Strong hydrodynamic limit for attractive particle systems on ℤ

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Abstract

We prove almost sure Euler hydrodynamics for a large class of attractive particle systems on ℤ starting from an arbitrary initial profile. We generalize earlier works by Seppäläinen (1999) and Andjel et al. (2004). Our constructive approach requires new ideas since the subadditive ergodic theorem (central to previous works) is no longer effective in our setting.

Key words: Strong (a.s.) hydrodynamics, attractive particle system, graphical construction, non-explicit invariant measures, non-convex or non-concave flux, entropy solution, Glimm scheme.

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1 Introduction

Hydrodynamic limit ([14; 40; 25; 42]) is a law of large numbers for the time evolution (usually described by a limiting PDE, called the hydrodynamic equation) of empirical density fields in interacting particle systems (*IPS*). In most known results, only a weak law of large numbers is established. In this description one need not have an explicit construction of the dynamics: the limit is shown in probability with respect to the law of the process, which is characterized in an abstract way by its Markov generator and Hille-Yosida's theorem ([30]). Nevertheless, when simulating particle systems, one naturally uses a pathwise construction of the process on a Poisson space-time random graph (the so-called "graphical construction"). In this description the dynamics is deterministically driven by a random space-time measure which tells when and where the configuration has a chance of being modified. It is of special interest to show that the hydrodynamic limit holds for almost every realization of the space-time measure, as this means a single simulation is enough to approximate solutions of the limiting PDE. In a sense this is comparable to the interest of proving strong (rather than weak) consistency for statistical estimators, by which one knows that a single path observation is enough to estimate parameters.

Most usual IPS can be divided into two groups, diffusive and hyperbolic. In the first group, which contains for instance the symmetric or mean-zero asymmetric exclusion process, the macroscopic→microscopic space-time scaling is $(x, t) \mapsto (Nx, N^2 t)$ with $N \to \infty$, and the limiting PDE is a diffusive equation. In the second group, which contains for instance the nonzero mean asymmetric exclusion process, the scaling is $(x, t) \mapsto (Nx, Nt)$, and the limiting PDE is of Euler type. In both groups this PDE often exhibits nonlinearity, either via the diffusion coefficient in the first group, or via the flux function in the second one. This raises special difficulties in the hyperbolic case, due to shocks and non-uniqueness for the solution of the PDE, in which case the natural problem is to establish convergence to the so-called "entropy solution" ([39]). In the diffusive class it is not so necessary to specifically establish strong laws of large numbers, because one has a fairly robust method ([26]) to establish large deviations from the hydrodynamic limit. Large deviation upper bound and Borel Cantelli's lemma imply a strong law of large numbers as long as the large deviation functional is lower-semicontinuous and has a single zero. The situation is quite different in the hyperbolic class, where large deviation principles are much more difficult to obtain. So far only the remarkable result of Jensen and Varadhan ([22] and [43]) is available, and it applies only to the one-dimensional totally asymmetric simple exclusion process. Besides, the fact that the resulting large deviation functional has a single zero is not at all obvious: it follows only from recent and refined work on conservation laws ([13]). An even more difficult situation occurs for particle systems with nonconvex and nonconcave flux function, for which the Jensen-Varadhan large deviation functional does not have a unique zero, due to the existence of non-entropic solutions satisfying a single entropy inequality ([21]). In this case a more complicated rate functional can actually be conjectured from [5; 32].

The derivation of hyperbolic equations as hydrodynamic limits began with the seminal paper [36], which established a strong law of large numbers for the totally asymmetric simple exclusion process on $\mathbb{Z}$, starting with 1's to the left of the origin and 0's to the right. This result was extended by [6] and [1] to nonzero mean exclusion process starting from product Bernoulli distributions with arbitrary densities λ to the left and ρ to the right (the so-called "Riemann" initial condition). The Bernoulli distribution at time 0 is related to the fact that uniform Bernoulli measures are

invariant for the process. For the one-dimensional totally asymmetric K-exclusion process, which does not have explicit invariant measures, a strong hydrodynamic limit was established in [37], starting from arbitrary initial profiles, by means of the so-called "variational coupling". These are the only strong laws available so far. A common feature of these works is the use of sub-additive ergodic theorem to exhibit some a.s. limit, which is then identified by additional arguments.

