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On the speed of coming down from infinity for Ξ-coalescent processes

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Abstract

The Ξ-coalescent processes were initially studied by Möhle and Sagitov (2001), and introduced by Schweinsberg (2000) in their full generality. They arise in the mathematical population genetics as the complete class of scaling limits for genealogies of Cannings' models. The Ξ-coalescents generalize Λ-coalescents, where now simultaneous multiple collisions of blocks are possible. The standard version starts with infinitely many blocks at time 0, and it is said to come down from infinity if its number of blocks becomes immediately finite, almost surely. This work builds on the technique introduced recently by Berestycki, Berestycki and Limic (2009), and exhibits a deterministic "speed" function – an almost sure small time asymptotic to the number of blocks process, for a large class of Ξ-coalescents that come down from infinity.

Key words: Exchangeable coalescents, small-time asymptotics, coming down from infinity, martingale technique.

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1 Introduction

Kingman's coalescent [15; 16] is one of the central models of mathematical population genetics. From the theoretical perspective, its importance is linked to the duality with the Fisher-Wright diffusion (and more generally with the Fleming-Viot process). Therefore the Kingman coalescent emerges in the scaling limit of genealogies of all evolutionary models that are asymptotically linked to Fisher-Wright diffusions. From the practical perspective, its elementary nature allows for exact computations and fast simulation, making it amenable to statistical analysis.

Assume that the original sample has m individuals, labeled $\{1, 2, \ldots, m\}$. One can identify each of the active ancestral lineages, at any particular time, with a unique equivalence class of $\{1, 2, \ldots, m\}$ that consists of all the individuals that descend from this lineage. In this way, the coalescent event of two ancestral lineages can be perceived as the merging event of two equivalence classes. Ignoring the partition structure information, one can now view the coalescent as a *block* (rather than equivalence class) merging process.

Kingman's coalescent corresponds to the dynamics where each pair of blocks coalesces at rate 1. Hence, if there are n blocks present in the current configuration, the total number of blocks decreases by 1 at rate $\binom{n}{2}$. Using this observation and elementary properties of exponential random variables, one can quickly construct the *standard version* of the process, which "starts" from a configuration containing an infinite number of individuals (particles, or blocks) at time 0, and has the property that its configuration consists of finitely many blocks at any positive time.

The fact that in the Kingman coalescent dynamics only pairs of blocks can merge at any given time makes it less suitable to model evolutions of marine populations or viral populations under strong selection. In fact, it is believed (and argued to be observed in experiments, see e.g. [17]) that in such settings the reproduction mechanism allows for a proportion of the population to have the same parent (i.e., first generation ancestor). This translates to having multiple collisions of the ancestral lineages in the corresponding coalescent mechanism.

A family of mathematical models with the above property was independently introduced and studied by Pitman [21] and Sagitov [22] under the name Λ-*coalescents* or *coalescents with multiple collisions*. Almost immediately emerged an even more general class of models, named Ξ-*coalescents* or *coalescents with simultaneous multiple collisions* or *exchangeable coalescents*. The Greek letter Ξ in the name is a reference to the *driving measure* Ξ (see Sections 2.2 and 2.4 for details). The Ξ-coalescent processes were initially studied by Möhle and Sagitov [19], and introduced by Schweinsberg [24] in their full generality. In particular, it is shown in [19] that any limit of genealogies arising from a population genetics model with exchangeable reproduction mechanism must be a Ξ-coalescent.

The current paper uses the setting and several of the results from [24] that will be recalled soon. Formally, under the Ξ-coalescent dynamics, several families of blocks (with two or more blocks in each family) may (and typically do) coalesce simultaneously. The Ξ-coalescents will be rigorously defined in the next section.

More recently, Bertoin and Le Gall [9] established a one-to-one correspondence between a class of processes called *stochastic flows of bridges* and Ξ-coalescents, and then constructed in [10] the *generalized Fleming-Viot (or* Λ*-Fleming-Viot) processes* that have Λ-coalescent processes as duals; Birkner et al. [12] recently extended this further by constructing for each driving measure Ξ the Ξ-*Fleming-Viot processes*, dual to the corresponding Ξ-coalescent process; Durrett and Schweinsberg [14] showed that genealogies during *selective sweeps* are well-approximated by certain Ξ-coalescents; and Birkner and Blath [11] initiated a statistical study of coalescents with multiple collisions.

Generalizations of Λ-coalescents to spatial (not a mean-field) setting are studied by Limic and Sturm [18], and more recently by Angel et al. [1] and Barton et al. [2]. The reader can find detailed information about these and related research areas in recent texts by Berestycki [3] and Bertoin [8].

Let $N^{\Xi} \equiv N := (N(t),\, t \geq 0)$ be the number of blocks process corresponding to a particular standard (meaning $\lim_{t\to 0+} N(t) = \infty$) Ξ-coalescent process. Moreover, suppose that this Ξ-coalescent *comes down from infinity*, or equivalently, assume that $P(N(t) < \infty, \forall t > 0) = 1$ (see the end of Section 2.4 for a formal discussion). From the practical perspective, it seems important to understand the nature of the divergence of $N(t)$ as t decreases to 0 (see [5] for further discussion and applications).

The main goal of this work is to exhibit a function $v_\Xi \equiv v : (0,\infty) \to (0,\infty)$ such that

$$\lim_{t\to 0+} \frac{N(t)}{v(t)} = 1, \text{ almost surely.} \tag{1}$$

We call any such v the *speed* of coming down from infinity (speed of CDI) for the corresponding Ξ-coalescent. The exact form of the function v is implicit and somewhat technical, see Theorems 1 or 10 for a precise statement. Moreover, the speed is obtained under (relatively weak) additional "regularity" condition.

The coming down from infinity property was already studied by Schweinsberg [24] in detail. The speed of coming down from infinity for general Ξ-coalescents has not been previously studied. In Berestycki et al. [4] the speed of CDI of any Λ-coalescent that comes down from infinity was found using a martingale-based technique. A modification of this technique will be used presently to determine the above v, and the steps in the argument that carry over directly to the current setting will only be sketched.

In the Λ-coalescent setting, weaker asymptotic results (than (1)) on $N^{\Xi}/v = N^{\Lambda}/v$ can be deduced by an entirely different approach, based on the theory of Lévy processes and superprocesses. This link was initially discovered in [6; 7] in the special case of so-called Beta-coalescents, and recently understood in the context of general Λ-coalescents in [5]. It is worthwhile pointing out, that for any "true" Ξ-coalescent (meaning that simultaneous multiple collisions are possible in the dynamics), an approach analogous to [5] seems rather difficult to implement (to start with, the expression (20), unlike (15), does not seem to be directly linked with any well-known stochastic process). Indeed, the martingale technique from [4] has at least three advantages: (i) it yields stronger forms of convergence, more precisely, (1) and its counterparts in the L^p-sense, for any $p \geq 1$ (cf. [4] Theorems 1 and 2); it yields explicit error estimates needed in the frequency spectrum analysis (see [4; 5] for details); and (iii) it extends to the Ξ-coalescent setting as will be explained shortly.

It is not surprising that the form of the "candidate" speed of CDI for the Ξ-coalescent is completely analogous to that for the Λ-coalescent. What may be surprising is that there are Ξ-coalescents that come down from infinity but their candidate speed is identically infinite. And also that there might be coalescents that come down from infinity but faster than their finite candidate speed. (Remark 6 in Section 2.4 explains how neither of these can occur under a Λ-coalescent mechanism.) The question of whether an asymptotic speed still exists in such cases remains open. This discussion will be continued in Section 3.2.

The rest of the paper is organized as follows: Section 2 introduces various processes of interest (including in Section 2.6 the novel "color-reduction" and "color-joining" constructions that might be of independent interest) and presents a “preview” of the main result as Theorem 1. Section 3 contains a “matured” statement of the main result (Theorem 10), followed by a discussion of some of its immediate consequences, and of the significance of a certain “regularity hypothesis”, while Section 4 is devoted to the proof of Theorem 10.

2 Definitions and preliminaries

2.1 Notation

In this section we recall some standard notation, as well as gather less standard notation that will be frequently used.

Denote the set of real numbers by $\mathbb{R}$ and set $\mathbb{R}_+ = (0, \infty)$. For $a, b \in \mathbb{R}$, denote by $a \wedge b$ (resp. $a \vee b$) the minimum (resp. maximum) of the two numbers. Let

$$\Delta := \{(x_1, x_2, \ldots) : x_1 \geq x_2 \geq \ldots \geq 0, \sum_i x_i \leq 1\}, \tag{2}$$

be the infinite unit simplex. If $\mathbf{x} = (x_1, x_2, \ldots) \in \Delta$ and $c \in \mathbb{R}$, let

$$c\,\mathbf{x} = (cx_1, cx_2, \ldots).$$

Denote by 0 the *zero* $(0, 0 \ldots,)$ in Δ.

