \in [0, T \wedge \tau_n^\delta]} \frac{\|V(s, \cdot) - V(s', \cdot)\|_p}{|s - s'|^\gamma}.$$

Then for $q \geq 1$,

$$\begin{aligned}
& \mathbf{P} \left(\Phi_n^{p,\gamma}(T) \geq \delta \right) \\
& \leq \mathbf{P} \left(\Phi_n^{p,\gamma}(T) \geq \delta, A_T(M - \delta) \cap \bar{\Omega}_{n,T}^\alpha \right) + \mathbf{P} \left(A_T^c(M - \delta) \right) + \mathbf{P} \left(\left[\bar{\Omega}_{n,T}^\alpha \right]^c \right) \\
& = \mathbf{P} \left(\|Y_n\|_{\gamma,p,\tau_n^\delta} \geq \delta, A_T(M - \delta) \cap \bar{\Omega}_{n,T}^\alpha \right) + \mathbf{P} \left(A_T^c(M - \delta) \right) + \mathbf{P} \left(\left[\bar{\Omega}_{n,T}^\alpha \right]^c \right) \\
& \leq \delta^{-q} \mathbf{E} \left[\mathbf{1}_{A_T(M - \delta) \cap \bar{\Omega}_{n,T}^\alpha} \|Y_n\|_{\gamma,p,\tau_n^\delta}^q \right] + \mathbf{P} \left(A_T^c(M - \delta) \right) + \mathbf{P} \left(\left[\bar{\Omega}_{n,T}^\alpha \right]^c \right),
\end{aligned}$$

However, Lemma 2.2 and Lemma 3.1 (the latter will be proved below) yield that

$$\mathbf{P} \left(A_T^c(M - \delta) \right) + \mathbf{P} \left(\left[\bar{\Omega}_{n,T}^\alpha \right]^c \right) \rightarrow 0, \text{ as } M \rightarrow \infty, n \rightarrow \infty.$$

Therefore, by Lemma A.1 in [7] and (3.6), in order to prove

$$\Phi_n^{p,\gamma}(T) \rightarrow 0, \text{ in probability, as } n \rightarrow \infty, \tag{3.7}$$

it suffices to check that there exist $q \geq p$ and $\theta > q\bar{\alpha}$ (we have set $\gamma = \bar{\alpha}$, where $\bar{\alpha}$ is the exponent presented in Proposition 3.1) such that

$$\begin{aligned}
\text{(C1)} \quad & \forall t \in [0, T], \lim_{n \rightarrow \infty} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|Y_n(t, \cdot)\|_p^q \right] = 0; \\
\text{(C2)} \quad & \forall s < t \in [0, T], \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|Y_n(t, \cdot) - Y_n(s, \cdot)\|_p^q \right] \leq C|t - s|^{1+\theta}.
\end{aligned}$$

Here the event $\bar{A}_n^M(t)$ is defined by

$$\bar{A}_n^M(t) = A_n^M(t) \cap \bar{\Omega}_{n,t}^\alpha, \quad t \in [0, T], \tag{3.8}$$

and which satisfies the order relation:

$$\bar{A}_n^M(t) \subset \bar{A}_n^M(r), \text{ if } r \leq t.$$

Lemma 3.1. *Let the event $A_T(M)$ be defined by (3.4). Then*

$$\lim_{M \rightarrow \infty} \mathbf{P} \left([A_T(M)]^c \right) = 0.$$

Proof. Note that for $p \geq 4$, $\beta \in [p, \infty[$ and $\beta \in]p, 6p/(6-p)^+[$ (if $d = 3$),

$$\mathbf{P} \left([A_T(M)]^c \right) = \mathbf{P} \left(\sup_{t \in [0, T]} \|X(t, \cdot)\|_p \geq M \right) \leq M^{-\beta} \mathbf{E} \left[\sup_{t \in [0, T]} \|X(t, \cdot)\|_p^\beta \right].$$

Therefore, it remains to prove

$$\mathbf{E} \left(\sup_{t \in [0, T]} \|X(t, \cdot)\|_p^\beta \right) < \infty. \tag{3.9}$$

Define for $(t, x) \in \mathcal{O}_T$,

$$L_1(u)(t, x) := \int_0^t \int_D G_{t-s}(x, y) [F + H](u(s, y)) W(dy, ds);$$

$$L_2(u)(t, x) := \int_0^t \int_D G_{t-s}(x, y) K(u(s, y)) \mathbf{h}(s, y) dy ds,$$

and set $Z = X - L(X)$ with $L = L_1 + L_2$. Then

$$\begin{cases} \frac{\partial Z}{\partial t} + \Delta^2 Z - \Delta f([Z + L(X)]) = 0, \\ Z(0) = \psi, \\ \partial Z / \partial \mathbf{n} = \partial [\Delta Z] / \partial \mathbf{n} = 0, \text{ on } \partial D. \end{cases} \quad (3.10)$$

Further, the Garsia-Rodemich-Rumsey lemma (see, e.g., Theorem B.1.1 and Theorem B.1.5 in [9]) yields that, if for any $q, \delta \in]1, \infty[$ and some $\gamma', \gamma'' \in]0, 1]$,

- (a) $\sup_{(t,x) \in \mathcal{O}_T} \mathbf{E} [|L(u)(t, x)|^{2q\delta}] < \infty,$
- (b) $\mathbf{E} [|L(u)(t, x) - L(u)(t', x')|^{2q}] \leq C [|t - t'|^{\gamma''} + |x - x'|^{\gamma'}]^q, \quad q > 1,$

then (3.9) holds. So we only need to prove (a) and (b). Note that K is bounded and $d \leq 3$. Then in light of (A.4),

$$\begin{aligned} \sup_{(t,x) \in \mathcal{O}_T} \mathbf{E} [|L_2(u)(t, x)|^{2q\delta}] &\leq \|K\|_\infty^{2q\delta} \sup_{(t,x) \in \mathcal{O}_T} \left| \int_0^t \int_D |G_{t-s}(x, y) \mathbf{h}(s, y)| dy ds \right|^{2q\delta} \\ &\leq \|K\|_\infty^{2q\delta} \|h\|_{\mathcal{H}}^{q\delta} T^{(1-\frac{d}{4})q\delta} < \infty. \end{aligned}$$

If set $t > t'$, then by (A.7)–(A.9) in Lemma A.2, we have for $\gamma' \in [0, 1 - d/4[$ and $\gamma'' \in [0, 2 \wedge (4 - d)[$,

$$\mathbf{E} [|L_2(u)(t, x) - L_2(u)(t', x')|^{2q}] \leq C [|t - t'|^{\gamma''} + |x - x'|^{\gamma'}]^q.$$

The estimate of L_1 is similar to that of L_2 (or see [6]). Thus the proof the lemma is complete. \square

4 Auxiliary lemmas

In this section, we present a sequence of auxiliary lemmas for checking the conditions (C1) and (C2) under (H1)–(H3) given in Section 1 and 2. Throughout Sections 4–6, (H1)–(H3) are assumed to be satisfied.

The following lemma tells us that, to check (C1), it suffices to show (C1) holds with $\Lambda_n(t, x)$ instead of $Y_n(t, x)$.

Lemma 4.1. *Assume $p \geq 4$, and $q \geq p$ if $d = 1, 2$, and $q \in]p, 6p/(6 - p)^+[$ if $d = 3$. Then for each $n \in \mathbf{N}$,*

$$\sup_{t \in [0, T]} \mathbf{E} [\mathbf{1}_{\bar{A}_n^M(t)} \|Y_n(t, \cdot)\|_p^q] \leq C \sup_{t \in [0, T]} \mathbf{E} [\mathbf{1}_{\bar{A}_n^M(t)} \|\Lambda_n(t, \cdot)\|_p^q], \quad (4.1)$$

where $\Lambda_n(t, x)$ is defined by (3.2).

Proof. Note that for each $t \in [0, T]$,

$$\mathbf{E} \left[\|Y_n(t, \cdot)\|_p^q \right] \leq C \mathbf{E} \left[\sum_{i=1}^3 \mathbf{1}_{\bar{A}_n^M(t)} \|\Gamma_n^i(t, \cdot)\|_p^q + \mathbf{1}_{\bar{A}_n^M(t)} \|\Lambda_n(t, \cdot)\|_p^q \right],$$

where $\Gamma^i(t, x)$ ($i = 1, 2, 3$) are defined in (3.1). Therefore by (A.16) in Lemma A.3, for $\frac{1}{r} = \frac{2}{q} - \frac{2}{p} + 1 \in [0, 1]$,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|\Gamma_n^1(t, \cdot)\|_p^q \right] \\ & \leq C \int_D \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} |\Gamma_n^1(t, x)|^q \right] dx \\ & \leq C \mathbf{E} \left[\left\| \int_0^t \int_D G_{t-s}^2(\cdot, y) \mathbf{1}_{\bar{A}_n^M(s)} |Y_n(s, y)|^2 dy ds \right\|_{\frac{q}{2}}^q \right] \\ & \leq C \mathbf{E} \left[\int_0^t (t-s)^{\frac{d}{4r} - \frac{d}{2}} \mathbf{1}_{\bar{A}_n^M(s)} \|Y_n(s, \cdot)\|_p^2 ds \right]^{\frac{q}{2}} \\ & \leq C \left[\int_0^t (t-s)^{\frac{d}{4r} - \frac{d}{2}} ds \right]^{\frac{q-2}{2}} \left[\int_0^t (t-s)^{\frac{d}{4r} - \frac{d}{2}} \sup_{r \in [0, s]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(r)} \|Y_n(r, \cdot)\|_p^q \right] ds \right]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|\Gamma_n^2(t, \cdot)\|_p^q \right] \\ & \leq C \|h\|_{\mathcal{H}}^q \mathbf{E} \left[\left\| \int_0^t \int_D G_{t-s}^2(\cdot, y) \mathbf{1}_{\bar{A}_n^M(s)} |Y_n(s, y)|^2 dy ds \right\|_{\frac{q}{2}}^q \right] \\ & \leq C \left[\int_0^t (t-s)^{\frac{d}{4r} - \frac{d}{2}} ds \right]^{\frac{q-2}{2}} \left[\int_0^t (t-s)^{\frac{d}{4r} - \frac{d}{2}} \sup_{r \in [0, s]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(r)} \|Y_n(r, \cdot)\|_p^q \right] ds \right]. \end{aligned}$$

As for $\Gamma_n^3(t, x)$, using (A.13) with $\frac{1}{r_2} = \frac{1}{q} - \frac{1}{\rho} + 1 \in [0, 1]$,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|\Gamma_n^3(t, \cdot)\|_p^q \right] \\ & \leq C \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left\| \int_0^t \int_D \Delta_y G(t-s, \cdot, y) [f(X_n(s, \cdot)) - f(X(s, \cdot))] dy ds \right\|_q^q \right] \\ & \leq C \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left[\int_0^t (t-s)^{\frac{d}{4r_2} - \frac{d+2}{4}} \|f(X_n(s, \cdot)) - f(X(s, \cdot))\|_\rho ds \right]^q \right]. \end{aligned} \quad (4.2)$$

Note that $u(s, \cdot), v(s, \cdot) \in L^p(D)$, for each $s \in [0, T]$, we have for $\rho = \frac{p}{3}$,

$$\begin{aligned} \|u(s, \cdot) - v(s, \cdot)\|_\rho & \leq C \|u(s, \cdot) - v(s, \cdot)\|_p, \\ \|u^2(s, \cdot) - v^2(s, \cdot)\|_\rho & \leq C \|u(s, \cdot) - v(s, \cdot)\|_p \left[\|u(s, \cdot)\|_p + \|v(s, \cdot)\|_p \right] \end{aligned}$$

$$\begin{aligned} \|u^3(s, \cdot) - v^3(s, \cdot)\|_p &\leq C \|u(s, \cdot) - v(s, \cdot)\|_p \\ &\quad \times \left[\|u(s, \cdot)\|_p^2 + \|v(s, \cdot)\|_p^2 + \|u(s, \cdot)\|_p \|v(s, \cdot)\|_p \right]. \end{aligned}$$

Hence from (4.2), it follows that

$$\begin{aligned} &\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|\Gamma_n^3(t, \cdot)\|_p^q \right] \\ &\leq C \mathbf{E} \left[\int_0^t (t-s)^{\frac{d}{4r_2} - \frac{d+2}{4}} \mathbf{1}_{\bar{A}_n^M(s)} \|Y_n(s, \cdot)\|_p ds \right]^q \\ &\leq C \left[\int_0^t (t-s)^{\frac{d}{4r_2} - \frac{d+2}{4}} ds \right]^{q-1} \left[\int_0^t (t-s)^{\frac{d}{4r_2} - \frac{d+2}{4}} \sup_{r \in [0, s]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(r)} \|Y_n(r, \cdot)\|_p^q \right] ds \right]. \end{aligned}$$

Note that the following equivalent relations holds:

$$\begin{aligned} \frac{d}{4r} - \frac{d}{2} + 1 > 0 &\Leftrightarrow \frac{1}{q} > \frac{1}{p} - \frac{1}{6}, \\ \frac{d}{4r_2} - \frac{d+2}{4} + 1 > 0 &\Leftrightarrow \frac{1}{q} > \frac{1}{p} + \frac{1}{2} - \frac{2}{d}. \end{aligned}$$

Then the desired result follows from the Gronwall's lemma. \square

Recall the \mathcal{F}_{t_n} -adapted process $X_n^-(t, x)$ defined by (3.3). Then we have

Lemma 4.2. *Let $q \geq p \geq 6$. Then there exists a constant $C := C_M > 0$ such that*

$$\sup_{(t, x) \in \mathcal{O}_T} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} |X_n(t, x) - X_n^-(t, x)|^q \right] \leq C 2^{-nqt}, \quad (4.3)$$

where $\iota := \frac{1}{2}(1 - \frac{d}{4})$.

