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Ratio of The Tail of An Infinitely Divisible Distribution on The Line to That of Its Lévy Measure*

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Abstract

A necessary and sufficient condition for the tail of an infinitely divisible distribution on the real line to be estimated by the tail of its Lévy measure is found. The lower limit and the upper limit of the ratio of the right tail $\overline{\mu}(r)$ of an infinitely divisible distribution μ to the right tail $\overline{\nu}(r)$ of its Lévy measure ν as $r \to \infty$ are estimated from above and below by reviving Teugels's classical method. The exponential class and the dominated varying class are studied in detail.

Key words: infinite divisibility, Lévy measure, O-subexponentiality, dominated variation, exponential class.

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*Dedicated to Professor Ken-iti Sato on his 75-th birthday

1 Introduction

The study of tails of distributions is important in theoretical and applied probability. In one direction, it has been developed in relation to infinitely divisible distributions and their Lévy measures since early works such as Feller [16; 17], Cohen [8], and Embrechts et al. [13]. It is known that the two-sided case is harder to analyze than the one-sided case (distributions on $\mathbb{R}_+ = [0,\infty)$). Let $\overline{\eta}(r)$ be the right tail of a measure η, that is, $\overline{\eta}(r) := \eta(r,\infty) = \eta((r,\infty))$. For positive functions $f(r)$ and $g(r)$, the relation $f(r) \sim g(r)$ means that $\lim_{r\to\infty} f(r)/g(r) = 1$. We define $\widehat{\rho}(s) := \int_{\mathbb{R}} e^{sx}\rho(dx)$ for $s \in \mathbb{R}$. Denote by $\rho * \eta$ the convolution of distributions ρ and η.

Definition 1.1. Let ρ be a distribution on $\mathbb{R}$. Suppose that $\overline{\rho}(r) > 0$ for all $r \in \mathbb{R}$. Let $\gamma \geq 0$.

(1) We say $\rho \in \mathscr{L}(\gamma)$ if $\overline{\rho}(r+a) \sim e^{-a\gamma}\overline{\rho}(r)$ for all $a \in \mathbb{R}$.

(2) We say $\rho \in \mathscr{S}(\gamma)$ if $\rho \in \mathscr{L}(\gamma)$, $\widehat{\rho}(\gamma) < \infty$, and $\overline{\rho * \rho}(r) \sim 2\widehat{\rho}(\gamma)\overline{\rho}(r)$.

(3) We say $\rho \in \mathscr{S}_D(\gamma)$ if, for some $a > 0$, ρ is a distribution on $a\mathbb{Z} := \{0, \pm a, \pm 2a, \ldots\}$ with $\rho(\{na\}) > 0$ for all sufficiently large $n \in \mathbb{Z}$, and

$$\lim_{n\to\infty} \frac{\rho(\{(n+1)a\})}{\rho(\{na\})} = e^{-\gamma a}$$

and if $\widehat{\rho}(\gamma) < \infty$ and

$$\lim_{n\to\infty} \frac{\rho * \rho(\{na\})}{\rho(\{na\})} = 2\widehat{\rho}(\gamma).$$

Let $\mathscr{L}$ and $\mathscr{S}$ denote $\mathscr{L}(0)$ and $\mathscr{S}(0)$, respectively. The distributions in $\mathscr{L}$ and $\mathscr{S}$, respectively, are called *long-tailed* and *subexponential*. Those in $\mathscr{S}(\gamma)$ are called *convolution equivalent*. The class $\mathscr{L}(\gamma)$ is often called *exponential class*.

We say $\mu \in \mathbf{ID}_+$ if μ is an infinitely divisible distribution on $\mathbb{R}$ with Lévy measure ν satisfying $\overline{\nu}(r) > 0$ for all $r \in \mathbb{R}$. Let $\mu \in \mathbf{ID}_+$. We define C_* and C^* as

$$C_* := \liminf_{r\to\infty} \frac{\overline{\mu}(r)}{\overline{\nu}(r)} \quad \text{and} \quad C^* := \limsup_{r\to\infty} \frac{\overline{\mu}(r)}{\overline{\nu}(r)}.$$

If $0 < C_* \leq C^* < \infty$, then we can estimate the tail $\overline{\mu}(r)$ by the tail $\overline{\nu}(r)$ in a weak sense. That is, for any $\epsilon \in (0,1)$, there is $R > 0$ such that

$$(1-\epsilon)C_*\overline{\nu}(r) \leq \overline{\mu}(r) \leq (1+\epsilon)C^*\overline{\nu}(r) \tag{1.1}$$

whenever $r > R$. To make the estimate (1.1) more meaningful, we should give the expression of C_* and C^*. But the expression is known only in some special cases. See Theorem 1.3 of [27] for one-sided strictly semistable distributions. However, there is an important case where the tail $\overline{\mu}(r)$ is estimated by the tail $\overline{\nu}(r)$ in a stronger sense. Namely, it is proved in Theorem 1.1 of [31] (Lemma 5.3 of Section 5 below) that if $\mu \in \mathscr{S}(\gamma)$ with $\gamma \geq 0$, then there is an explicit $C \in (0,\infty)$ such that

$$\overline{\mu}(r) \sim C\overline{\nu}(r). \tag{1.2}$$

Prior to [31], the case $\gamma = 0$ is already treated in the one-sided case by [13] and in the two-sided case by [24]. There is a lattice-version of the statement above. That is, if $\mu \in \mathscr{S}_D(\gamma)$ on $\mathbb{R}_+$ with

$\gamma \geq 0$, then (1.2) holds with some $C \in (0,\infty)$. See Theorem 2 of [14]. We do not know whether the converse is true. Namely, there might be $\mu \notin \bigcup_{\gamma\geq 0}(\mathscr{S}(\gamma)\cup\mathscr{S}_D(\gamma))$ such that (1.2) holds with $C \in (0,\infty)$. Thus we are led to the following two problems.

Problem 1. What is a necessary and sufficient condition in order that $0 < C_* \leq C^* < \infty$? In the case where $0 < C_* \leq C^* < \infty$, what are the expressions of C_* and C^* or the lower and upper bounds of them ?

Problem 2. What is a necessary and sufficient condition in order that (1.2) holds with $0 < C < \infty$? In the case where (1.2) holds with $0 < C < \infty$, what is the expression of C ?

We will give answers to Problem 1 and to the second question of Problem 2. A partial answer to the first question of Problem 2 will be given too. Our results are important from the viewpoint of the asymptotic estimates of the transition probabilities of Lévy processes.

In Section 2 we give definitions of classes such as $\mathscr{OS}$, $\mathscr{OL}$, $\mathscr{H}$, and $\mathscr{D}$ and formulate our main results in Theorems 2.1, 2.2, and 2.3 and two corollaries. Relations with other works are given in detail. Section 3 discusses the meaning of $\mathscr{OS}$ in infinite divisibility. Section 4 studies a bound separating C_* and C^* for $\mu \in \mathscr{H}$. In Section 5 we prove Theorem 2.1 and its corollaries. In Sections 6 and 7, we prove Theorems 2.2 and 2.3, respectively.

2 Main results

For positive functions $f(x)$ and $g(x)$, the relation $f(r) \asymp g(r)$ means that $\liminf_{r\to\infty} f(r)/g(r) > 0$ and $\limsup_{r\to\infty} f(r)/g(r) < \infty$. We introduce some basic classes of distributions on $\mathbb{R}$ in addition to $\mathscr{L}(\gamma)$, $\mathscr{S}(\gamma)$, and $\mathscr{S}_D(\gamma)$ in Definition 1.1.

Definition 2.1. Let ρ be a distribution on $\mathbb{R}$ satisfying $\overline{\rho}(r) > 0$ for all $r \in \mathbb{R}$.
(1) We say $\rho \in \mathscr{OS}$ if $\overline{\rho * \rho}(r) \asymp \overline{\rho}(r)$.
(2) We say $\rho \in \mathscr{H}$ if $\widehat{\rho}(s) = \infty$ for all $s > 0$.
(3) We say $\rho \in \mathscr{D}$ if $\overline{\rho}(2r) \asymp \overline{\rho}(r)$.
(4) We say $\rho \in \mathscr{OL}$ if $\overline{\rho}(r+a) \asymp \overline{\rho}(r)$ for all $a \in \mathbb{R}$.

The distributions in $\mathscr{OS}$ are called *O-subexponential* and those in $\mathscr{H}$ are sometimes called *heavy-tailed*. Those in $\mathscr{D}$ are called *dominatedly varying*.

Remark 2.1. Let $\gamma \geq 0$. The classes in Definitions 1.1 and 2.1 satisfy the following inclusion relations:
(i) $\mathscr{D}\cap\mathscr{L} \subset \mathscr{S} \subset \mathscr{OS}\cap\mathscr{L}$.
(ii) $\mathscr{D}\cup\mathscr{L} \subset \mathscr{H}$.
(iii) $\mathscr{D}\cup\bigcup_{\gamma\geq 0}(\mathscr{S}(\gamma)\cup\mathscr{S}_D(\gamma)) \subset \mathscr{OS} \subset \mathscr{OL}$.
(iv) $\bigcup_{\gamma\geq 0}\mathscr{L}(\gamma) \subset \mathscr{OL}$.

Refer to [15; 19; 27] in the one-sided case. The proofs in the two-sided case are similar.

Let $\mu \in \mathbf{ID}_+$ with Lévy measure ν. Denote by μ^{t*} t-th convolution power of μ for $t > 0$. We define the normalized Lévy measure $\nu_{(1)}$ on $(1,\infty)$ by

$$\nu_{(1)}(dx) := c_0^{-1} 1_{\{x>1\}}(x)\nu(dx),$$

where $c_0 := \nu(1,\infty)$. Define quantities d^* and γ^* as

$$d^* := \limsup_{r\to\infty} \frac{\overline{\nu_{(1)} * \nu_{(1)}}(r)}{\overline{\nu_{(1)}}(r)},$$

and

$$\gamma^* := \sup\{\gamma \geq 0 : \widehat{\mu}(\gamma) < \infty\}.$$

Under the assumption that $\nu_{(1)} \in \mathscr{OL}$, we define the following. Let Λ be the totality of increasing sequences $\{\lambda_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \lambda_n = \infty$ such that, for every $x \in \mathbb{R}$, the following $m(x;\{\lambda_n\})$ exists and is finite:

$$m(x;\{\lambda_n\}) := \lim_{n\to\infty} \frac{\overline{\nu}(\lambda_n - x)}{\overline{\nu}(\lambda_n)}.$$

The idea of the use of the function $m(x;\{\lambda_n\})$ goes back to Teugels [29]. See Remark 5.1 in Section 5 for richness of the set Λ. Given a distribution ρ, define

$$I_*(\rho) := \inf_{\{\lambda_n\}\in\Lambda} \int_{-\infty}^{\infty} m(x;\{\lambda_n\})\rho(dx),$$

$$I^*(\rho) := \sup_{\{\lambda_n\}\in\Lambda} \int_{-\infty}^{\infty} m(x;\{\lambda_n\})\rho(dx).$$

Let $B := d^* - I_*(\nu_{(1)}) - I^*(\nu_{(1)})$. Note that $0 \leq B < \infty$ whenever $\nu_{(1)} \in \mathscr{OS}$, as will be shown in Lemma 5.2 (iv) of Section 5. Define μ_1 as the compound Poisson distribution with Lévy measure $1_{\{x>1\}}\nu(dx)$ and let μ_2 be the infinitely divisible distribution satisfying $\mu = \mu_1 * \mu_2$. Then, under the assumption that $\nu_{(1)} \in \mathscr{OS}$, define

$$J(\mu) := \begin{cases} I^*(\mu_2)\exp(c_0(I^*(\nu_{(1)})-1)) & \text{if } B = 0, \\ I^*(\mu_2)\exp(c_0(I^*(\nu_{(1)})-1))\dfrac{\exp(c_0 B)-1}{c_0 B} & \text{if } B > 0. \end{cases}$$

We answer Problem 1 in the main theorem (Theorem 2.1 below). Shimura and Watanabe [27] solved the first question of Problem 1 in the one-sided case. We will give a method to reduce the two-sided case to the one-sided case. Concerning the second question, the existing knowledge is that $C_* \geq 1$ in the one-sided case and that if moreover $\mu \in \mathscr{H}$, then $C_* = 1$. See Proposition 2 of [11] and the proof of Theorem 7 of [9]. We can give general lower and upper bounds of C_* and C^* by evolving the theory of O-subexponentiality with the help of Teugels's idea. The most involved part of our discussion is in finding the upper bound of C^*.

Theorem 2.1. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν.*

(i) $0 < C_* \leq C^* < \infty$ *if and only if* $\nu_{(1)} \in \mathscr{OS}$.

(ii) *Let* $\nu_{(1)} \in \mathscr{OS}$. *Then* $0 \le \gamma^* < \infty$ *and* $0 < I_*(\mu) \le \widehat{\mu}(\gamma^*) \le J(\mu) < \infty$. *Moreover*

$$I_*(\mu) \le C_* \le \widehat{\mu}(\gamma^*), \tag{2.1}$$

and

$$\widehat{\mu}(\gamma^*) \le C^* \le J(\mu). \tag{2.2}$$

Our results concerning Problem 2 are as follows. The first question in Problem 2 is hard to solve and our answer (Corollary 2.2 below) is only partial, which is an improvement of Corollary 1.3 of [27]. It is a long-standing problem for thirty years since [13; 14] and is still open. Moreover, we do not know yet whether $\mu \in \mathscr{S}_D(\gamma)$ implies $\nu_{(1)} \in \mathscr{S}_D(\gamma)$ in the two-sided case. The second question is solved by Theorem 1.1 of [31] under the assumption that $\nu_{(1)} \in \mathscr{L}(\gamma)$ with $\gamma \ge 0$. We show in Corollary 2.1 below that such an additional assumption is not needed.