Let $\mathbb{N} := \{1, 2, \ldots\}$, and $\mathscr{P}$ be the set of partitions of $\mathbb{N}$. Furthermore, for $n \in \mathbb{N}$ denote by $\mathscr{P}_n$ the set of partitions of $[n] := \{1, \ldots, n\}$.

If f is a function, defined in a left-neighborhood $(s - \varepsilon, s)$ of a point s, denote by $f(s-)$ the left limit of f at s. Given two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, write $f = O(g)$ if $\limsup f(x)/g(x) < \infty$, $f = o(g)$ if $\limsup f(x)/g(x) = 0$, and $f \sim g$ if $\lim f(x)/g(x) = 1$. Furthermore, write $f = \Theta(g)$ if both $f = O(g)$ and $g = O(f)$. The point at which the limits are taken is determined from the context.

If $\mathscr{F} = (\mathscr{F}_t, t \geq 0)$ is a filtration, and T is a stopping time relative to $\mathscr{F}$, denote by $\mathscr{F}_T$ the standard filtration generated by T, see for example [13], page 389.

For ν a finite or σ-finite measure on Δ or on $[0, 1]$, denote the support of ν by $\mathrm{supp}(\nu)$.

2.2 Ξ-coalescents

Let Ξ be a finite measure on Δ, and write

$$\Xi = \Xi_0 + a\delta_0,$$

where $a \geq 0$ and $\Xi_0((0, 0, \ldots)) = 0$. As noted in [24], we may assume without loss of generality that Ξ is a probability measure. The Ξ-*coalescent* driven by the above Ξ is a Markov process $(\Pi_t, t \geq 0)$ with values in $\mathscr{P}$ (the set of partitions of $\mathbb{N}$), characterized in the following way. If $n \in \mathbb{N}$, then the restriction $(\Pi_t^{(n)}, t \geq 0)$ of $(\Pi_t, t \geq 0)$ to $[n]$ is a Markov chain, taking values in $\mathscr{P}_n$, such that while $\Pi_t^{(n)}$ consists of b blocks, any given k_1-tuple, k_2-tuple,..., and k_r-tuple of its blocks (here $\sum_{i=1}^r k_i \leq b$ and $k_i \geq 2$, $i = 1, \ldots, r$) merge simultaneously (each forming one new block) at rate

$$\lambda_{b;k_1,\ldots,k_r;\bar{s}} = \int_\Delta \frac{\sum_{l=0}^{\bar{s}} \sum_{i_1,\ldots,i_{r+l}} \binom{\bar{s}}{l} x_{i_1}^{k_1} \cdots x_{i_r}^{k_r} x_{i_{r+1}} \cdots x_{i_{r+l}} (1 - \sum_{i=1}^\infty x_i)^{\bar{s}-l}}{\sum_{i=1}^\infty x_i^2} \Xi(d\mathbf{x}),$$

where $\bar{s} := b - \sum_{i=1}^{r} k_i$ is the number of blocks that do not participate in the merger event, and where the sum $\sum_{i_1,\ldots,i_{r+l}}$ in the above summation stands for the infinite sum $\sum_{i_1=1}^{\infty}\sum_{i_2=1,i_2\neq i_1}^{\infty}\cdots\sum_{i_{r+l}=1,i_{r+l}\notin\{i_1,\ldots,i_{r+l-1}\}}^{\infty}$ over $r+l$ different indices. It is easy to verify that each such coalescent process has the same rate of pairwise merging

$$\lambda_{2;2;0} = \Xi(\Delta) = 1. \tag{3}$$

2.3 Preview of the small-time asymptotics

One can now state the central result of this paper. Given a probability measure Ξ as above, for each $t > 0$ denote by $N^{\Xi}(t)$ the number of blocks at time t in the corresponding (standard) Ξ-coalescent process. Define

$$\psi_{\Xi}(q) := \int_{\Delta} \frac{\sum_{i=1}^{\infty}(e^{-qx_i} - 1 + qx_i)}{\sum_{i=1}^{\infty} x_i^2}\,\Xi(d\mathbf{x}),\ q \geq 0,$$

and

$$v_{\Xi}(t) := \inf\left\{ s > 0 : \int_s^{\infty} \frac{1}{\psi_{\Xi}(q)}\,dq < t \right\},\ t > 0.$$

Theorem 1. *If both*

$$\Xi(\{\mathbf{x} \in \Delta : \sum_{i=1}^{n} x_i = 1 \textit{ for some finite } n\}) = 0$$

and

$$\int_{\Delta} \frac{(\sum_{i=1}^{\infty} x_i)^2}{\sum_{i=1}^{\infty} x_i^2}\,\Xi(d\mathbf{x}) < \infty,$$

then

$$\lim_{t\to 0+} \frac{N^{\Xi}(t)}{v_{\Xi}(t)} = 1,\ \textit{almost surely},$$

where $\infty/\infty \equiv 1$*. In particular, under the above assumptions, the quantity* $v_{\Xi}(t)$ *is finite for (one and then for) all* $t > 0$*, if and only if the* Ξ*-coalescent comes down from infinity.*

Most of the sequel is devoted to explaining the above implicit definition of the speed v_{Ξ}, as well as the significance of the two hypotheses in Theorem 1.

As already mentioned, Theorem 1 is restated as Theorem 10 in Section 3, which is proved in Section 4. The additional condition $\Xi(\{0\}) = 0$ in Theorem 10 is not really restrictive, since the case where $\Xi(\{0\}) > 0$ is already well-understood (cf. Remark 3 below).

2.4 Basic properties of Ξ-coalescents

Recall the setting and notation of Section 2.2.

If $\mathrm{supp}(\Xi) \subset \{(x, 0, 0, \ldots) : x \in [0, 1]\}$, the resulting Ξ-coalescent is usually called the Λ-*coalescent*, where Λ is specified by

$$\Lambda(dx) := \Xi(d(x, 0, \ldots)). \tag{4}$$

The transition mechanism simplifies as follows: whenever $\Pi_t^{(n)}$ consists of b blocks, the rate at which any given k-tuple of its blocks merges into a single block equals

$$\lambda_{b,k} = \int_{[0,1]} x^{k-2}(1-x)^{b-k}\Lambda(dx). \tag{5}$$

Note that mergers of several blocks into one are still possible here, but multiple mergers cannot occur simultaneously.

We recall several properties of the Ξ-coalescents carefully established in [24], the reader is referred to this article for details. The Ξ-coalescents can be constructed via a Poisson point process in the following way. Assume that $\Xi = \Xi_0$, or equivalently that Ξ does not have an atom at 0 (see also Remark 3 below). Let

$$\pi(\cdot) = \sum_{k\in\mathbb{N}} \delta_{t_k,\mathbf{x}_k}(\cdot) \tag{6}$$

be a Poisson point process on $\mathbb{R}_+ \times \Delta$ with intensity measure $dt \otimes \Xi(d\mathbf{x})/\sum_{i=1}^\infty x_i^2$. Each atom $(t,\mathbf{x})$ of π influences the evolution of the process Π as follows: to each block of $\Pi(t-)$ assign a random "color" in an i.i.d. fashion (also independently of the past) where the colors take values in $\mathbb{N}\cup(0,1)$ and their common distribution $P_\mathbf{x}$ is specified by

$$P_\mathbf{x}(\{i\}) = x_i,\ i \geq 1 \ \text{ and } P_\mathbf{x}(du) = (1-\sum_{i=1}^\infty x_i)\,du,\ u \in (0,1); \tag{7}$$

given the colors, merge immediately and simultaneously all the blocks of equal color into a single block (note that this can happen only for integral colors), while leaving the blocks of unique color unchanged.