Proof. Recall (2.2) and (3.3), and we get

$$X_n(t, x) - X_n^-(t, x) = \sum_{k=1}^4 T_n^k(t, x),$$

where

$$\begin{aligned} T_n^1(t, x) &:= \int_{t_n}^t \int_D G_{t-s}(x, y) F(X_n(s, y)) W(dy, ds), \\ T_n^2(t, x) &:= \int_{t_n}^t \int_D G_{t-s}(x, y) H(X_n(s, y)) W_n(dy, ds), \\ T_n^3(t, x) &:= \int_{t_n}^t \int_D \Delta_y G_{t-s}(x, y) f(X_n(s, y)) dy ds, \\ T_n^4(t, x) &:= \int_{t_n}^t \int_D G_{t-s}(x, y) K_n(s, y) dy ds, \end{aligned}$$

and

$$K_n(s, y) = K(X_n(s, y))\mathbf{h}(s, y) - (F\dot{H})(X_n(s, y))\alpha_n(s, y) - (H\dot{H})(X_n(s, y))\beta_n(s, y).$$

From (2.4) and (2.5), it follows that for $h \in \mathcal{A}_b$,

$$\sup_{(s, y) \in \mathcal{O}_T} |K_n(s, y)| \leq Cn^d. \quad (4.4)$$

On the other hand, using (A.4) and the boundedness of F ,

$$\begin{aligned} \mathbf{E} \left[|T_n^1(t, x)|^q \right] &= \mathbf{E} \left[\left| \int_{t_n}^t \int_D G_{t-s}(x, y) F(X_n(s, y)) W(dy, ds) \right|^q \right] \\ &\leq C \mathbf{E} \left[\left| \int_{t_n}^t \int_D G_{t-s}^2(x, y) F^2(X_n(s, y)) dy ds \right|^{q/2} \right] \\ &\leq C \left[\int_{t_n}^t \int_D G_{t-s}^2(x, y) dy ds \right]^{q/2} \\ &\leq C \left[\int_{t_n}^t (t-s)^{-\frac{d}{4}} ds \right]^{q/2} \\ &\leq C 2^{-\frac{1}{2}nq(1-\frac{d}{4})}. \end{aligned} \quad (4.5)$$

Further, the Hölder inequality, Lemma 2.1 and the boundedness of H jointly imply that

$$\begin{aligned} &\mathbf{E} \left[|T_n^2(t, x)|^q \right] \\ &= \mathbf{E} \left[\left| \int_{t_n}^t \int_D G_{t-s}(x, y) H(X_n(s, y)) W_n(dy, ds) \right|^q \right] \\ &\leq C \mathbf{E} \left[\left| \int_{t_n}^t \int_D |G_{t-s}(x, y)| |\dot{W}_n(y, s)| dy ds \right|^q \right] \\ &\leq C \mathbf{E} \left[\left[\int_{t_n}^t \int_D |G_{t-s}(x, y)| |\dot{W}_n(y, s)|^q dy ds \right] \cdot \left[\int_{t_n}^t \int_D |G_{t-s}(x, y)| dy ds \right]^{q-1} \right] \\ &\leq C \left[\int_{t_n}^t \int_D |G_{t-s}(x, y)| dy ds \right]^q \sup_{(t, x) \in \mathcal{O}_T} \mathbf{E} \left[|\dot{W}_n(x, t)|^q \right] \\ &\leq C n^{\frac{dq}{2}} 2^{\frac{nq}{2}} |t - t_n|^q \\ &\leq C n^{\frac{dq}{2}} 2^{-\frac{nq}{2}}. \end{aligned} \quad (4.6)$$

Note that f is a polynomial of degree 3. Then by virtue of (A.13) with $\kappa = \frac{1}{\infty} - \frac{3}{p} + 1 = 1 - \frac{3}{p} \in [0, 1]$,

$$\mathbf{E} \left[\mathbf{1}_{A_n^M(t)} |T_n^3(t, x)|^q \right]$$

$$\begin{aligned}
&= \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left| \int_{t_n}^t \int_D \Delta_y G_{t-s}(x, y) f(X_n(s, y)) dy ds \right|^q \right] \\
&\leq \mathbf{E} \left[\left[\int_{t_n}^t (t-s)^{\frac{d}{4}\kappa - \frac{d+2}{4}} \mathbf{1}_{\bar{A}_n^M(s)} \|f(X_n(s, \cdot))\|_{\frac{p}{3}} ds \right]^q \right] \\
&\leq \mathbf{E} \left[\left[\int_{t_n}^t (t-s)^{\frac{d}{4}(1-\frac{3}{p}) - \frac{d+2}{4}} \mathbf{1}_{\bar{A}_n^M(s)} \left[\|X_n(s, \cdot)\|_p + \|X_n(s, \cdot)\|_p^2 + \|X_n(s, \cdot)\|_p^3 \right] ds \right]^q \right] \\
&\leq C \left[\int_{t_n}^t (t-s)^{-\left(\frac{1}{2} + \frac{3d}{4p}\right)} ds \right]^q \\
&\leq C 2^{-nq\left(\frac{1}{2} - \frac{3d}{4p}\right)}. \tag{4.7}
\end{aligned}$$

Thanks to (4.4), we conclude that

$$\begin{aligned}
\mathbf{E} \left[|T_n^4(t, x)|^q \right] &= \mathbf{E} \left[\left| \int_{t_n}^t \int_D G_{t-s}(x, y) K_n(s, y) dy ds \right|^q \right] \\
&\leq C n^{dq} \left[\int_{t_n}^t \int_D |G_{t-s}(x, y)| dy ds \right]^q \\
&\leq C n^{dq} |t - t_n|^q \\
&\leq C n^{dq} 2^{-nq}. \tag{4.8}
\end{aligned}$$

Thus the estimate (4.3) follows from (4.5)–(4.8). \square

Remark 4.1. We can easily check that there exists a constant $C_M > 0$ such that for $q \geq p \geq 4$,

$$\sup_{t \in [0, T]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|X_n(t, \cdot) - X_n^-(t, \cdot)\|_p^q \right] \leq C_M 2^{-nq}, \tag{4.9}$$

where $\iota = \frac{1}{2}(1 - \frac{d}{4})$. In fact, for the proof of (4.9), the only estimate that needs to be checked is that of T_n^4 . Apply (A.13) with $\kappa = \frac{1}{p} - \frac{3}{p} + 1 = 1 - \frac{2}{p} \in [0, 1]$ to T_n^4 and we can get (4.9).

Next we prove a useful lemma, which will be used frequently later. For $(t, x), (s, y) \in \mathcal{O}_T$, set

$$[\eta(t, s, x, y)](\cdot, *) := \mathbf{1}_{[0, t]}(\cdot) G_{t-\cdot}(x, *) - \mathbf{1}_{[0, s]}(\cdot) G_{s-\cdot}(y, *), \tag{4.10}$$

$$[\tilde{\eta}(t, s, x, y)](\cdot, *) := \mathbf{1}_{[0, t]}(\cdot) \Delta G_{t-\cdot}(x, *) - \mathbf{1}_{[0, s]}(\cdot) \Delta G_{s-\cdot}(y, *). \tag{4.11}$$

Let V be an $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ -predictable process. For $0 \leq s \leq t \leq T$ and $x, y \in D$, define

$$\lambda_n^k(V)(t, s, x, y) := \int_{\frac{kT}{2^n} \wedge t}^{\frac{(k+1)T}{2^n} \wedge t} \int_D [\eta(t, s, x, y)](r, z) V(r, z) W_n(dz, dr), \tag{4.12}$$

with $k = 0, 1, \dots, 2^n - 1$. Then we have

Lemma 4.3. Let $V : \Omega \times \mathcal{O}_T \rightarrow \mathbf{R}$ be an \mathbb{F} -predictable process. If for each $p \in [1, \infty[$, there exist some constants $\tilde{\alpha} > 0$, $a > 0$ and $C > 0$ such that

$$\sup_{t \in [0, T]} \left\| \|V(t, \cdot)\|_{L^{r_1 p}(D)} \right\|_{L^{r_1 p}(\Omega)} \leq C 2^{-n\tilde{\alpha}} n^a, \quad \forall r_1 \geq 2. \quad (4.13)$$

Then for $\theta < 1 - \frac{d}{4}$, $\theta' < 4 - d$ and $\theta' \leq 2$, there exists $C > 0$ such that for $0 \leq s \leq t \leq T$ and $x, y \in D$,

$$\left\| \sum_{k=0}^{\lfloor 2^{n t/T} \rfloor - 1} |\lambda_n^k(V)(t, s, x, y)|^2 \right\|_{L^p(\Omega)} \leq C n^{(d+2a)} 2^{-n\alpha'} [|t-s|^\theta + |x-y|^{\theta'}], \quad (4.14)$$

where $\alpha' = 2\tilde{\alpha} - 2 + \frac{1}{p} + \frac{2}{\gamma\beta} > 0$ with $\gamma, \beta \geq 1$ such that $1 < \gamma\beta < 2$.

Proof. As in (4.10), define

$$\lambda_n^k(V)(t, x, y) := \int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}} \int_D [G_{t-r}(x, z) - G_{t-r}(y, z)] V(r, z) W_n(dz, dr). \quad (4.15)$$

Applying Hölder inequality thrice with the respective exponents $(2p\delta, \gamma)$, (δ, δ') and $(\beta, \frac{2p\delta'}{\gamma})$. Then by (4.13) and Lemma 2.1,

$$\begin{aligned} & \mathbf{E} \left[|\lambda_n^k(V)(t, x, y)|^{2p} \right] \\ &= \mathbf{E} \left[\left| \int_{\frac{T}{2^n} k}^{\frac{T}{2^n} (k+1)} \int_D [G_{t-r}(x, z) - G_{t-r}(y, z)] V(r, z) \dot{W}_n(z, r) dz dr \right|^{2p} \right] \\ &\leq \mathbf{E} \left[\left[\int_{\frac{T}{2^n} k}^{\frac{T}{2^n} (k+1)} \int_D |V(r, z)|^{2p\delta} dz dr \right]^{\frac{1}{\delta}} \right. \\ &\quad \times \left. \left[\int_{\frac{T}{2^n} k}^{\frac{T}{2^n} (k+1)} \int_D |\dot{W}_n(z, r)|^\gamma |G_{t-r}(x, z) - G_{t-r}(y, z)|^\gamma dz dr \right]^{\frac{2p}{\gamma}} \right] \\ &\leq \left[\mathbf{E} \int_{\frac{T}{2^n} k}^{\frac{T}{2^n} (k+1)} \int_D |V(r, z)|^{2p\delta} dz dr \right]^{\frac{1}{\delta}} \left[\mathbf{E} \left[\int_{\frac{T}{2^n} k}^{\frac{T}{2^n} (k+1)} \int_D |\dot{W}_n(z, r)|^\gamma \right. \right. \\ &\quad \times \left. \left. |G_{t-r}(x, z) - G_{t-r}(y, z)|^\gamma dz dr \right]^{\frac{2p\delta'}{\gamma}} \right]^{\frac{1}{\delta'}} \\ &\leq \left[\mathbf{E} \int_{\frac{T}{2^n} k}^{\frac{T}{2^n} (k+1)} \int_D |V(r, z)|^{2p\delta} dz dr \right]^{\frac{1}{\delta}} \left[\mathbf{E} \left[\int_{\frac{T}{2^n} k}^{\frac{T}{2^n} (k+1)} \int_D |\dot{W}_n(z, r)|^{2p\delta'} dz dr \right] \right. \\ &\quad \times \left. \left[\int_{\frac{T}{2^n} k}^{\frac{T}{2^n} (k+1)} \int_D |G_{t-r}(x, z) - G_{t-r}(y, z)|^{\gamma\beta} dz dr \right]^{\frac{2p\delta'}{\gamma\beta}} \right]^{\frac{1}{\delta'}} \end{aligned}$$

$$\leq Cn^{dp+2ap}2^{np-2np\hat{\alpha}-n} \left[\int_{\frac{T}{2^n}k}^{\frac{T}{2^n}(k+1)} \int_D |G_{t-r}(x,z) - G_{t-r}(y,z)|^{\gamma\beta} dz dr \right]^{\frac{2p}{\gamma\beta}}. \quad (4.16)$$

In light of (A.24) in Lemma A.4, there exists a $\theta' \in]0, 4-d[$ and $\theta' < 2$ such that

$$\left\| \sum_{k=0}^{\lfloor 2^n t/T \rfloor - 1} \left| \lambda_n^k(V)(t, x, y) \right|^2 \right\|_{L^p(\Omega)} \leq Cn^{d+2a}2^{-n(2\hat{\alpha}-2+\frac{2}{\gamma\beta}+\frac{1}{p})} |x-y|^{\theta'}. \quad (4.17)$$

Next we turn to the time increment. Set

$$\lambda_n^k(V)(t, s, x) := \int_{\frac{kT}{2^n} \wedge s}^{\frac{(k+1)T}{2^n} \wedge t} \int_D [G_{t-r}(x, z) - G_{s-r}(x, z)] V(r, z) W_n(dz, dr). \quad (4.18)$$

Then for $0 \leq s \leq t \leq T$ and $x \in D$,

$$\begin{aligned} & \left| \lambda_n^k(V)(t, s, x) \right| \\ & \leq \left| \int_{\frac{Tk}{2^n} \wedge s}^{\frac{T}{2^n}(k+1) \wedge t} \int_D [G_{t-r}(x, z) - G_{s-r}(x, z)] V(r, z) \dot{W}_n(z, r) dz dr \right| \\ & \quad + \left| \int_{\frac{Tk}{2^n} \wedge s}^{\frac{T}{2^n}(k+1) \wedge t} \int_D G_{t-r}(x, z) V(r, z) \dot{W}_n(z, r) dz dr \right| \\ & =: A_{1,n}^k(V)(t, s, x) + A_{2,n}^k(V)(t, s, x), \end{aligned} \quad (4.19)$$

with the definition

$$A_{2,n}^k(V)(t, s, x) = 0, \text{ whenever } [(k+1)T2^{-n} \wedge t] < [kT2^{-n} \vee s].$$

By the similar proof to that of the space increment, there exists a $\theta \in]0, 1 - \frac{d}{4}[$ such that

$$\left\| \sum_{k=0}^{\lfloor 2^n t/T \rfloor - 1} \left| A_{1,n}^k(V)(t, s, x) \right|^2 \right\|_{L^p(\Omega)} \leq Cn^{d+2a}2^{-n(2\hat{\alpha}-2+\frac{1}{p}+\frac{2}{\gamma\beta})} |t-s|^\theta. \quad (4.20)$$

Using the proof of (4.15), we have for each $k = 0, 1, \dots, \lfloor 2^n t/T \rfloor$,

$$\begin{aligned} & \mathbf{E} \left[|A_{2,n}^k(V)(t, s, x)|^{2p} \right] \\ & \leq Cn^{(d+2a)p}2^{-np(2\hat{\alpha}-1+\frac{1}{p})} \left[\int_{\frac{Tk}{2^n} \wedge s}^{\frac{T}{2^n}(k+1) \wedge t} \int_D |G_{t-r}(x, z)|^{\gamma\beta} dz dr \right]^{\frac{2p}{\gamma\beta}}. \end{aligned} \quad (4.21)$$

Then from (A.4) and the Hölder inequality, it follows that

$$\left\| \sum_{k=0}^{\lfloor 2^n t/T \rfloor - 1} \left| A_{2,n}^k(V)(t, s, x) \right|^2 \right\|_{L^p(\Omega)}$$

$$\begin{aligned}
&\leq Cn^{d+2a}2^{-n(2\hat{\alpha}-1+\frac{1}{p})} \left[\int_{\frac{Tk}{2^n} \vee s}^{\frac{T}{2^n}(k+1) \wedge t} (t-r)^{-\frac{d}{4}(\gamma\beta-1)} dr \right]^{\frac{2}{\gamma\beta}} \\
&\leq Cn^{d+2a}2^{-n(2\hat{\alpha}-1+\frac{1}{p})} 2^{-n(\frac{2}{\gamma\beta}-1)} \left[\int_s^t (t-r)^{-\frac{d}{4}(\gamma\beta-1)} dr \right] \\
&\leq Cn^{d+2a}2^{-n(2\hat{\alpha}-2+\frac{1}{p}+\frac{2}{\gamma\beta})} |t-s|^{-\frac{d}{4}(\gamma\beta-1)+1}. \tag{4.22}
\end{aligned}$$

Thus from (4.17), (4.20) and (4.22), we get (4.14). Hence the proof of the lemma is complete. \square

Remark 4.2. From the proof of Lemma 4.3, the conclusion of the lemma still holds if we replace $\lambda_n^k(V)(t, s, x, y)$ by $\mathbf{E} \left[\lambda_n^k(V)(t, s, x, y) | \mathcal{F}_{\frac{(k-1)T}{2^n}} \right]$.