Corollary 2.1. *Let* μ *be a distribution in* $\mathbf{ID}_+$ *with Lévy measure* ν. *If* (1.2) *holds with* $0 < C < \infty$, *then* $0 \le \gamma^* < \infty$, $\widehat{\mu}(\gamma^*) < \infty$, *and* $C = \widehat{\mu}(\gamma^*)$.

Corollary 2.2. *Let* μ *be a distribution in* $\mathbf{ID}_+$ *with Lévy measure* ν. *Suppose that there are real numbers* a_1, a_2 *with* $a_2 \ne 0$ *and* a_1/a_2 *being irrational such that, for* $a = 0, a_1, a_2$, *there is* $C(a) \in (0, \infty)$ *satisfying*

$$\overline{\mu}(r + a) \sim C(a)\overline{\nu}(r).$$

Then $0 \le \gamma^* < \infty$ *and* $\mu \in \mathscr{S}(\gamma^*)$.

Remark 2.2. Suppose that all assumptions of Corollary 2.2 are satisfied except the irrationality of a_1/a_2. Then $0 \le \gamma^* < \infty$ and $\widehat{\mu}(\gamma^*) < \infty$. In case $\gamma^* = 0$ we have $\mu \in \mathscr{S}$, but in case $\gamma^* > 0$ we may have $\mu \in \mathscr{S}_D(\gamma^*)$ and $\mu \notin \mathscr{S}(\gamma^*)$.

We present explicit lower and upper bounds of C_* and C^* for μ in $(\mathscr{L}(\gamma) \cup \mathscr{D}) \cap \mathbf{ID}_+$ in Theorems 2.2 and 2.3 below. The class $\mathscr{L}(\gamma)$ is extensively studied by [7; 10; 12] in the one-sided case and by [1; 5; 24] also in the two-sided case. Concerning Theorem 2.2 (i), a recent paper [1] of Albin contains an assertion that if $\nu_{(1)} \in \mathscr{L}(\gamma)$, then $\mu \in \mathscr{L}(\gamma)$. However, his proof for $\gamma > 0$ depends on an incorrect lemma, as will be explained in Remark 6.1 of Section 6. Braverman [5] also proved that if $\nu_{(1)} \in \mathscr{L}(\gamma)$ and $\widehat{\nu}_{(1)}(\gamma) = \infty$, then $\mu \in \mathscr{L}(\gamma)$ for $\gamma > 0$. Applications of the class $\mathscr{L}(\gamma)$ to Lévy processes are found in [2; 5].

Theorem 2.2. *Let* μ *be a distribution in* $\mathbf{ID}_+$ *with Lévy measure* ν. *Let* $\gamma \ge 0$.

(i) *If* $\nu_{(1)} \in \mathscr{L}(\gamma)$, *then* $\mu^{t*} \in \mathscr{L}(\gamma)$ *for every* $t > 0$. *In the converse direction, if* $\mu^{t*} \in \mathscr{L}(\gamma)$ *for every* $t > 0$ *and* $\nu_{(1)} \in \mathscr{OS}$, *then* $\nu_{(1)} \in \mathscr{L}(\gamma)$.

(ii) *Suppose that* $\nu_{(1)} \in \mathscr{L}(\gamma)$. *Then* $0 < C_* = \widehat{\mu}(\gamma) \le \infty$ *and the following are true:*
(1) *If* $d^* = 2\widehat{\nu}_{(1)}(\gamma) < \infty$, *then* $\nu_{(1)} \in \mathscr{S}(\gamma)$ *and* $C^* = C_* = \widehat{\mu}(\gamma) < \infty$.

(2) *If* $2\widehat{\nu}_{(1)}(\gamma) < d^* < \infty$, *then* $\nu_{(1)} \in (\mathscr{L}(\gamma) \cap \mathscr{OS}) \backslash \mathscr{S}(\gamma)$ *and*

$$\widehat{\mu}(\gamma) < C^* \le \widehat{\mu}(\gamma) \frac{\exp(c_0(d^* - 2\widehat{\nu}_{(1)}(\gamma))) - 1}{c_0(d^* - 2\widehat{\nu}_{(1)}(\gamma))} < \infty. \tag{2.3}$$

(3) *If* $d^* = \infty$, *then* $\nu_{(1)} \in \mathscr{L}(\gamma) \backslash \mathscr{OS}$ *and* $C^* = \infty$.

Remark 2.3. Let $\gamma \ge 0$. There exists μ in $\mathbf{ID}_+$ such that $\nu_{(1)} \notin \mathscr{L}(\gamma)$ but $\mu^{t*} \in \mathscr{L}(\gamma)$ for all $t > 0$. We shall show this in a forthcoming paper. Embrechts and Goldie [10; 12] conjectured that the class $\mathscr{L}(\gamma)$ is closed under convolution roots. However, Shimura and Watanabe [28] disproved their conjecture for all $\gamma \ge 0$. We do not know yet whether the class $\mathscr{L}(\gamma) \cap \mathbf{ID}_+$ is closed under convolution roots.

Remark 2.4. Let $\gamma \ge 0$. We see from Lemma 5.4 of [24] that $\nu_{(1)} \in \mathscr{L}(\gamma)$ implies $d^* \ge 2\widehat{\nu}_{(1)}(\gamma)$. The class $(\mathscr{OS} \cap \mathscr{L}(\gamma)) \backslash \mathscr{S}(\gamma)$ is not empty. We know that there are ρ and ρ' both in $\mathscr{S}(\gamma)$ such that $\rho * \rho'$ is not in $\mathscr{S}(\gamma)$. As the classes $\mathscr{OS}$ and $\mathscr{L}(\gamma)$ are closed under convolution, this $\rho * \rho'$ is in $\mathscr{OS} \cap \mathscr{L}(\gamma)$. See Lemma 3.1 (iii) of Section 3 and Lemma 2.5 of [31]. Hence this $\rho * \rho'$ is in $(\mathscr{OS} \cap \mathscr{L}(\gamma)) \backslash \mathscr{S}(\gamma)$. For example, in the case of $\gamma = 0$, we can take the distributions in Section 6 of [23] as ρ and ρ'. In the case of $\gamma > 0$, we can take the distributions in the proof of Theorem 2 of [22] as ρ and ρ'. The class $\mathscr{L}(\gamma) \backslash \mathscr{OS}$ is not empty. For example, in the case of $\gamma = 0$, a distribution on $\mathbb{R}_+$ in Section 3 of [10] belongs to $\mathscr{L} \backslash \mathscr{OS}$. In the case of $\gamma > 0$, any distribution $\rho \in \mathscr{L}(\gamma)$ with $\widehat{\rho}(\gamma) = \infty$ can be taken as $\rho \in \mathscr{L}(\gamma) \backslash \mathscr{OS}$, because $\rho \in \mathscr{OS} \cap \mathscr{L}(\gamma)$ implies $\widehat{\rho}(\gamma) < \infty$ by Lemma 6.4 of [31]. Thus none of the cases (1)–(3) in Theorem 2.2 (ii) is empty.

Feller [16] started the study of dominated variation of infinitely divisible distributions. But his assertion that $\nu_{(1)} \in \mathscr{D}$ implies $\overline{\mu}(r) \sim \overline{\nu}(r)$ is not true. In the following Theorem 2.3 we clarify the role of dominated variation in our problems. In the one-sided case, Watanabe [30] proved assertion (i) by preparing a Tauberian theorem similar to Theorem 1 of [20] and Shimura and Watanabe [27] gave an alternative proof by employing O-subexponentiality. However, they did not discuss the values of C_* and C^* for μ in $\mathscr{D} \cap \mathbf{ID}_+$. A result weaker than assertion (ii) is given in Theorem 1 of Yakimiv [33]. An application of the class $\mathscr{D}$ to selfsimilar processes with independent increments is found in [30].

Define constants Q_* and Q^* as

$$Q_* := \lim_{N\to\infty} \liminf_{r\to\infty} \frac{\overline{\nu}(r+N)}{\overline{\nu}(r)},$$
$$Q^* := \lim_{N\to\infty} \limsup_{r\to\infty} \frac{\overline{\nu}(r-N)}{\overline{\nu}(r)} = (Q_*)^{-1}.$$

Theorem 2.3. *Let* μ *be a distribution in* $\mathbf{ID}_+$ *with Lévy measure* ν.

(i) $\mu \in \mathscr{D}$ *if and only if* $\nu_{(1)} \in \mathscr{D}$.

(ii) *If* $\nu_{(1)} \in \mathscr{D}$, *then* $0 < Q_* \le 1 \le Q^* < \infty$,

$$1 - (1 - Q_*)\mu(-\infty, 0) \le C_* \le 1, \tag{2.4}$$

and

$$1 \le C^* \le Q^*. \tag{2.5}$$

Remark 2.5. The class $\mathscr{D}\cap\mathscr{S}$ contains all distributions with regularly varying tails. The lognormal distributions are infinitely divisible and belong to the class $\mathscr{S}\setminus\mathscr{D}$. On the other hand, the Peter and Paul distribution belongs to the class $\mathscr{D}\setminus\mathscr{S}$. One-sided strictly semistable distributions with discrete Lévy measures are infinitely divisible distributions in the class $\mathscr{D}\setminus\mathscr{S}$ with $C_*=1$ and $C^*=Q^*>1$. See Theorem 1.3 of [27]. We show in Example 7.1 of Section 7 that there exists $\mu\in\mathscr{D}\cap\mathbf{ID}_+$ such that $Q_*<C_*<1<C^*<Q^*$.

3 Class $\mathscr{O}\mathscr{S}$ and Infinite Divisibility

The class $\mathscr{O}\mathscr{S}$ was introduced by Shimura and Watanabe [27]. They studied the asymptotic relation between an infinitely divisible distribution on $\mathbb{R}_+$ and its Lévy measure by using O-subexponentiality. In this section, we extend their results to the two-sided case. Let $\delta_a(dx)$ be the delta measure at $a\in\mathbb{R}$. For a distribution ρ on $\mathbb{R}$, we define

$$\rho_+(dx):=\rho(-\infty,0]\delta_0(dx)+1_{(0,\infty)}(x)\rho(dx).$$

Denote by ρ^{n*} n-th convolution power of ρ with the understanding that $\rho^{0*}(dx):=\delta_0(dx)$. Let $\mu\in\mathbf{ID}_+$ with Lévy measure ν. In what follows, define μ_3 as the compound Poisson distribution with Lévy measure $1_{\{x>c\}}\nu(dx)$ with $c>1$ and let μ_4 be the infinitely divisible distribution satisfying $\mu=\mu_3*\mu_4$. We define the distribution $\nu_{(c)}$ by

$$\nu_{(c)}(dx):=(\bar{\nu}(c))^{-1}1_{\{x>c\}}(x)\nu(dx).$$

We choose sufficiently large $c>1$ such that $\mu_4(0,\infty)>0$.

Lemma 3.1. *Let ρ and η be distributions on $\mathbb{R}$.*
(i) *If $\overline{\rho}(r)\asymp\overline{\eta}(r)$ for some $\eta\in\mathscr{O}\mathscr{S}$, then $\rho\in\mathscr{O}\mathscr{S}$.*
(ii) *$\rho\in\mathscr{O}\mathscr{S}$ if and only if $\rho_+\in\mathscr{O}\mathscr{S}$.*
(iii) *If $\rho,\eta\in\mathscr{O}\mathscr{S}$, then $\rho*\eta\in\mathscr{O}\mathscr{S}$. In particular, if $\rho\in\mathscr{O}\mathscr{S}$, then $\rho^{n*}\in\mathscr{O}\mathscr{S}$ for all $n\geq 1$.*

Proof. First we prove (i). Suppose that $\overline{\rho}(r)\asymp\overline{\eta}(r)$ for some $\eta\in\mathscr{O}\mathscr{S}$. Then we have

$$\begin{aligned}\overline{\rho*\rho}(r) &= \int_{\mathbb{R}}\overline{\rho}(r-y)\rho(dy)\\ &\asymp \int_{\mathbb{R}}\overline{\eta}(r-y)\rho(dy)=\int_{\mathbb{R}}\overline{\rho}(r-y)\eta(dy)\\ &\asymp \int_{\mathbb{R}}\overline{\eta}(r-y)\eta(dy)=\overline{\eta*\eta}(r)\\ &\asymp \overline{\eta}(r)\asymp\overline{\rho}(r). \end{aligned}\tag{3.1}$$

Thus we have $\rho\in\mathscr{O}\mathscr{S}$. We have $\overline{\rho}(r)=\overline{\rho_+}(r)$ for $r>0$. It follows from (i) that (ii) holds. Suppose that $\rho,\eta\in\mathscr{O}\mathscr{S}$. We see as in (3.1) that

$$\overline{\rho*\eta}(r)\asymp\overline{\rho_+*\eta_+}(r).\tag{3.2}$$

Thus we obtain from (i), (ii), and Proposition 2.5 of [27] that $\rho_+ * \eta_+ \in \mathcal{OS}$ and thereby $\rho * \eta \in \mathcal{OS}$. The second assertion of (iii) is clear. $\square$