Note that in order to make this construction rigorous, one should first consider the restrictions $(\Pi^{(n)}(t), t \geq 0)$, since the measure $\Xi(d\mathbf{x})/\sum_{i=1}^\infty x_i^2$ may have (and typically will have in the cases of interest) infinite total mass. Given a fixed time $s > 0$, a small $\varepsilon > 0$ and any $n \in \mathbb{N}$, it is straightforward to run the above procedure using only the finite number of atoms of π that are contained in $[0,s] \times \{\mathbf{x} \in \Delta : \sum_{i=1}^\infty x_i^2 > \varepsilon\}$. Denote the resulting process by $(\widetilde{\Pi}^{(n),\varepsilon}(t), t \in [0,s])$. The following observation is essential: the atoms of π contained in the "complement" $[0,s] \times \{\mathbf{x} \in \Delta : \sum_{i=1}^\infty x_i^2 \leq \varepsilon\}$, together with the coloring procedure, influence further the state of $(\widetilde{\Pi}^{(n),\varepsilon}(t), t \in [0,s])$ on the event $A_{\varepsilon;s,n}$ (by causing additional mergers during $[0,s]$), where

$$P(A_{\varepsilon;s,n}) \leq 1 - \exp\left(\int_0^s dt \int_{\Delta\cap\{\sum_{i=1}^\infty x_i^2 \leq \varepsilon\}} \frac{R_\mathbf{x}(n)}{\sum_{i=1}^\infty x_i^2}\,\Xi(d\mathbf{x})\right),$$

and where the right-hand-side goes to 0 as $\varepsilon \to 0$. Indeed, $R_\mathbf{x}(n)$ is the probability that under $P_\mathbf{x}$ (assuming n blocks present at time $t-$) at least two of the blocks are colored by the same color. For the benefit of the reader we include the exact expression for this probability:

$$\begin{aligned}
R_{\mathbf{x}}(n) &= 1-(1-\sum_{i=1}^{\infty}x_i)^n - n\sum_{i=1}^{\infty}x_i(1-x_i)^{n-1} \\
&- \binom{n}{2}\sum_{i_1=1}^{\infty}\sum_{i_2=1, i_2\neq i_1}^{\infty} x_{i_1}x_{i_2}(1-x_{i_1}-x_{i_2})^{n-2} - \dots \\
&- n\sum_{i_1=1}^{\infty}\dots\sum_{i_{n-1}=1, i_{n-1}\notin\{i_1,i_2,\dots,i_{n-2}\}}^{\infty}\prod_{\ell=1}^{n-1}x_{i_\ell}(1-\sum_{\ell=1}^{n-1}x_{i_\ell}) \\
&- \sum_{i_1=1}^{\infty}\dots\sum_{i_n=1, i_n\notin\{i_1,i_2,\dots,i_{n-1}\}}^{\infty}\prod_{\ell=1}^{n}x_{i_\ell}.
\end{aligned}$$

Note that $R_{\mathbf{x}}(n) \leq \binom{n}{2}\sum_{i=1}^{\infty}x_i^2$, so that

$$\int_0^s dt\int_{\Delta\cap\{\sum_{i=1}^{\infty}x_i^2\leq\varepsilon\}}\frac{R_{\mathbf{x}}(n)}{\sum_{i=1}^{\infty}x_i^2}\Xi(d\mathbf{x}) \leq s\binom{n}{2}\int_{\Delta\cap\{\sum_{i=1}^{\infty}x_i^2\leq\varepsilon\}}\Xi(d\mathbf{x}) \to 0, \text{ as } \varepsilon\to 0.$$

In this way one obtains a coupling (that is, a simultaneous construction on a single probability space) of the family of processes $(\widetilde{\Pi}^{(n),\varepsilon}(t),\ t\in[0,s])$, as $\varepsilon\in(0,1)$, and can define $\Pi^{(n)}$ as the limit $\Pi^{(n)} := \lim_{\varepsilon\to 0}\widetilde{\Pi}^{(n),\varepsilon}$ on [0,s]. Moreover, the above construction is amenable to appending particles/blocks to the initial configuration, hence it yields a coupling of

$$(\widetilde{\Pi}^{(n),\varepsilon}(t),\ t\in[0,s]), \text{ as } \varepsilon\in(0,1) \text{ and } n\in\mathbb{N}. \tag{8}$$

An interested reader is invited to check (or see [24]) that the limit

$$\Pi := \lim_{n\to\infty}\Pi^{(n)} \tag{9}$$

is a well-defined realization of the Ξ-coalescent, corresponding to the measure Ξ. We will denote its law simply by P (rather than by P_Ξ).

Consider the above Ξ-coalescent process Π. Let $E = \{N(t) = \infty$ for all $t \geq 0\}$, and $F = \{N(t) < \infty$ for all $t > 0\}$.

Definition 2. *We say that a* Ξ*-coalescent* comes down from infinity *if* $P(F) = 1$.

Let

$$\Delta_f := \{\mathbf{x}\in\Delta : \sum_{i=1}^{n}x_i = 1 \text{ for some finite } n\}. \tag{10}$$

Lemma 31 [24] extends Proposition 23 of Pitman [21] to the Ξ-coalescent setting. It says that provided $\Xi(\Delta_f) = 0$, there are two possibilities for the evolution of N: either $P(E) = 1$ or $P(F) = 1$.

Remark 3. *A careful reader will note that the above Poisson point process (PPP) construction assumed* $\Xi(\{0\}) = 0$*. It is possible to enrich it with extra pairwise mergers if* $\Xi(\{0\}) > 0$*, see [24] for details. For the purposes of the current study this does not seem to be necessary. Indeed, by the argument of [4] Section 4.2, one can easily see that if* $\Xi((0,0,\dots,)) = a > 0$*, then the corresponding* Ξ*-coalescent comes down from infinity, and moreover its speed of CDI is determined by* a*. More precisely, such a* Ξ*-coalescent comes down faster than the* Λ*-coalescent* $(\Pi_a(s), s\geq 0)$ *corresponding to* $\Lambda(dx) = a\delta_0(dx)$ *(note that* Π_a *is just a time-changed Kingman coalescent), and slower than* $(\Pi_a((1+\varepsilon)s), s\geq 0)$*, for any* $\varepsilon > 0$*.*

In the rest of this paper we will assume that $\Xi(\{0\}) = 0$, or equivalently, that $\Xi = \Xi_0$.

Remark 4. *The condition $\Xi(\Delta_f) = 0$ is similar (but not completely analogous, see next remark) to the condition $\Lambda(\{1\}) = 0$ for Λ-coalescents. It is not difficult to construct a Ξ-coalescent, such that $\Xi(\Delta_f) > 0$ and $P(E) = P(F) = 0$. Take some probability measure Ξ' on Δ such that the corresponding Ξ-coalescent does not come from infinity, and define $\Xi = (1-a)\Xi' + a\nu$, for some $a \in (0,1)$ and some probability measure ν on Δ_f (for example $\nu(d\mathbf{x}) = \delta_{(1/2,1/2,0,\dots)}(d\mathbf{x})$). Then its block counting process stays infinite for all times strictly smaller than T_*, and it is finite for all times larger than or equal to T_*, where*

$$T_* := \inf\{s : \pi(\{s\} \times \Delta_f) > 0\}$$

has exponential (rate a) distribution, hence is strictly positive with probability 1.

Remark 5. *It may be surprising that there are measures Ξ satisfying $\Xi(\Delta_f) = 1$, and such that the corresponding Ξ-coalescent comes down from infinity. Note that there is no analogy in the setting of Λ-coalescents, since if $\Lambda(\{1\}) = 1$ (and therefore $\Lambda([0,1)) = 0$), the only such "Λ-coalescent" will contain a single block for all times.*

It was already observed by Schweinsberg [24] Section 5.5 that if the quantity

$$\int_{\Delta_f} \frac{1}{\sum_{i=1}^{\infty} x_i^2}\, \Xi(d\mathbf{x})$$

is infinite, the corresponding Ξ-coalescent comes down from infinity. Moreover, if the above quantity is positive and finite, the corresponding Ξ-coalescent comes down from infinity if and only if the Ξ-coalescent corresponding to $\Xi'(d\mathbf{x}) = \Xi(d\mathbf{x})\mathbf{1}_{\{\mathbf{x}\in\Delta\setminus\Delta_f\}}$ comes down from infinity, and the speed of CDI is determined by Ξ'. This type of coalescent was already mentioned in Remark 4. The reader should note that if such a Ξ-coalescent does not come down from infinity, then $P(E) = P(F) = 0$.

Henceforth we will mostly assume that $\Xi(\Delta_f) = 0$.

2.5 Coming down from infinity revisited

In this section we assume that $\Xi(\Delta_f) = 0$, as well as $\Xi(\{0\}) = 0$. A sufficient condition for a Ξ-coalescent to come down from infinity was given by Schweinsberg [24]. For k_i, $i = 1, \dots, r$ such that $k_i \geq 2$ and $\bar{s} := b - \sum_{i=1}^{r} k_i \geq 0$, define $N(b; k_1, \dots, k_r; \bar{s})$ to be the number of different simultaneous choices of a k_1-tuple, a k_2-tuple,… and a k_r-tuple from a set of b elements. The exact expression for $N(b; k_1, \dots, k_r; \bar{s})$ is not difficult to find (also given in [24] display (3)), but is not important for the rest of the current analysis. Let

$$\gamma_b := \sum_{r=1}^{\lfloor b/2 \rfloor} \sum_{\{k_1,\dots,k_r\}} (b - r - \bar{s}) N(b; k_1, \dots, k_r; \bar{s}) \lambda_{b;k_1,\dots,k_r;\bar{s}}$$

be the total rate of decrease in the number of blocks for the Ξ-coalescent, when the current configuration has precisely b blocks.