Lemma 4.4. For $(s, y) \in \mathcal{O}_T$, it holds that

$$\begin{aligned}
&\mathbf{E} \left[\dot{W}_n(y, s) \int_{s_n}^s \int_D G_{s-r}(y, z) F(X_n^-(r, z)) W(dz, dr) | \mathcal{F}_{s_n} \right] \\
&= (T\pi^d)^{-1} n^d 2^n \int_{s_n}^{s_n} \int_{D_k(y)} G_{s-r}(y, z) F(X_n^-(r, z)) dz dr.
\end{aligned}$$

Proof. For $u \geq s_n$, set

$$\begin{aligned}
N_u(s, y) &:= (T\pi^d)^{-1} n^d 2^n W(D_k(y) \times]s_n, u]), \\
M_u(s, y) &:= \int_{s_n}^u \int_D G_{s-r}(y, z) F(X_n^-(r, z)) W(dz, dr).
\end{aligned}$$

Then the martingale property of $\{M_u(s, y); u \geq s_n\}$ and Itô formula jointly imply that

$$\begin{aligned}
&\mathbf{E} \left[\dot{W}_n(y, s) \int_{s_n}^s \int_D G_{s-r}(y, z) F(X_n^-(r, z)) W(dz, dr) | \mathcal{F}_{s_n} \right] \\
&= (T\pi^d)^{-1} n^d 2^n \mathbf{E} \left[N_{s_n}(s, y) M_s(s, y) | \mathcal{F}_{s_n} \right] \\
&= (T\pi^d)^{-1} n^d 2^n \mathbf{E} \left[\mathbf{E} \left[N_{s_n}(s, y) M_s(s, y) | \mathcal{F}_{s_n} \right] | \mathcal{F}_{s_n} \right] \\
&= (T\pi^d)^{-1} n^d 2^n \mathbf{E} \left[N_{s_n}(s, y) \mathbf{E} [M_s(s, y) | \mathcal{F}_{s_n}] | \mathcal{F}_{s_n} \right] \\
&= (T\pi^d)^{-1} n^d 2^n \mathbf{E} \left[N_{s_n}(s, y) M_{s_n}(s, y) | \mathcal{F}_{s_n} \right] \\
&= (T\pi^d)^{-1} n^d 2^n \int_{s_n}^{s_n} \int_{D_k(y)} G_{s-r}(y, z) F(X_n^-(r, z)) dz dr,
\end{aligned}$$

follows from the fact that $F(X_n^-(r, z))$ is \mathcal{F}_{s_n} -measurable when $r \leq s$. This proves the lemma. \square

The following lemma shows the local Hölder continuity of \mathbb{F} -adapted process $X_n(t, x)$ defined by (2.2). Recall the assumption (H2), in which the initial function ψ is ϱ -Hölder continuous ($\varrho \in]0, 1]$).

Lemma 4.5. Let $q \geq p \geq 6$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$ and $\bar{\beta} \in]0, \min\{2 - \frac{d}{2}, \varrho\}[$, there exists a $C > 0$ such that

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} |X_n(t, x) - X_n(s, y)|^q \right] \leq C n^{dq} \left[|t - s|^{\bar{\alpha}} + |x - y|^{\bar{\beta}} \right]^q,$$

with $(t, x), (s, y) \in \mathcal{O}_T$. In addition, if $q \geq p \geq 4$, then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$, there exists a $C > 0$ such that

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \|X_n(t, \cdot) - X_n(s, \cdot)\|_p^q \right] \leq C n^{dq} |t - s|^{\bar{\alpha}q}.$$

Proof. Recall (2.2), (4.11) and (4.12). We have for $(t, x), (s, y) \in \mathcal{O}_T$,

$$X_n(t, x) - X_n(s, y) = \sum_{i=1}^5 J_i(s, t, x, y),$$

where

$$\begin{aligned} J_1(t, s, x, y) &:= G_t * \psi(x) - G_s * \psi(y), \\ J_2(t, s, x, y) &:= \int_0^t \int_D [\eta(t, s, x, y)](r, z) \left[F(X_n(r, z)) W(dz, dr) \right. \\ &\quad \left. + H(X_n^-(r, z)) W_n(dz, dr) \right], \\ J_3(t, s, x, y) &:= \int_0^t \int_D [\eta(t, s, x, y)](r, z) \left[(H(X_n(r, z)) - H(X_n^-(r, z))) W_n(dz, dr) \right. \\ &\quad \left. - \dot{H}(X_n(r, z)) [\alpha_n(r, z) F(X_n(r, z)) + \beta_n(r, z) H(X_n(r, z))] dz dr \right], \\ J_4(t, s, x, y) &:= \int_0^t \int_D [\eta(t, s, x, y)](r, z) K(X_n(r, z)) \dot{h}(r, z) dz dr, \\ J_5(t, s, x, y) &:= \int_0^t \int_D [\tilde{\eta}(t, s, x, y)](r, z) f(X_n(r, z)) dz dr. \end{aligned}$$

Note that the initial function $\psi(x)$ is ϱ -Hölder continuous in $x \in \mathbf{R}^d$. Then by Lemma 2.2 in [6],

$$\mathbf{E} \left[|J_1(s, t, x, y)|^q \right] \leq C \left[|t - s|^{\varrho/4} + |x - y|^{\varrho} \right]^q. \quad (4.23)$$

Since F, H are bounded, by Burkholder's inequality and Lemma A.2, there exist $\gamma' \in]0, 4 - d[, \gamma' \leq 2$ and $\gamma'' \in]0, 1 - d/4[$ such that

$$\mathbf{E} \left[|J_2(t, s, x, y)|^q \right] \leq C \left[|x - y|^{\gamma'} + |t - s|^{\gamma''} \right]^{\frac{q}{2}}. \quad (4.24)$$

On the other hand, since K is bounded, the Schwarz's inequality yields that for $h \in \mathcal{H}_b$,

$$\mathbf{E} \left[|J_4(t, s, x, y)|^q \right] \leq C \left[|x - y|^{\gamma'} + |t - s|^{\gamma''} \right]^{\frac{q}{2}}, \quad (4.25)$$

where γ', γ'' are presented in (4.24). Similar as in the proof of (4.7), by (A.14) and (A.15) with $\frac{1}{r} = \frac{1}{\infty} - \frac{3}{p} + 1 \in [0, 1]$, we have for $\alpha \in]0, 1/2 - \frac{3d}{4p}[$ and $\beta \in]0, \min\{2 - \frac{3d}{p}, \frac{2}{d}, 1\}[$,

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} |J_5(t, s, x, y)|^q \right] \leq C \left[|x - y|^\beta + |t - s|^\alpha \right]^q. \quad (4.26)$$

and using (A.14) with $\kappa = \frac{1}{p} - \frac{3}{p} + 1 = 1 - \frac{2}{p} \in [0, 1]$, for $\alpha \in]0, 1/2 - \frac{d}{2p}[$,

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \|J_5(t, s, \cdot, \cdot)\|_p^q \right] \leq C [|t - s|^\alpha]^q. \quad (4.27)$$

In the following, we are going to estimate $J_3(t, s, x, y)$. Using the Taylor expansion of H at point $X_n^-(t, x)$,

$$H(X_n(t, x)) - H(X_n^-(t, x)) = \dot{H}(X_n^-(t, x)) [X_n(t, x) - X_n^-(t, x)] + \rho_n(t, x), \quad (4.28)$$

and

$$\rho_n(t, x) \leq C |X_n(t, x) - X_n^-(t, x)|^2.$$

Recall the above J_3 , and we have

$$J_3(t, s, x, y) = \sum_{i=1}^5 J_3^i(t, s, x, y),$$

where

$$\begin{aligned} J_3^1(t, s, x, y) &:= \int_0^t \int_D [\eta(t, s, x, y)](r, z) \left[T_n^1(r, z) \dot{H}(X_n^-(r, z)) \dot{W}_n(z, r) \right. \\ &\quad \left. - \dot{H}(X_n(r, z)) \alpha_n(r, z) F(X_n(r, z)) \right] dz dr, \\ J_3^2(t, s, x, y) &:= \int_0^t \int_D [\eta(t, s, x, y)](r, z) \left[T_n^2(r, z) \dot{H}(X_n^-(r, z)) \dot{W}_n(z, r) \right. \\ &\quad \left. - \dot{H}(X_n(r, z)) \beta_n(r, z) F(X_n(r, z)) \right] dz dr, \\ J_3^3(t, s, x, y) &:= \int_0^t \int_D [\eta(t, s, x, y)](r, z) \dot{H}(X_n^-(r, z)) T_n^3(r, z) \dot{W}_n(z, r) dz dr, \\ J_3^4(t, s, x, y) &:= \int_0^t \int_D [\eta(t, s, x, y)](r, z) \dot{H}(X_n^-(r, z)) T_n^4(r, z) \dot{W}_n(z, r) dz dr, \\ J_3^5(t, s, x, y) &:= \int_0^t \int_D [\eta(t, s, x, y)](r, z) \rho_n(r, z) \dot{W}_n(z, r) dz dr. \end{aligned}$$

Next we decompose $J_3^1(t, s, x, y)$ as follows:

$$J_3^1(t, s, x, y) = \sum_{j=1}^5 J_3^{1,j}(t, s, x, y),$$

with

$$\begin{aligned} J_3^{1,1}(t, s, x, y) &= \int_0^t \int_D [\eta(t, s, x, y)](r, z) \dot{H}(X_n^-(r, z)) \dot{W}_n(z, r) \\ &\quad \times \left[\int_{r_n}^r \int_D G_{r-u}(z, v) [F(X_n(u, v)) - F(X_n^-(u, v))] W(dv, du) \right] dz dr, \end{aligned}$$

$$\begin{aligned}
J_3^{1,2}(t,s,x,y) &= \int_0^t \int_D [\eta(t,s,x,y)](r,z) \dot{H}(X_n^-(r,z)) \\
&\quad \times \left[\int_{r_n}^r \int_D G_{r-u}(z,v) F(X_n^-(r,z)) W(dv,du) \dot{W}_n(z,r) \right. \\
&\quad \left. - \mathbf{E} \left[\int_{r_n}^r \int_D G_{r-u}(z,v) F(X_n^-(u,v)) W(dv,du) \dot{W}_n(z,r) \middle| \mathcal{F}_{r_n} \right] \right] dz dr \\
J_3^{1,3}(t,s,x,y) &= \int_0^t \int_D [\eta(t,s,x,y)](r,z) \dot{H}(X_n^-(r,z)) \\
&\quad \times \left[\mathbf{E} \left[\int_{r_n}^r \int_D G_{r-u}(z,v) F(X_n^-(u,v)) W(dv,du) \dot{W}_n(z,r) \middle| \mathcal{F}_{r_n} \right] \right. \\
&\quad \left. - 2^n n^{-d} T^{-1} \pi^{-d} \int_{r_n}^{r_n} \int_{D_k(z)} G_{r-u}(z,v) F(X_n^-(u,v)) dv du \right] dz dr, \\
J_3^{1,4}(t,s,x,y) &= 2^n T^{-1} n^{-d} \pi^{-d} \int_0^t \int_D [\eta(t,s,x,y)](r,z) \dot{H}(X_n^-(r,z)) \\
&\quad \times \left[\int_{r_n}^{r_n} \int_{D_k(z)} G_{r-u}(z,v) F(X_n^-(u,v)) dv du - \alpha_n(r,z) F(X_n(r,z)) \right] dz dr, \\
J_3^{1,5}(t,s,x,y) &= \int_0^t \int_D [\eta(t,s,x,y)](r,z) \left[\dot{H}(X_n^-(r,z)) - \dot{H}(X_n(r,x)) \right] \\
&\quad \times \alpha_n(r,z) F(X_n(r,z)) dz dr.
\end{aligned}$$

We first estimate the term $J_3^{1,1}$. Define

$$\begin{aligned}
V(r,z) &= \mathbf{1}_{A_n^M(r)} \dot{H}(X_n^-(r,z)) \\
&\quad \times \int_{r_n}^r \int_D G_{r-u}(z,v) \left[F(X_n(u,v)) - F(X_n^-(u,v)) \right] W(dv,du). \tag{4.29}
\end{aligned}$$

Then

$$\begin{aligned}
J_3^{1,1}(t,s,x,y) &= \int_0^t \int_D [\eta(t,s,x,y)](r,z) V(r,z) \dot{W}_n(dz,dr) \\
&= \sum_{k=0}^{\lfloor 2^n t/T \rfloor - 1} \lambda_n^k(V)(t,s,x,y), \tag{4.30}
\end{aligned}$$

where $\lambda_n^k(V)(t,s,x,y)$ is defined by (4.12). By virtue of the Burkholder inequality, (A.16) with $\kappa = \frac{2}{q} - \frac{2}{q} + 1 = 1$ and Lemma 4.2,

$$\begin{aligned}
\mathbf{E} \left[\|V(r,\cdot)\|_q^q \right] &\leq C 2^{-nq(1-\frac{d}{4})} \sup_{r \in [0,T]} \mathbf{E} \left[\mathbf{1}_{A_n^M(r)} \|X_n(r,\cdot) - X_n^-(r,\cdot)\|_q^q \right] \\
&\leq C 2^{-nq(1-\frac{d}{4}+\sigma)}. \tag{4.31}
\end{aligned}$$

Then using Lemma 4.3, there exist $\theta' < 4 - d$ and $\theta' \leq 2$, $\theta < 1 - \frac{d}{4}$ such that

$$\begin{aligned}
& \mathbf{E} \left[\mathbf{1}_{\bar{A}_M^n(t) \cap \bar{A}_M^n(s)} \left| J_3^{1,1}(t, s, x, y) \right|^q \right] \\
& \leq 2^{\frac{nq}{2}} \mathbf{E} \left[\mathbf{1}_{\bar{A}_M^n(t) \cap \bar{A}_M^n(s)} \left[\sum_{k=0}^{\lceil 2^n t/T \rceil - 1} |\lambda_n^k(t, s, x, y)|^2 \right]^{\frac{q}{2}} \right] \\
& \leq C n^{\frac{dq}{2}} 2^{-\frac{nq}{2}(1+2\sigma-\frac{d}{2}+\frac{1}{q}+\frac{2}{\gamma\beta})} [|x-y|^{\theta'} + |t-s|^\theta]^{\frac{q}{2}}.
\end{aligned} \tag{4.32}$$

Next we turn to estimate the term $J_3^{1,2}(t, s, x, y)$. If we set

$$V(r, z) = \mathbf{1}_{\bar{A}_n^M(r)} \dot{H}(X_n^-(r, z)) \int_{r_n}^r \int_D G_{r-u}(z, v) F(X_n^-(u, v)) W(dv, du), \tag{4.33}$$

then

$$J_3^{1,2}(t, s, x, y) = \sum_{k=0}^{\lceil 2^n t/T \rceil - 1} \left[\lambda_n^k(V)(t, s, x, y) - \mathbf{E} \left[\lambda_n^k(V)(t, s, x, y) \middle| \mathcal{F}_{\frac{(k-1)T}{2^n}} \right] \right]. \tag{4.34}$$

Applying the discrete Burkholder inequality and Jensen's inequality to conclude that

$$\begin{aligned}
& \mathbf{E} \left[|J_3^{1,2}(t, s, x, y)|^q \right] \\
& \leq C \mathbf{E} \left[\left| \sum_{k=0}^{\lceil 2^n t/T \rceil - 1} \left| \lambda_n^k(V)(t, s, x, y) - \mathbf{E} \left[\lambda_n^k(V)(t, s, x, y) \middle| \mathcal{F}_{\frac{(k-1)T}{2^n}} \right] \right| \right|^2 \right]^{\frac{q}{2}} \\
& \leq C \mathbf{E} \left[\sum_{k=0}^{\lceil 2^n t/T \rceil - 1} |\lambda_n^k(V)(t, s, x, y)|^2 \right]^{\frac{q}{2}} \\
& \quad + C \mathbf{E} \left[\sum_{k=0}^{\lceil 2^n t/T \rceil - 1} \left| \mathbf{E} \left[\lambda_n^k(V)(t, s, x, y) \middle| \mathcal{F}_{\frac{(k-1)T}{2^n}} \right] \right|^2 \right]^{\frac{q}{2}}.
\end{aligned} \tag{4.35}$$

According to a similar proof to that of (4.31), for $V(r, z)$ defined by (4.33), one gets

$$\sup_{r \in [0, T]} \mathbf{E} \left[\|V(r, \cdot)\|_q^q \right] \leq C 2^{-nq(1-d/4)}.$$