Lemma 3.2. *Let ρ be a distribution on $\mathbb{R}$.*
(i) *If $\rho \in \mathcal{OS}$, then $\rho \in \mathcal{OL}$.*
(ii) *If $\rho \in \mathcal{OL}$, then there is $\gamma_1 \in (0,\infty)$ such that $\widehat{\rho}(\gamma_1) = \infty$.*
(iii)*If $\rho \in \mathcal{OS}$, there is $K > 0$ such that, for all $n \geq 1$ and $r \in \mathbb{R}$,*

$$\overline{\rho^{n*}}(r) \leq K^n \overline{\rho}(r).$$

Proof. Suppose that $\rho \in \mathcal{OS}$. Then $\rho_+ \in \mathcal{OS}$. Let $r > 0$. By Proposition 2.1 (ii) of [27], we have $\rho_+ \in \mathcal{OL}$ and $\overline{\rho}(r+a) = \overline{\rho_+}(r+a) \asymp \overline{\rho_+}(r) = \overline{\rho}(r)$ for any $a \geq 0$ and thereby $\rho \in \mathcal{OL}$. Thus we have proved (i). We see from Proposition 2.2 of [27] that if $\rho \in \mathcal{OL}$, then there is $\gamma_1 \in (0,\infty)$ such that $\widehat{\rho}_+(\gamma_1) = \infty$, that is, $\widehat{\rho}(\gamma_1) = \infty$. Thus assertion (ii) is true. Finally assertion (iii) is due to Lemma 6.3 (ii) of [31]. $\square$

Lemma 3.3. *Let μ be a distribution in $\mathbf{ID}_+$. Then $\mu \in \mathcal{OS}$ if and only if $\mu_1 \in \mathcal{OS}$. Furthermore, if $\mu \in \mathcal{OS}$, then $\overline{\mu}(r) \asymp \overline{\mu_1}(r)$ and $\overline{\mu_2}(r) = o(\overline{\mu_1}(r)) = o(\overline{\mu}(r))$.*

Proof. Suppose that $\mu_1 \in \mathcal{OS}$. We see from Lemma 3.2 (i) that $\overline{\mu_1}(\log r)$ is in *OR*. As regards the definition of the class *OR*, see [3]. By virtue of Theorem 2.2.7 of [3], there is $c_1 > 0$ such that $e^{c_1 r}\overline{\mu_1}(r) \to \infty$ as $r \to \infty$. Furthermore, by Theorem 26.8 of [26], there is $c_2 > 0$ such that $\overline{\mu_2}(r) = o(e^{-c_2 r \log r})$. Thus $\overline{\mu_2}(r) = o(\overline{\mu_1}(r))$ and there is $c_3 > 0$ such that $\overline{\mu_2}(r) \leq c_3\overline{\mu_1}(r)$. We obtain that

$$\begin{aligned}\overline{\mu}(r) &= \int_{\mathbb{R}} \overline{\mu_2}(r-y)\mu_1(dy) \\ &\leq c_3 \int_{\mathbb{R}} \overline{\mu_1}(r-y)\mu_1(dy) = c_3\overline{\mu_1 * \mu_1}(r) \asymp \overline{\mu_1}(r).\end{aligned}$$

Choose $b \in \mathbb{R}$ such that $c_4 := \mu_2(b,\infty) > 0$. Then we have

$$\overline{\mu}(r) \geq \int_{b+}^{\infty} \overline{\mu_1}(r-y)\mu_2(dy) \geq c_4\overline{\mu_1}(r-b). \tag{3.3}$$

Hence by Lemma 3.2 (i) we have $\overline{\mu}(r) \asymp \overline{\mu_1}(r)$. It follows from Lemma 3.1 (i) that $\mu \in \mathcal{OS}$.

Conversely, suppose that $\mu \in \mathcal{OS}$. It follows that $\overline{\mu_2}(r) = o(\overline{\mu}(r-b))$ in the same way as above. Let $\epsilon > 0$. There is $a > 0$ such that $\overline{\mu_2}(r) \leq \epsilon\overline{\mu}(r-b)$ for $r \geq a$. Hence we have

$$\begin{aligned}\overline{\mu}(r+a) &\leq \int_{-\infty}^{r+} \overline{\mu_2}(r+a-y)\mu_1(dy) + \overline{\mu_1}(r) \\ &\leq \epsilon \int_{-\infty}^{r+} \overline{\mu}(r+a-b-y)\mu_1(dy) + \overline{\mu_1}(r)\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon\overline{\mu * \mu_1}(r+a-b)+\overline{\mu_1}(r)\\
&= \epsilon\int_{\mathbb{R}}\overline{\mu_1}(r+a-b-y)\mu(dy)+\overline{\mu_1}(r)\\
&\leq \epsilon c_4^{-1}\int_{\mathbb{R}}\overline{\mu}(r+a-y)\mu(dy)+\overline{\mu_1}(r)\\
&= \epsilon c_4^{-1}\overline{\mu * \mu}(r+a)+\overline{\mu_1}(r).
\end{aligned}
$$

Here we used (3.3) in the last inequality. Since $\mu \in \mathcal{OS}$, there is $c_5 > 0$ such that $\overline{\mu * \mu}(r) \leq c_5\overline{\mu}(r)$. Hence we have $(1-\epsilon c_4^{-1}c_5)\overline{\mu}(r+a) \leq \overline{\mu_1}(r)$. Here we can take ϵ satisfying $\epsilon c_4^{-1}c_5 < 1$. Therefore we see from Lemma 3.2 (i) that there is $c_6 > 0$ such that $\overline{\mu}(r+b) \leq c_6\overline{\mu_1}(r)$. Combining this inequality with (3.3), we have $\overline{\mu}(r) \asymp \overline{\mu}(r+b) \asymp \overline{\mu_1}(r)$. Furthermore, we have $\mu_1 \in \mathcal{OS}$ by Lemma 3.1 (i). Finally, we see that $\overline{\mu}(r) \asymp \overline{\mu_1}(r)$ and $\overline{\mu_2}(r) = o(\overline{\mu_1}(r)) = o(\overline{\mu}(r))$. □

Lemma 3.4. (Theorem 1.1 of [27]) *Let μ be a distribution on $\mathbb{R}_+$ in $\mathbf{ID}_+$ with Lévy measure ν.*
(i) *$\overline{\mu}(r) \asymp \overline{\nu}(r)$ if and only if $\nu_{(1)} \in \mathcal{OS}$.*
(ii) *The following statements are equivalent:*
(1) *$\mu \in \mathcal{OS}$;*
(2) *$(\nu_{(1)})^{n*} \in \mathcal{OS}$ for some $n \geq 1$;*
(3) *$\overline{\mu}(r) \asymp \overline{(\nu_{(1)})^{n*}}(r)$ for some $n \geq 1$.*

Next we present the main result of this section.

Propsition 3.1. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν.*
(i) *We have $0 < C_*$.*
(ii) *$C^* < \infty$ if and only if $\nu_{(1)} \in \mathcal{OS}$.*
(iii) *The following statements are equivalent:*
(1) *$\mu \in \mathcal{OS}$;*
(2) *$(\nu_{(1)})^{n*} \in \mathcal{OS}$ for some $n \geq 1$;*
(3) *$\overline{\mu}(r) \asymp \overline{(\nu_{(1)})^{n*}}(r)$ for some $n \geq 1$.*

Remark 3.1. In Proposition 3.1 (iii), we cannot replace statement (2) by the statement $\nu_{(1)} \in \mathcal{OS}$. The class $\mathcal{OS}$ is not closed under convolution roots. See Remark 1.3 and Proposition 1.1 of [27] for further details. We see from Proposition 3.1 (iii) that $\mathcal{OS} \cap \mathbf{ID}_+$ is closed under convolution roots.

Proof of Proposition 3.1. We prove (i). Notice that, for $r > 0$,

$$
\overline{\mu_3}(r) = e^{-a}\sum_{n=1}^{\infty} a^n (n!)^{-1}\overline{(\nu_{(c)})^{n*}}(r),
$$

with $a := \nu(c,\infty)$. Then we have

$$
\frac{\overline{\mu}(r)}{\overline{\nu}(r)} \quad \geq \quad \int_{\mathbb{R}} e^{-a}\frac{\overline{\nu_{(c)}}(r-y)}{\overline{\nu_{(c)}}(r)}\mu_4(dy)
$$

$$\geq \quad e^{-a}\overline{\mu_4}(0) > 0.$$

Hence $C_* > 0$.

Next we show (ii). We see from Lemma 3.4 that $\nu_{(1)} \in \mathscr{OS}$ if and only if $\overline{\mu_1}(r) \asymp \overline{\nu}(r)$. We find from (i) that $C^* < \infty$ if and only if $\overline{\mu}(r) \asymp \overline{\nu}(r)$. Suppose that $\nu_{(1)} \in \mathscr{OS}$. Then $\overline{\mu_1}(r) \asymp \overline{\nu}(r)$ and $\mu_1 \in \mathscr{OS}$. We see from Lemma 3.3 that $\mu \in \mathscr{OS}$ and $\overline{\mu}(r) \asymp \overline{\mu_1}(r)$. It follows that $\overline{\mu}(r) \asymp \overline{\nu}(r)$, that is, $C^* < \infty$. Conversely, suppose that $C^* < \infty$, that is, $\overline{\mu}(r) \asymp \overline{\nu}(r)$. Let $r > 0$ and $c_7 := \mu_4(0, \infty) > 0$. We have

$$\overline{\mu}(r) \geq \int_{(0,\infty)} \overline{\mu_3}(r - y)\mu_4(dy) \geq c_7\overline{\mu_3}(r). \tag{3.4}$$

As μ_3 is compound Poisson, there is $c_8 > 0$ such that $\overline{\mu_3}(r) \geq c_8\overline{\nu}(r)$. Hence we obtain from (3.4) that $c_7\overline{\mu_3}(r) \leq \overline{\mu}(r) \asymp \overline{\nu}(r) \leq c_8^{-1}\overline{\mu_3}(r)$. This implies that $\overline{\mu_3}(r) \asymp \overline{\nu}(r)$. Hence we see from Lemmas 3.1 (i) and 3.4 (i) that $(\nu_{(c)})_{(1)} = \nu_{(c)} \in \mathscr{OS}$ and thereby $\nu_{(1)} \in \mathscr{OS}$.

Lastly, we prove (iii). Lemma 3.3 states that $\mu \in \mathscr{OS}$ if and only if $\mu_1 \in \mathscr{OS}$. We see from Lemma 3.4 that $\mu_1 \in \mathscr{OS}$ if and only if $(\nu_{(1)})^{n*} \in \mathscr{OS}$ for some $n \geq 1$. Thus we have proved that (1) is equivalent to (2). Suppose that $\overline{\mu}(r) \asymp \overline{(\nu_{(1)})^{n*}}(r)$. Since μ_3 is compound Poisson, there is $c_9 > 0$ such that $\overline{\mu_3}(r) \geq c_9\overline{(\nu_{(c)})^{n*}}(r)$. Hence we obtain from (3.4) that

$$c_7\overline{\mu_3}(r) \leq \overline{\mu}(r) \asymp \overline{(\nu_{(1)})^{n*}}(r) \asymp \overline{(\nu_{(c)})^{n*}}(r) \leq c_9^{-1}\overline{\mu_3}(r).$$

This implies that $\overline{\mu_3}(r) \asymp \overline{(\nu_{(c)})^{n*}}(r)$ and thereby we see from Lemma 3.4 that $\mu_3 \in \mathscr{OS}$ and $(\nu_{(c)})^{n*} \in \mathscr{OS}$. It follows that $(\nu_{(1)})^{n*} \in \mathscr{OS}$ and thereby $\mu_1 \in \mathscr{OS}$. Thus we find from Lemma 3.3 that $\mu \in \mathscr{OS}$. Conversely, suppose that $\mu \in \mathscr{OS}$. We obtain from Lemmas 3.3 and 3.4 (ii) that

$$\overline{\mu}(r) \asymp \overline{\mu_1}(r) \asymp \overline{(\nu_{(1)})^{n*}}(r)$$

for some $n \geq 1$. We have proved that (1) is equivalent to (3). $\square$

4 Bound separating C_* and C^* for μ in $\mathscr{H}$

The study of the lower and upper limits of ratios of tails of distributions on $\mathbb{R}_+$ was initiated by [11; 25] and progressed by [9; 18]. Watanabe and Yamamuro [32] discussed them for distributions on $\mathbb{R}$ and showed basic diferrences between the one-sided and two-sided cases. In this section, we study a bound separating C_* and C^* for $\mu \in \mathscr{H}$. We need the following lemma of Denisov et al. [9].

Lemma 4.1. (Lemma 2 of [9]) *If a distribution ρ on $\mathbb{R}_+$ is in $\mathscr{H}$, then there exists an increasing concave function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\int_{\mathbb{R}_+} e^{h(x)}\rho(dx) < \infty \quad \textit{and} \quad \int_{\mathbb{R}_+} xe^{h(x)}\rho(dx) = \infty.$$

Now we present the main result of this section.