Given a configuration consisting of b blocks and an $\mathbf{x} \in \Delta$, consider the coloring procedure (7), and define

$$Y_\ell^{(b)} := \sum_{j=1}^{b} \mathbf{1}_{\{i\text{th block has color } \ell\}},\ \ell \in \mathbb{N}, \tag{11}$$

so that $Y_\ell^{(b)}$ has Binomial(b, x_ℓ) distribution. Due to the PPP construction of the previous subsection, we then have

$$\begin{aligned}
\gamma_b &= \int_\Delta \frac{\sum_{\ell=1}^\infty E(Y_\ell^{(b)} - \mathbf{1}_{\{Y_\ell^{(b)}>0\}})}{\sum_{i=1}^\infty x_i^2}\,\Xi(d\mathbf{x}) \\
&= \int_\Delta \frac{\sum_{\ell=1}^\infty (bx_\ell - 1 + (1-x_\ell)^b)}{\sum_{i=1}^\infty x_i^2}\,\Xi(d\mathbf{x})
\end{aligned} \tag{12}$$

Proposition 32 in [24] says that the Ξ-coalescent comes down from infinity if

$$\sum_{b=2}^\infty \gamma_b^{-1} < \infty. \tag{13}$$

Let $\varepsilon \in (0,1)$ be fixed. Recall (2) and define

$$\Delta^\varepsilon := \{\mathbf{x} \in \Delta : \sum_i x_i \le 1-\varepsilon\}. \tag{14}$$

Proposition 33 in [24] says that (13) is necessary for coming down from infinity if also

$$\int_{\Delta\setminus\Delta^\varepsilon} \frac{1}{\sum_{i=1}^\infty x_i^2}\,\Xi(d\mathbf{x}) < \infty, \quad \text{for some } \varepsilon > 0.$$

Moreover [24] provides an example of a Ξ-coalescent that comes down from infinity, but does not satisfy (13). More details are given in Section 3.2.

The CDI property for Λ-coalescents is, in comparison, completely understood. Define

$$\psi_\Lambda(q) := \int_{[0,1]} \frac{(e^{-qx} - 1 + qx)}{x^2}\Lambda(dx), \tag{15}$$

and note that γ_b simplifies to $\sum_{k=2}^b (k-1)\binom{b}{k}\lambda_{b,k}$, with $\lambda_{b,k}$ as in (5). The original sharp criteria is due to Schweinsberg [23]: a particular Λ-coalescent comes down from infinity if and only if (13) holds. Bertoin and Le Gall [10] observed that

$$\gamma_b = \Theta(\psi_\Lambda(b)), \tag{16}$$

and that therefore the CDI happens if and only if

$$\int_a^\infty \frac{dq}{\psi_\Lambda(q)} < \infty, \tag{17}$$

for some (and then automatically for all) $a > 0$.

Remark 6. *A variation of the argument from Berestycki et al. [4] provides an independent (probabilistic) derivation of the last claim. More precisely, let $N^n(t) = \#\Pi^{(n)}(t)$, $t \ge 0$ and let v^n be the unique solution of the following Cauchy problem*

$$v'(t) = -\psi(v(t)), \quad v(0) = n.$$

Use the argument of [4] Theorem 1 (or see Part I in Section 4.2 for analogous argument in the Ξ-coalescent setting) to find $n_0 < \infty$, $\alpha \in (0,1/2)$, and $C < \infty$ such that

$$\bigcap_{n \geq n_0} \left\{ \sup_{t \in [0,s]} \left| \frac{N^n(t)}{\nu^n(t)} - 1 \right| \leq C s^\alpha \right\}$$

happens with overwhelming (positive would suffice) probability, uniformly in small s. Finally, note that ν^n satisfies the identity

$$\int_{\nu^n(s)}^{n} \frac{dq}{\psi(q)} = s, \; s \geq 0, \tag{18}$$

therefore $\lim_n \nu^n(s) < \infty$ *if and only if (17) holds.*

Moreover, it was shown in [4] that under condition (17), a speed $t \mapsto \nu(t)$ of CDI is specified by

$$\int_{\nu(t)}^{\infty} \frac{dq}{\psi_\Lambda(q)} = t, \; t \geq 0. \tag{19}$$

Consider again the general Ξ-coalescent setting. In analogy to (15), define

$$\psi(q) \equiv \psi_\Xi(q) := \int_\Delta \frac{\sum_{i=1}^\infty (e^{-qx_i} - 1 + qx_i)}{\sum_{i=1}^\infty x_i^2} \Xi(d\mathbf{x}). \tag{20}$$

Note that the above integral converges since

$$e^{-z} - 1 + z \leq z^2/2, \text{ for all } z \geq 0, \tag{21}$$

and so in particular $\psi_\Xi(q) \leq q^2/2$, for any probability measure Ξ on Δ. It is easy to check that $q \mapsto \psi_\Xi(q)$ is an infinitely differentiable, strictly increasing, and convex function on $\mathbb{R}_+$, as well as that $\psi_\Xi(q) \sim q^2/2$ as $q \to 0$. Therefore, if

$$\int_a^\infty \frac{dq}{\psi(q)} < \infty \tag{22}$$

for some $a > 0$, the same will be true for all $a > 0$, and irrespectively of that $\int_0^a dq/\psi(q) = \infty$, for any $a > 0$.

Lemma 7. *The function $q \mapsto \psi(q)/q$ is strictly increasing.*

The proof (straightforward and left to the reader) is analogous to that for [4] Lemma 9.

Lemma 8. *The conditions (13) and (22) are equivalent.*

Proof. It suffices to show the order of magnitude equivalence (16) in the current setting. Use expression (12) for γ_b. Note that if $x \in [0,1]$ and $b \geq 1$, then $e^{-bx} \geq (1-x)^b$, in fact for $x \in [0,1)$

$$e^{-bx} - (1-x)^b = e^{-bx} \left(1 - \exp\left\{-b\left(\frac{x^2}{2} + \frac{x^3}{3} + \dots\right)\right\}\right).$$

If $x \geq 1/4$ and $b \geq 16$ it is clearly true that $e^{-bx} - (1-x)^b \leq bx^2$. For $x \leq 1/4$ we have $\sum_{j\geq 2} x^j \leq 4x^2/3$, and so

$$1 - \exp\left\{-b\left(\frac{x^2}{2} + \frac{x^3}{3} + \dots\right)\right\} \leq 1 - \exp\left(-\frac{2}{3}bx^2\right) \leq \frac{2}{3}bx^2.$$

We conclude that

$$0 \leq e^{-bx} - (1-x)^b \leq bx^2, \text{ for all } b \geq 16 \text{ and } x \in [0,1]. \tag{23}$$

As a consequence,

$$\sum_{\ell=1}^{\infty} E(Y_\ell^{(b)} - \mathbf{1}_{\{Y_\ell^{(b)}>0\}}) = \sum_{\ell=1}^{\infty}(bx_\ell - 1 + e^{-bx_\ell}) + b\,O(\sum_{i=1}^{\infty} x_i^2), \tag{24}$$

where $O(\sum_{i=1}^{\infty} x_i^2) \in [-\sum_{i=1}^{\infty} x_i^2, 0]$. By integrating over $\Xi(d\mathbf{x})/\sum_{i=1}^{\infty} x_i^2$, we get

$$\gamma_b = \psi(b) + O(b), \text{ for some } O(b) \in [-b, 0],$$

implying $\gamma_b = O(\psi(b))$. It is easy to check directly from (12) that $\gamma_{b+1} - \gamma_b \geq \gamma_b - \gamma_{b-1} \geq 0$, for any $b \geq 3$ implying $b = O(\gamma_b)$. Using convexity of ψ, we now have that either $\psi(b) = O(b)$, or $b = o(\psi(b))$ so that

$$\frac{\gamma_b}{\psi(b)} = 1 + o(1).$$

In both cases we have $\gamma_b = \Theta(\psi(b))$. □

Assuming (22) (or equivalently, (13)), one can define

$$u_\Xi(t) \equiv u(t) := \int_t^\infty \frac{dq}{\psi(q)} \in \mathbb{R}_+, \ t > 0, \tag{25}$$

and its càdlàg inverse

$$v_\Xi(t) \equiv v(t) := \inf\left\{s > 0 : \int_s^\infty \frac{1}{\psi(q)} dq < t\right\}, \ t > 0. \tag{26}$$

Call thus defined v_Ξ the *candidate speed*. In fact, due to the continuity and strict monotonicity of u, v_Ξ is again specified by (19), with ψ_Ξ replacing ψ_Λ. If (1) holds with $v = v_\Xi$, we will sometimes refer to the candidate speed v_Ξ as the *true speed of CDI*.

Note that (26) makes sense regardless of (22), and yields $v_\Xi(t) = \infty$, for each $t > 0$, if (and only if) (22) fails. We will say that that the Ξ-coalescent "has an infinite candidate speed" in this setting.