Also using Lemma 4.3,

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_M^n(t) \cap \bar{A}_M^n(s)} \left| J_3^{1,2}(t, s, x, y) \right|^q \right] \leq C n^{\frac{dq}{2}} 2^{-\frac{nq\delta'}{2}} [|x-y|^{\theta'} + |t-s|^\theta]^{\frac{q}{2}}, \tag{4.36}$$

where $\delta' = -\frac{d}{2} + \frac{1}{q} + \frac{2}{\gamma\beta} > 0$. On the other hand, Lemma 4.4 yields that

$$\mathbf{E} \left[\dot{W}_n(z, r) \int_{r_n}^r \int_D G_{r-u}(z, v) F(X_n^-(u, v)) W(dv, du) \middle| \mathcal{F}_{r_n} \right]$$

$$= 2^n T^{-1} n^d \pi^{-d} \int_{r_n}^{r_n} \int_{D_k(z)} G_{r-u}(z, v) F(X_n^-(u, v)) dv du.$$

This implies that

$$J_3^{1,3}(t, s, x, y) \equiv 0.$$

Since \dot{H} and F are bounded, from (2.4), (A.4) and (A.7)–(A.9), it follows that

$$\begin{aligned} \left| J_3^{1,4}(t, s, x, y) \right|^q &\leq C n^{dq} \left[\int_0^t \int_D |\eta(t, s, x, y)(r, z)| dz dr \right]^q \\ &\quad \times \left[2^n \sup_{(r, z) \in \mathcal{O}_T} \int_{r_n}^{r_n} \int_{D_k(z)} |G_{r-u}(z, v)| dv du \right]^q \\ &\leq C n^{dq} \left[|t-s|^{\gamma''} + |x-y|^{\gamma'} \right]^q, \end{aligned} \quad (4.37)$$

where $0 < \gamma' < 2 - \frac{d}{2}$ and $0 < \gamma'' < \frac{1}{2}(1 - \frac{d}{4})$. From (2.4), Lemma 4.2 and (A.11)–(A.12) with $\kappa = \frac{1}{\infty} - \frac{1}{p} + 1 = 1 \in]0, 1[$, we conclude that there exist $\alpha \in]0, 1 - \frac{d}{4p}[$ and $\beta \in]0, 1[$ such that

$$\begin{aligned} &\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| J_3^{1,5}(t, s, x, y) \right|^q \right] \\ &\leq C n^{dq} \left[|t-s|^\alpha + |x-y|^\beta \right]^q \sup_{r \in [0, T]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(r)} \|X_n(r, \cdot) - X_n^-(r, \cdot)\|_q^q \right] \\ &\leq C n^{dq} 2^{-nq\sigma} \left[|t-s|^\alpha + |x-y|^\beta \right]^q. \end{aligned} \quad (4.38)$$

From the above estimations, it follows that there exist $\bar{\alpha} \in]0, \min\{\frac{1}{2} - \frac{3d}{4p}, \frac{1}{2}(1 - \frac{d}{4}), \frac{\rho}{4}\}[$ and $\bar{\beta} \in]0, \min\{2 - \frac{3d}{p}, 2 - \frac{d}{2}, \frac{2}{d}, \rho\}[$ such that for $(t, x), (s, y) \in \mathcal{O}_T$,

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| J_3^1(t, s, x, y) \right|^q \right] \leq C n^{dq} \left[|t-s|^{\bar{\alpha}} + |x-y|^{\bar{\beta}} \right]^q. \quad (4.39)$$

Now we turn to estimate the term $J_3^2(t, s, x, y)$. The procedure is similar to that of $J_3^1(t, s, x, y)$. Replace $F(X_n(r, z))$ by $H(X_n(r, z))$, $F(X_n^-(r, z))$ by $H(X_n^-(r, z))$, $W(dz, dr)$ by $W_n(dz, dr)$, and $\alpha_n(r, z, X_n(r, z))$ by $\beta_n(r, z, X_n(r, z))$, respectively. Then there exist $\bar{\alpha}, \bar{\beta}$ presented in (4.39) such that

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| J_3^2(t, s, x, y) \right|^q \right] \leq C n^q \left[|t-s|^{\bar{\alpha}} + |x-y|^{\bar{\beta}} \right]^q. \quad (4.40)$$

As for the term $J_3^3(t, s, x, y)$, we have

$$\begin{aligned} &\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| J_3^3(t, s, x, y) \right|^q \right] \\ &= \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) \dot{H}(X_n^-(r, z)) T_n^3(r, z) \dot{W}_n(z, r) dz dr \right|^q \right] \\ &= B_n^1(t, s, x, y) + B_n^2(t, s, x, y), \end{aligned} \quad (4.41)$$

where

$$B_n^1(t, s, x, y) = \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) \dot{H}(X_n^-(r, z)) \dot{W}_n(z, r) \right|^q \right]$$

$$\begin{aligned}
& \times \left[\int_{r_n}^r \int_D \Delta_\nu G_{r-u}(z, \nu) [f(X_n(u, \nu)) - f(X_n^-(u, \nu))] d\nu du \right] dz dr \Big|^q, \\
B_n^2(t, s, x, y) &= \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) \dot{H}(X_n^-(r, z)) \dot{W}_n(z, r) \right. \right. \\
& \times \left. \left. \left[\int_{r_n}^r \int_D \Delta_\nu G_{r-u}(z, \nu) f(X_n^-(u, \nu)) d\nu du \right] dz dr \right|^q \right].
\end{aligned}$$

By virtue of (A.11) and (A.12) with $\kappa = \frac{1}{\infty} - \frac{1}{p} + 1 = 1 - \frac{1}{p} \in [0, 1]$, (A.13) with $\kappa_1 = \frac{1}{p} - \frac{1}{p} + 1 = 1$ and (4.9), we have for $\alpha \in]0, 1 - \frac{d}{4p}[$ and $\beta \in]0, 1[$,

$$\begin{aligned}
& B_n^1(t, s, x, y) \\
& \leq Cn^{dq} 2^{\frac{1}{2}nq} [|t-s|^\alpha + |x-y|^\beta]^q \\
& \times \mathbf{E} \left[\left\| \int_{r_n}^r \int_D |\Delta_\nu G_{r-u}(\cdot, \nu)| \mathbf{1}_{\bar{A}_n^M(u)} |X_n(u, \nu) - X_n^-(u, \nu)| d\nu du \right\|_q^p \right] \\
& \leq Cn^{dq} 2^{\frac{1}{2}nq} [|t-s|^\alpha + |x-y|^\beta]^q 2^{-\frac{1}{2}nq} \sup_{r \in [0, T]} \mathbf{E} \left[\|X_n(r, \cdot) - X_n^-(r, \cdot)\|_p^q \right] \\
& \leq Cn^{dq} 2^{-nqu} [|t-s|^\alpha + |x-y|^\beta]^q. \tag{4.42}
\end{aligned}$$

For $r \leq t$, set

$$V(r, z) := \int_{r_n}^r \int_D \Delta_\nu G_{r-u}(z, \nu) \mathbf{1}_{\bar{A}_n^M(u)} f(X_n^-(u, \nu)) d\nu du. \tag{4.43}$$

Using the B-D-G inequality

$$B_n^2(t, s, x, y) \leq \mathbf{E} \left[\sum_{k=0}^{\lfloor 2^n t/T \rfloor - 1} |\lambda_n^k(V)(t, s, x, y)|^2 \right]^{\frac{q}{2}}. \tag{4.44}$$

Since $X_n^-(u, \nu) = G_{u-u_n}(\nu, X_n(u_n, \cdot))$, it is obvious that

$$\|X_n^-(u, \cdot)\|_q \leq C \|X_n(u, \cdot)\|_q.$$

By virtue of (4.7),

$$\sup_{r \in [0, T]} \mathbf{E} \left[\|V(r, \cdot)\|_q^q \right] \leq C 2^{-nq(\frac{1}{2} - \frac{3d}{4q})}. \tag{4.45}$$

Again by Lemma 4.3, there exist $\theta' < 4 - d$ and $\theta' \leq 2$, $\theta < 1 - \frac{d}{4}$ such that

$$B_n^2(t, s, x, y) \leq Cn^{\frac{1}{2}dq} 2^{-nq\alpha'} [|t-s|^\theta + |x-y|^{\theta'}]^q, \tag{4.46}$$

where $\alpha' = -1 - \frac{3d}{2q} + \frac{1}{q} + \frac{2}{\gamma\beta}$. So that for $\alpha \in]0, \frac{1}{2}(1 - \frac{d}{4})[$ and $\beta \in]0, (2 - d/2) \wedge 1[$,

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} |J_3^3(t, s, x, y)|^q \right] \leq Cn^{dq} 2^{-nq\delta'} [|t-s|^\alpha + |x-y|^\beta]^q. \tag{4.47}$$

with $\delta' = \iota \wedge \frac{\alpha'}{2}$. Finally we consider $J_3^4(t, s, x, y)$ and $J_3^5(t, s, x, y)$. From (A.11) and (A.12) with $\kappa := \frac{1}{\infty} - \frac{1}{p} + 1 \in [0, 1]$ and (4.8), it follows that for $\alpha \in]0, 1 - \frac{d}{4p}[$ and $\beta \in]0, 1[$,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} |J_3^4(t, s, x, y)|^q \right] \\ & \leq Cn^{dq} 2^{\frac{1}{2}nq} \left[|t-s|^\alpha + |x-y|^\beta \right]^q \sup_{r \in [0, T]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(r)} \|T_n^4(r, \cdot)\|_p^q \right] \\ & \leq Cn^{2dq} 2^{-\frac{1}{2}nq} \left[|t-s|^\alpha + |x-y|^\beta \right]^q. \end{aligned} \quad (4.48)$$

As for $J_3^5(t, s, x, y)$, using (A.11) and (A.12) with $\kappa := \frac{1}{\infty} - \frac{2}{p} + 1 := 1 - \frac{2}{p} \in [0, 1]$ and Lemma 4.2, we get for $\alpha \in]0, 1 - \frac{d}{2p}[$ and $\beta \in]0, \min\{4 - \frac{2d}{p}, 1\}[$,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} |J_3^5(t, s, x, y)|^q \right] \\ & = \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) \rho_n(r, z) \dot{W}_n(z, r) dz dr \right|^q \right] \\ & \leq \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) |X_n(r, z) - X_n^-(r, z)|^2 \dot{W}_n(z, r) dz dr \right|^q \right] \\ & \leq Cn^{\frac{dq}{2}} 2^{(\frac{1}{2} - \frac{d}{4})nq} \left[|t-s|^\alpha + |x-y|^\beta \right]^q. \end{aligned}$$

When $d = 3$, we can obtain a more precise estimate than the cases of $d = 1, 2$, which will be concluded in Lemma 6.1 of Section 6. Thus we complete the proof of the lemma. \square

5 The proof of (C1)

The condition (C1) presented by Section 3 shall be verified in this section. By Lemma 4.1, to check (C1), we only need to prove the right hand side of (4.1) approaches 0, when $n \rightarrow \infty$. Note that for each fixed $n \in \mathbf{N}$,

$$\left\{ \frac{\mathbf{1}_{\Delta_{j,k}}(x, t)}{\sqrt{|\Delta_{j,k}|}}; j = 0, 1, \dots, 2^n - 1, \mathbf{k} \in \mathbf{I}_n^d \right\}$$

forms a CONS of $L^2([0, T] \times D)$. Let π_n be the orthogonal projection of above basis and for any mapping $g : \mathbf{R} \rightarrow \mathbf{R}$, define

$$\tau_n g(s) = g\left((s + T2^{-n}) \wedge T\right), \quad s \in [0, T].$$

Then for each $t \in [0, T]$ and \mathbb{F} -predictable process $(\psi(t, x); x \in D)_{0 \leq t \leq T}$,

$$\int_0^t \int_D \psi(s, y) W_n(dy, ds) = \int_0^T \int_D \pi_n \left[\tau_n \left(\mathbf{1}_{[0, t]}(\cdot) \psi(\cdot, *) \right) \right] (s, y) W(dy, ds).$$

Recall $\Lambda_n(t, x)$ defined by (3.2) and we have

$$\Lambda_n(t, x) = \sum_{i=1}^3 \tilde{\Lambda}_n^i(t, x),$$

where

$$\begin{aligned} \tilde{\Lambda}_n^1(t, x) &:= \int_0^t \int_D G_{t-s}(x, y) H(X_n^-(s, y)) [W_n(dy, ds) - W(dy, ds)], \\ \tilde{\Lambda}_n^2(t, x) &:= \int_0^t \int_D G_{t-s}(x, y) [H(X_n^-(s, y)) - H(X_n(s, y))] W(dy, ds), \\ \tilde{\Lambda}_n^3(t, x) &:= \int_0^t \int_D G_{t-s}(x, y) [H(X_n(s, y)) - H(X_n^-(s, y))] W_n(dy, ds) \\ &\quad - \int_0^t \int_D G_{t-s}(x, y) \dot{H}(X_n(s, y)) \\ &\quad \times [\alpha_n(s, y) F(X_n(s, y)) + \beta_n(s, y) H(X_n(s, y))] dy ds. \end{aligned}$$

We begin with the estimation of the term $\tilde{\Lambda}_n^1(t, x)$. Note that

$$\begin{aligned} \tilde{\Lambda}_n^1(t, x) &= \int_0^t \int_D \left[\pi_n \left[\tau_n \left(\mathbf{1}_{[0,t]}(\cdot) G_{t-\cdot}(x, *) H(X_n^-(\cdot, *)) \right) \right] (s, y) \right. \\ &\quad \left. - \pi_n \left[\mathbf{1}_{[0,t]}(\cdot) G_{t-\cdot}(x, *) H(X_n^-(\cdot, *)) \right] (s, y) \right] W(dy, ds), \\ &\quad + \int_0^t \int_D \left[\pi_n \left[\mathbf{1}_{[0,t]}(\cdot) G_{t-\cdot}(x, *) H(X_n^-(\cdot, *)) \right] (s, y) \right. \\ &\quad \left. - \mathbf{1}_{[0,t]}(s) G_{t-s}(x, y) H(X_n^-(s, y)) \right] W(dy, ds) \\ &=: \tilde{\Lambda}_n^{1,1}(t, x) + \tilde{\Lambda}_n^{1,2}(t, x). \end{aligned}$$

Then by the Hölder inequality and Burkholder's inequality, and note that π_n is an orthogonal projection of $L^2([0, t] \times D)$, we have for $q \geq p$,

$$\begin{aligned} &\mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \tilde{\Lambda}_n^{1,1}(t, \cdot) \right\|_p^q \right] \\ &\leq C \int_D \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left| \int_0^T \int_D \left[\tau_n \left[\mathbf{1}_{[0,t]}(s) G_{t-s}(x, y) H(X_n^-(s, y)) \right] \right. \right. \right. \\ &\quad \left. \left. - \mathbf{1}_{[0,t]}(s) G_{t-s}(x, y) H(X_n^-(s, y)) \right] W(dy, ds) \right|^q \right] dx \\ &\leq C \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \int_0^T \int_D \left[\tau_n \left[\mathbf{1}_{[0,t]}(s) G_{t-s}(\bullet, y) H(X_n^-(s, y)) \right] \right. \right. \right. \right. \\ &\quad \left. \left. - \mathbf{1}_{[0,t]}(s) G_{t-s}(\bullet, y) H(X_n^-(s, y)) \right]^2 dy ds \right\|_{\frac{q}{2}}^{\frac{q}{2}} \right] \\ &= C \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \int_0^T \int_D \left[\mathbf{1}_{[0,t]}(s + T2^{-n}) G_{t-(s+T2^{-n})}(\bullet, y) H(X_n^-(s + T2^{-n}, y)) \right. \right. \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\mathbf{1}_{[0,t]}(s)G_{t-s}(\bullet, y)H(X_n^-(s, y)) \Big] dy ds \Big\|_{\frac{q}{2}}^2 \Big] \\
\leq & \mathbf{CE} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \int_0^{(t-T2^{-n})^+} \int_D [G_{t-(s+T2^{-n})}(\bullet, y) - G_{t-s}(\bullet, y)]^2 \right. \right. \\
& \times H^2(X_n^-(s+T2^{-n}, y)) dy ds \Big\|_{\frac{q}{2}}^2 \Big] \\
& + \mathbf{CE} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \int_0^{(t-T2^{-n})^+} \int_D |G_{t-s}(\bullet, y)|^2 \right. \right. \\
& \times [H(X_n^-(s+T2^{-n}, y)) - H(X_n^-(s, y))]^2 dy ds \Big\|_{\frac{q}{2}}^2 \Big] \\
& + \mathbf{CE} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \int_{(t-T2^{-n})^+}^t \int_D G_{t-s}^2(\bullet, y) H^2(X_n^-(s, y)) dy ds \right\|_{\frac{q}{2}}^2 \right] \\
\leq & \mathbf{CE} \left[\left\| \int_0^{(t-T2^{-n})^+} \int_D [G_{t-(s+T2^{-n})}(\bullet, y) - G_{t-s}(\bullet, y)]^2 dy ds \right\|_{\frac{q}{2}}^2 \right] \\
& + \mathbf{CE} \left[\left\| \int_0^{(t-T2^{-n})^+} \int_D |G_{t-s}(\bullet, y)|^2 \right. \right. \\
& \times \mathbf{1}_{\tilde{A}_n^M(s)} [X_n^-(s+T2^{-n}, y) - X_n^-(s, y)]^2 dy ds \Big\|_{\frac{q}{2}}^2 \Big] \\
& + \mathbf{CE} \left[\left\| \int_{(t-T2^{-n})^+}^t \int_D G_{t-s}^2(\bullet, y) H^2(X_n^-(s, y)) dy ds \right\|_{\frac{q}{2}}^2 \right] \\
= &: C \sum_{i=1}^3 \tilde{\Lambda}_n^{1,1,i}(t).
\end{aligned}$$