Proposition 4.1. *Let μ be a distribution in* $\mathbf{ID}_+$ *with Lévy measure ν.*
(i) *$\mu \in \mathscr{H}$ if and only if $\nu_{(1)} \in \mathscr{H}$.*
(ii) *If $\nu_{(1)} \in \mathscr{H}$, then $0 < C_* \leq 1 \leq C^* \leq \infty$.*

Proof. Assertion (i) is obvious by virtue of Theorem 25.17 of [26]. We prove only (ii). Suppose that $\nu_{(1)} \in \mathscr{H}$. First we show that $C^* \geq 1$. We have

$$\frac{\overline{\mu}(r)}{\overline{\nu}(r)} = \int_{\mathbb{R}} \frac{\overline{\mu_3}(r-y)}{\overline{\nu}(r)}\mu_4(dy) \geq \frac{\overline{\mu_3}(r+a)}{\overline{\nu}(r)}\overline{\mu_4}(-a) \tag{4.1}$$

for any $a > 0$. Let $r > 0$. We have for any $k \geq 1$

$$\overline{(\nu_{(c)})^{k*}}(r) \geq \overline{\nu_{(c)}}(r).$$

Notice that $\overline{\mu_3}(r) = e^{-\overline{\nu}(c)} \sum_{n=1}^{\infty} \overline{\nu}(c)^n (n!)^{-1} \overline{(\nu_{(c)})^{n*}}(r)$. It follows that

$$\begin{aligned} \overline{\mu_3}(r+a) &= e^{-\overline{\nu}(c)} \sum_{n=1}^{\infty} \overline{\nu}(c)^n (n!)^{-1} \overline{(\nu_{(c)})^{n*}}(r+a) \\ &\geq e^{-\overline{\nu}(c)} \sum_{n=1}^{\infty} \overline{\nu}(c)^n (n!)^{-1} \overline{\nu_{(c)}}(r+a). \end{aligned} \tag{4.2}$$

As $\nu_{(1)} \in \mathscr{H}$, we have $\nu_{(c)} \in \mathscr{H}$. Suppose that, for some $a > 0$ and $\delta > 0$,

$$\limsup_{r\to\infty} \frac{\overline{\nu_{(c)}}(r+a)}{\overline{\nu_{(c)}}(r)} \leq e^{-\delta}.$$

Then there is sufficiently large $r_0 > 0$ such that $\overline{\nu_{(c)}}(r_0 + ak) \leq e^{-k\delta/2}\overline{\nu_{(c)}}(r_0)$ for all integers $k \geq 1$. Thus we have $\widehat{\nu}_{(c)}(\gamma_1) = \gamma_1 \int_{-\infty}^{\infty} e^{\gamma_1 x}\overline{\nu_{(c)}}(x)dx < \infty$ for $\gamma_1 := \delta/(4a) > 0$. This is a contradiction. Thus we have, for any $a > 0$,

$$\limsup_{r\to\infty} \frac{\overline{\nu_{(c)}}(r+a)}{\overline{\nu_{(c)}}(r)} = 1.$$

Hence we see from (4.1) and (4.2) that

$$\begin{aligned} C^* &\geq \limsup_{r\to\infty} \frac{\overline{\mu_3}(r+a)}{\overline{\nu}(c)\overline{\nu_{(c)}}(r)}\overline{\mu_4}(-a) \\ &\geq e^{-\overline{\nu}(c)} \sum_{n=1}^{\infty} \overline{\nu}(c)^n (n!)^{-1} \frac{\overline{\mu_4}(-a)}{\overline{\nu}(c)} = \frac{1-e^{-\overline{\nu}(c)}}{\overline{\nu}(c)}\overline{\mu_4}(-a). \end{aligned}$$

As $a \to \infty$ and $c \to \infty$, the right-hand side goes to 1. We have proved that $C^* \geq 1$.

Next we show that $C_* \leq 1$. The proof is suggested by that of Theorem 4 of [9]. Lemma 4.1 is applied to $\nu_{(c)}$. Then we can take a function $h(x)$ of Lemma 4.1. For any $b > 0$, we consider a concave function $h_b(x) := \min\{h(x), bx\}$. Hence there is $x_1 > 0$ such that $h_b(x) = h(x)$ for all $x \geq x_1$. Thus we have

$$\int_{\mathbb{R}_+} e^{h_b(x)} \nu_{(c)}(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}_+} x e^{h_b(x)} \nu_{(c)}(dx) = \infty. \tag{4.3}$$

Furthermore, we see from Theorem 25.17 of [26] that $\widehat{\mu}_4(\gamma) < \infty$ for all $\gamma > 0$. Hence

$$\int_{\mathbb{R}} (y \vee 0) e^{h_b(y \vee 0)} \mu_4(dy) \leq \int_{\mathbb{R}_+} y e^{by} \mu_4(dy) < \infty. \tag{4.4}$$

For real a and t, we use the notation of [9] and put $a^{[t]} = \min\{a, t\}$ and $a \vee t = \max\{a, t\}$. Let $t > 0$. We note that $(x+y)^{[t]} \leq x^{[t]} + y^{[t]}$ for $x, y \geq 0$. Hence we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}} (x \vee 0)^{[t]} e^{h_b(x \vee 0)} \mu(dx) = \iint ((x+y) \vee 0)^{[t]} e^{h_b((x+y)\vee 0)} \mu_3(dx) \mu_4(dy) \\
&\leq \iint (x \vee 0)^{[t]} e^{h_b((x+y)\vee 0)} \mu_3(dx) \mu_4(dy) \\
&\quad + \iint (y \vee 0)^{[t]} e^{h_b((x+y)\vee 0)} \mu_3(dx) \mu_4(dy) = J_1 + J_2.
\end{aligned}$$

By concavity of the function $h_b(x)$, we have $h_b(x+y) \leq h_b(x) + h_b(y)$ for $x, y \geq 0$. Hence

$$J_1 \leq \int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \mu_3(dx) \int_{\mathbb{R}} e^{h_b(y \vee 0)} \mu_4(dy).$$

Here we see that, for any positive integer n,

$$\begin{aligned}
&\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} (\nu_{(c)})^{n*}(dx) \\
&\leq n \int \cdots \int_{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+} x_1^{[t]} e^{h_b(x_1 + \cdots + x_n)} \nu_{(c)}(dx_1) \cdots \nu_{(c)}(dx_n) \\
&\leq n \int_{\mathbb{R}_+} x_1^{[t]} e^{h_b(x_1)} \nu_{(c)}(dx_1) \left(\int_{\mathbb{R}_+} e^{h_b(x)} \nu_{(c)}(dx) \right)^{n-1}.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \mu_3(dx) \\
&= e^{-\overline{\nu}(c)} \sum_{n=1}^{\infty} \frac{\overline{\nu}(c)^n}{n!} \int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} (\nu_{(c)})^{n*}(dx) \\
&\leq \overline{\nu}(c) \exp\left[\overline{\nu}(c) \left(\int_{\mathbb{R}_+} e^{h_b(x)} \nu_{(c)}(dx) - 1 \right) \right] \int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \nu_{(c)}(dx).
\end{aligned} \tag{4.5}$$

Therefore we obtain from (4.3) and (4.4) that

$$\limsup_{b\to 0} \frac{J_1}{\displaystyle\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \nu_{(c)}(dx)} \le \overline{\nu}(c). \tag{4.6}$$

Furthermore, we have

$$\int_{\mathbb{R}_+} e^{h_b(x)} \mu_3(dx) \le \exp\left[\overline{\nu}(c)\left(\int_{\mathbb{R}_+} e^{h_b(x)} \nu_{(c)}(dx) - 1\right)\right].$$

Thus it follows from (4.3) and (4.4) that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{J_2}{\displaystyle\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \nu_{(c)}(dx)} \\
&\le \lim_{t\to\infty} \frac{\displaystyle\int_{\mathbb{R}} e^{h_b(x\vee 0)} \mu_3(dx) \int_{\mathbb{R}_+} y e^{h_b(y)} \mu_4(dy)}{\displaystyle\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \nu_{(c)}(dx)} = 0.
\end{aligned} \tag{4.7}$$

Let $\delta_1 > 0$. Consequently, we obtain from (4.6) and (4.7) that, for sufficiently small b and sufficiently large t,

$$\frac{\displaystyle\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \mu(dx)}{\displaystyle\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \nu_{(c)}(dx)} < \overline{\nu}(c) + \delta_1. \tag{4.8}$$

Let $x_0 > 0$. Suppose that $\overline{\mu}(x) \ge \overline{\nu_{(c)}}(x)(\overline{\nu}(c) + \delta_1)$ for all $x \ge x_0$. Here we notice that

$$\begin{aligned}
\int_{\mathbb{R}_+} x e^{h_b(x)} \mu(dx) &\ge \iint_{\mathbb{R}_+\times\mathbb{R}_+} (x+y) e^{h_b(x+y)} \mu_3(dx)\mu_4(dy) \\
&\ge \mu_4(\mathbb{R}_+) \int_{\mathbb{R}_+} x e^{h_b(x)} \mu_3(dx) \\
&\ge \mu_4(\mathbb{R}_+) e^{-\overline{\nu}(c)} \overline{\nu}(c) \int_{\mathbb{R}_+} x e^{h_b(x)} \nu_{(c)}(dx) = \infty.
\end{aligned} \tag{4.9}$$

We see from (4.3) and (4.9) that both the numerator and the denominator of (4.8) goes to infinity as $t \to \infty$. Hence we have

$$\liminf_{t\to\infty} \frac{\displaystyle\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \mu(dx)}{\displaystyle\int_{\mathbb{R}_+} x^{[t]} e^{h_b(x)} \nu_{(c)}(dx)} = \liminf_{t\to\infty} \frac{\displaystyle\int_{x_0+}^{\infty} x^{[t]} e^{h_b(x)} \mu(dx)}{\displaystyle\int_{x_0+}^{\infty} x^{[t]} e^{h_b(x)} \nu_{(c)}(dx)}$$

$$
\begin{aligned}
&= \liminf_{t\to\infty} \frac{\displaystyle\int_{x_0+}^{\infty} \overline{\mu}(x) d(x^{[t]} e^{h_b(x)})}{\displaystyle\int_{x_0+}^{\infty} \overline{\nu_{(c)}}(x) d(x^{[t]} e^{h_b(x)})} \\
&\geq \overline{\nu}(c) + \delta_1.
\end{aligned}
$$

This contradicts (4.8). Hence we have $\liminf_{r\to\infty} \overline{\mu}(r)/\overline{\nu_{(c)}}(r) \leq \overline{\nu}(c) + \delta_1$, because x_0 is arbitrary. Letting $\delta_1 \downarrow 0$, we obtain that $C_* \leq 1$. We see from Proposition 3.1 (i) that $0 < C_*$. $\square$

5 Proofs of Theorem 2.1 and Its Corollaries

Let $\gamma \in \mathbb{R}$. We define the class $\mathscr{M}(\gamma)$ as the totality of distributions ρ satisfying $\widehat{\rho}(\gamma) < \infty$. For $\rho \in \mathscr{M}(\gamma)$, we define the exponential tilt $\rho_{\langle\gamma\rangle}$ of ρ as

$$
\rho_{\langle\gamma\rangle}(dx) := \frac{1}{\widehat{\rho}(\gamma)} e^{\gamma x} \rho(dx). \tag{5.1}
$$

Note that the exponential tilt conserves convolution. That is, for $\rho, \eta \in \mathscr{M}(\gamma)$, $(\rho * \eta)_{\langle\gamma\rangle} = \rho_{\langle\gamma\rangle} * \eta_{\langle\gamma\rangle}$. Let $\{X_j\}_{j=0}^{\infty}$ be i.i.d. random variables with distribution $\nu_{(1)}$. Let Y be a random variable with distribution μ_2 independent of $\{X_j\}$. Define a random walk $\{S_n\}_{n=0}^{\infty}$ as $S_n := \sum_{j=1}^{n} X_j$ for $n \geq 1$ and $S_0 := 0$. Recall that $c_0 := \nu(1,\infty)$.

Lemma 5.1. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν.*

(i) *If $0 \leq \gamma^* < \infty$ and $\widehat{\mu}(\gamma^*) = \infty$, then $C^* = \infty$.*

(ii) *If $0 \leq \gamma^* < \infty$ and $\widehat{\mu}(\gamma^*) < \infty$, then $C_* \leq \widehat{\mu}(\gamma^*) \leq C^*$.*

Proof. Suppose that $0 < \gamma^* < \infty$ and $\widehat{\mu}(\gamma^*) = \infty$. Since $\mu \in \mathscr{M}(\gamma)$ for $0 < \gamma < \gamma^*$, $\mu_{\langle\gamma\rangle}$ exists. We have $\mu_{\langle\gamma\rangle} = (\mu_3)_{\langle\gamma\rangle} * (\mu_4)_{\langle\gamma\rangle}$ and $\widehat{\mu}_3(\gamma^*) = \infty$. Define $\nu_1(dx) := e^{\gamma x}\nu(dx)$. Note that ν_1 is the Lévy measure of $\mu_{\langle\gamma\rangle}$. Since $(\mu_3)_{\langle\gamma\rangle}$ is one-sided, we have

$$
\liminf_{r\to\infty} \frac{\overline{(\mu_3)_{\langle\gamma\rangle}}(r)}{\overline{\nu_1}(r)} \geq 1.
$$

By using integration by parts, we have

$$
\bar{\mu}_{\langle\gamma\rangle}(r) = \frac{e^{\gamma r}\bar{\mu}(r)}{\widehat{\mu}(\gamma)} + \frac{\gamma}{\widehat{\mu}(\gamma)} \int_r^{\infty} e^{\gamma t}\bar{\mu}(t) dt, \tag{5.2}
$$

and

$$
\overline{\nu_1}(r) = e^{\gamma r}\bar{\nu}(r) + \gamma \int_r^{\infty} e^{\gamma t}\bar{\nu}(t) dt. \tag{5.3}
$$