On the other hand, many *weak* laws of large numbers were established for attractive particle systems. A first series of results treated systems with product invariant measures and product initial distributions. In [2], for a particular zero-range model, a weak law was deduced from conservation of local equilibrium under Riemann initial condition. It was then extended in [3] to the misanthrope's process of [12] under an additional convexity assumption on the flux function. These were substantially generalized (using Kružkov's entropy inequalities, see [28]) in [34] to multidimensional attractive systems with arbitrary Cauchy data, without any convexity requirement on the flux. For systems with unknown invariant measures, the result of [37] was later extended to other models, though only through weak laws. In [35], using semigroup point of view, hydrodynamic limit was established for the one-dimensional nearest-neighbor K-exclusion process. In [8] we studied fairly general attractive systems, employing a constructive approach we had initiated in [7]. This method is based on an approximation scheme (to go from Riemann to Cauchy initial data) and control of some distance between the particle system and the entropy solution to the hydrodynamic equation.

In this paper we prove a strong law of large numbers, starting from arbitrary initial profiles, for finite-range attractive particle systems on $\mathbb{Z}$ with bounded occupation number. We need no assumption on the flux, invariant measures or microscopic structure of initial distributions. This includes finite-range exclusion and K-exclusion processes, more general misanthrope-type models and k-step exclusion processes. We proceed in the constructive spirit of [7; 8], which is adapted to a.s. convergence, though this requires a novel approach for the Riemann part, and new error analysis for the Cauchy part. The implementation of an approximation scheme involves Riemann hydrodynamics from *any space-time shifted initial position*, which cannot be obtained through subadditive ergodic theorem, and thus prevents us from using the a.s. result of [1]. This can be explained as follows (see Appendix B for more details). Assume that on a probability space we have a shift operator θ that preserves the probability measure, and a sequence $(X_n)_{n \in \mathbb{N}}$ of real-valued random variables converging a.s. to a constant x. Then we can conclude that the sequence $(Y_n := X_n \circ \theta^n)_n$ converges *in probability* to x, but not necessarily almost surely. Indeed, for fixed n, Y_n has the same distribution as X_n, but the sequence $(Y_n)_n$ need not have the same distribution as $(X_n)_n$. So the derivation of a.s. convergence for Y_n is a case by case problem depending on how it was obtained for X_n. In particular, if convergence of X_n was established from the subadditivity property

$$X_{n+m} \leq X_n + X_m \circ \theta^n$$

this property is no longer satisfied by Y_n. In contrast, if convergence of X_n was established by large deviation estimates for X_n, these estimates carry over to Y_n, thus also implying a.s. convergence of $(Y_n)_n$. In our context the random variable X_n is a current in the Riemann setting. In order to handle the Cauchy problem we have to establish a.s. convergence for a shifted version Y_n of this current. We derive asymptotics for Y_n by means of large deviation arguments.

The paper is organized as follows. In Section 2, we define the model, give its graphical representation, its monotonicity properties, and state strong hydrodynamics. In Section 3, we derive almost sure Riemann hydrodynamics. In Section 4 we prove the result starting from any initial profile.