Take the boundedness of the mapping H into account, from (A.8) and (A.9) in Lemma A.2, it follows that for $\gamma'' < 1 - \frac{d}{4}$,

$$\sup_{t \in [0, T]} [|\tilde{\Lambda}_n^{1,1,1}(t)| + |\tilde{\Lambda}_n^{1,1,3}(t)|] \leq C 2^{-\frac{1}{2}\gamma'' n q}. \quad (5.1)$$

On the other hand, applying (A.16) with $\kappa = \frac{2}{q} - \frac{2}{q} + 1 = 1$, the Hölder inequality, Lemma 4.2, and Lemma 6.1 (in Section 6) to conclude that

$$\begin{aligned}
|\tilde{\Lambda}_n^{1,1,2}(t)| & \leq C \sup_{s \in [0, T]} \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(s) \cap \tilde{A}_n^M(s+T/2^n)} \left\| X_n^-(s+T2^{-n}, \cdot) - X_n^-(s, \cdot) \right\|_q^q \right] \\
& \leq C \sup_{(s,y) \in \mathcal{O}_T} \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(s+T/2^n)} \left| X_n^-(s+T2^{-n}, y) - X_n(s+T2^{-n}, y) \right|^q \right] \\
& \quad + C \sup_{(s,y) \in \mathcal{O}_T} \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(s) \cap \tilde{A}_n^M(s+T/2^n)} \left| X_n(s+T2^{-n}, y) - X_n(s, y) \right|^q \right]
\end{aligned}$$

$$\begin{aligned}
& +C \sup_{(s,y) \in \mathcal{O}_T} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(s)} |X_n(s,y) - X_n^-(s,y)|^q \right] \\
& \leq C \left[2^{-\iota nq} + 2^{-\xi nq} + 2^{-\iota nq} \right].
\end{aligned} \tag{5.2}$$

Therefore for $q \geq p > \frac{3d}{2}$,

$$\sup_{t \in [0, T]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|\tilde{\Lambda}_n^{1,1}(t, \cdot)\|_p^q \right] \rightarrow 0, \quad n \rightarrow \infty. \tag{5.3}$$

As for the term $\tilde{\Lambda}_n^{1,2}(t, x)$, note that

$$\begin{aligned}
& \int_0^t \int_D \pi_n \left[\mathbf{1}_{[0,t]}(\cdot) G_{t-\cdot}(x, *) H(X_n^-(\cdot, *)) \right] (s, y) W(dy, ds) \\
& = \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k} \in \mathbf{I}_n^d} \int_0^t \int_D \frac{\mathbf{1}_{\Delta_{j,\mathbf{k}}}(y, s)}{|\Delta_{j,\mathbf{k}}|} \left[\int_{\Delta_{j,\mathbf{k}}} G_{t-r}(x, z) H(X_n^-(r, z)) dz dr \right] W(dy, ds).
\end{aligned} \tag{5.4}$$

Then the B-D-G inequality yields that for $q \geq p$,

$$\begin{aligned}
& \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \|\tilde{\Lambda}_n^{1,2}(t, \cdot)\|_p^q \right] \\
& \leq C \int_D \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left\| \int_0^T \int_D \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k} \in \mathbf{I}_n^d} \frac{\mathbf{1}_{\Delta_{j,\mathbf{k}}}(y, s)}{|\Delta_{j,\mathbf{k}}|} \right. \right. \\
& \quad \times \left. \left. \left[\int_{\Delta_{j,\mathbf{k}}} [G_{t-r}(x, z) H(X_n^-(r, z)) - G_{t-s}(x, y) H(X_n^-(s, y))] dz dr \right] \right. \right. \\
& \quad \left. \left. \times W(dy, ds) \right\|_p^q dx \right] \\
& \leq C \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left\| \int_0^T \int_D \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k} \in \mathbf{I}_n^d} \frac{\mathbf{1}_{\Delta_{j,\mathbf{k}}}(y, s)}{|\Delta_{j,\mathbf{k}}|^2} \right. \right. \\
& \quad \times \left. \left. \left[\int_{\Delta_{j,\mathbf{k}}} [G_{t-r}(\bullet, z) - G_{t-s}(\bullet, y)] H(X_n^-(r, z)) dz dr \right]^2 dy ds \right\|_{\frac{q}{2}}^{\frac{q}{2}} \right. \\
& \quad \left. + C \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left\| \int_0^T \int_D \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k} \in \mathbf{I}_n^d} \frac{\mathbf{1}_{\Delta_{j,\mathbf{k}}}(y, s)}{|\Delta_{j,\mathbf{k}}|^2} \right. \right. \right. \\
& \quad \times \left. \left. \left[\int_{\Delta_{j,\mathbf{k}}} G_{t-s}(\bullet, y) [H(X_n^-(r, z)) - H(X_n^-(s, y))] dz dr \right]^2 dy ds \right\|_{\frac{q}{2}}^{\frac{q}{2}} \right. \right. \\
& \quad \left. \left. =: C [H_n^1(t) + H_n^2(t)] \right].
\end{aligned}$$

By the boundedness of H and Dini's theorem, we have for $t \in [0, T]$,

$$0 \leq H_n^1(t) \leq C \left\| \int_0^T \int_D \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k} \in \mathbf{I}_n^d} \frac{\mathbf{1}_{\Delta_{j,\mathbf{k}}}(y, s)}{|\Delta_{j,\mathbf{k}}|^2} \right.$$

$$\begin{aligned}
& \times \left[\int_{\Delta_{j,k}} [G_{t-r}(\bullet, z) - G_{t-s}(\bullet, y)] dz dr \right]^2 dy ds \Bigg\|_{\frac{q}{2}}^{\frac{q}{2}} \\
& = C \left\| \int_0^t \int_D |\pi_n [G_{t-\cdot}(\bullet, *)](s, y) - G_{t-s}(\bullet, y)|^2 dy ds \right\|_{\frac{q}{2}}^{\frac{q}{2}} \\
& \leq C \sup_{x \in D} \left\| \int_0^t \int_D |\pi_n [G_{t-\cdot}(x, *)](s, y) - G_{t-s}(x, y)|^2 dy ds \right\|_{\frac{q}{2}}^{\frac{q}{2}} \\
& \rightarrow 0, \quad n \rightarrow \infty,
\end{aligned}$$

follows from the fact that when $n \rightarrow \infty$, it holds that

$$\left\| \pi_n [G_{t-\cdot}(x, *)] - G_{t-\cdot}(x, *) \right\|_{L^2([0, T] \times D)} \rightarrow 0,$$

for all $(t, x) \in \mathcal{O}_T$ (a compact set in \mathbf{R}^{d+1}). Because H is Lipschitzian continuous, using the Hölder inequality

$$\begin{aligned}
H_n^2(t) & \leq \mathbf{CE} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left\| \int_0^T \int_D \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k} \in \mathbb{I}_n^d} \frac{\mathbf{1}_{\Delta_{j,k}}(y, s)}{|\Delta_{j,k}|^2} \left[\int_{\Delta_{j,k}} G_{t-s}^2(\bullet, y) dz dr \right] \right. \right. \\
& \quad \times \left. \left. \int_{\Delta_{j,k}} |H(X_n^-(r, z)) - H(X_n^-(s, y))|^2 dz dr \right] dy ds \right\|_{\frac{q}{2}}^{\frac{q}{2}} \Bigg] \\
& \leq \mathbf{CE} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left\| \int_0^T \int_D \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k} \in \mathbb{I}_n^d} \frac{\mathbf{1}_{\Delta_{j,k}}(y, s)}{|\Delta_{j,k}|^2} \left[\int_{\Delta_{j,k}} G_{t-s}^2(\bullet, y) dz dr \right] \right. \right. \\
& \quad \times \left. \left. \int_{\Delta_{j,k}} |H(X_n^-(r, z)) - H(X_n^-(s, y))|^2 dz dr \right] dy ds \right\|_{\frac{q}{2}}^{\frac{q}{2}} \Bigg] \\
& \leq C \sup_{(s, y) \in \mathcal{O}_T} \sum_{j=0}^{[2^n t]} \sum_{\mathbf{k} \in \mathbb{I}_n^d} \sup_{\mathbf{z}, r \in \Delta_{j,k}} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(r) \cap \bar{A}_n^M(s)} |X_n^-(r, z) - X_n^-(s, y)|^q \right] \\
& \quad \times \mathbf{1}_{\Delta_{j,k}}(y, s). \tag{5.5}
\end{aligned}$$

Note that

$$\begin{aligned}
|X_n^-(r, z) - X_n^-(s, y)| & \leq |X_n^-(r, z) - X_n(r, z)| + |X_n(r, z) - X_n(s, y)| \\
& \quad + |X_n(s, y) - X_n^-(s, y)|.
\end{aligned}$$

Then from Lemma 4.2, and Lemma 6.1 (in Section 6), it follows that

$$H_n^2(t) \leq C \left[2^{-\iota n q} + 2^{-\xi n q} + 2^{-\iota n q} \right] \leq C 2^{-\lambda n q}, \tag{5.6}$$

where $\lambda = \min\{\iota, \xi\}$. This further implies that for $q \geq p > \frac{3d}{2}$,

$$\sup_{t \in [0, T]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t)} \left\| \tilde{\Lambda}_n^{1,2}(t, \cdot) \right\|_p^q \right] \rightarrow 0, \quad n \rightarrow \infty,$$

and hence for $q \geq p > \frac{3d}{2}$,

$$\sup_{t \in [0, T]} \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \tilde{\Lambda}_n^1(t, \cdot) \right\|_p^q \right] \rightarrow 0, \quad n \rightarrow \infty. \quad (5.7)$$

Finally, we turn to the estimation of the term $\tilde{\Lambda}_n^2(t, x)$. In light of the B-D-G inequality and (A.16) with $\kappa = \frac{2}{p} - \frac{2}{p} + 1 = 1$,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \tilde{\Lambda}_n^2(t, \cdot) \right\|_p^q \right] \\ & \leq \mathbf{C} \mathbf{E} \left\| \mathbf{1}_{\tilde{A}_n^M(t)} \left[\int_0^t \int_D G_{t-s}^2(\cdot, y) |H(X_n(s, y)) - H(X_n^-(s, y))|^2 dy ds \right] \right\|_{\frac{p}{2}}^{\frac{q}{2}} \\ & \leq \mathbf{C} \mathbf{E} \left[\int_0^t (t-s)^{-\frac{d}{4}} \|X_n(s, \cdot) - X_n^-(s, \cdot)\|_q^p ds \right] \\ & \leq C 2^{-tnq}. \end{aligned} \quad (5.8)$$

Observe that

$$\tilde{\Lambda}_n^3(t, x) = J_3(t, 0, x, 0), \quad (5.9)$$

for the term $J_3(t, s, x, y)$ defined in Lemma 4.5. Then there exists a $\lambda > 0$ such that for $q \geq p > \frac{3d}{2}$,

$$\sup_{t \in [0, T]} \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t)} \left\| \tilde{\Lambda}_n^3(t, \cdot) \right\|_p^q \right] \leq C 2^{-\lambda nq}. \quad (5.10)$$

Thus we prove that the condition (C1) holds. \square

6 The proof of (C2)

The aim of this section is to check the validity of the condition (C2) presented in Section 3. Note that for all $s < t \in [0, T]$, we have for $q \geq p > \frac{3d}{2}$,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t) \cap \tilde{A}_n^M(s)} \left\| Y_n(t, \cdot) - Y_n(s, \cdot) \right\|_p^q \right] \\ & \leq \mathbf{C} \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t) \cap \tilde{A}_n^M(s)} \left\| X_n(t, \cdot) - X_n(s, \cdot) \right\|_p^q \right] + \mathbf{C} \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t) \cap \tilde{A}_n^M(s)} \left\| X(t, \cdot) - X(s, \cdot) \right\|_p^q \right]. \end{aligned}$$

Observe the forms of Equations (2.2) and (2.3), X is a particular case of X_n . Hence in order to prove that (C2) holds, it suffices to check that for all $0 \leq s < t \leq T$, there exist $q \geq p$ and $\theta > q\bar{\alpha}$ ($\bar{\alpha}$ is presented in Theorem 1.1) such that for each $n \in \mathbf{N}$,

$$(C2)' \quad \mathbf{E} \left[\mathbf{1}_{\tilde{A}_n^M(t) \cap \tilde{A}_n^M(s)} \left\| X_n(t, \cdot) - X_n(s, \cdot) \right\|_p^q \right] \leq C |t - s|^{1+\theta}.$$

From (2.2), it follows that for $(s, y), (t, x) \in \mathcal{O}_T$,

$$X_n(t, x) - X_n(s, y)$$

$$\begin{aligned}
&= G_t * \psi(x) - G_s * \psi(y) \\
&+ \int_0^T \int_D [\eta(s, t, x, y)](r, z) [F(X_n(r, z))W(dz, dr) + H(X_n(r, z))W_n(dz, dr)] \\
&+ \int_0^T \int_D [\eta(s, t, x, y)](r, z) \left[K(X_n(r, z))\dot{h}(r, z) \right. \\
&\quad \left. - [\alpha_n(r, z)(F\dot{H})(X_n(r, z)) + \beta_n(r, z)(H\dot{H})(X_n(r, z))] \right] dz dr \\
&+ \int_0^T \int_D [\tilde{\eta}(s, t, x, y)](r, z) f(X_n(r, z)) dz dr. \tag{6.1}
\end{aligned}$$

To prove (C2)', we will sharp the estimations in Lemma 4.5 by the following lemma.

Lemma 6.1. *It holds that*

(i) *Let $q \geq p > 6$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$ and $\bar{\beta} \in]0, \min\{2 - \frac{d}{2}, \varrho\}[$, there exists a constant $C > 0$ such that*

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(s) \cap \bar{A}_n^M(t)} |X_n(t, x) - X_n(s, y)| \right]^q \leq C n^{dq} 2^{-nq\xi} \left[|t - s|^{\bar{\alpha}} + |x - y|^{\bar{\beta}} \right]^q, \tag{6.2}$$

for some $\xi \in]0, \min\{\frac{1}{2}, \sigma, \frac{1}{2}(-1 - \frac{3d}{2q} + \frac{1}{q} + \frac{2}{\gamma\beta})[$ with σ, γ, β defined in Lemma 4.2 and Lemma 4.3, respectively.