Thus we see from Fatou's lemma that

$$
\frac{C^*}{\widehat{\mu}(\gamma)} \geq \liminf_{r\to\infty} \frac{\overline{\mu_{\langle\gamma\rangle}}(r)}{\overline{\nu_1}(r)}
$$

$$\begin{aligned}
&\geq \int_{0-}^{\infty} \liminf_{r\to\infty} \frac{\overline{(\mu_3)_{\langle\gamma\rangle}}(r-y)}{\overline{\nu_1}(r-y)} (\mu_4)_{\langle\gamma\rangle}(dy) \\
&\geq \frac{1}{\widehat{\mu}_4(\gamma)} \int_{0-}^{\infty} e^{\gamma y} \mu_4(dy).
\end{aligned}$$

Hence we have

$$C^* \geq \liminf_{\gamma\uparrow\gamma^*} \widehat{\mu}_3(\gamma) \int_{0-}^{\infty} e^{\gamma y} \mu_4(dy) = \infty.$$

Thus assertion (i) is true. It is clear from Proposition 4.1 that assertion (ii) is true for $\gamma^* = 0$. Suppose that $0 \leq \gamma^* < \infty$ and $\widehat{\mu}(\gamma^*) < \infty$. Then we see that $\mu_{\langle\gamma^*\rangle} \in \mathscr{H}$. Define $\nu_2(dx) := e^{\gamma^* x}\nu(dx)$. Thus we obtain from Proposition 4.1 and (5.2) and (5.3) with replacing γ by γ^* that

$$\frac{C^*}{\widehat{\mu}(\gamma^*)} \quad \geq \quad \limsup_{r\to\infty} \frac{\overline{\mu_{\langle\gamma^*\rangle}}(r)}{\overline{\nu_2}(r)} \geq 1,$$

and

$$\frac{C_*}{\widehat{\mu}(\gamma^*)} \leq \liminf_{r\to\infty} \frac{\overline{\mu_{\langle\gamma^*\rangle}}(r)}{\overline{\nu_2}(r)} \leq 1.$$

Thus we have proved the lemma. $\square$

Proposition 5.1. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν. Suppose that $\nu_{(1)} \in \mathscr{OS}$. Then we have $0 \leq \gamma^* < \infty$, $\widehat{\mu}(\gamma^*) < \infty$, and $0 < C_* \leq \widehat{\mu}(\gamma^*) \leq C^* < \infty$.*

Proof. Suppose that $\nu_{(1)} \in \mathscr{OS}$. Then we find from Proposition 3.1 that $\mu \in \mathscr{OS} \subset \mathscr{OL}$ and $0 < C_* \leq C^* < \infty$. We see from Lemma 3.2 (ii) that $0 \leq \gamma^* < \infty$. We see from Lemma 5.1 (i) that $\widehat{\mu}(\gamma^*) < \infty$. Thus the proposition follows from Lemma 5.1 (ii). $\square$

Remark 5.1. Suppose that $\nu_{(1)} \in \mathscr{OL}$. As is mentioned in the proof of Theorem 1 of [29], there exists an increasing subsequence $\{\lambda_n\} \in \Lambda$ of $\{x_n\}$ for each sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = \infty$.

Proof. Define $T_n(y)$ as

$$T_n(y) := \frac{\overline{\nu_{(1)}}(x_n - y)}{\overline{\nu_{(1)}}(x_n)}.$$

Since $\{T_n(y)\}_{n=1}^{\infty}$ is a sequence of increasing functions, uniformly bounded on all finite intervals, by the selection principle (see Chap. VIII of [17]) there exists an increasing subsequence $\{\lambda_n\}$ of $\{x_n\}$ with $\lim_{n\to\infty} \lambda_n = \infty$ such that everywhere on $\mathbb{R}$

$$\lim_{n\to\infty} \frac{\overline{\nu_{(1)}}(\lambda_n - y)}{\overline{\nu_{(1)}}(\lambda_n)} = \lim_{n\to\infty} \frac{\bar{\nu}(\lambda_n - y)}{\bar{\nu}(\lambda_n)} =: m(y; \{\lambda_n\}).$$

The limit function $m(x; \{\lambda_n\})$ is increasing and is finite. That is, $\{\lambda_n\} \in \Lambda$. $\square$

Proposition 5.2. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν. Suppose that $\nu_{(1)} \in \mathscr{OL}$. Then we have $0 < I_*(\mu) \leq C_* \leq \infty$.*

Proof. Suppose that $\nu_{(1)} \in \mathscr{OL}$. Define $h_*(x) := \liminf_{r\to\infty} \overline{\nu}(r-x)/\overline{\nu}(r)$. Since $\nu_{(1)} \in \mathscr{OL}$, we have for $\{\lambda_n\} \in \Lambda$ and $x \in \mathbb{R}$,

$$0 < h_*(x) \le m(x; \{\lambda_n\}) < \infty.$$

Thus it follows that $0 < \int_{-\infty}^{\infty} h_*(x)\mu(dx) \le I_*(\mu)$. Choose $\{\lambda_n\} \in \Lambda$ such that

$$C_* = \lim_{k\to\infty} \sum_{n=0}^{\infty} e^{-c_0} \frac{c_0^{n-1} P(Y + S_n > \lambda_k)}{n! P(X_0 > \lambda_k)}. \tag{5.4}$$

Define the events A_j for $1 \le j \le n$ and $b > 0$ as

$$A_j := \{X_j > \lambda_k - b \text{ and } Y + S_n > \lambda_k\}.$$

Let $B_n := \{(i,j) : j \ne i, 1 \le i \le n, 1 \le j \le n\}$. We have

$$P(Y + S_n > \lambda_k) \ge P(\bigcup_{j=1}^{n} A_j) \ge \sum_{j=1}^{n} P(A_j) - \sum_{(i,j)\in B_n} P(A_i \cap A_j). \tag{5.5}$$

We obtain from Fatou's lemma that, for $1 \le j \le n$,

$$\begin{aligned} & \liminf_{k\to\infty} \frac{P(A_j)}{P(X_0 > \lambda_k)} \\ \ge \quad & \int_{-\infty}^{b+} \liminf_{k\to\infty} \frac{P(X_j > \lambda_k - u) P(Y + S_n - X_j \in du)}{P(X_0 > \lambda_k)} \\ = \quad & \int_{-\infty}^{b+} m(u; \{\lambda_n\}) P(Y + S_{n-1} \in du). \end{aligned}$$

Letting $b \to \infty$, we have

$$\liminf_{b\to\infty} \liminf_{k\to\infty} \frac{P(A_j)}{P(X_0 > \lambda_k)} \ge \int_{-\infty}^{\infty} m(u; \{\lambda_n\}) P(Y + S_{n-1} \in du). \tag{5.6}$$

Moreover, we have, for $i \ne j$,

$$\begin{aligned} & \limsup_{k\to\infty} \frac{P(A_i \cap A_j)}{P(X_0 > \lambda_k)} \\ \le \quad & \limsup_{k\to\infty} \frac{P(X_i > \lambda_k - b) P(X_j > \lambda_k - b)}{P(X_0 > \lambda_k)} \\ = \quad & m(b; \{\lambda_n\}) \cdot 0 = 0. \end{aligned} \tag{5.7}$$

We see from Lemma 3.3 that

$$\lim_{k\to\infty} P(Y > \lambda_k)/P(X_0 > \lambda_k) = 0. \tag{5.8}$$

Thus we established from (5.4)-(5.8) that

$$C_* \quad = \quad \lim_{k\to\infty} \sum_{n=1}^{\infty} e^{-c_0} \frac{c_0^{n-1} P(Y + S_n > \lambda_k)}{n! P(X_0 > \lambda_k)}$$

$$\begin{aligned}
&\geq \sum_{n=1}^{\infty} e^{-c_0} \frac{c_0^{n-1} n}{n!} \int_{-\infty}^{\infty} m(u;\{\lambda_k\}) P(Y+S_{n-1} \in du) \\
&\geq \int_{-\infty}^{\infty} m(u;\{\lambda_k\}) \sum_{n=0}^{\infty} e^{-c_0} \frac{c_0^{n}}{n!} P(Y+S_{n} \in du) \\
&= \int_{-\infty}^{\infty} m(u;\{\lambda_k\}) \mu(du) \geq I_*(\mu).
\end{aligned}$$

Thus we have proved the proposition. □

Lemma 5.2. *Suppose that $\nu_{(1)} \in \mathscr{OS}$. Then we have the following:*

(i) *There are some $b_1, b_2 > 0$ such that $m(x;\{\lambda_n\}) \leq b_1 e^{b_2(x \vee 0)}$ for all $\{\lambda_n\} \in \Lambda$ and for all $x \in \mathbb{R}$.*

(ii) $I^*(\mu_2) < \infty$.

(iii) *For distributions ρ and η, $I^*(\rho * \eta) \leq I^*(\rho) I^*(\eta)$.*

(iv) $2I^*(\nu_{(1)}) \leq d^* < \infty$.

Proof. Suppose that $\nu_{(1)} \in \mathscr{OS}$. Let $\{\lambda_n\} \in \Lambda$. Define

$$h^*(x) := \limsup_{r \to \infty} \frac{\bar{\nu}(r-x)}{\bar{\nu}(r)}.$$

Since $\mathscr{OS} \subset \mathscr{OL}$, we have $m(x;\{\lambda_k\}) \leq h^*(x) < \infty$ and $h^*(x+y) \leq h^*(x)h^*(y)$. Thus $h^*(x)$ is so-called submultiplicative and by Lemma 25.5 of [26] there are $b_1, b_2 > 0$ such that $h^*(x) \leq b_1 e^{b_2(x \vee 0)}$. Thus (i) is true. Since we see from Theorem 25.17 of [26] that $\widehat{\mu}_2(\gamma) < \infty$ for any $\gamma > 0$, we have (ii) by (i). We see that, for $x,\ y \in \mathbb{R}$,

$$\lim_{n \to \infty} \frac{\bar{\nu}(\lambda_n - y - x)}{\bar{\nu}(\lambda_n - y)} = \frac{m(x+y;\{\lambda_n\})}{m(y;\{\lambda_n\})}.$$

Hence $\{\lambda_n - y\} \in \Lambda$ for any $y \in \mathbb{R}$. Thus we have

$$\begin{aligned}
I^*(\rho * \eta) &= \sup_{\{\lambda_n\} \in \Lambda} \int_{-\infty}^{\infty} m(x;\{\lambda_n\}) \rho * \eta(dx) \\
&= \sup_{\{\lambda_n\} \in \Lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(x+y;\{\lambda_n\}) \rho(dx) \eta(dy) \\
&= \sup_{\{\lambda_n\} \in \Lambda} \int_{-\infty}^{\infty} m(x;\{\lambda_n - y\}) \rho(dx) \int_{-\infty}^{\infty} m(y;\{\lambda_n\}) \eta(dy) \\
&\leq \sup_{\{\lambda'_n\} \in \Lambda} \int_{-\infty}^{\infty} m(x;\{\lambda'_n\}) \rho(dx) \sup_{\{\lambda_n\} \in \Lambda} \int_{-\infty}^{\infty} m(y;\{\lambda_n\}) \eta(dy) \\
&= I^*(\rho) I^*(\eta).
\end{aligned}$$

We have, for $\{\lambda_n\} \in \Lambda$ and $s > 0$,

$$\infty \quad > \quad d^* \geq \limsup_{k \to \infty} \frac{P(X_0 + X_1 > \lambda_k)}{P(X_0 > \lambda_k)}$$

$$
\begin{aligned}
&\geq \quad 2\limsup_{k\to\infty}\int_{-\infty}^{s+}\frac{P(X_0>\lambda_k-u)}{P(X_0>\lambda_k)}P(X_1\in du)\\
&\geq \quad 2\int_{-\infty}^{s+}\liminf_{k\to\infty}\frac{P(X_0>\lambda_k-u)}{P(X_0>\lambda_k)}P(X_1\in du)\\
&= \quad 2\int_{-\infty}^{s+}m(u;\{\lambda_n\})P(X_1\in du).
\end{aligned}
$$

Letting $s\to\infty$, we see that $2I^*(\nu_{(1)})\leq d^*<\infty$. □

Proposition 5.3. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν. Suppose that $\nu_{(1)}\in\mathcal{OS}$. Then we have $C^*\leq J(\mu)<\infty$.*

Proof. Suppose that $\rho:=\nu_{(1)}\in\mathcal{OS}$. We find from Lemma 5.2 (ii) and (iv) that $J(\mu)<\infty$. Define, for $n\geq 0$,

$$d_n:=\limsup_{x\to\infty}\frac{P(S_n+Y>x)}{P(X_0>x)}.$$

Since $\mu\in\mathcal{OS}$ by Proposition 3.1, we see from Lemma 3.3 that $d_0=0$. Choose $\{\lambda_n\}\in\Lambda$ such that

$$d_n=\lim_{k\to\infty}\frac{P(S_n+Y>\lambda_k)}{P(X_0>\lambda_k)}.$$

For $s>1$ and $n\geq 1$, define

$$I_1:=\int_{-\infty}^{(\lambda_k-s)+}P(S_{n-1}+Y>\lambda_k-u)P(X_n\in du),$$

$$I_2:=\int_{-\infty}^{s+}P(X_n>\lambda_k-u)P(S_{n-1}+Y\in du),$$

and

$$I_3:=P(S_{n-1}+Y>s)P(X_n>\lambda_k-s).$$

Then we have $P(S_n+Y>\lambda_k)=\sum_{j=1}^{3}I_j$. For any $\epsilon>0$, we can take sufficiently large $s>1$ such that

$$
\begin{aligned}
&\limsup_{k\to\infty}\frac{I_1}{P(X_0>\lambda_k)}\\
\leq\quad &(d_{n-1}+\epsilon)\limsup_{k\to\infty}\int_{-\infty}^{(\lambda_k-s)+}\frac{P(X_0>\lambda_k-u)}{P(X_0>\lambda_k)}P(X_n\in du),
\end{aligned}
\tag{5.9}
$$

and

$$
\begin{aligned}
&\limsup_{k\to\infty}\frac{I_3}{P(X_0>\lambda_k)}\\
\leq\quad &(d_{n-1}+\epsilon)\lim_{k\to\infty}\frac{P(X_n>\lambda_k-s)P(X_0>s)}{P(X_0>\lambda_k)}.
\end{aligned}
\tag{5.10}
$$