Due to the fact $\psi_\Xi(q) \leq q^2/2$ (cf. discussion following (20)) we have

Corollary 9. *If (3) holds, then $v_\Xi(t) \geq 2/t$, for $t > 0$.*

One could try to rephrase the corollary by saying that among all the Ξ-coalescents (satisfying (3)), the Kingman coalescent is the fastest to come down from infinity at speed $t \mapsto 2/t$ (as is the case in the Λ-coalescent setting, cf. [4] Corollary 3). However, there are examples of Ξ-coalescents with infinite candidate speed that do come down from infinity. Moreover, there are coalescents that come down from infinity, and that have finite candidate speed v_Ξ, but the methods of this article break in the attempt of associating $N = N^\Xi$ and v_Ξ at small times. The existence of a (deterministic) speed, and its relation to the function $t \mapsto 2/t$ in these situations, are open problems. Vaguely speaking, such "difficult cases" correspond to measures Ξ for which there exists a set $\Delta_f^* \approx \Delta_f$ such that

$$\int_{\Delta_f^*} \frac{1}{\sum_{i=1}^\infty x_i^2} \Xi(d\mathbf{x}) = \infty.$$

For rigorous statements see Section 3.

2.6 Two operations on Ξ-coalescents

In this section we consider two variations of the PPP construction (8)–(9), each of which gives a probabilistic coupling of the original Ξ-coalescent with a simpler Ξ-coalescent.

Given a realization of the Poisson point process (6) and the coloring of (7), define the *δ-reduction* or (*δ-color-reduction*) Π_δ^r (r stands for "reduction") of Π to be the partition valued process constructed as follows: immediately after each coloring step (and before the merging) run Bernoulli(δ) random variable for each block, independently over the blocks and the rest of the randomness, and for each of the blocks having this new value 1, resample its color from the uniform $U[0,1]$ distribution, again independently from everything else. Note that the "reduction" in the name refers to reducing the coloring (atom) weights, however this has the opposite effect on the number of blocks. Indeed, the above procedure makes some blocks that share (integer) color with others in the construction of Π become uniquely colored in the construction of Π_δ^r. With a little extra care, one can obtain a coupling of $\Pi^{(n),\varepsilon}$ (resp. Π) and its reduction $\Pi_\delta^{(n),\varepsilon,r}(t)$ (resp. Π_δ^r), so that there are fewer blocks contained in $\Pi^{(n),\varepsilon}$ (resp. Π) than in $\Pi_\delta^{(n),\varepsilon,r}$ (resp. Π_δ^r) at all times.

Note that Π_δ^r is also a Ξ-coalescent, and that its driving measure is

$$\Xi_\delta(d\mathbf{x}) := (1-\delta)\Xi(d\tfrac{\mathbf{x}}{1-\delta})1_{\{\mathbf{x}\in\Delta^\delta\}}$$

Then $\Xi_\delta(\Delta) = \Xi_\delta(\Delta^\delta) = (1-\delta)^2\Xi(\Delta)$. Let ψ_δ^r be defined as in (20), but corresponding to Π_δ^r,

$$\begin{aligned}
\psi_\delta^r(q) \equiv \psi_{\Xi_\delta^r}(q) &= \int_{\Delta^\delta} \frac{\sum_{i=1}^\infty (e^{-qx_i} - 1 + qx_i)}{\sum_{i=1}^\infty x_i^2}(1-\delta)\Xi(d\tfrac{\mathbf{x}}{1-\delta}) \\
&= \int_\Delta \frac{\sum_{i=1}^\infty (e^{-qx_i(1-\delta)} - 1 + qx_i(1-\delta))}{\sum_{i=1}^\infty x_i^2}\Xi(d\mathbf{x}). \qquad (27)
\end{aligned}$$

Since $z \mapsto e^{-z} - 1 + z$ is an increasing function on $[0,\infty)$ we have $\psi_\delta^r(q) \le \psi(q)$. In fact, (27) states that $\psi_\delta^r(q) = \psi((1-\delta)q)$, $q \ge 0$, hence

$$\int_a^\infty \frac{dq}{\psi(q)} < \infty \iff \int_a^\infty \frac{dq}{\psi_\delta^r(q)} < \infty.$$

It is perhaps not a priori clear why Π^r_δ is a simpler process. We will soon see that because its Ξ-measure is concentrated on Δ^δ, the criterion of [24] for CDI is sharp, and under an additional condition, its asymptotic speed can be found in a way analogous to [4].

The second variation is as follows: given realizations of (6) and (7) as before, define the *color-joining* Π^j (j stands for "joining") of Π to be the partition valued process where all the blocks with integral color are immediately merged together into one block. As for the δ-reduction, one can obtain a coupling of $\Pi^{(n),\varepsilon}$ (resp. Π) and its color-joining $\Pi^{(n),\varepsilon,j}(t)$ (resp. Π^j), so that there are fewer blocks contained in $\Pi^{(n),\varepsilon,j}(t)$ (resp. Π^j) than in $\Pi^{(n),\varepsilon}$ (resp. Π) at all times.

The coalescent Π^j should be a Λ-coalescent, with its corresponding ψ_Λ from (15) given by

$$\psi^j(q) = \int_\Delta \frac{(e^{-q\sum_{i=1}^\infty x_i} - 1 + q\sum_{i=1}^\infty x_i)}{\sum_{i=1}^\infty x_i^2}\,\Xi(d\mathbf{x}). \tag{28}$$

The slick point is that the right-hand side in (28) may be infinite. The existence of the integral in (28) is equivalent to (cf. the condition (R) in the next section)

$$\int_\Delta \frac{(\sum_{i=1}^\infty x_i)^2}{\sum_{i=1}^\infty x_i^2}\,\Xi(d\mathbf{x}) < \infty.$$

Indeed, we have

$$\left(\sum_{i=1}^\infty x_i\right)^2 \frac{q^2\wedge q}{10} \le e^{-q\sum_{i=1}^\infty x_i} - 1 + q\sum_{i=1}^\infty x_i \le \left(\sum_{i=1}^\infty x_i\right)^2 \frac{q^2}{2}. \tag{29}$$

The upper bound is just an application of (21). For the lower bound, assume that $q \ge 1$, the argument is simpler otherwise. Note that $e^{-z} - 1 + z \ge z^2/2 - z^3/3 \ge z^2/10$ for $z < 5/4$, and that $e^{-z} - 1 + z \ge z/10$ for $z \ge 5/4$. Substituting $z = q\sum_{i=1}^\infty x_i$ we arrive at (29). As a consequence, the right-hand-side in (28) is finite for one $q \in \mathbb{R}_+$ if and only if it is finite for all $q \in \mathbb{R}_+$.

3 Main results

3.1 Regular case

In this subsection assume that (in addition to $\Xi(\{0\}\cup\Delta_f) = 0$) the measure Ξ satisfies the *regularity* condition

$$\int_\Delta \frac{(\sum_{i=1}^\infty x_i)^2}{\sum_{i=1}^\infty x_i^2}\,\Xi(d\mathbf{x}) < \infty. \tag{R}$$

The complementary setting is discussed in Section 3.2.

Denote by $(N^\Xi(t), t \ge 0)$ the number of blocks process for the Ξ-coalescent $(\Pi(t), t \ge 0)$, and recall definition (26). Regularity (R) implies

$$\int_{\Delta\setminus\Delta^{1-a}} \frac{1}{\sum_{i=1}^\infty x_i^2}\,\Xi(d\mathbf{x}) < \infty,$$

for any $a \in (0,1)$. In particular, in the PPP construction an atom $(t, \mathbf{x})$ satisfying $\sum_i x_i > a$ appears at a strictly positive random (exponential) time. Therefore, if $\Xi^a(d\mathbf{x}) = \Xi(d\mathbf{x})\mathbf{1}_{\{\sum_i x_i \le a\}}$, the Ξ-coalescent and the Ξ^a-coalescent have the same small time behavior.

As already indicated in Section 2.3, the central result of this paper is

Theorem 10. *If both $\Xi(\{0\} \cup \Delta_f) = 0$ and (R) hold, then*

$$\lim_{t\to 0} \frac{N^{\Xi}(t)}{v_{\Xi}(t)} = 1, \text{ almost surely,}$$

where $\infty/\infty \equiv 1$. In particular, under these assumptions, the candidate speed is finite if and only if the Ξ-coalescent comes down from infinity, which happens if and only if it is the true speed of CDI.

The proof is postponed until Section 4. The importance of condition (R) will become evident in view of Lemma 18, that implies Proposition 17 (see also (33)–(34)), which is an essential ingredient in the martingale analysis of Section 4. It is interesting to note that (R) arises independently in the context of the color-joining construction in Section 2.6.