(ii) *Let $q \geq p \geq 4$. Then for $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{\varrho}{4}\}[$ there exists a $C > 0$ such that*

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(s) \cap \bar{A}_n^M(t)} \|X_n(t, \cdot) - X_n(s, \cdot)\|_p^q \right] \leq C n^{dq} 2^{-nq\xi} |t - s|^{\bar{\alpha}q}. \tag{6.3}$$

where ξ is the same as in (i).

Proof. Recall the entire proof of Lemma 4.5. We only need to re-estimate $J_3^1(t, s, x, y)$, $J_3^2(t, s, x, y)$ and $J_3^5(t, s, x, y)$ given in Lemma 4.5. To improve the estimation of $J_3^1(t, s, x, y)$, we only need to estimate the term $J_3^{1,4}(t, s, x, y)$. Note that, for $(s, y), (t, x) \in \mathcal{O}_T$,

$$\begin{aligned}
&J_3^{1,4}(t, s, x, y) \\
&\leq C 2^{\frac{n}{2}d} \int_0^t \int_D [\eta(s, t, x, y)](r, z) \dot{H}(X_n^-(r, z)) \\
&\quad \times \left[\int_{r_n}^{r_n} \int_{D_k(z)} G_{r-u}(z, v) [F(X_n^-(u, v)) - F(X_n(u, v))] dv du \right] dz dr \\
&+ C 2^{\frac{n}{2}d} \int_0^t \int_D [\eta(s, t, x, y)](r, z) \dot{H}(X_n^-(r, z)) \\
&\quad \times \left[\int_{r_n}^{r_n} \int_{D_k(z)} G_{r-u}(z, v) [F(X_n(u, v)) - F(X_n(r, z))] dv du \right] dz dr \\
&=: K_n^1(t, s, x, y) + K_n^2(t, s, x, y). \tag{6.4}
\end{aligned}$$

Using a similar argument to that of (4.42), we get for $\alpha, \beta \in (0, 1)$,

$$\mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| K_n^1(t, s, x, y) \right|^q \right] \leq C n^{dq} 2^{-nq(\frac{1}{2} + t)} \left[|t - s|^\alpha + |x - y|^\beta \right]^q. \quad (6.5)$$

In order to estimate $K_n^2(t, s, x, y)$, we apply the Hölder inequality w.r.t the measures dy and $G_{r-u}(z, v)dvdu$, respectively. Then (A.11), (A.12) and Lemma 4.5 jointly imply that, for $\alpha, \beta \in]0, 1[$,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| K_n^2(t, s, x, y) \right|^q \right] \\ & \leq C 2^{\frac{nq}{2}} n^{dq} \left[|t - s|^\alpha + |x - y|^\beta \right]^q \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \int_0^t \left\| \int_{r_n}^{r_n} \int_{D_k(\cdot)} G_{r-u}(\cdot, v) \right. \right. \\ & \quad \left. \left. \times [F(X_n(u, v)) - F(X_n(r, \cdot))] dvdu \right\|_q^q dr \right] \\ & \leq C 2^{\frac{nq}{2}} n^{2dq} 2^{-(q-1)n} \left[|t - s|^\alpha + |x - y|^\beta \right]^q \\ & \quad \times \left[\int_0^t \left[\int_{r_n}^{r_n} \int_{D_k(z)} G_{r-u}(z, v) \left[|r - u|^{\bar{\alpha}} + |z - v|^{\bar{\beta}} \right]^q dvdu \right] dr \right] \\ & \leq C 2^{n-nq/2} n^{2dq} \left[|t - s|^\alpha + |x - y|^\beta \right]^q \\ & \quad \times \left[\int_0^t \left[\int_{r_n}^{r_n} \int_{\mathbf{R}^d} \left[|r - u|^{-\frac{d}{4}} |v|^{\bar{\beta}q} \exp \left(-C \frac{|v|^{4/3}}{|r - u|^{1/3}} \right) \right. \right. \right. \\ & \quad \left. \left. \left. + 2^{-n\bar{\alpha}q} G_{r-u}(z, v) \right] dvdu \right] dr \right] \\ & \leq C 2^{n-nq/2} n^{2dq} \left[|t - s|^\alpha + |x - y|^\beta \right]^q \int_0^t \left[\int_{r_n}^{r_n} |r - u|^{\frac{\bar{\beta}q}{4}} du + 2^{-n-\bar{\alpha}qn} \right] dr \\ & \leq C 2^{-nq(\frac{1}{2} + \alpha')} n^{2dq} \left[|t - s|^\alpha + |x - y|^\beta \right]^q, \end{aligned} \quad (6.6)$$

where $\alpha' = \min\{\bar{\alpha}, \frac{\bar{\beta}}{4}\}$. The estimation of $J_3^2(t, s, x, y)$ is similar to that of $J_3^1(t, s, x, y)$.

Finally, we improve the estimation of the term $J_3^5(t, s, x, y)$. Using the decomposition of $X_n(r, z) - X_n^-(r, z)$,

$$\begin{aligned} & \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| J_3^5(t, s, x, y) \right|^q \right] \\ & \leq \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) |X_n(r, z) - X_n^-(r, z)|^2 \dot{W}_n(z, r) dz dr \right|^q \right] \\ & := \sum_{i=1}^4 \sum_{j=1}^4 R_n^{i,j}(t, s, x, y), \end{aligned} \quad (6.7)$$

where

$$\begin{aligned} & R_n^{i,j}(t, s, x, y) \\ & = \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) T_n^i(r, z) T_n^j(r, z) \dot{W}_n(z, r) dz dr \right|^q \right]. \end{aligned}$$

Using (A.11) and (A.12) with $\kappa := \frac{1}{\infty} - \frac{2}{p} + 1 = 1 - \frac{2}{p} \in [0, 1]$, we get for $\alpha \in]0, 1 - \frac{d}{2p}[$, $\beta \in]0, \min\{4 - \frac{2d}{p}, 1\}[$, and $\gamma > \max\left\{\frac{1}{1 + \frac{d}{4}(\kappa-1) - \alpha}, \frac{1}{1 + \frac{d}{4}(\kappa-1) - \frac{\beta}{4}}, \frac{1}{1 - \frac{d}{4}\beta}\right\}$,

$$\begin{aligned} R_n^{i,j}(t, s, x, y) &\leq C n^{\frac{dq}{2}} 2^{\frac{nq}{2}} [|t-s|^\alpha + |x-y|^\beta]^q \sup_{r \in [0, t]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \|T_n^i(r, \cdot)\|_p^{q\gamma} \right]^{\frac{1}{\gamma}} \\ &\quad \times \sup_{r \in [0, t]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \|T_n^j(r, \cdot)\|_p^{q\gamma} \right]^{\frac{1}{\gamma}}. \end{aligned}$$

Thanks to (4.5)–(4.8), we deduce that there exists $\delta > 0$ such that for all (i, j) with either i or j not belonging to $\{1, 4\}$,

$$R_n^{i,j}(t, s, x, y) \leq C 2^{-nq\delta} [|t-s|^\alpha + |x-y|^\beta]^q. \quad (6.8)$$

Next we improve the estimate of $R_n^{i,j}$ when $i, j \in \{1, 4\}$. We first deal with $R_n^{1,4}$. By the Hölder inequality with $\frac{1}{2} + \frac{1}{2} = 1$, for $p > 6$,

$$\begin{aligned} R_n^{i,j}(t, s, x, y) &\leq C n^{\frac{dq}{2}} 2^{\frac{nq}{2}} [|t-s|^\alpha + |x-y|^\beta]^q \sup_{r \in [0, t]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \|T_n^i(r, \cdot)\|_{2p}^{q\gamma} \right]^{\frac{1}{\gamma}} \\ &\quad \times \sup_{r \in [0, t]} \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \|T_n^j(r, \cdot)\|_{2p}^{q\gamma} \right]^{\frac{1}{\gamma}} \\ &\leq C 2^{-nq(\frac{1}{2} - \frac{3d}{4p})} [|t-s|^\alpha + |x-y|^\beta]^q. \end{aligned} \quad (6.9)$$

The estimation of the term $R_n^{4,4}$ is similar to that of $R_n^{1,4}$. To improve the estimation of $R_n^{1,1}(t, s, x, y)$, we introduce the process

$$T_n^{1-}(r, z) = \int_{r_n}^r \int_D G_{r-u}(z, v) F(X_n^-(u, v)) W(dv, du). \quad (6.10)$$

Then

$$\begin{aligned} &R_n^{1,1}(t, s, x, y) \\ &\leq \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) \left[(T_n^1(r, z))^2 - (T_n^{1-}(r, z))^2 \right] \dot{W}_n(z, r) dz dr \right|^q \right] \\ &\quad + \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) (T_n^{1-}(r, z))^2 \dot{W}_n(z, r) dz dr \right|^q \right] \\ &:= R_n^{1,1,1}(t, s, x, y) + R_n^{1,1,2}(t, s, x, y). \end{aligned} \quad (6.11)$$

Similarly to the proof of the term $J^{1,1}(t, s, x, y)$ in (4.32), we conclude that there exists a $\delta' := 2 - d + \frac{1}{q} + \frac{2}{\gamma\beta} > 0$ with $\gamma\beta \in]1, 2[$ such that

$$R_n^{1,1,1}(t, s, x, y) \leq C 2^{-\frac{1}{2}nq\delta'} [|t-s|^\alpha + |x-y|^\beta]^q. \quad (6.12)$$

As for the term $R_n^{1,1,2}(t, s, x, y)$, we have

$$\begin{aligned}
& R_n^{1,1,2}(t, s, x, y) \\
\leq & \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) \right. \right. \\
& \times \left. \left. [(T_n^{1-}(r, z))^2 \dot{W}_n(z, r) - \mathbf{E}[(T_n^{1-}(r, z))^2 \dot{W}_n(z, r) | \mathcal{F}_{r_n}]] dz dr \right|^q \right] \\
& + \mathbf{E} \left[\mathbf{1}_{\bar{A}_n^M(t) \cap \bar{A}_n^M(s)} \left| \int_0^t \int_D [\eta(t, s, x, y)](r, z) \mathbf{E}[(T_n^{1-}(r, z))^2 \dot{W}_n(z, r) | \mathcal{F}_{r_n}] dz dr \right|^q \right] \\
:= & Z_n^1(t, s, x, y) + Z_n^2(t, s, x, y). \tag{6.13}
\end{aligned}$$

Similarly to the proof of $J_3^{1,2}(t, s, x, y)$ in (4.36), there exists a $\delta := -\frac{d}{2} + \frac{1}{q} + \frac{1}{\alpha\beta} > 0$ with $\alpha\beta \in]1, 2[$ such that

$$Z_n^1(t, s, x, y) \leq C 2^{-\frac{1}{2}nq\delta} [|t-s|^\alpha + |x-y|^\beta]^q. \tag{6.14}$$

Since $\dot{W}_n(z, r)$ is \mathcal{F}_{r_n} -measurable and we have

$$\begin{aligned}
& \mathbf{E} \left[\dot{W}_n(z, r) \left| \int_{r_n}^r \int_D G_{r-u}(z, v) F(X_n^-(u, v)) W(dv, du) \right|^2 \middle| \mathcal{F}_{r_n} \right] \\
= & \mathbf{E} \left[\dot{W}_n(z, r) \mathbf{E} \left[\int_{r_n}^r \int_D G_{r-u}^2(z, v) F^2(X_n^-(u, v)) dv du \middle| \mathcal{F}_{r_n} \right] \middle| \mathcal{F}_{r_n} \right] = 0,
\end{aligned}$$

and

$$\begin{aligned}
& \mathbf{E} \left[\dot{W}_n(z, r) \left[\int_{r_n}^r \int_D G_{r-u}(z, v) F(X_n^-(u, v)) W(dv, du) \right] \right. \\
& \quad \times \left. \left[\int_{r_n}^{r_n} \int_D G_{r-u}(z, v) F(X_n^-(u, v)) W(dv, du) \right] \middle| \mathcal{F}_{r_n} \right] = 0.
\end{aligned}$$

Applying Itô rule to conclude that

$$\begin{aligned}
& \mathbf{E} \left[(T_n^{1-}(r, z))^2 \dot{W}_n(z, r) \middle| \mathcal{F}_{r_n} \right] \\
\leq & C n^{\frac{1}{2}d} 2^{\frac{1}{2}n} \mathbf{E} \left[\int_{r_n}^{r_n} \left[\int_{r_n}^w \int_D G_{r-u}(z, v) F(X_n^-(u, v)) W(dv, du) \right] \right. \\
& \times \left. \left[\int_{D_{\mathbf{k}}(z)} G_{r-u}(z, v) F(X_n^-(u, v)) dv \right] dw \middle| \mathcal{F}_{r_n} \right] \\
& + C n^{\frac{1}{2}d} 2^{\frac{1}{2}n} \mathbf{E} \left[\int_{r_n}^{r_n} W(D_{\mathbf{k}}(z) \times (r_n, w)) \right. \\
& \times \left. \left[\int_D G_{r-u}^2(z, v) F^2(X_n^-(u, v)) dv \right] dw \middle| \mathcal{F}_{r_n} \right]
\end{aligned}$$

$$:= Z_n^{2,1}(t, s, x, y) + Z_n^{2,2}(t, s, x, y). \quad (6.15)$$

However the Fubini's theorem yields that

$$\begin{aligned} & \mathbf{E} \left[\left| Z_n^{2,1}(t, s, x, y) \right|^q \right] \\ & \leq \mathbf{E} \left[\left| \int_{r_n}^{t_n} \left[\int_{r_n}^w \int_D G_{r-u}(z, v) F(X_n^-(u, v)) W(dv, du) \right] dw \right|^q \right] \\ & \leq C 2^{-nq(\frac{3}{2} - \frac{d}{8})}. \end{aligned} \quad (6.16)$$

Note that the integral $\int_D G_{r-u}^2(z, v) F^2(X_n^-(u, v)) dv$ is \mathcal{F}_{s_n} -measurable. Hence

$$Z_n^{2,2}(t, s, x, y) = 0.$$

From (A.11) and (A.12) with $\kappa := \frac{1}{\infty} - \frac{1}{p} + 1$, it follows that for $\alpha \in]0, 1 - \frac{d}{2p}[$ and $\beta \in]0, \min\{4 - \frac{2d}{p}, 1\}[$,

$$Z_n^2(t, s, x, y) \leq C 2^{-nq(1 - \frac{d}{8})} [|t - s|^\alpha + |x - y|^\beta]^q. \quad (6.17)$$

Using the estimations (6.11)–(6.14) and (6.17), we conclude that there exists a $\delta > 0$ such that

$$R_n^{1,1}(t, s, x, y) \leq C 2^{-nq\delta} [|t - s|^\alpha + |x - y|^\beta]^q. \quad (6.18)$$

Thus we complete the proof of the lemma. \square

Now we prove the skeleton process $S(h)$ to Equation (1.3) is continuous for all $h \in \mathcal{H}$.