By virtue of the dominated convergence theorem, we see that

$$\begin{aligned} & \limsup_{k\to\infty} \frac{I_2}{P(X_0>\lambda_k)} \\ = & \int_{-\infty}^{s+} m(u;\{\lambda_k\}) P(S_{n-1}+Y\in du) \\ \le & \int_{-\infty}^{\infty} m(u;\{\lambda_k\}) P(S_{n-1}+Y\in du) \le I^*(\rho^{(n-1)*}*\mu_2). \end{aligned} \tag{5.11}$$

Note that

$$\begin{aligned} & \limsup_{k\to\infty} \frac{P(X_0+X_n>\lambda_k)}{P(X_0>\lambda_k)} \\ \ge & \limsup_{k\to\infty} \int_{-\infty}^{(\lambda_k-s)+} \frac{P(X_0>\lambda_k-u)}{P(X_0>\lambda_k)} P(X_n\in du) \\ + & \lim_{k\to\infty} \frac{P(X_n>\lambda_k-s)P(X_0>s)}{P(X_0>\lambda_k)} \\ + & \liminf_{k\to\infty} \int_{-\infty}^{s+} \frac{P(X_n>\lambda_k-u)}{P(X_0>\lambda_k)} P(X_0\in du). \end{aligned} \tag{5.12}$$

Thus we obtain from (5.9)-(5.12) that, for $n\ge 1$,

$$\begin{aligned} d_n \le & (d_{n-1}+\epsilon) \limsup_{k\to\infty} \frac{P(X_0+X_n>\lambda_k)}{P(X_0>\lambda_k)} + I^*(\rho^{(n-1)*}*\mu_2) \\ - & (d_{n-1}+\epsilon) \liminf_{k\to\infty} \int_{-\infty}^{s+} \frac{P(X_n>\lambda_k-u)}{P(X_0>\lambda_k)} P(X_0\in du) \\ \le & (d_{n-1}+\epsilon)\left(d^* - \int_{-\infty}^{s+} m(u;\{\lambda_k\})\rho(du)\right) + I^*(\rho^{(n-1)*}*\mu_2). \end{aligned}$$

Letting $s\to\infty$ and $\epsilon\to 0$, we have by Lemma 5.2 (ii)

$$d_n \le d_{n-1}(d^* - I_*(\rho)) + I^*(\rho)^{n-1} I^*(\mu_2). \tag{5.13}$$

Noting that $d_0=0$, we see from (5.13) that

$$d_n \le I^*(\mu_2) \sum_{k=0}^{n-1} I^*(\rho)^k (d^* - I_*(\rho))^{n-1-k} < \infty. \tag{5.14}$$

By Lemma 3.2 (iii), we have for $x\ge 0$

$$\begin{aligned} \frac{P(S_n+Y>x)}{P(X_0>x)} & \le \int_{-\infty}^{\infty} \frac{P(S_n>x-u)}{P(X_0>x)} P(Y\in du) \\ & \le \int_{-\infty}^{\infty} K^n \frac{P(X_0>x-u)}{P(X_0>x)} P(Y\in du) \end{aligned}$$

$$= \quad K^n \frac{P(S_1+Y>x)}{P(X_0>x)} \le K^n(d_1+\delta),$$

with some positive constants K and δ. Thus we can use Fatou's lemma and establish from (5.14) that

$$\begin{aligned} C^* &= \limsup_{x\to\infty} e^{-c_0} \sum_{n=1}^{\infty} \frac{c_0^{n-1}}{n!} \frac{P(S_n+Y>x)}{P(X_0>x)} \\ &\le e^{-c_0} \sum_{n=1}^{\infty} \frac{c_0^{n-1}}{n!} d_n \\ &\le e^{-c_0} \sum_{n=1}^{\infty} \frac{c_0^{n-1}}{n!} I^*(\mu_2) \sum_{k=0}^{n-1} I^*(\rho)^k (d^* - I_*(\rho))^{n-1-k} = J(\mu). \end{aligned}$$

We have proved the proposition. □

Proof of Theorem 2.1. Assertion (i) is due to Proposition 3.1. Assertion (ii) is due to Propositions 5.1-5.3. □

Proof of Corollary 2.1. We see from Theorem 2.1 (ii) that $0 \le \gamma^* < \infty$, $\widehat{\mu}(\gamma^*) < \infty$ and $C = C_* = C^* = \widehat{\mu}(\gamma^*)$. □

The following is due to Theorem 1 of [13] and Theorem 3.1 of [24], and conclusively to Theorem 1.1 of [31]. An interesting history of the establishment of this result is found in [31]. It has an application to the local subexponentiality of an infinitely divisible distribution. See [32]. Applications of the class $\mathscr{S}(\gamma)$ to Lévy processes are found in [4; 6; 21].

Lemma 5.3. *Let $\gamma \ge 0$. Let μ be a distribution in* $\mathbf{ID}_+$ *with Lévy measure ν. Then the following are equivalent:*

(1) *$\mu \in \mathscr{S}(\gamma)$.*

(2) *$\nu_{(1)} \in \mathscr{S}(\gamma)$.*

(3) *$\nu_{(1)} \in \mathscr{L}(\gamma)$, $\widehat{\mu}(\gamma) < \infty$, and $\bar{\mu}(x) \sim \widehat{\mu}(\gamma)\bar{\nu}(x)$.*

(4) *$\nu_{(1)} \in \mathscr{L}(\gamma)$ and, for some $C \in (0,\infty)$, $\bar{\mu}(x) \sim C\bar{\nu}(x)$.*

Remark 5.2. Let $\gamma \ge 0$. We see from the above lemma that $\mathscr{S}(\gamma) \cap \mathbf{ID}_+$ is closed under convolution roots. The class $\mathscr{S}$ is closed under convolution roots. Refer to Theorem 2 of [13] in the one-sided case and see Prposition 2.7 of [31] in the two-sided case. However, we do not know whether the class $\mathscr{S}(\gamma)$ is closed under convolution roots for $\gamma > 0$, so far. We find from Theorem 2.1 of [32] that $\mathscr{S}(\gamma)$ on $\mathbb{R}_+$ is closed under convolution roots for some (equivalently for all) $\gamma > 0$ if and only if so is the locally subexponential class on $\mathbb{R}_+$.

Proof of Corollary 2.2. Let $a \in \mathbb{R}$. Note that $\mu * \delta_{-a}$ is also an infinitely divisible distribution on $\mathbb{R}$ with the same Lévy measure as that of μ and that $\overline{\mu * \delta_{-a}}(x) = \overline{\mu}(x+a)$. Hence we see from

Corollary 2.1 that $0 \le \gamma^* < \infty$, $\widehat{\mu}(\gamma^*) < \infty$ and $C(a_j) = \widehat{\mu}(\gamma^*)\exp(-\gamma^* a_j)$ for $j = 1, 2$. Define a set E as $E := \{ma_1 + na_2 : m,\ n \in \mathbb{Z}\}$. Then we have, for any $a \in E$,

$$\overline{\mu}(x+a) \sim \exp(-\gamma^* a)\overline{\mu}(x). \tag{5.15}$$

Note that E is a dense set in $\mathbb{R}$ because a_1/a_2 is irrational. Thus we have (5.15) for any $a \in \bar{E} = \mathbb{R}$. It follws that $\mu \in \mathscr{L}(\gamma^*)$ and $\overline{\mu}(x) \sim \widehat{\mu}(\gamma^*)\overline{\nu}(x)$. Thus we conclude from Lemma 5.3 that $\mu \in \mathscr{S}(\gamma^*)$. $\square$

6 Proof of Theorem 2.2

Albin [1] asserted that if $\nu_{(1)} \in \mathscr{L}(\gamma)$, then $\mu \in \mathscr{L}(\gamma)$. His proof in the case $\gamma = 0$ is complete. However, his proof in the case $\gamma > 0$ depends on an incomplete lemma which is stated without precise proof.

Assertion (Lemma 2.1 of [1]) *Let $\rho \in \mathscr{L}(\gamma)$ be supported on $[0, \infty)$ for some $\gamma \ge 0$. Given constants $\epsilon > 0$ and $t \in \mathbb{R}$, pick a constant $x_0 \in \mathbb{R}$ such that*

$$\frac{\overline{\rho}(x-t)}{\overline{\rho}(x)} \le (1+\epsilon)e^{\gamma t} \quad \text{for } x \ge x_0. \tag{6.1}$$

Then

$$\frac{\overline{\rho^{n*}}(x-t)}{\overline{\rho^{n*}}(x)} \le (1+\epsilon)e^{\gamma t} \quad \text{for } x \ge n(x_0 - t) + t. \tag{6.2}$$

Remark 6.1. The assertion above is correct for $\gamma = 0$. But, we can make the following counterexample of this assertion for $\gamma > 0$. Indeed, let $\overline{\rho}(x) := \int_x^\infty e^{-u} du$. Then $\rho \in \mathscr{L}(1)$ and the condition (6.1) is satisfied for $x_0 = 0$. Notice that $\overline{\rho^{n*}}(x) = ((n-1)!)^{-1} \int_x^\infty u^{n-1} e^{-u} du$. Let $t < 0$ and put $c := 1 - t$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\overline{\rho^{n*}}(cn-t)}{\overline{\rho^{n*}}(cn)} &= \lim_{n\to\infty} \frac{\int_0^\infty (cn-t+u)^{n-1} e^{-(cn-t+u)} du}{\int_0^\infty (cn+u)^{n-1} e^{-(cn+u)} du} \\
&= e^t \lim_{n\to\infty} \frac{\int_0^\infty (1+\frac{-t+u}{cn})^{n-1} e^{-u} du}{\int_0^\infty (1+\frac{u}{cn})^{n-1} e^{-u} du} = e^{-c^{-1}t} \cdot e^t.
\end{aligned}$$

Take ϵ satisfying $1+\epsilon < e^{-c^{-1}t}$. The inequality (6.2) does not hold for sufficiently large n. The example for a general $\gamma > 0$ is analogous and is omitted.

Lemma 6.1. *Let ρ and η be distributions on $\mathbb{R}$. Let $\gamma \ge 0$.*

(i) (Lemma 2.5 of [31]) *If $\rho, \eta \in \mathscr{L}(\gamma)$, then $\rho * \eta \in \mathscr{L}(\gamma)$. In particular, if $\rho \in \mathscr{L}(\gamma)$, then $\rho^{n*} \in \mathscr{L}(\gamma)$ for all $n \ge 1$.*

(ii) (Lemma 1 of [5]) *If $\rho \in \mathscr{L}(\gamma)$, then, for every $a \in \mathbb{R}$, there is a positive constant $M(a)$ which depends only on a such that, for every $x \in \mathbb{R}$ and every $n \geq 1$,*

$$\overline{\rho^{n*}}(x-a) \leq M(a)\overline{\rho^{n*}}(x). \tag{6.3}$$

The following lemma is suggested by an argument in [5].

Lemma 6.2. *Let $\gamma \geq 0$, $a \in \mathbb{R}$. Suppose that $\rho \in \mathscr{L}(\gamma)$. For any ϵ with $0 < \epsilon < 1$, there is $b > 0$ such that for all $n \geq 1$ and for all $x \in \mathbb{R}$,*

$$\overline{\rho^{(n+1)*}}(x+a) \leq (1+\epsilon)e^{-\gamma a}\overline{\rho^{(n+1)*}}(x) + \overline{\rho^{n*}}(x+a-b), \tag{6.4}$$

and

$$\overline{\rho^{(n+1)*}}(x+a) \geq (1-\epsilon)e^{-\gamma a}(\overline{\rho^{(n+1)*}}(x) - \overline{\rho^{n*}}(x-b)). \tag{6.5}$$

Proof. We have

$$\begin{aligned} \overline{\rho^{(n+1)*}}(x) &= \int_{-\infty}^{(x-b)+} \overline{\rho}(x-y)\rho^{n*}(dy) + \int_{(x-b)+}^{\infty} \overline{\rho}(x-y)\rho^{n*}(dy) \\ &= K_1(x) + K_2(x). \end{aligned}$$

Let $0 < \epsilon < 1$. Since $\rho \in \mathscr{L}(\gamma)$, it follows that

$$(1-\epsilon)e^{-\gamma a} \leq \frac{\overline{\rho}(x-y+a)}{\overline{\rho}(x-y)} \leq (1+\epsilon)e^{-\gamma a}$$

for sufficiently large $b > 0$ and $y \leq x - b$. Notice that

$$K_1(x+a) = \int_{-\infty}^{(x-b)+} \frac{\overline{\rho}(x-y+a)}{\overline{\rho}(x-y)}\overline{\rho}(x-y)\rho^{n*}(dy).$$

Hence we obtain that, for sufficiently large $b > 0$,

$$(1-\epsilon)e^{-\gamma a}K_1(x) \leq K_1(x+a) \leq (1+\epsilon)e^{-\gamma a}K_1(x).$$

Here we have $K_1(x) \leq \overline{\rho^{(n+1)*}}(x)$ and $K_2(x) \leq \overline{\rho^{n*}}(x-b)$. Hence we get (6.4). Furthermore, we have

$$K_1(x) = \overline{\rho^{(n+1)*}}(x) - K_2(x) \geq \overline{\rho^{(n+1)*}}(x) - \overline{\rho^{n*}}(x-b).$$