Remark 11. *Once given Theorem 10, by straightforward copying of arguments from [4], one could obtain the convergence of N^{Ξ}/v_{Ξ} in the L^p sense for $p \ge 1$, as well as the convergence of the total length of the genealogical tree in the regular setting.*

Lemma 12. *Under (R) the color-joining Π^j is a Λ-coalescent corresponding to ψ^j from (28). If Π comes down from infinity, then Π^j comes down from infinity at least as fast as Π, meaning that Π^j has the speed of CDI $v_{\Xi}^j(t)$ determined by*

$$\int_{v_{\Xi}^j(t)}^{\infty} \frac{dq}{\psi^j(q)} = t,$$

where $v_{\Xi}^j(t) \le v_{\Xi}(t)$, for any $t > 0$.

Proof. Consider the process $\Pi^{(n)} = \lim_{\varepsilon\to 0} \widetilde{\Pi}^{(n),\varepsilon}$ from the PPP construction of Π. It suffices to show that $\Pi^{(n),j}$ is a Λ-coalescent (started from a configuration of n blocks) corresponding to

$$\Lambda(dy) = y^2 \int_{\Delta\cap\{\sum_i x_i = y\}} \frac{\Xi(d\mathbf{x})}{\sum_{i=1}^{\infty} x_i^2}$$

This is an immediate consequence of elementary properties of the Poisson point process π from (6). From the coupling of Π and Π^j, where Π has at least as many blocks as Π^j at any positive time, it is clear that if Π comes down from infinity, then also does Π^j. Alternatively, the reader can verify analytically that

$$(\psi^j(q) - \psi_{\Xi}(q))' \ge 0, \ q \ge 0.$$

Since Π^j is a Λ-coalescent, we know that v_{Ξ}^j is its speed of CDI. Then again due to the above coupling of Π and Π^j we conclude that $v_{\Xi}^j(t) \le v_{\Xi}(t)$, for any $t > 0$. □

Remark 13. *Note that all the Ξ-coalescents with Ξ of the form (4) are regular, and more generally, if Ξ is supported on any "finite" subsimplex $\{\mathbf{x} : x_k = 0, \forall\, k \ge n\}$ of Δ, then the corresponding Ξ-coalescent is regular. In particular, the Ξ-coalescents featuring in the selective sweep approximation of [14; 25] are regular.*

3.2 Non-regular case

Assume $\Xi(\{0\} \cup \Delta_f) = 0$ as in the previous subsection. The setting where

$$\int_{\Delta} \frac{(\sum_{i=1}^{\infty} x_i)^2}{\sum_{i=1}^{\infty} x_i^2} \, \Xi(d\mathbf{x}) = \infty \tag{NR}$$

is more complicated, and the small time asymptotics for such Ξ-coalescents is only partially understood.

Due to observations made in the previous section, (NR) is equivalent to the fact that the integral in (28) diverges.

Lemma 14. *Under (NR) the color-joining Π^j is a trivial process containing one block at all positive times.*

Proof. As for Lemma 12, consider the prelimit coalescents $\Pi^{(n)}$ and their color-joinings $\Pi^{(n),j}$. It is easy to verify that (NR) implies instantaneous coalescence of any two blocks of $\Pi^{(n),j}$, almost surely. Indeed, the rate of coalescence for a pair of blocks is given by the integral in (NR). □

Remark 15. *The last lemma holds even if Π does not come down from infinity.*

The following illuminating example was given in [24]. Suppose Ξ has an atom of mass $1/2^n$ at

$$\mathbf{x}^n := (x_1^n, \ldots, x_{2^n-1}^n, 0, \ldots),$$

where $x_i^n = 1/2^n$, $i = 1, \ldots, 2^n - 1$, and $n \in \mathbb{N}$. Then $\psi_\Xi(q) = \Theta(q \log(q))$ so v_Ξ is infinite, but the corresponding Π comes down from infinity. Due to Theorem 10 we see that (R) cannot hold in this case.

It is useful to consider a generalization as follows: for a sequence $f : \mathbb{N} \to (0,1)$, let Ξ have atom of mass $1/2^n$ at

$$\mathbf{x}^n := (x_1^n, \ldots, x_{\lfloor f(n)2^n \rfloor}^n, 0, \ldots),$$

where again $x_i^n = 1/2^n$, $i = 1, \ldots, \lfloor f(n)2^n \rfloor$, and where we assume that $\lfloor f(n)2^n \rfloor \in \{1, \ldots, 2^{n-1}-1\}$, $n \geq 1$, so that $\Xi(\Delta_f) = 0$. It turns out that again $\psi_\Xi(q) = \Theta(q \log(q))$ (in fact, this asymptotic behavior is uniform in the above choice of f), while the integral in (R) (or (NR)) is asymptotic to

$$\sum_n f(n).$$

Due to Theorem 10 we see that as soon as the above series converges, the corresponding Ξ-coalescent does not come down from infinity. However, its color-joining will in many cases come down from infinity, for example if $f(n) = n^{-2}$, then $\psi^j(q) = \Theta(q^{3/2})$.

Proposition 16. *Suppose that (NR) holds and that the the corresponding (standard) Ξ-coalescent Π has an infinite candidate speed (or equivalently, that (22) fails). Then for any $\delta \in (0,1)$, its δ-reduction Π_δ^r does not come down from infinity.*

Proof. If Π does not come down from infinity, then Π^r_δ does not either, due to the monotone coupling of Π and Π^r_δ.

Even if Π comes down from infinity, we have that

$$\int_{\Delta\setminus\Delta^\delta} \frac{1}{\sum_{i=1}^\infty x_i^2}\,\Xi^r_\delta(d\mathbf{x}) = 0 < \infty,$$

and, as already observed, that $\int_a^\infty 1/\psi^r_\delta(q)\,dq = \infty$ (for one and then all $a \in (0,\infty)$), so due to Proposition 33 of [24], Π^r_δ does not come down from infinity. □

The last result and Lemma 14 indicate the level of opacity of the non-regular setting. Indeed, a Ξ-coalescent Π that comes down from infinity, but has infinite candidate speed and satisfies (NR), can be formally "sandwiched" between its corresponding Π^j and Π^r_δ, where $\delta > 0$ is very small, however the lower bound Π^j is trivial, and the upper bound Π^r_δ does not come down from infinity, so one gains no pertinent information from the coupling.

To end this discussion, let us mention another class of frustrating examples. Suppose that Ξ_1 is a probability measure on Δ satisfying both (R) and (22) and denote by v_1 the speed of CDI for the corresponding Ξ_1-coalescent. Let Ξ_2 be a probability measure on Δ satisfying (NR). Define

$$\Xi := \frac{1}{2}(\Xi_1 + \Xi_2),$$

so that Ξ satisfies both (22) and (NR). Due to easy coupling, the Ξ-coalescent comes down from infinity, and moreover

$$\limsup_{t\to 0} \frac{2N^\Xi(t)}{v_1(t)} = 1.$$

The martingale technique however breaks in the non-regular setting, and we have no further information on the small time asymptotics of N^Ξ. It seems reasonable to guess that N^Ξ is asymptotic to $v_1/2$ as $t \to 0$.

Remark 22 discusses an approach that might be helpful in resolving the question of speed for Ξ-coalescents that come down from infinity in the non-regular setting.

4 The arguments

The goal of this section is to prove Theorem 10 (that is, Theorem 1) by adapting the technique from [4].

As already noted, the function ψ defined in (20) is strictly increasing and convex. Furthermore, it is easy to check that $v'(s) = -\psi(v(s))$ where $v = v_\Xi$ is defined in (26), so that both v and $|v'|$ are decreasing functions.

Due to the observation preceding the statement of Theorem 10, we can suppose without loss of generality that $\mathrm{supp}(\Xi) \subset \Delta^{3/4}$ (recall notation (14)). As in [4], this will simplify certain technical estimates.

To shorten notation, write N instead of N^{Ξ}. Note that the function v is the unique solution of the following integral equation

$$\log(v(t)) - \log(v(z)) + \int_z^t \frac{\psi(v(r))}{v(r)}\, dr = 0,\ \forall 0 < z < t, \tag{30}$$

with the “initial condition” $v(0+) = \infty$. If Ξ can be identified with a probability measure Λ on $[0,1]$ as in (4), then (30) is identical to the starting observation in the proof of Theorem 10 for Λ-coalescents (cf. proof of [4] Theorem 1).

Indeed, the rest of the argument is analogous to the one from [4] for Λ-coalescents, the general regular Ξ-coalescent setting being only slightly more complicated. The few points of difference will be treated in detail, while the rest of the argument is only sketched.