2   Notation and results

Throughout this paper $\mathbb{N} = \{1, 2, ...\}$ will denote the set of natural numbers, $\mathbb{Z}^+ = \{0, 1, 2, ...\}$ the set of non-negative integers, and $\mathbb{R}^{+*} = \mathbb{R}^+ - \{0\}$ the set of positive real numbers. The integer part of $x \in \mathbb{R}$, denoted by $[x] \in \mathbb{Z}$, is uniquely defined by $[x] \leq x < [x]+1$. We consider particle systems on $\mathbb{Z}$ with at most K particles per site, $K \in \mathbb{N}$. Thus the state space of the process is $\mathbf{X} = \{0, 1, \cdots, K\}^{\mathbb{Z}}$, which we endow with the product topology, that makes $\mathbf{X}$ a compact metrisable space. A function defined on $\mathbf{X}$ is called *local* if it depends on the variable $\eta \in \mathbf{X}$ only through $(\eta(x), x \in \Lambda)$ for some finite subset Λ of $\mathbb{Z}$. We denote by τ_x, either the spatial translation operator on the real line for $x \in \mathbb{R}$, defined by $\tau_x y = x + y$, or its restriction to $\mathbb{Z}$ for $x \in \mathbb{Z}$. By extension, we set $\tau_x f = f \circ \tau_x$ if f is a function defined on $\mathbb{R}$; $\tau_x \eta = \eta \circ \tau_x$, for $x \in \mathbb{Z}$, if $\eta \in \mathbf{X}$ is a particle configuration (that is a particular function on $\mathbb{Z}$); $\tau_x \mu = \mu \circ \tau_x^{-1}$ if μ is a measure on $\mathbb{R}$ or on $\mathbf{X}$. We let $\mathscr{M}^+(\mathbb{R})$ denote the set of positive measures on $\mathbb{R}$ equipped with the metrizable topology of vague convergence, defined by convergence on continuous test functions with compact support. The set of probability measures on $\mathbf{X}$ is denoted by $\mathscr{P}(\mathbf{X})$. If η is an $\mathbf{X}$-valued random variable and $\nu \in \mathscr{P}(\mathbf{X})$, we write $\eta \sim \nu$ to specify that η has distribution ν. The notation $\nu(f)$, where f is a real-valued function and $\nu \in \mathscr{P}(\mathbf{X})$, will be an alternative for $\int_{\mathbf{X}} f \, d\nu$. We say a sequence $(\nu_n, n \in \mathbb{N})$ of probability measures on $\mathbf{X}$ converges weakly to some $\nu \in \mathscr{P}(\mathbf{X})$, if and only if $\nu_n(f) \to \nu(f)$ as $n \to \infty$ for every continuous function f on $\mathbf{X}$. The topology of weak convergence is metrizable and makes $\mathscr{P}(\mathbf{X})$ compact.

2.1   The system and its graphical construction

We consider Feller processes on $\mathbf{X}$ whose transitions consist of particle jumps encoded by the Markov generator

$$Lf(\eta) = \sum_{x,y \in \mathbb{Z}} p(y - x) b(\eta(x), \eta(y)) \left[f\left(\eta^{x,y}\right) - f(\eta) \right] \tag{1}$$

for any local function f, where $\eta^{x,y}$ denotes the new state after a particle has jumped from x to y (that is $\eta^{x,y}(x) = \eta(x) - 1$, $\eta^{x,y}(y) = \eta(y) + 1$, $\eta^{x,y}(z) = \eta(z)$ otherwise), p is the particles' jump kernel, that is a probability distribution on $\mathbb{Z}$, and $b : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$ is the jump rate. We assume that p and b satisfy :

(A1) The semigroup of $\mathbb{Z}$ generated by the support of p is $\mathbb{Z}$ itself (irreducibility);
(A2) p has a finite first moment, that is $\sum_{z \in \mathbb{Z}} |z| p(z) < +\infty$;
(A3) $b(0, .) = 0$, $b(., K) = 0$ (no more than K particles per site), and $b(1, K - 1) > 0$;
(A4) b is nondecreasing (nonincreasing) in its first (second) argument.

We denote by $(S(t), t \in \mathbb{R}^+)$ the semigroup generated by L. Without additional algebraic relations satisfied by b (see [12]), the system in general has no explicit invariant measures. This is the case even for *K-exclusion process* with $K \geq 2$ (see [37]), where $b(n, m) = \mathbf{1}_{\{n > 0, m < K\}}$.