Hence we get (6.5). ☐

Lemma 6.3. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν. Let $\gamma \geq 0$. Then $\mu \in \mathscr{L}(\gamma)$ if and only if $\mu_1 \in \mathscr{L}(\gamma)$.*

Proof. As in the proof of Theorem 1.1 of [31], we see from Lemma 2.5 of [24] that if $\mu \in \mathscr{L}(\gamma)$, then $\mu_1 \in \mathscr{L}(\gamma)$. Conversely, if $\mu_1 \in \mathscr{L}(\gamma)$, then we have $\mu \in \mathscr{L}(\gamma)$ by Lemma 2.1 of [24]. □

Proposition 6.1. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν. Let $\gamma \geq 0$. If $\nu_{(1)} \in \mathscr{L}(\gamma)$, then $\mu^{t*} \in \mathscr{L}(\gamma)$ for all $t > 0$. In the converse direction, if $\mu^{t*} \in \mathscr{L}(\gamma)$ for all $t > 0$ and $\nu_{(1)} \in \mathscr{OS}$, then $\nu_{(1)} \in \mathscr{L}(\gamma)$.*

Proof. We prove the first assertion. Suppose that $\nu_{(1)} \in \mathscr{L}(\gamma)$. Without loss of generality, we can assume that $t = 1$. Denote $\rho := \nu_{(1)}$ and $\lambda_n := e^{-c_0}c_0^n(n!)^{-1}$ with $c_0 := \nu(1,\infty)$. Note that $\lim_{n\to\infty} \lambda_{n+1}/\lambda_n = 0$. Let $0 < \epsilon < 1$. Let N be a positive integer satisfying $\lambda_{n+1} \leq \epsilon\lambda_n$ for all $n \geq N$. Define

$$I_N(x) := \sum_{n=1}^{N} \lambda_n \overline{\rho^{n*}}(x) \quad \text{and} \quad J_N(x) := \sum_{n=N+1}^{\infty} \lambda_n \overline{\rho^{n*}}(x).$$

Then $\overline{\mu_1}(x) = I_N(x) + J_N(x)$ for $x > 0$. Since $\rho \in \mathscr{L}(\gamma)$, we have by Lemma 6.1 (i)

$$(1-\epsilon)e^{-\gamma a} I_N(x) \leq I_N(x+a) \leq (1+\epsilon)e^{-\gamma a} I_N(x)$$

for sufficiently large x. We see from Lemma 6.2 that, for all $x \in \mathbb{R}$,

$$J_N(x+a) \leq \sum_{n=N+1}^{\infty} \lambda_n \overline{\rho^{n*}}(x)(1+\epsilon)e^{-\gamma a} + \sum_{n=N+1}^{\infty} \lambda_n \overline{\rho^{(n-1)*}}(x+a-b).$$

By Lemma 6.1 (ii), we have

$$\begin{aligned} \sum_{n=N+1}^{\infty} \lambda_n \overline{\rho^{(n-1)*}}(x+a-b) &= \sum_{n=N}^{\infty} \lambda_{n+1} \overline{\rho^{n*}}(x+a-b) \\ &\leq \epsilon M(b-a) \sum_{n=N}^{\infty} \lambda_n \overline{\rho^{n*}}(x). \end{aligned}$$

Hence, for all $x \in \mathbb{R}$,

$$J_N(x+a) \leq (1+\epsilon)e^{-\gamma a} J_N(x) + \epsilon M(b-a)\overline{\mu_1}(x).$$

Furthermore, we obtain from Lemmas 6.1 (ii) and 6.2 that, for all $x \in \mathbb{R}$,

$$\begin{aligned} J_N(x+a) &\geq (1-\epsilon)e^{-\gamma a}\Big\{ \sum_{n=N+1}^{\infty} \lambda_n \overline{\rho^{n*}}(x) - \sum_{n=N+1}^{\infty} \lambda_n \overline{\rho^{(n-1)*}}(x-b)\Big\} \\ &\geq (1-\epsilon)e^{-\gamma a}\Big\{ \sum_{n=N+1}^{\infty} \lambda_n \overline{\rho^{n*}}(x) - \epsilon M(b) \sum_{n=N}^{\infty} \lambda_n \overline{\rho^{n*}}(x)\Big\} \\ &\geq (1-\epsilon)e^{-\gamma a}(J_N(x) - \epsilon M(b)\overline{\mu_1}(x)). \end{aligned}$$

In consequence, it follows that, for sufficiently large x,

$$\overline{\mu_1}(x+a) \leq (1+\epsilon)e^{-\gamma a} I_N(x) + (1+\epsilon)e^{-\gamma a} J_N(x) + \epsilon M(b-a)\overline{\mu_1}(x)$$

$$= \Big((1+\epsilon)e^{-\gamma a} + \epsilon M(b-a)\Big)\overline{\mu_1}(x),$$

and

$$\begin{aligned}\overline{\mu_1}(x+a) &\geq (1-\epsilon)e^{-\gamma a}I_N(x) + (1-\epsilon)e^{-\gamma a}(J_N(x) - \epsilon M(b)\overline{\mu_1}(x)) \\ &= \Big((1-\epsilon)e^{-\gamma a} - \epsilon M(b)\Big)\overline{\mu_1}(x).\end{aligned}$$

Thus we obtain that

$$\begin{aligned}(1-\epsilon)e^{-\gamma a} - \epsilon M(b) &\leq \liminf_{x\to\infty} \frac{\overline{\mu_1}(x+a)}{\overline{\mu_1}(x)} \\ &\leq \limsup_{x\to\infty} \frac{\overline{\mu_1}(x+a)}{\overline{\mu_1}(x)} \leq (1+\epsilon)e^{-\gamma a} + \epsilon M(b-a).\end{aligned}$$

Letting $\epsilon \to 0$, we can show $\mu_1 \in \mathscr{L}(\gamma)$. Hence we have $\mu \in \mathscr{L}(\gamma)$ by Lemma 6.3.

Next we prove the second assertion. Suppose that $\mu^{t*} \in \mathscr{L}(\gamma)$ for any $t > 0$ and $\nu_{(1)} \in \mathscr{OS}$. We obtain from Lemma 6.3 that $\mu_1^{t*} \in \mathscr{L}(\gamma)$ for any $t > 0$. Define $C_*(t)$ and $C^*(t)$ for $t > 0$ as

$$C_*(t) := \liminf_{r\to\infty} \frac{\overline{\mu_1^{t*}}(r)}{t\overline{\nu}(r)},$$

and

$$C^*(t) := \limsup_{r\to\infty} \frac{\overline{\mu_1^{t*}}(r)}{t\overline{\nu}(r)}.$$

Since μ_1^{t*} is one-sided, we find from Proposition 2 of [11] that $1 \leq C_*(t)$. We see from Theorem 2.1 (ii) that $C^*(t) \leq J(\mu_1^{t*})$. Since $I^*(\delta_0(dx)) = 1$, we can represent $J(\mu_1^{t*})$, for $B = 0$, as

$$J(\mu_1^{t*}) = \exp(c_0 t(I^*(\nu_{(1)}) - 1)),$$

and, for $B > 0$, as

$$J(\mu_1^{t*}) = \exp(c_0 t(I^*(\nu_{(1)}) - 1))\frac{\exp(c_0 Bt) - 1}{c_0 Bt}.$$

Thus we have $\lim_{t\to 0} J(\mu_1^{t*}) = 1$. Hence we obtain that

$$\lim_{t\to 0} C_*(t) = \lim_{t\to 0} C^*(t) = 1.$$

Thus we see that, for any $a \in \mathbb{R}$,

$$\begin{aligned}e^{-\gamma a} &= \lim_{t\to 0}\liminf_{r\to\infty} \frac{\overline{\mu_1^{t*}}(r+a)}{\overline{\mu_1^{t*}}(r)} = \liminf_{r\to\infty} \frac{\overline{\nu}(r+a)}{\overline{\nu}(r)} \\ &\leq \limsup_{r\to\infty} \frac{\overline{\nu}(r+a)}{\overline{\nu}(r)} = \lim_{t\to 0}\limsup_{r\to\infty} \frac{\overline{\mu_1^{t*}}(r+a)}{\overline{\mu_1^{t*}}(r)} = e^{-\gamma a}.\end{aligned}$$

That is, $\nu_{(1)} \in \mathscr{L}(\gamma)$. $\square$

Now we prove Theorem 2.2. We can obtain the same upper bound of C^* as in (2.3) again by using Corollary 2.6 (ii) of [7].

Proof of Theorem 2.2. Note that $\gamma^* = \gamma$. Assertion (i) is due to Proposition 6.1.

Next we prove (ii). Since $\nu_{(1)} \in \mathscr{L}(\gamma) \subset \mathscr{OL}$, we see that $m(x;\{\lambda_n\}) = e^{\gamma x}$ for any $\{\lambda_n\} \in \Lambda$ and thereby obtain from Proposition 5.2 that

$$C_* \geq I_*(\mu) = \widehat{\mu}(\gamma).$$

Hence we see from Lemma 5.1 (ii) that $C_* = \widehat{\mu}(\gamma) \in (0, \infty]$.

Lastly, we prove assertions (1)-(3) in (ii). If $d^* = 2\widehat{\nu}_{(1)}(\gamma) < \infty$, then $\nu_{(1)} \in \mathscr{S}(\gamma)$. It follows from Lemma 5.3 that $C_* = C^* = \widehat{\mu}(\gamma)$. Suppose that $2\widehat{\nu}_{(1)}(\gamma) < d^* < \infty$. Then $\nu_{(1)} \in \mathscr{OS} \setminus \mathscr{S}(\gamma)$. We have $C^* \geq \widehat{\mu}(\gamma)$ by Theorem 2.1. Next suppose that $C^* = \widehat{\mu}(\gamma)$. Since $C_* = \widehat{\mu}(\gamma)$, we have $\overline{\mu}(r) \sim \widehat{\mu}(\gamma)\overline{\nu}(r)$. As we have $\nu_{(1)} \in \mathscr{L}(\gamma)$, we obtain from Lemma 5.3 that $\nu_{(1)} \in \mathscr{S}(\gamma)$. This is a contradiction. Hence $C^* > \widehat{\mu}(\gamma)$. Since $I_*(\nu_{(1)}) = I^*(\nu_{(1)}) = \widehat{\nu}_{(1)}(\gamma)$ and $I^*(\mu_2) = \widehat{\mu}_2(\gamma)$, we have $B = d^* - 2\widehat{\nu}_{(1)}(\gamma)$ and

$$J(\mu) = \widehat{\mu}(\gamma)\frac{\exp(c_0(d^* - 2\widehat{\nu}_{(1)}(\gamma))) - 1}{c_0(d^* - 2\widehat{\nu}_{(1)}(\gamma))}.$$

Thus we have (2.3) by (2.2). If $d^* = \infty$, then $\nu_{(1)} \notin \mathscr{OS}$. We see from Proposition 3.1 (ii) that $C^* = \infty$. $\square$

Remark 6.2. Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν. Suppose that $\nu_{(1)} \in \mathscr{OS}$. Then $\mu \in \mathscr{L}$ if and only if $\nu_{(1)} \in \mathscr{L}$.

Proof. Suppose that $\mu \in \mathscr{L}$ and $\nu_{(1)} \in \mathscr{OS}$. Let $\rho := \nu_{(1)}$. Then we have $\mu_1 \in \mathscr{L}$ by Lemma 6.3. Thus we see that

$$0 = \lim_{r\to\infty} \frac{\overline{\mu_1}(r) - \overline{\mu_1}(r+1)}{\overline{\mu_1}(r)} \geq e^{-c_0} c_0 \limsup_{r\to\infty} \frac{\overline{\rho}(r) - \overline{\rho}(r+1)}{\overline{\mu_1}(r)} \geq 0. \tag{6.6}$$

Since $\rho := \nu_{(1)} \in \mathscr{OS}$, we have $\overline{\rho}(r) \asymp \overline{\mu_1}(r)$ by Proposition 3.1. Hence we obtain from (6.6) that

$$\lim_{r\to\infty} \frac{\overline{\rho}(r) - \overline{\rho}(r+1)}{\overline{\rho}(r)} = 0,$$

that is, $\nu_{(1)} = \rho \in \mathscr{L}$. Conversely, we see from Proposition 6.1 that if $\nu_{(1)} \in \mathscr{L}$, then $\mu \in \mathscr{L}$. $\square$

7 Proof of Theorem 2.3.

Among the classes in Definitions 1.1 and 2.1, only the classes $\mathscr{D}$ and $\mathscr{H}$ are closed under convolution and under convolution roots, simultaneously. The proof for $\mathscr{H}$ is easy and that for $\mathscr{D}$ is as follows.