Lemma 6.2. *Let $\alpha \in]0, 1[$ and $q \geq 4$. Then for each $a \geq 0$, there exists a $C > 0$ such that when $\|h_1\|_{\mathcal{H}} \vee \|h_2\|_{\mathcal{H}} \leq a$,*

$$\|S(h_1) - S(h_2)\|_{\alpha, q} \leq C \|h_1 - h_2\|_{\mathcal{H}}.$$

Proof. A similar proof to that of Theorem 1.4 in [6] yields that, for $q \geq 4$,

$$\sup_{h \in \{h \in \mathcal{H}; \|h\|_{\mathcal{H}} \leq a\}} \sup_{t \in [0, T]} \|S(h)(t, \cdot)\|_q = C_a < \infty.$$

Recall the skeleton equation (1.3) and we have

$$\begin{aligned} & S(h_2)(t, x) - S(h_1)(t, x) \\ & = \int_0^t \int_D \Delta G_{t-s}(x, y) [f(S(h_2)(s, y)) - f(S(h_1)(s, y))] dy ds \\ & \quad + \int_0^t \int_D G_{t-s}(x, y) \sigma(S(h_2)(s, y)) [\mathbf{h}_2(s, y) - \mathbf{h}_1(s, y)] dy ds \\ & \quad + \int_0^t \int_D G_{t-s}(x, y) [\sigma(S(h_2)(s, y)) - \sigma(S(h_1)(s, y))] \mathbf{h}_1(s, y) dy ds \end{aligned}$$

$$=: \sum_{i=1}^3 R_i(t, x)$$

Note that $u(s, \cdot), v(s, \cdot) \in L^q(D)$ for each $s \in [0, T]$, we have for $\rho = q/3$,

$$\begin{aligned} \|u(s, \cdot) - v(s, \cdot)\|_\rho &\leq C \|u(s, \cdot) - v(s, \cdot)\|_q, \\ \|u^2(s, \cdot) - v^2(s, \cdot)\|_\rho &\leq C \|u(s, \cdot) - v(s, \cdot)\|_q [\|u(s, \cdot)\|_q + \|v(s, \cdot)\|_q] \\ \|u^3(s, \cdot) - v^3(s, \cdot)\|_\rho &\leq C \|u(s, \cdot) - v(s, \cdot)\|_q \\ &\quad \times [\|u(s, \cdot)\|_q^2 + \|v(s, \cdot)\|_q^2 + \|u(s, \cdot)\|_q \|v(s, \cdot)\|_q]. \end{aligned}$$

Then by virtue of (A.13) with $\kappa = \frac{1}{q} - \frac{3}{q} + 1 = 1 - \frac{2}{q} \in [0, 1]$,

$$\|R_1(t, \cdot)\|_q \leq \int_0^t (t-s)^{-\frac{1}{2} - \frac{d}{2q}} \|S(h_2)(s, y) - S(h_1)(s, y)\|_q ds.$$

On the other hand, under (H1), the Schwarz's inequality implies that

$$\|R_2(t, \cdot)\|_q \leq C \|h_2 - h_1\|_{\mathcal{H}} \left\| \int_0^t \int_D [G_{t-s}(\cdot, y)]^2 dy ds \right\|_q \leq C \|h_2 - h_1\|_{\mathcal{H}}.$$

Again by the Schwarz's inequality and (A.10) with $\kappa = \frac{1}{q} - \frac{1}{q} + 1 = 1 \in [0, 1]$,

$$\|R_3(t, \cdot)\|_q \leq C \int_0^t (t-s)^{-\frac{d}{4}} \|S(h_2)(s, y) - S(h_1)(s, y)\|_q ds.$$

Hence the Gronwall's lemma yields that

$$\|S(h_2) - S(h_1)\|_q \leq C \|h_2 - h_1\|_{\mathcal{H}}.$$

While for $(t, x) \in \mathcal{O}_T$ and $t' \in [0, T]$,

$$\begin{aligned} &S(h_2)(t, x) - S(h_1)(t, x) - S(h_2)(t', x) + S(h_1)(t', x) \\ &= \int_0^t \int_D [\tilde{\eta}(t, t', x, x)](s, y) [f(S(h_2)(s, y)) - f(S(h_1)(s, y))] dy ds \\ &\quad + \int_0^{t'} \int_D [\eta(t, t', x, x)](s, y) \sigma(S(h_2)(s, y)) [\mathbf{h}_2(s, y) - \mathbf{h}_1(s, y)] dy ds \\ &\quad + \int_0^{t'} \int_D [\eta(t, t', x, x)](s, y) [\sigma(S(h_2)(s, y)) - \sigma(S(h_1)(s, y))] \mathbf{h}_1(s, y) dy ds, \end{aligned}$$

and then by (A.11) and (A.14) with $\rho = q$ and $\rho = \frac{q}{2}$, we get for $\alpha \in]0, \frac{1}{2}[$,

$$\|S(h_2)(t, \cdot) - S(h_1)(t, \cdot) - S(h_2)(t', \cdot) + S(h_1)(t', \cdot)\|_{\alpha, q} \leq C \|h_2 - h_1\|_{\mathcal{H}} |t - t'|^\alpha,$$

where the Hölder norm $\|\cdot\|_{\alpha, q} := \|\cdot\|_{\alpha, q, \infty}$, which is defined in Section 3. Thus we complete the proof of the lemma. \square

Now we are at the position to prove Theorem 1.1.

Proof of Theorem 1.1. We adopt the method used in Theorem 2.1 of [1]. We only provide a sketch for this proof. For parsimony, we only prove the part (b) of Theorem 1.1, since the proof of the part (a) is similar. Given $h \in \mathcal{H}_b$, let X_n^h be the solution to Equation (2.2) with $h = F = K = 0$ and $H = \sigma$. Define $H_n : \Omega \rightarrow \mathcal{H}_b$ by

$$\dot{H}_n(s, y) = \dot{W}_n(y, s) - \dot{\sigma}(X_n(s, y))\beta_n(s, y).$$

By virtue of the uniqueness of the solution to (2.2), we have $X_n = S(H_n)$. Moreover, Proposition 3.1 yields that $X_n \rightarrow u$ in probability in $C^{\bar{\alpha}, 0}([0, T], L^p(D))$. Using Proposition 2.1-(i) in [13], it holds that

$$\text{supp}(\mathbf{P} \circ u^{-1}) \subset \overline{S(\mathcal{H}_b)}.$$

On the other hand, we fix $h \in \mathcal{H}_b$ and let X_n be the solution to (2.2) with $F = \sigma$, $H = -\sigma$ and $K = \sigma$. Set

$$\dot{K}_n(s, y) := \dot{h}(s, y) - \dot{\sigma}(X_n(s, y))(\alpha_n(s, y) - \beta_n(s, y)),$$

where α_n, β_n are defined in (2.3). Let $\Gamma_n^h : \Omega \rightarrow \Omega$ and

$$\Gamma_n^h(\omega) = \omega - \omega_n + K_n(\omega).$$

Then the Girsanov's theorem implies that $\mathbf{P} \circ (\Gamma_n^h)^{-1} \ll \mathbf{P}$. Note that $X_n = u \circ \Gamma_n^h$ and $X = S(h)$ and then Proposition 3.1 concludes that $u \circ \Gamma_n^h$ converges in probability to $S(h)$ in $C^{\bar{\alpha}, 0}([0, T]; L^p(D))$. By Proposition 2.2-(ii) in [13],

$$\overline{S(\mathcal{H}_b)} \subset \text{supp}(\mathbf{P} \circ u^{-1}).$$

Next we have to check that $\overline{S(\mathcal{H}_b)} = \overline{S(\mathcal{H})}$. Since \mathcal{H}_b is dense in \mathcal{H} , it suffices to check that for any $0 < M < \infty$ and $\bar{\alpha} \in]0, \min\{\frac{1}{2}(1 - \frac{d}{4}), \frac{d}{4}\}[$, there exists a $C > 0$ such that

$$\|S(h_2) - S(h_1)\|_{\bar{\alpha}, p} \leq C \|h_1 - h_2\|_{\mathcal{H}}, \quad (6.19)$$

when $\|h_1\|_{\mathcal{H}} \vee \|h_2\|_{\mathcal{H}} < M$. Hence the desired result follows from Lemma 6.2. Thus we finish the entire proof of Theorem 1.1. \square

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Appendix

In this section, we present some new estimates on the Green kernel $G_t(\cdot, *)$ corresponding to the operator $\partial/\partial t + \Delta^2$ with the homogeneous Neumann's boundary condition in (1.1). As in Da Prato and Debussche [8], the Green kernel admits the following expansion.

Let $A = -\Delta$ be defined on $\mathcal{D}(A) = \{u \in H^2(D); \partial u / \partial \mathbf{n} = 0, \text{ on } \partial D\}$ and $\{\Theta_k\}_{k \in \mathbb{N}^d}$ be the basis of eigenfunctions of A in $L^2(D)$, which can be written as

$$\Theta_k(x) = \prod_{i=1}^d \theta_{k_i}(x_i), \quad (\text{A.1})$$

where $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, $x = (x_1, \dots, x_d) \in D$ and

$$\begin{cases} \theta_{k_i}(x_i) = \sqrt{\frac{2}{\pi}} \cos(k_i x_i), & k_i \neq 0, \\ \theta_0(x_i) = \frac{1}{\sqrt{\pi}}, & k_i = 0, \end{cases} \quad (\text{A.2})$$

for $i = 1, \dots, d$. Here $\{\lambda_k = \sum_{i=1}^d k_i^2\}_{k \in \mathbb{N}^d}$ are the eigenvalues corresponding to the eigenfunctions. Hence the Green kernel G on $[0, \infty) \times D$ can be expressed as

$$G_t(x, y) = \sum_{k \in \mathbb{N}^d} e^{-\lambda_k t} \Theta_k(x) \Theta_k(y), \quad (\text{A.3})$$

with $(t, x, y) \in [0, \infty) \times D^2$.

The following estimates concerning the Green kernel G are quoted from [6].

Lemma A.1 *There exist $C, K > 0$ such that for $t \in]0, T]$ and $x, y \in D$,*

$$|G_t(x, y)| \leq \frac{K}{t^{\frac{d}{4}}} \exp\left(-C \frac{|x-y|^{\frac{4}{3}}}{|t|^{\frac{1}{3}}}\right), \quad (\text{A.4})$$

$$|\Delta_y G_t(x, y)| \leq \frac{K}{t^{\frac{d+2}{4}}} \exp\left(-C \frac{|x-y|^{\frac{4}{3}}}{|t|^{\frac{1}{3}}}\right), \quad (\text{A.5})$$

$$\left| \frac{\partial G_t(x, y)}{\partial t} \right| \leq \frac{K}{t^{\frac{d+4}{4}}} \exp\left(-C \frac{|x-y|^{\frac{4}{3}}}{|t|^{\frac{1}{3}}}\right). \quad (\text{A.6})$$

Lemma A.2 *For $\gamma' < 4-d$ and $\gamma' \leq 2$, $\gamma'' < 1 - \frac{d}{4}$, there exists $C > 0$ such that for $0 \leq s < t \leq T$ and $x, z \in D$,*

$$\int_0^t \int_D |G_{t-u}(x, y) - G_{t-u}(z, y)|^2 dy du \leq C|x-z|^{\gamma'}, \quad (\text{A.7})$$

$$\int_0^s \int_D |G_{t-u}(x, y) - G_{s-u}(x, y)|^2 dy du \leq C|t-s|^{\gamma''}, \quad (\text{A.8})$$

$$\int_s^t \int_D |G_{t-u}(x, y)|^2 dy du \leq C|t-s|^{\gamma''}. \quad (\text{A.9})$$

The proofs of the following lemma are similar to that of Lemma 3.1 in Gyöngy [11], which further improve the estimates in [6].

Lemma A.3 *Assume that $\rho \in [1, \infty[$, $p \in [\rho, \infty[$, $\gamma > 1$ and $\kappa := \frac{1}{p} - \frac{1}{\rho} + 1 \in [0, 1]$. For $v \in L^\gamma([0, T], L^p(D))$, $0 \leq t_0 \leq t \leq T$ and $x \in D$, define*

$$J(v)(t_0, t, x) := \int_{t_0}^t \int_D H(t-r, x, y) v(r, y) dy dr.$$

Then J is a bounded operator from $L^\gamma([0, T], L^p(D))$ into $L^\infty([0, T], L^p(D))$ such that the following conclusions hold.

(a) *Let $H(t, x, y) = G_t(x, y)$. Then there exists $C > 0$ such that*

(1) For all $\gamma > \frac{1}{1+\frac{d}{4}(\kappa-1)}$,

$$\begin{aligned} \|J(v)(t_0, t, *)\|_p &\leq C \int_{t_0}^t (t-r)^{\frac{d}{4}(\kappa-1)} \|V(r, *)\|_\rho dr \\ &\leq C t^{1+\frac{d}{4}(\kappa-1)-\frac{1}{\gamma}} \|v(\cdot, *)\|_{L^\gamma([t_0, T], L^\rho(D))}. \end{aligned} \quad (\text{A.10})$$

(2) Let $\alpha \in]0, 1 + \frac{d}{4}(\kappa-1)[$. For all $\gamma > \frac{1}{1+\frac{d}{4}(\kappa-1)-\alpha}$,

$$\|J(v)(0, t, *) - J(v)(0, s, *)\|_p \leq C |t-s|^\alpha \|v(\cdot, *)\|_{L^\gamma([0, T], L^\rho(D))}, \quad (\text{A.11})$$

with $0 \leq s \leq t \leq T$.

(3) Let $\beta \in]0, \min\{4 - (1-\kappa)d, 1\}[$. For all $\gamma > \max\left\{\frac{1}{1+\frac{d}{4}(\kappa-1)-\frac{\beta}{4}}, \frac{1}{1-\frac{d}{4}\beta}\right\}$,

$$\|J(v)(0, t, *) - J(v)(0, t, *+z)\|_p \leq C |z|^\beta \|v(\cdot, *)\|_{L^\gamma([0, T], L^\rho(D))}, \quad (\text{A.12})$$

with $t \in [0, T]$, where we set $J(v)(t, y) := 0$ when $y \in D^c$.

(b) If $H(t, x, y) = \Delta_y G_t(x, y)$ (if $d = 3$, we also need $\kappa^{-1} < 3$; if $d = 2$, $\kappa^{-1} \neq \infty$), there exists $C > 0$ such that

(1) For all $\gamma > \frac{1}{\frac{1}{2}+\frac{d}{4}(\kappa-1)}$,

$$\begin{aligned} \|J(v)(t_0, t, *)\|_p &\leq C \int_{t_0}^t (t-r)^{-\frac{1}{2}+\frac{d}{4}(\kappa-1)} \|v(r, *)\|_\rho dr \\ &\leq C t^{\frac{1}{2}+\frac{d}{4}(\kappa-1)-\frac{1}{\gamma}} \|v(\cdot, *)\|_{L^\gamma([t_0, T], L^\rho(D))}. \end{aligned} \quad (\text{A.13})$$

(2) Let $\alpha \in]0, \frac{1}{2} + \frac{d}{4}(\kappa-1)[$. For all $\gamma > \frac{1}{\frac{1}{2}+\frac{d}{4}(\kappa-1)-\alpha}$,

$$\|J(v)(0, t, *) - J(v)(0, s, *)\|_p \leq C |t-s|^\alpha \|v(\cdot, *)\|_{L^\gamma([t_0, T], L^\rho(D))}, \quad (\text{A.14})$$

with $0 \leq s \leq t \leq T$.

(3) Let $\beta \in]0, \min\{2 - (1-\kappa)d, \frac{2}{d}, 1\}[$. For all $\gamma > \max\left\{\frac{1}{\frac{1}{2}+\frac{d}{4}(\kappa-1)-\frac{\beta}{4}}, \frac{1}{\frac{1}{2}-\frac{d}{4}\beta}\right\}$,

$$\|J(v)(0, t, *) - J(v)(0, t, *+z)\|_p \leq C |z|^\beta \|v(\cdot, *)\|_{L^\gamma([t_0, T], L^\rho(D))}, \quad (\text{A.15})$$

with $t \in [0, T]$, where we set $J(v)(t, y) := 0$ when $y \in D^c$.