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Remark 2.1. *For the sake of simplicity, we restrict our attention to processes of "misanthrope type", for which particles leave more likely crowded sites for less occupied ones (cf. [12]). But our method can be applied to a much larger class of attractive models that can be constructed through a graphical representation, such as k-step exclusion (see [18] and [7]).*

We now describe the graphical construction of the system given by (1), which uses the Harris representation ([19; 20], [30, p. 172], [9, p. 119], [31, p. 215]); see for instance [1] for details and justifications. This enables us to define the evolution from different initial configurations simultaneously on the same probability space. We consider the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ of measures ω on $\mathbb{R}^+ \times \mathbb{Z}^2 \times [0,1]$ of the form

$$\omega(dt, dx, dz, du) = \sum_{m \in \mathbb{N}} \delta_{(t_m, x_m, z_m, u_m)}$$

where $\delta_{(.)}$ denotes Dirac measure, and $(t_m, x_m, z_m, u_m)_{m \geq 0}$ are pairwise distinct and form a locally finite set. The σ-field $\mathcal{F}$ is generated by the mappings $\omega \mapsto \omega(S)$ for Borel sets S. The probability measure $\mathbf{P}$ on Ω is the one that makes ω a Poisson process with intensity

$$m(dt, dx, dz, du) = ||b||_\infty \lambda_{\mathbb{R}^+}(dt) \times \lambda_{\mathbb{Z}}(dx) \times p(dz) \times \lambda_{[0,1]}(du)$$

where λ denotes either the Lebesgue or the counting measure. We denote by $\mathbf{E}$ the corresponding expectation. Thanks to assumption *(A2)*, for $\mathbf{P}$-a.e. ω, there exists a unique mapping

$$(\eta_0, t) \in \mathbf{X} \times \mathbb{R}^+ \mapsto \eta_t = \eta_t(\eta_0, \omega) \in \mathbf{X} \tag{2}$$

satisfying: (a) $t \mapsto \eta_t(\eta_0, \omega)$ is right-continuous; (b) $\eta_0(\eta_0, \omega) = \eta_0$; (c) for $t \in \mathbb{R}^+$, $(x, z) \in \mathbb{Z}^2$, $\eta_t = \eta_{t^-}^{x, x+z}$ if

$$\exists u \in [0,1] : \omega\{(t, x, z, u)\} = 1 \text{ and } u \leq \frac{b(\eta_{t^-}(x), \eta_{t^-}(x+z))}{||b||_\infty} \tag{3}$$

and (d) for all $s, t \in \mathbb{R}^{+*}$ and $x \in \mathbb{Z}$,

$$\omega\{[s, t] \times Z_x \times (0, 1)\} = 0 \Rightarrow \forall v \in [s, t], \eta_v(x) = \eta_s(x) \tag{4}$$

where

$$Z_x := \left\{ (y, z) \in \mathbb{Z}^2 : y = x \text{ or } y + z = x \right\}$$

In short, (3) tells how the state of the system can be modified by an "ω-event", and (4) says that the system cannot be modified outside ω-events. This defines a Feller process with generator (1): that is for any $t \in \mathbb{R}^+$ and $f \in C(\mathbf{X})$ (the set of continuous functions on $\mathbf{X}$), $S(t)f \in C(\mathbf{X})$ where $S(t)f(\eta_0) = \mathbf{E}[f(\eta_t(\eta_0, \omega))]$.

An equivalent formulation is the following. For each $(x, z) \in \mathbb{Z}^2$, let $\{T_n^{x,z}, n \geq 1\}$ be the arrival times of mutually independent rate $||b||_\infty p(z)$ Poisson processes, let $\{U_n^{x,z}, n \geq 1\}$ be mutually independent (and independent of the Poisson processes) random variables, uniform on $[0, 1]$. At time $t = T_n^{x,z}$, the configuration η_{t^-} becomes $\eta_{t^-}^{x, x+z}$ if $U_n^{x,z} \leq \dfrac{b(\eta_{t^-}(x), \eta_{t^-}(x+z))}{||b||_\infty}$, and stays unchanged otherwise.