4.1 Preliminary calculations

Assume that the given Ξ-coalescent has a finite number of blocks at some positive time z. Consider the process

$$M(t) := \log(N(t)) - \log(N(z)) + \int_z^t \frac{\psi(N(r))}{N(r)} dr,\ t \geq z.$$

Let $n_0 \geq 1$ be fixed. Define

$$\tau_{n_0} := \inf\{s > 0 : N(s) \leq n_0\}. \tag{31}$$

It turns out that, under the regularity hypothesis (R), $M(t \wedge \tau_{n_0})$ is “almost” (up to a bounded drift correction) a local martingale, with respect to the natural filtration $(\mathscr{F}_t, t \geq 0)$ generated by the underlying Ξ-coalescent process.

Proposition 17. *There exists some deterministic $n_0 \in \mathbb{N}$ and $C < \infty$ such that*

$$E[d\log(N(s))|\mathscr{F}_s] = \left(-\frac{\psi(N(s))}{N(s)} + h(s)\right) ds, \tag{32}$$

where $(h(s), s \geq z)$ is an $\mathscr{F}$-adapted process such that $\sup_{s\in[z,z\wedge\tau_{n_0}]} |h(s)| \leq C$, and

$$E[[d\log(N(s))]^2|\mathscr{F}_s]\mathbf{1}_{\{s\leq\tau_{n_0}\}} \leq C\, ds,\ \textit{almost surely.}$$

Both estimates are valid uniformly over $z > 0$.

Restricting the analysis to n larger than n_0 is a consequence of the following estimate, whose proof is given immediately after the proof of the proposition. Recall $Y_\ell^{(n)}$ defined in (11). When taking probabilities or expectations with respect to the joint law of $(Y_\ell^{(n)}, \ell \geq 1)$, we include subscript $\mathbf{x}$ to indicate the dependence of the law on $\mathbf{x}$. Define

$$S(\mathbf{x}) := \sum_{i=1}^{\infty} x_i^2 + \left(\sum_{i=1}^{\infty} x_i\right)^2.$$

Lemma 18. *There exists $n_0 \in \mathbb{N}$ and $C_0 < \infty$ such that for all $n \geq n_0$ and all $\mathbf{x} \in \Delta^{3/4}$, we have*

$$\left| E_{\mathbf{x}}\left(\log\left[n - \sum_{\ell=1}^{\infty}(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}})\right] - \log n\right) + \frac{\sum_{\ell=1}^{\infty} nx_\ell - 1 + (1-x_\ell)^n}{n}\right| \leq C_0 S(\mathbf{x}),$$

and

$$E_{\mathbf{x}}\left(\log\left[n - \sum_{\ell=1}^{\infty}(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}})\right] - \log n\right)^2 \leq C_0 S(\mathbf{x}).$$

Proof. [*of Proposition 17*] Since (R) holds, it suffices to show that for each $s > 0$, we have on $\{N(s) \geq n_0\}$

$$\left|\frac{E(d\log(N(s))|\mathscr{F}_s)}{ds} + \frac{\psi(N(s))}{N(s)}\right| = |h(s)| = O\left(\int_{\Delta^{3/4}} \frac{S(\mathbf{x})}{\sum_{i=1}^{\infty} x_i^2}\, \Xi(d\mathbf{x})\right), \tag{33}$$

and

$$E([d\log(N(s))]^2|\mathscr{F}_s) = O\left(\int_{\Delta^{3/4}} \frac{S(\mathbf{x})}{\sum_{i=1}^{\infty} x_i^2}\, \Xi(d\mathbf{x})\right) ds, \tag{34}$$

where $O(\cdot)$ can be taken uniformly in s. Note that the finite integrals above are in fact taken over Δ, since Ξ is supported on $\Delta^{3/4}$.

Recall the PPP construction of Section 2.4 and fix $n \geq n_0$. On the event $\{N(s) = n\}$, an atom carrying value $\mathbf{x} \in \Delta$ arrives at rate $1/(\sum_{i=1}^{\infty} x_i^2)\Xi(d\mathbf{x})\, ds$, and given its arrival, $\log N(s) = \log n$ jumps to $\log(n - \sum_{\ell=1}^{\infty}(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}}))$. Therefore,

$$E(d\log(N(s))|\mathscr{F}_s) = \int_{\Delta} E_{\mathbf{x}}\left[\log\frac{n - \sum_{\ell=1}^{\infty}(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}})}{n}\right]\frac{1}{\sum_{i=1}^{\infty} x_i^2}\,\Xi(d\mathbf{x})\, ds.$$

Due to Lemma 18 and (23) we can now derive (33).

To bound the infinitesimal variance on the event $\{N(s) = n\}$, use the second estimate in Lemma 18, together with the fact

$$\frac{E([d\log(N(s))]^2|\mathscr{F}_s)}{ds} \leq \int_{\Delta} E_{\mathbf{x}}\left[\log^2\left(\frac{n - \sum_{\ell=1}^{\infty}(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}})}{n}\right)\right]\frac{1}{\sum_{i=1}^{\infty} x_i^2}\,\Xi(d\mathbf{x}).$$

Finally, note that both (33) and (34) are uniform upper bounds over s. □

Proof. [*of Lemma 18*] The argument is almost the same as that for [4] Lemma 19 in the Λ-coalescent setting. Since the regularity "dichotomy" is a consequence of some more complicated expressions (arising in the calculations) in the current setting, most of the steps are included. Abbreviate

$$Z^{(n)} := \frac{\sum_{\ell=1}^{\infty}(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}})}{n},$$

and note that $Z^{(n)}$ is stochastically bounded by a Binomial$(n, \sum_i x_i)$ random variable. Let

$$T \equiv T_n := \log\left(1 - Z^{(n)}\right).$$

Split the computation according to the event

$$A_n = \{Z^{(n)} \le 1/2\},$$

whose complement has probability bounded by

$$\exp\left\{-n\left(\frac{1}{2}\log\frac{1}{2p} + \frac{1}{2}\log\frac{1}{2(1-p)}\right)\right\} = 2^n p^{n/2}(1-p)^{n/2},$$

uniformly in $p := \sum_i x_i \le 1/4$ and n, due to a large deviation bound (for sums of i.i.d. Bernoulli random variables). On A_n^c we have $|T| \le \log n$, and on A_n we apply a calculus fact, $|\log(1-y)+y| \le \frac{y^2}{2(1-y)} \le y^2,\ y \in [0, 1/2]$, to obtain

$$\left|E[T] + E\left[Z^{(n)}\mathbf{1}_{A_n}\right]\right| \le (\log n)P(A_n^c) + E\left[(Z^{(n)})^2\mathbf{1}_{A_n}\right].$$

Since $Z^{(n)} \le 1$, we conclude

$$\left|E[T] + E[Z^{(n)}]\right| \le (\log n + 1)P(A_n^c) + E[(Z^{(n)})^2].$$

Note that $|E[T] + E[Z^{(n)}]|$ is precisely the left-hand side of the first estimate stated in the lemma. Due to the estimate (52) in the proof of [4] Lemma 19 we have

$$(\log n)P(A_n^c) \le (\log n)2^n p^{n/2}(1-p)^{n/2} \le Cp^2 < CS(\mathbf{x}),$$

for some $C < \infty$, all $p \in [0, 1/4]$, and all n large.

Until this point the argument is identical to the one for Λ-coalescents. The new step is verifying that

$$E[(Z^{(n)})^2] \le S(\mathbf{x}). \tag{35}$$

It is easy to check (see for example [4] Corollary 18) that

$$E[(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}})^2] \le Cn^2(x_\ell)^2, \tag{36}$$

for some constant $C < \infty$. For two different indices k, ℓ, use Cauchy-Schwartz inequality together with the above bound to get

$$|E[(Y_k^{(n)} - \mathbf{1}_{\{Y_k^{(n)}>0\}})(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}})]| \le \sqrt{C^2 n^4 (x_k)^2 (x_\ell)^2} = Cn^2 x_k x_\ell. \tag{37}$$

One obtains (35) from (36)–(37) after rewriting $E[(Z^{(n)})^2]$ as

$$\frac{1}{n^2}\left(\sum_\ell E[(Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}})^2] + \sum_k \sum_{\ell \ne k} E[(Y_\ell^{(n)} - Y_\ell^{(n)} > 0)(Y_k^{(n)} - \mathbf{1}_{\{Y_k^{(n)}>0\}})]\right).$$

The second estimate is proved exactly as in [4]. □

Remark 19. *The expectation of the product of $Y_k^{(n)} - \mathbf{1}_{\{Y_k^{(n)}>0\}}$ and $Y_\ell^{(n)} - \mathbf{1}_{\{Y_\ell^{(n)}>0\}}$ can be computed explicitly, and one can verify that its absolute value has the order of magnitude $n^2 x_k x_\ell$ as x_k and (or) x_ℓ tend to* 0.

4.2 Proof of Theorem 10

Part I. Suppose that a given regular Ξ-coalescent starts from n blocks, where $n \in \mathbb{N}$ is large and finite. In other words, consider the prelimit process $\Pi^{(n)}$.