(c) If $H(t, x, y) = G^2(t, x, y)$ (if $d = 3$, we also need $\kappa^{-1} < \frac{3}{2}$; if $d = 2$, $\kappa^{-1} \neq \infty$), there is a constant $C > 0$ such that

(1) For all $\gamma > \frac{1}{1+\frac{d}{4}(\kappa-2)}$,

$$\begin{aligned} \|J(v)(t_0, t, \cdot)\|_p &\leq C \int_{t_0}^t (t-r)^{\frac{d}{4}(\kappa-2)} \|V(r, *)\|_\rho dr \\ &\leq C t^{1+\frac{d}{4}(\kappa-2)-\frac{1}{\gamma}} \|v(\cdot, *)\|_{L^\gamma([t_0, T], L^\rho(D))}. \end{aligned} \quad (\text{A.16})$$

(2) Let $\alpha \in]0, 1 + \frac{d}{4}(\kappa-2)[$, for all $\gamma > \frac{1}{1+\frac{d}{4}(\kappa-2)-\alpha}$,

$$\|J(v)(0, t, \cdot) - J(v)(0, s, \cdot)\|_p \leq C |t-s|^\alpha \|v(\cdot, *)\|_{L^\gamma([t_0, T], L^\rho(D))}, \quad (\text{A.17})$$

for all $0 \leq s \leq t \leq T$.

(3) Let $\beta \in]0, \min\{4 - (2-\kappa)d, \frac{4-d}{d}, 1\}[$, for all $\gamma > \max\left\{\frac{1}{1+\frac{d}{4}(\kappa-2)-\frac{\beta}{4}}, \frac{1}{1-\frac{d}{4}(\beta+1)}\right\}$,

$$\|J(v)(0, t, \cdot) - J(v)(0, t, \cdot+z)\|_p \leq C |z|^\beta \|v(\cdot, *)\|_{L^\gamma([t_0, T], L^\rho(D))}, \quad (\text{A.18})$$

for all $t \in [0, T]$, where we set $J(v)(t, y) := 0$ when $y \in D^c$.

Proof. We only prove the case (a), since the proofs of the cases (b) and (c) are similar. We first treat (A.10). By (A.4) and the Minkowski's inequality

$$\begin{aligned} \|J(v)(t_0, t, *)\|_p &\leq \int_{t_0}^t \left\| G(t-r, *, y)v(r, y)dy \right\|_p dr \\ &\leq C \int_{t_0}^t |t-r|^{-d/4} \left\| \left[\exp\left(-C \frac{|\cdot|^{4/3}}{|t-r|^{1/3}}\right) * |v(r, \cdot)| \right] (\cdot) \right\|_p dr. \end{aligned}$$

Let $\kappa = \frac{1}{p} - \frac{1}{\rho} + 1 \in [0, 1]$. From the Young's inequality and Hölder's inequality, it follows that, for $\gamma > \frac{1}{1+\frac{d}{4}(\kappa-1)}$,

$$\|J(v)(t_0, t, *)\|_p \leq C \int_{t_0}^t |t-r|^{\frac{d}{4}(\kappa-1)} \|v(r, *)\|_\rho dr \leq C t^{1+\frac{d}{4}(\kappa-1)-\frac{1}{\gamma}} \left[\int_{t_0}^t \|v(r, *)\|_\rho^\gamma dr \right]^{1/\gamma},$$

where we have used the equality

$$\int_{\mathbf{R}^d} \exp\left(-C \frac{|x|^{4/3}}{t^{1/3}}\right) dx = K t^{\frac{d}{4}}, \quad (\text{A.19})$$

for some constant $K > 0$. This proves (A.10). Next we turn to the proof of (A.11). For any $0 \leq s \leq t \leq T$,

$$\|J(v)(0, t, *) - J(v)(0, s, *)\|_p \leq A_1 + A_2,$$

where

$$\begin{aligned} A_1 &= \left\| \int_0^s \int_D (G_{t-r}(*, y) - G_{s-r}(*, y)) v(r, y) dy dr \right\|_p, \\ A_2 &= \left\| \int_s^t \int_D G_{t-r}(*, y) v(r, y) dy dr \right\|_p. \end{aligned}$$

Note that for $\alpha \in]0, 1[$ and $h_1, h_2 \in \mathbf{R}$, it holds that

$$|h_1 - h_2| \leq [|h_1 - h_2|^\alpha] [|h_1|^{1-\alpha} + |h_2|^{1-\alpha}].$$

Then by the Mean-Value theorem, for $\theta \in [s, t]$,

$$\begin{aligned} A_1 &\leq \int_0^s \left\| \int_D |G_{t-r}(*, y) - G_{s-r}(*, y)| v(r, y) dy \right\|_p dr \\ &\leq C |t-s|^\alpha \int_0^s \left\| \int_D \frac{1}{(\theta-r)^{(\frac{d}{4}+1)\alpha}} \exp\left(-C \frac{|\cdot|^{4/3}}{|\theta-r|^{1/3}}\right) \frac{v(r, y)}{(s-r)^{\frac{d}{4}(1-\alpha)}} dy \right\|_p dr. \end{aligned}$$

For any $\ell \in \mathbf{R}^+$ and $\epsilon > 0$, we have $\exp\{-\ell\} \leq \left(\frac{\epsilon}{\ell}\right)^\epsilon$. Then

$$\begin{aligned} A_1 &\leq |t-s|^\alpha \int_0^s \left\| \int_D \frac{1}{(s-r)^{\alpha(1+\frac{d}{4})+\frac{d}{4}(1-\alpha)}} \frac{|\theta-r|^{1/3\epsilon}}{|\cdot-y|^{4/3\epsilon}} v(r, y) dy \right\|_p dr \\ &\leq |t-s|^\alpha \int_0^s \left\| \int_D \frac{1}{(s-r)^{\alpha+\frac{d}{4}-\frac{1}{3}\epsilon}} \frac{1}{|\cdot-y|^{4/3\epsilon}} v(r, y) dy \right\|_p dr. \end{aligned}$$

Apply the Young's inequality for $\kappa = \frac{1}{p} - \frac{1}{\rho} + 1 \in [0, 1]$ to conclude that

$$A_1 \leq |t-s|^\alpha \int_0^s \frac{1}{(s-r)^{\alpha+\frac{d}{4}-\frac{1}{3}\epsilon}} \left[\int_D |y|^{-\frac{4}{3\kappa}\epsilon} dy \right]^\kappa \|v(r, *)\|_\rho dr.$$

Let $\epsilon \in]0, \frac{3\kappa d}{4}[$ be sufficiently close to $\frac{3\kappa d}{4}$. For $\alpha \in]0, 1 + \frac{d}{4}(\kappa - 1)[$ and $\gamma > \frac{1}{1 + \frac{d}{4}(\kappa - 1) - \alpha}$, we have

$$A_1 \leq |t - s|^\alpha \int_0^s \frac{1}{(s - r)^{\alpha + \frac{d}{4} - \frac{1}{3}\epsilon}} \|v(r, *)\|_\rho dr \leq C |t - s|^\alpha \left[\int_0^s \|v(r, *)\|_\rho^\gamma dr \right]^{\frac{1}{\gamma}}. \quad (\text{A.20})$$

As for A_2 , if $\gamma > \frac{1}{\frac{d}{4}(\kappa - 1) + 1}$, then

$$\begin{aligned} A_2 &\leq \int_s^t \left\| \int_D G_{t-r}(*, y) v(r, y) dy \right\|_p dr \leq C \int_s^t |t - r|^{\frac{d}{4}(\kappa - 1)} \|v(r, *)\|_\rho dr \\ &\leq C |t - s|^{1 + \frac{d}{4}(\kappa - 1) - \frac{1}{\gamma}} \left[\int_0^t \|v(r, *)\|_\rho^\gamma dr \right]^{\frac{1}{\gamma}}. \end{aligned} \quad (\text{A.21})$$

The estimates (A.20) and (A.21) imply (A.11). Finally we prove (A.12). Note that

$$\|J(v)(t, *) - J(v)(t, * + z)\|_p \leq B_1 + B_2 + R,$$

where R admits the similar expression as that on the right hand side of (A.10) and

$$\begin{aligned} B_1 &= \int_0^t \left[\int_D \mathbf{1}_{\{x+z \in D\}} \left(\int_D \mathbf{1}_{\{|x-y| \leq |z|\}} |G_{t-r}(x+z, y) - G_{t-r}(z, y)| v(r, y) dy \right)^p dx \right]^{\frac{1}{p}} dr, \\ B_2 &= \int_0^t \left[\int_D \mathbf{1}_{\{x+z \in D\}} \left(\int_D \mathbf{1}_{\{|x-y| > |z|\}} |G_{t-r}(x+z, y) - G_{t-r}(z, y)| v(r, y) dy \right)^p dx \right]^{1/p} dr. \end{aligned}$$

By (A.4),

$$\begin{aligned} B_1 &\leq C \int_0^t \left[\int_D \left(\int_D \frac{1}{(t-r)^{d/4}} \mathbf{1}_{\{|x-y| \leq |z|\}} \left[\exp\left(-C \frac{|x-y|^{4/3}}{|t-r|^{1/3}}\right) \right. \right. \right. \\ &\quad \left. \left. \left. + \exp\left(-C \frac{|x+z-y|^{4/3}}{|t-r|^{1/3}}\right) \right] |v(r, y)| dy \right)^p dx \right]^{1/p} dr. \end{aligned}$$

The Young's inequality for $\kappa = \frac{1}{p} - \frac{1}{p} + 1 \in [0, 1]$ further yields that

$$B_1 \leq C \int_0^t \frac{1}{(t-r)^{\frac{d}{4}}} \left[\int_0^{|z|} \left[\exp\left(-C \frac{|x-y|^{4/3}}{|t-r|^{1/3}}\right) + \exp\left(-C \frac{|x+z-y|^{4/3}}{|t-r|^{1/3}}\right) \right]^{\kappa^{-1}} dy \right]^\kappa \|v(r, \cdot)\|_\rho dr.$$

Let $r_1, r_2 \geq 1$ satisfy $\frac{1}{r_1} + \frac{1}{r_2} = 1$ and set $\beta = \frac{d\kappa}{r_1}$. Then

$$\begin{aligned} B_1 &\leq C \int_0^t \frac{1}{(t-r)^{\frac{d}{4}}} |z|^\beta \left[\int_0^{|z|} \left[\exp\left(-C \frac{|x-y|^{4/3}}{|t-r|^{1/3}}\right) + \exp\left(-C \frac{|x+z-y|^{4/3}}{|t-r|^{1/3}}\right) \right]^{\frac{r_2}{\kappa}} dy \right]^{\frac{\kappa}{r_2}} \|v(r, *)\|_\rho dr \\ &\leq C |z|^\beta \int_0^t \frac{1}{(t-r)^{\frac{d}{4}}} |t-r|^{\frac{d}{4}(1-\beta)} \|v(r, *)\|_\rho dr \leq C |z|^\beta \int_0^t \frac{1}{(t-r)^{\frac{d}{4}\beta}} \|v(r, *)\|_\rho dr \\ &\leq C |z|^\beta \left[\int_0^t \|v(r, *)\|_\rho^\gamma dr \right]^{\frac{1}{\gamma}}, \end{aligned} \quad (\text{A.22})$$

for $\gamma > \frac{1}{1 - \frac{d}{4}\beta}$. Using the Mean-Value theorem, for $\beta < 1$ and $\phi \in [x, x+z]$,

$$\begin{aligned} B_2 &= \int_0^t \left[\int_D \mathbf{1}_{\{x+z \in D\}} \left(\int_D \mathbf{1}_{\{|x-y| > |z|\}} |G_{t-r}(x+z, y) - G_{t-r}(z, y)|^\beta \right. \right. \\ &\quad \left. \left. \times |G_{t-r}(x+z, y) - G_{t-r}(z, y)|^{1-\beta} |v(r, y)| dy \right)^p dx \right]^{\frac{1}{p}} dr \end{aligned}$$

$$\begin{aligned} &\leq C \int_0^t \left[\int_D \left(\int_D \mathbf{1}_{\{|x-y|>|z|\}} \frac{1}{|t-r|^{\frac{\beta(d+1)}{4}}} \exp\left(-C \frac{|\phi-y|^{4/3}}{|t-r|^{1/3}}\right) |z|^\beta \right. \right. \\ &\quad \left. \left. \times |t-r|^{\frac{d}{4}(1-\beta)} |v(r,y)| dy \right)^p dx \right]^{\frac{1}{p}} dr \\ &\leq C |z|^\beta \int_0^t \frac{1}{|t-r|^{\frac{d}{4}+\frac{\beta}{4}}} \left[\int_D \left(\int_D \mathbf{1}_{\{|x-y|>|z|\}} \exp\left(-C \frac{|\phi-y|^{4/3}}{|t-r|^{1/3}}\right) |v(r,y)| dy \right)^p dx \right]^{\frac{1}{p}} dr. \end{aligned}$$

Note that if $|x-y| > |z|$ and $|x| \leq |\phi| \leq |x+z|$, then

$$|\phi-y| \leq \frac{1}{\sqrt{2}} |x-z+y|, \text{ or } |\phi-y| \leq \frac{1}{\sqrt{2}} |x+z+y|.$$

Hence for $\beta \in]0, \min\{4-(1-\kappa)d, 1\}[$ and $\gamma > \max\left\{\frac{1}{1+\frac{d}{4}(\kappa-1)-\frac{\beta}{4}}, \frac{1}{1-\frac{d}{4}\beta}\right\}$,

$$B_2 \leq C |z|^\beta \int_0^t \frac{1}{|t-r|^{\frac{d}{4}(1-\kappa)+\frac{\beta}{4}}} \|v(r,*)\|_\rho dr \leq C |z|^\beta \left[\int_0^t \|v(r,*)\|_\rho^\gamma dr \right]^{\frac{1}{\gamma}}. \quad (\text{A.23})$$

This further yields (A.12). □

Lemma A.4 *Let $1 < \gamma < 2$ and ℓ denote the positive integer such that $\ell 2^{-n} \leq t \leq (\ell+1)2^{-n}$. Then for $\theta' < 4-d$ and $\theta' \leq 2$, $\theta < 1 - \frac{d}{4}$, there exists $C > 0$ such that for $0 \leq s < t \leq T$ and $x, z \in D$,*

$$\sum_{k=0}^{\ell} \left(\int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}} \int_D |G_{t-r}(x, z) - G_{t-r}(y, z)|^\gamma dz dr \right)^{\frac{2}{\gamma}} \leq C 2^{-n(\frac{2}{\gamma}-1)} |x-y|^{\theta'} \quad (\text{A.24})$$

and

$$\sum_{k=0}^{\ell} \left(\int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}} \int_D |G_{t-r}(x, z) - G_{s-r}(x, z)|^\gamma dz dr \right)^{\frac{2}{\gamma}} \leq C 2^{-n(\frac{2}{\gamma}-1)} |t-s|^\theta. \quad (\text{A.25})$$

Proof. We only prove (A.24), since the proof of (A.25) is very similar. By (A.8), (A.9) and the Hölder inequality, we have for $\theta' < 4-d$ and $\theta' \leq 2$,

$$\begin{aligned} &\sum_{k=0}^{\ell} \left(\int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}} \int_D |G_{t-r}(x, z) - G_{t-r}(y, z)|^\gamma dz dr \right)^{\frac{2}{\gamma}} \\ &\leq C 2^{-n(\frac{2}{\gamma}-1)} \left(\sum_{k=0}^{\ell} \int_{\frac{kT}{2^n}}^{\frac{(k+1)T}{2^n}} \int_D |G_{t-r}(x, z) - G_{t-r}(y, z)|^2 dz dr \right) \\ &\leq C 2^{-n(\frac{2}{\gamma}-1)} \left(\int_0^t \int_D |G_{t-r}(x, z) - G_{t-r}(y, z)|^2 dz dr \right) \\ &\leq C 2^{-n(\frac{2}{\gamma}-1)} |x-y|^{\theta'}. \end{aligned} \quad (\text{A.26})$$

Thus we finish the proof of the lemma. □

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