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Remark 2.2. *For the K-exclusion process, b takes its values in $\{0,1\}$, the probability space can be reduced to measures $\omega(dt, dx, dz)$ on $\mathbb{R}^+ \times \mathbb{Z}^2$, and (3) to $\eta_t = \eta_{t^-}^{x,x+z}$ if $\eta_{t^-}(x) > 0$ and $\eta_{t^-}(x+z) < K$. One recovers exactly the graphical construction presented in e.g. [38] or [1].*

One may further introduce an "initial" probability space $(\Omega_0, \mathscr{F}_0, \mathbf{P}_0)$, large enough to construct random initial configurations $\eta_0 = \eta_0(\omega_0)$ for $\omega_0 \in \Omega_0$. The general process with random initial configurations is constructed on the enlarged space $(\widetilde{\Omega} = \Omega_0 \times \Omega, \widetilde{\mathscr{F}} = \sigma(\mathscr{F}_0 \times \mathscr{F}), \widetilde{\mathbf{P}} = \mathbf{P}_0 \otimes \mathbf{P})$ by setting

$$\eta_t(\widetilde{\omega}) = \eta_t(\eta_0(\omega_0), \omega)$$

for $\widetilde{\omega} = (\omega_0, \omega) \in \widetilde{\Omega}$. If η_0 has distribution μ_0, then the process thus constructed is Feller with generator (1) and initial distribution μ_0. By a *coupling* of two systems, we mean a process $(\eta_t, \zeta_t)_{t \geq 0}$ defined on $\widetilde{\Omega}$, where each component evolves according to (2)–(4), and the random variables η_0 and ζ_0 are defined simultaneously on Ω_0.

We define on Ω the *space-time shift* θ_{x_0,t_0}: for any $\omega \in \Omega$, for any (t,x,z,u)

$$(t,x,z,u) \in \theta_{x_0,t_0}\omega \text{ if and only if } (t_0 + t, x_0 + x, z, u) \in \omega$$

where $(t,x,z,u) \in \omega$ means $\omega\{(t,x,z,u)\} = 1$. By its very definition, the mapping introduced in (2) enjoys the following properties, for all $s, t \geq 0$, $x \in \mathbb{Z}$ and $(\eta, \omega) \in \mathbf{X} \times \Omega$:

$$\eta_s(\eta_t(\eta, \omega), \theta_{0,t}\omega) = \eta_{t+s}(\eta, \omega) \tag{5}$$

which implies Markov property, and

$$\tau_x \eta_t(\eta, \omega) = \eta_t(\tau_x \eta, \theta_{x,0}\omega) \tag{6}$$

which implies that $S(t)$ and τ_x commute.

2.2 Main result

We give here a precise statement of strong hydrodynamics. Let $N \in \mathbb{N}$ be the scaling parameter for the hydrodynamic limit, that is the inverse of the macroscopic distance between two consecutive sites. The empirical measure of a configuration η viewed on scale N is given by

$$\alpha^N(\eta)(dx) = N^{-1} \sum_{y \in \mathbb{Z}} \eta(y) \delta_{y/N}(dx) \in \mathscr{M}^+(\mathbb{R}) \tag{7}$$

We now state our main result.

Theorem 2.1. *Assume $p(.)$ is finite range, that is there exists $M > 0$ such that $p(x) = 0$ for all $|x| > M$. Let $(\eta_0^N, N \in \mathbb{N})$ be a sequence of $\mathbf{X}$-valued random variables on Ω_0. Assume there exists a measurable $[0,K]$-valued profile $u_0(.)$ on $\mathbb{R}$ such that*

$$\lim_{N \to \infty} \alpha^N(\eta_0^N)(dx) = u_0(.)dx, \quad \mathbf{P}_0\text{-a.s.} \tag{8}$$

that is,

$$\lim_{N \to \infty} \int_{\mathbb{R}} \psi(x)\alpha^N(\eta_0^N)(dx) = \int \psi(x)u_0(x)dx, \quad \mathbf{P}_0\text{-a.s.} \tag{9}$$

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for every continuous function ψ on $\mathbb{R}$ with compact support. Let $(x, t) \mapsto u(x, t)$ denote the unique entropy solution to the scalar conservation law

$$\partial_t u + \partial_x [G(u)] = 0 \tag{10}$$

with initial condition u_0, where G is a Lipschitz-continuous flux function (defined in (13) below) determined by $p(.)$ and $b(.,.)$. Then, with $\widetilde{\mathbf{P}}$-probability one, the convergence

$$\lim_{N \to \infty} \alpha^N(\eta_{Nt}(\eta_0^N(\omega_0), \omega))(dx) = u(., t)dx \tag{11}$$

holds uniformly on all bounded time intervals. That is, for every continuous function ψ on $\mathbb{R}$ with compact support, the convergence

$$\lim_{N \to \infty} \int_{\mathbb{R}} \psi(x) \alpha^N(\eta_{Nt}^N)(dx) = \int \psi(x) u(x, t)dx \tag{12}$$

holds uniformly on all bounded time intervals.