Recall Remark 6. Define a family of deterministic functions $(v^n, n \in \mathbb{N})$ as in (18), where $\psi = \psi_\Xi$, and note that v^n satisfies $v^n(0) = n$ and

$$\log(v^n(t)) - \log(n) + \int_0^t \frac{\psi(v^n(r))}{v^n(r)}\,dr = 0,\ \forall t > 0. \tag{38}$$

It is easy to see that the following is true.

Lemma 20. *We have $v^n(t) \le v^{n+1}(t)$ and $\lim_{n\to\infty} v^n(t) = v_\Xi(t)$, for each $t > 0$.*

For each $n \ge n_0$ (where n_0 is the parameter from Proposition 17) define the process

$$M_n(t) := \log\frac{N^n(t \wedge \tau^n_{n_0})}{v^n(t \wedge \tau^n_{n_0})} + \int_0^{t\wedge\tau^n_{n_0}} \left[\frac{\psi(N^n(r))}{N^n(r)} - \frac{\psi(v^n(r))}{v^n(r)} + h(r)\right] dr,\ t \ge 0,$$

where $h = h^n$ is given in (32), and $\tau^n_{n_0} := \inf\{s > 0 : N^n(s) \le n_0\}$ in analogy to (31).

Due to Proposition 17 and (38), we know that M_n is a martingale (note that $M_n(0) = 0$), such that

$$E[(M_n(s) - M_n(u))^2 | \mathscr{F}_s] \le C(s-u),$$

uniformly over $n \ge n_0$ and u, s such that $s \ge u \ge 0$. Fix any $\alpha \in (0, 1/2)$. Doob's L^2-inequality therefore implies

$$P(\sup_{t\in[0,s]} |M_n(t)| > s^\alpha) = O(s^{1-2\alpha}), \tag{39}$$

where $O(\cdot)$ term is uniform over $n \ge n_0$. Due to Proposition 17, the term

$$\int_0^{t\wedge\tau^n_{n_0}} h(r)dr$$

is of smaller order $O(s)$, again uniformly in $n \ge n_0$. Hence we obtain from (39) that

$$P\left(\sup_{t\in[0,s]}\left|\log\frac{N^n(t\wedge\tau^n_{n_0})}{v^n(t\wedge\tau^n_{n_0})} + \int_0^{t\wedge\tau^n_{n_0}}\left[\frac{\psi(N^n(r))}{N^n(r)} - \frac{\psi(v^n(r))}{v^n(r)}\right]dr\right| > s^\alpha\right) = O(s^{1-2\alpha}).$$

Due to Lemma 7 and [4] Lemma 10, the last estimate implies in turn

$$P\left(\sup_{t\in[0,s]}\left|\log\frac{N^n(t\wedge\tau^n_{n_0})}{v^n(t\wedge\tau^n_{n_0})}\right| > 2s^\alpha\right) = O(s^{1-2\alpha}). \tag{40}$$

Assume that the corresponding regular (standard) Ξ-coalescent Π comes down from infinity. Since $N^n(t) \nearrow N(t)$, for each $t > 0$, $\tau^n_{n_0} \nearrow \tau_{n_0}$, almost surely, and since

$$P(\tau_{n_0} > 0) = 1, \tag{41}$$

we obtain due to Lemma 20 that the candidate speed $v(t) := \lim_n v^n(t)$ is finite for each $t > 0$.

Conversely, if this Ξ-coalescent does not come down from infinity, then it must be $v(t) := \lim_n v^n(t) = \infty$.

Part II. Suppose that the Ξ-coalescent from part I comes down from infinity. It is tempting to let $n \to \infty$ in (40) in order to obtain

$$P\left(\sup_{t\in[0,s]}\left|\log\frac{N(t\wedge\tau_{n_0})}{v(t\wedge\tau_{n_0})}\right| > 2s^{\alpha}\right) = O(s^{1-2\alpha}). \tag{42}$$

However, this step would not be rigorous without additional information on the family of events in (40), indexed by $n \geq n_0$. An alternative approach is discussed next.

From part I we know that the corresponding candidate speed is finite. Using this fact, a variation of the previous argument yields (42). Define a family of deterministic functions $(v_x,\, x\in\mathbb{R})$ by

$$v_x(t) = v(t+x),\ t \geq -x,$$

and note that each v_x satisfies an appropriate analogue of (30) on its entire domain, more precisely, $v_x(-x+) = \infty$ and

$$\log(v_x(t)) - \log(v_x(z)) + \int_z^t \frac{\psi(v_x(r))}{v_x(r)}\,dr = 0,\ \forall -x < z < t. \tag{43}$$

Due to (41), one can assume that $z \geq \tau_{n_0}$. For each $x > -z$ define

$$M_{z,x}(t) := \log\frac{N(t\wedge\tau_{n_0})}{v_x(t\wedge\tau_{n_0})} - \log\frac{N(z)}{v_x(z)} + \int_z^{t\wedge\tau_{n_0}}\left[\frac{\psi(N(r))}{N(r)} - \frac{\psi(v_x(r))}{v_x(r)} + h(r)\right]dr,\ t\geq z,$$

where h is given in (32).

It will be convenient to consider for each fixed $z > 0$ a process $M_{z,X}$, where $X \in \mathscr{F}_z$ such that $P(X > -z) = 1$. Note that such $M_{z,X}$ is adapted to the filtration $(\mathscr{F}_r,\, r\geq z)$. More precisely, let X_z be the random variable defined by

$$N(z) = v(X_z + z) = v_{X_z}(z).$$

It is easy to see that $X_z + z$ is decreasing to 0 as z decreases to 0, and that therefore the following is true.

Lemma 21. *We have* $\lim_{z\to 0} X_z = 0$*, hence* $\lim_{z\to 0} v_{X_z}(t) = v(t)$ *for all* $t > 0$*, almost surely.*

Due to Proposition 17 and (43), we know that M_{z,X_z} is a martingale (note that $M_{z,X_z}(0) = 0$), such that

$$E[(M_{z,X_z}(s) - M_{z,X_z}(u))^2|\mathscr{F}_s] \leq C(s-u),$$

uniformly over u, s such that $s \geq u \geq z$. As in part I, we obtain

$$P(\sup_{t\in[z,s]}|M_{z,X_z}(t)| > s^{\alpha}) = O(s^{1-2\alpha}),$$

where $O(\cdot)$ term is uniform over $z > 0$. Again, due to Proposition 17, the term

$$\int_z^{t\wedge\tau_{n_0}} h(r)dr$$

is of smaller order $O(s)$, uniformly in z. Hence

$$P\left(\sup_{t\in[z,s]}\left|\log\frac{N(t\wedge\tau_{n_0})}{v_{X_z}(t\wedge\tau_{n_0})}+\int_z^{t\wedge\tau_{n_0}}\left[\frac{\psi(N(r))}{N(r)}-\frac{\psi(v_{X_z}(r))}{v_{X_z}(r)}\right]dr\right|>s^\alpha\right)=O(s^{1-2\alpha}).$$

As before, due to Lemma 7 and [4] Lemma 10, the last estimate implies

$$P\left(\sup_{t\in[z,s]}\left|\log\frac{N(t\wedge\tau_{n_0})}{v_{X_z}(t\wedge\tau_{n_0})}\right|>2s^\alpha\right)=O(s^{1-2\alpha}),$$

and therefore for any $z' < z$

$$P\left(\sup_{t\in[z,s]}\left|\log\frac{N(t\wedge\tau_{n_0})}{v_{X_{z'}}(t\wedge\tau_{n_0})}\right|>2s^\alpha\right)=O(s^{1-2\alpha}).$$

Let $z' \to 0$ and use Lemma 21, and then let $z \to 0$ to obtain (42). This together with (41) shows that in this setting the candidate speed is the true speed of CDI.

Remark 22. *As already mentioned, the above argument works only under the assumption (R). However, regularity is only needed in linking $E(d\log N(t)|\mathscr{F}_t)$ to $-\psi(N(t))/N(t)$, and in uniformly bounding the infinitesimal variance of $\log N(t)$. For irregular Ξ-coalescents that have an infinite candidate speed, but also come down from infinity, a relation of similar kind*

$$E(d\log N(t)|\mathscr{F}_t)=-\frac{\psi_1(N(t))}{N(t)}+h(t)$$

might be possible, where h is still a uniformly bounded process, and where ψ_1 is an increasing, convex function satisfying Lemma 7 and

$$\int_a \frac{dq}{\psi_1(q)}<\infty,\ a>0.$$

It is natural to guess that $v_1:\mathbb{R}_+\to\mathbb{R}_+$, determined by $\int_{v_1(t)}^\infty dq/\psi_1(q)=t$, is then the speed of CDI.

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