Lemma 7.1. *Let ρ and η be distributions on $\mathbb{R}$.*
(i) *If $\overline{\rho}(r) \asymp \overline{\eta}(r)$ for some $\eta \in \mathscr{D}$, then $\rho \in \mathscr{D}$.*

(ii) *If $\rho^{n*} \in \mathscr{D}$ for some $n \geq 1$, then $\rho \in \mathscr{D}$.*
(iii) *If $\rho, \eta \in \mathscr{D}$, then $\rho * \eta \in \mathscr{D}$. In particular, if $\rho \in \mathscr{D}$, then $\rho^{n*} \in \mathscr{D}$ for all $n \geq 1$.*

Proof. Assertion (i) is obvious by the definition of the class $\mathscr{D}$. For (ii), suppose that $\rho^{n*} \in \mathscr{D}$. Since $\overline{\rho^{n*}}(r) \asymp \overline{(\rho_+)^{n*}}(r)$, we have $(\rho_+)^{n*} \in \mathscr{D}$ and, by Proposition 1.1 (iii) of [27], $\rho_+ \in \mathscr{D}$ and hence $\rho \in \mathscr{D}$. If $\rho, \eta \in \mathscr{D} \subset \mathscr{OS}$, then $\rho_+, \eta_+ \in \mathscr{D} \subset \mathscr{OS}$ and (3.2) holds. Thus we see from Proposition 2.3 (ii) of [27] that $\rho_+ * \eta_+ \in \mathscr{D}$ and thus from (3.2) and (i) that $\rho * \eta \in \mathscr{D}$. The second assertion of (iii) is obvious. □

Lemma 7.2. *Let ρ be a distribution on $\mathbb{R}$.*
(i) *If $\rho \in \mathscr{D}$, then, for all $n \geq 1$ and $a \geq 0$,*

$$\limsup_{r\to\infty} \frac{\overline{\rho^{n*}}(r-a)}{\overline{\rho}(r)} \quad \leq \quad n \cdot \lim_{N\to\infty} \limsup_{r\to\infty} \frac{\overline{\rho}(r-N)}{\overline{\rho}(r)}. \tag{7.1}$$

(ii) *If $\rho \in \mathscr{D}$, then there are two positive constants c_1 and c_2 such that, for all $n \geq 1$ and $r > 0$,*

$$\frac{\overline{\rho^{n*}}(r)}{\overline{\rho}(r)} \leq c_1 \cdot n^{c_2+1}. \tag{7.2}$$

Proof. First we prove (i). Let $\{X_j\}_{j=1}^{\infty}$ be i.i.d. random variables with a common distribution ρ. Let $r > a + (n-1)N_1$ with $N_1 > 0$. We have

$$\begin{aligned}
&\overline{\rho^{n*}}(r-a) \\
&= P\left(\sum_{j=1}^{n} X_j > r-a,\ X_{j_0} > n^{-1}(r-a) \text{ for some } j_0 \text{ with } 1 \leq j_0 \leq n\right) \\
&\leq \sum_{j_0=1}^{n} P\left(\sum_{j=1}^{n} X_j > r-a,\ X_{j_0} > n^{-1}(r-a),\ X_i \leq N_1 \text{ for any } i \neq j_0\right) \\
&+ \sum_{j_0=1}^{n} P\left(\sum_{j=1}^{n} X_j > r-a,\ X_{j_0} > n^{-1}(r-a),\ X_i > N_1 \text{ for some } i \neq j_0\right) \\
&\leq n\overline{\rho}(r-a-(n-1)N_1) + n(n-1)\overline{\rho}(N_1)\overline{\rho}(n^{-1}(r-a)).
\end{aligned} \tag{7.3}$$

Here, if $\rho \in \mathscr{D}$, then there are two positive constants c_1 and c_2 such that, for any $n \geq 1$ and $r > 0$,

$$\frac{\overline{\rho}(n^{-1}r)}{\overline{\rho}(r)} \leq c_1 n^{c_2}. \tag{7.4}$$

Thus we see from (7.3) and (7.4) that if $\rho \in \mathscr{D}$, then

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\overline{\rho^{n*}}(r-a)}{\overline{\rho}(r)} \quad &\leq \quad \lim_{N_1\to\infty} \limsup_{r\to\infty} \frac{n\overline{\rho}(r-a-(n-1)N_1)}{\overline{\rho}(r)} \\
&= \quad n \cdot \lim_{N\to\infty} \limsup_{r\to\infty} \frac{\overline{\rho}(r-N)}{\overline{\rho}(r)}.
\end{aligned}$$

As we have

$$\overline{\rho^{n*}}(r) = P\left(\sum_{j=1}^{n} X_j > r\right) \le \sum_{j=1}^{n} P(X_j > n^{-1}r) = n\overline{\rho}(n^{-1}r),$$

it follows from (7.4) that (ii) holds. □

Lemma 7.3. *Let μ be a distribution in $\mathbf{ID}_+$ with Lévy measure ν.*
(i) *If $\nu_{(1)} \in \mathscr{D}$, then $0 < Q_* \le 1 \le Q^* < \infty$.*
(ii) *If $\nu_{(1)} \in \mathscr{D}$ and μ is a compound Poisson distribution on $\mathbb{R}_+$, then*

$$C^* \le \lim_{N\to\infty} \limsup_{r\to\infty} \frac{\overline{\mu}(r-N)}{\overline{\nu}(r)} \le Q^*. \tag{7.5}$$

Proof. We prove (i). Suppose that $\nu_{(1)} \in \mathscr{D}$. Then we have

$$1 \ge Q_* \ge \liminf_{r\to\infty} \frac{\overline{\nu}(2r)}{\overline{\nu}(r)} > 0,$$

and

$$1 \le Q^* = (Q_*)^{-1} < \infty.$$

Next we prove (ii). Let $r > 0$. Notice that $\overline{\mu}(r) = e^{-a}\sum_{n=1}^{\infty} a^n (n!)^{-1}\overline{\rho^{n*}}(r)$, where $a := \nu(\mathbb{R}_+) = \overline{\nu}(0)$ and $\rho := a^{-1}\nu$. Suppose that $\nu_{(1)} \in \mathscr{D}$. Let $N_1 > 0$. Now we can use Fatou's lemma by virtue of Lemma 7.2 (ii). Thus we see from Lemma 7.2 (i) that

$$\begin{aligned} C^* &\le \limsup_{r\to\infty} \frac{\overline{\mu}(r-N_1)}{\overline{\nu}(r)} \\ &\le e^{-a}\sum_{n=1}^{\infty} \frac{a^n}{n!} \limsup_{r\to\infty} \frac{\overline{\rho^{n*}}(r-N_1)}{a\overline{\rho}(r)} \le Q^*. \end{aligned}$$

As $N_1 \to \infty$, we get (7.5). □

Now we prove Theorem 2.3.

Proof of Theorem 2.3. We prove (i). If $\nu_{(1)} \in \mathscr{D} \subset \mathscr{OS}$, we see from Theorem 2.1 and Lemma 7.1 (i) that $\overline{\mu}(r) \asymp \overline{\nu}(r)$ and $\mu \in \mathscr{D}$. Conversely, suppose that $\mu \in \mathscr{D} \subset \mathscr{OS}$. Then we see from Proposition 3.1 (iii) that $(\nu_{(1)})^{k*} \in \mathscr{OS}$ for some $k \ge 1$ and $\overline{\mu}(r) \asymp \overline{(\nu_{(1)})^{k*}}(r)$. It follows from Lemma 7.1 that $(\nu_{(1)})^{k*} \in \mathscr{D}$ and $\nu_{(1)} \in \mathscr{D}$.
We show (ii). Let Y_1 and Y_2 be independent random variables with distribution μ_1 and μ_2, respectively. Suppose that $\nu_{(1)} \in \mathscr{D}$. It follows from Lemma 7.3 (i) that $0 < Q_* \le 1 \le Q^* < \infty$. Since $\mathscr{D} \subset \mathscr{H}$, we see from Proposition 4.1 that $C_* \le 1 \le C^*$. Let $r > 2N > 0$. Then we have

$$\overline{\mu}(r) = P(Y_1 + Y_2 > r) \le \sum_{k=1}^{3} J_k,$$

where $J_1 = P(Y_1 + Y_2 > r,\ |Y_2| \le N)$, $J_2 = P(Y_1 + Y_2 > r,\ |Y_1| \le N)$, and $J_3 = P(Y_1 + Y_2 > r,\ |Y_1| > N,\ |Y_2| > N)$. Now we see from Lemma 7.3 (ii) that

$$\lim_{N\to\infty} \limsup_{r\to\infty} \frac{J_1}{\overline{\nu}(r)} \quad \le \quad \lim_{N\to\infty} \limsup_{r\to\infty} \frac{P(Y_1 > r - N)}{\overline{\nu}(r)} \le Q^*.$$

Since $\nu_{(1)} \in \mathscr{D} \subset \mathscr{OS} \subset \mathscr{OL}$, we see from Proposition 3.1 and Lemma 3.3 that

$$\limsup_{r\to\infty} \frac{J_2}{\overline{\nu}(r)} \quad \le \quad \limsup_{r\to\infty} \frac{P(Y_2 > r - N)}{\overline{\mu_1}(r-N)} \frac{\overline{\mu_1}(r-N)}{\overline{\nu}(r-N)} \frac{\overline{\nu}(r-N)}{\overline{\nu}(r)} = 0.$$

Using Lemma 7.3 (ii) again, we obtain that

$$\begin{aligned} & \lim_{N\to\infty} \limsup_{r\to\infty} \frac{J_3}{\overline{\nu}(r)} \\ \le \quad & \lim_{N\to\infty} \limsup_{r\to\infty} \left\{ \frac{P(Y_1 > 2^{-1}r)}{\overline{\nu}(2^{-1}r)} \frac{\overline{\nu}(2^{-1}r)}{\overline{\nu}(r)} P(|Y_2| > N) \right. \\ & \left. \qquad + \frac{P(Y_2 > 2^{-1}r)}{\overline{\mu_1}(2^{-1}r)} \frac{\overline{\mu_1}(2^{-1}r)}{\overline{\nu}(2^{-1}r)} \frac{\overline{\nu}(2^{-1}r)}{\overline{\nu}(r)} P(|Y_1| > N) \right\} = 0. \end{aligned}$$

Consequently, we have obtained (2.5). We have, for $\{\lambda_n\} \in \Lambda$, $Q_* \le m(x;\{\lambda_n\})$ for $x < 0$ and $1 \le m(x;\{\lambda_n\})$ for $x \ge 0$. Thus we obtain from Theorem 2.1 (ii) that

$$C_* \ge I_*(\mu) \ge \mu[0,\infty) + Q_*\mu(-\infty,0) = 1 - (1 - Q_*)\mu(-\infty,0).$$

Thus we have got (2.4). $\square$

As is mentioned in Section 2, Denisov et al. [9] proved that if μ is a distribution on $\mathbb{R}_+$ in $\mathscr{H} \cap \mathbf{ID}_+$, then $C_* = 1$. We show in the following example that there exists $\mu \in \mathscr{H} \cap \mathbf{ID}_+$ such that $C_* < 1$.

Example 7.1. Let ρ be the Peter and Paul distribution, that is,

$$\rho(dx) := \sum_{n=1}^{\infty} 2^{-n} \delta_{2^n}(dx).$$

Let $0 < \lambda < 1$. Let μ_1 be a compound Poisson distribution with Lévy measure $\lambda\rho$. Let μ_2 be a continuous infinitely divisible distribution on $(-\infty, 0]$. Define $\mu \in \mathbf{ID}_+$ as $\mu = \mu_1 * \mu_2$. Then we have $\mu \in \mathscr{D}$ and $Q^* = 2$. Moreover, we can take λ sufficiently small such that $Q_* < C_* < 1 < C^* < Q^*$.

Proof. Since $\nu_{(1)} = \rho \in \mathscr{D}$, we have $\mu \in \mathscr{D}$ by Theorem 2.3 and, obviously, $Q^* = 2$. Note that μ is continuous, because μ_2 is so. Thus we can take a positive sequence $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\lim_{n\to\infty} \frac{\bar{\mu}(2^n - \epsilon_n)}{\bar{\mu}(2^n)} = 1.$$

Hence we have

$$C_* \quad \le \quad \liminf_{n\to\infty} \frac{\bar{\mu}(2^n - \epsilon_n)}{\lambda\bar{\rho}(2^n - \epsilon_n)}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} \frac{\bar{\rho}(2^n)}{\bar{\rho}(2^n - \epsilon_n)} \limsup_{n\to\infty} \frac{\bar{\mu}(2^n)}{\lambda \bar{\rho}(2^n)} \\
&\leq 2^{-1} C^*. \qquad (7.6)
\end{aligned}
$$

Let $\{\lambda_n\} \in \Lambda$. Note that $m(x; \{\lambda_n\})$ and $I^*(\mu_2)$ do not depend on the value of λ. Since $m(x; \{\lambda_n\}) \leq 1$ for $x \leq 0$, we see that $I^*(\mu_2) \leq 1$. Thus we have, for $B = 0$,

$$
\lim_{\lambda\to 0} J(\mu) = \lim_{\lambda\to 0} I^*(\mu_2) \exp(\lambda(I^*(\rho) - 1)) = I^*(\mu_2) \leq 1,
$$

and, for $B > 0$,

$$
\lim_{\lambda\to 0} J(\mu) = \lim_{\lambda\to 0} I^*(\mu_2) \exp(\lambda(I^*(\rho) - 1)) \frac{\exp(\lambda B) - 1}{\lambda B} = I^*(\mu_2) \leq 1.
$$

Hence we obtain from Theorem 2.1 (ii) that

$$
1 \leq \lim_{\lambda\to 0} C^* \leq \lim_{\lambda\to 0} J(\mu) = I^*(\mu_2) \leq 1,
$$

that is, $I^*(\mu_2) = 1$ and $\lim_{\lambda\to 0} C^* = 1$. Thus we conclude from (7.6) and Theorem 2.3 (ii) that, for sufficiently small $\lambda > 0$,

$$
2^{-1} < 1 - 2^{-1}\mu(-\infty, 0) \leq C_* \leq 2^{-1}C^* < 1 < 2C_* \leq C^* < 2.
$$

We have proved all the assertion. □

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