Remark 2.3. *This result has been recently extended to kernels with infinite range and finite third moment (see [33]).*

As a byproduct we derive in Appendix A the main result of [8]:

Corollary 2.1. *The strong law of large numbers (11) implies the weak law of large numbers established in [8].*

We recall from [8, pp.1346–1347 and Lemma 4.1] the definition of the Lipschitz-continuous macroscopic flux function G. In Proposition 2.1 below, we define a closed subset $\mathscr{R}$ of $[0, K]$ depending on $p(.)$ and $b(.,.)$ (the set of possible equilibrium densities for the system). We let

$$G(\rho) = \nu^\rho \left[\sum_{z \in \mathbb{Z}} z p(z) b(\eta(0), \eta(z)) \right] \tag{13}$$

This represents the expectation, under the shift invariant equilibrium measure with density ρ (see Proposition 2.1), of the microscopic current through site 0. On the complement of $\mathscr{R}$, which is a finite or countably infinite union of disjoint open intervals, G is interpolated linearly.

Remark 2.4. *Very little is known about the set $\mathscr{R}$. That $\{0, K\} \subset \mathscr{R}$ is a trivial consequence of Assumption (A3). When $p(.)$ and $b(.,.)$ satisfy the algebraic relations of [12], the system has a whole continuum of product equilibrium measures, and $\mathscr{R} = [0, K]$. Whether $\mathscr{R} = [0, K]$ outside this case is a difficult open problem. We do not know of any definite conjecture regarding this question. The only known result is for the totally asymmetric K-exclusion process, that is $p(1) = 1$ and $b(n, m) = \mathbf{1}_{n>0, m<K}$. It is established in [8, Corollary 2.1] that $\mathscr{R}$ contains at least one point in $[1/3, K - 1/3]$, and 0 and K are limit points of $\mathscr{R}$. In particular, $\mathscr{R}$ is infinite. This result is derived from the (weak) hydrodynamic limit.*

A Lipschitz constant V of G is determined by the rates b, p in (1):

$$
\begin{aligned}
V &= 2B \sum_{z \in \mathbb{Z}} |z| p(z), \qquad \text{with} \\
B &= \sup_{0 \le a \le K, 0 \le k < K} \{ b(a, k) - b(a, k+1), b(k+1, a) - b(k, a) \}
\end{aligned}
$$

This constant will be a key tool for the finite propagation property of the particle system (see Proposition 4.1 below).

Remark 2.5. *When the system has product invariant measures (see [12]), (13) can be decomposed into a mean drift and a scalar flux:*

$$G(\rho) = \left[\sum_{z \in \mathbb{Z}} z p(z) \right] \nu^\rho [b(\eta(0), \eta(1))] \tag{14}$$

In Assumption (A2) we only assumed integrability of the jump kernel $p(.)$, but no non-zero mean or even asymmetry condition. Thus our result applies also when $\sum_z z p(z) = 0$, but then the flux is identically zero and the hydrodynamic limit is trivial, $u(x, t) = u_0(x)$ for all $t > 0$ and $x \in \mathbb{R}$. In this case the relevant time scale to observe nontrivial behavior is the diffusive scale. When the system has no explicit invariant measures, there is a priori no reason for (14) to hold, so the drift $\sum_z z p(z)$ does not play a special role. In particular, a mean-zero or even symmetry property for $p(.)$ need not imply a trivial flux.

In [8, Section 2.2] we discussed the different definitions of *entropy solutions* (loosely speaking : the physically relevant solutions) to an equation such as (10), and the ways to prove their existence and uniqueness. Therefore we just briefly recall the definition of entropy solutions based on shock admissibility conditions (Oleĭnik's entropy condition), valid only for solutions with locally bounded space-time variation ([44]). A weak solution to (10) is an entropy solution if and only if, for a.e. $t > 0$, all discontinuities of $u(., t)$ are entropy shocks. A discontinuity (u^-, u^+), with $u^\pm := u(x \pm 0, t)$, is called an entropy shock, if and only if:

> The chord of the graph of G between u^- and u^+ lies
> below the graph if $u^- < u^+$, above the graph if $u^+ < u^-$.

In the above condition, "below" or "above" is meant in wide sense, that is the graph and chord may coincide at some points between u^- and u^+.

If the initial datum $u_0(.)$ has locally bounded space-variation, there exists a unique entropy solution to (10), within the class of functions of locally bounded space-time variation ([44]).

2.3 Monotonicity and invariant measures

Assumption *(A4)* implies the following monotonicity property, crucial in our approach. For the coordinatewise partial order on **X**, defined by $\eta \le \zeta$ if and only if $\eta(x) \le \zeta(x)$ for every $x \in \mathbb{Z}$, we have

$$(\eta_0, t) \mapsto \eta_t(\eta_0, \omega) \text{ is nondecreasing w.r.t. } \eta_0 \tag{15}$$

for every ω such that this mapping is well defined. The partial order on **X** induces a partial stochastic order on $\mathscr{P}(\mathbf{X})$; namely, for $\mu_1, \mu_2 \in \mathscr{P}(\mathbf{X})$, we write $\mu_1 \le \mu_2$ if the following equivalent conditions hold (see *e.g.* [30], [41]):

i) For every non-decreasing nonnegative function f on **X**, $\mu_1(f) \le \mu_2(f)$.
ii) There exists a coupling measure $\widetilde{\mu}$ on $\widetilde{\mathbf{X}} = \mathbf{X} \times \mathbf{X}$ with marginals μ_1 and μ_2, such that $\widetilde{\mu}\{(\eta, \xi) : \eta \le \xi\} = 1$.

It follows from this and (15) that

$$\mu_1 \leq \mu_2 \Rightarrow \forall t \in \mathbb{R}^+, \, \mu_1 S(t) \leq \mu_2 S(t) \tag{16}$$

Either property (15) or (16) is usually called *attractiveness*.

Let $\mathscr{I}$ and $\mathscr{S}$ denote respectively the set of invariant probability measures for L, and the set of shift-invariant probability measures on $\mathbf{X}$. We derived in [8, Proposition 3.1], that

Proposition 2.1.

$$(\mathscr{I} \cap \mathscr{S})_e = \{\nu^\rho, \, \rho \in \mathscr{R}\} \tag{17}$$

with $\mathscr{R}$ a closed subset of $[0, K]$ containing 0 and K, and ν^ρ a shift-invariant measure such that $\nu^\rho [\eta(0)] = \rho$. (The index e denotes extremal elements.)
The measures ν^ρ are stochastically ordered:

$$\rho \leq \rho' \Rightarrow \nu^\rho \leq \nu^{\rho'} \tag{18}$$

Moreover we have, quoting [35, Lemma 4.5]:

Proposition 2.2. *The measure ν^ρ has* a.s. *density ρ, that is*

$$\lim_{l \to \infty} \frac{1}{2l+1} \sum_{x=-l}^{l} \eta(x) = \rho, \quad \nu^\rho - a.s.$$

By [23, Theorem 6], (18) implies existence of a probability space on which one can define random variables satisfying

$$\eta^\rho \sim \nu^\rho \tag{19}$$

and, with probability one,

$$\eta^\rho \leq \eta^{\rho'}, \, \forall \rho, \rho' \in \mathscr{R} \text{ such that } \rho \leq \rho' \tag{20}$$

Proceeding as in the proof of [24, Theorem 7], one can also require (but we shall not use this property) the joint distribution of $(\eta^\rho : \rho \in \mathscr{R})$ to be invariant by the spatial shift τ_x. In the special case where ν^ρ are product measures, that is when the function $b(.,.)$ satisfies assumptions of [12], such a family of random variables can be constructed explicitly: if $(U_x)_{x \in \mathbb{Z}}$ is a family of i.i.d. random variables uniformly distributed on $(0, 1)$, one defines

$$\eta^\rho(x) = F_\rho^{-1}(U_x) \tag{21}$$

where F_ρ is the cumulative distribution function of the single site marginal distribution of ν^ρ. We will assume without loss of generality (by proper enlargement) that the "initial" probability space Ω_0 is large enough to define a family of random variables satisfying (19)–(20).

An important result for our approach is a *space-time* ergodic theorem for particle systems mentioned in [35], which we state here in a general form, and prove in Appendix C.

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Proposition 2.3. *Let $(\eta_t)_{t \geq 0}$ be a Feller process on* $\mathbf{X}$ *with a translation invariant generator L, that is*

$$\tau_1 L \tau_{-1} = L \tag{22}$$

Assume further that

$$\mu \in (\mathscr{I}_L \cap \mathscr{S})_e$$

where $\mathscr{I}_L$ denotes the set of invariant measures for L. Then, for any local function f on $\mathbf{X}$, *and any $a > 0$*

$$\lim_{\ell \to \infty} \frac{1}{a\ell^2} \int_0^{a\ell} \sum_{i=0}^{\ell} \tau_i f(\eta_t) dt = \int f d\mu = \lim_{\ell \to \infty} \frac{1}{a\ell^2} \int_0^{a\ell} \sum_{i=-\ell}^{-1} \tau_i f(\eta_t) dt \tag{23}$$

a.s. with respect to the law of the process with initial distribution μ.

Remark 2.6. *It follows from* (23) *that, more generally,*

$$\lim_{\ell \to \infty} \frac{1}{(b-a)(d-c)\ell^2} \int_{a\ell}^{b\ell} \sum_{i \in \mathbb{Z} \cap [c\ell, d\ell]} \tau_i f(\eta_t) dt = \int f d\mu \tag{24}$$

for every $0 \leq a < b$ and $c < d$ in $\mathbb{R}$, as can be seen by decomposing the space-time rectangle $[a\ell, b\ell] \times [c\ell, d\ell]$ into rectangles containing the origin.

3   Almost sure Riemann hydrodynamics

By definition, the Riemann problem with values $\lambda, \rho \in [0, K]$ for (10) is the Cauchy problem for the particular initial condition

$$R^0_{\lambda,\rho}(x) = \lambda \mathbf{1}_{\{x < 0\}} + \rho \mathbf{1}_{\{x \geq 0\}} \tag{25}$$

The entropy solution for this Cauchy datum will be denoted in the sequel by $R_{\lambda,\rho}(x, t)$. In this section, we derive the corresponding almost sure hydrodynamic limit when $\lambda, \rho \in \mathscr{R}$. We will use the following variational representation of the Riemann problem. We henceforth assume $\lambda < \rho$ (for the case $\lambda > \rho$, replace minimum with maximum below and make subsequent changes everywhere in the section).

Proposition 3.1. *([8, Proposition 4.1]). Let $\lambda, \rho \in [0, K]$, $\lambda < \rho$.*
i) There is a countable set $\Sigma_{low} \subset [\lambda, \rho]$ (depending only on the differentiability properties of the convex envelope of G on $[\lambda, \rho]$) such that, for every $v \in \mathbb{R} \setminus \Sigma_{low}$, $G(\cdot) - v\cdot$ achieves its minimum over $[\lambda, \rho]$ at a unique point $h_c(v)$. Then $R_{\lambda,\rho}(x, t) = h_c(x/t)$ whenever $x/t \notin \Sigma_{low}$.
ii) Suppose $\lambda, \rho \in \mathscr{R}$. Then the previous minimum is unchanged if restricted to $[\lambda, \rho] \cap \mathscr{R}$. As a result, the Riemann entropy solution is a.e. *$\mathscr{R}$-valued.*

Remark 3.1. *Property ii) holds if we replace $\mathscr{R}$ by any closed subset of $[0, K]$ on the complement of which G is affine. We stated it for $\mathscr{R}$ because it is the set of densities relevant to the particle system.*

To state Riemann hydrodynamics, we define a particular initial distribution for the particle system. We introduce a transformation $T : \mathbf{X}^2 \to \mathbf{X}$ by

$$T(\eta, \xi)(x) = \eta(x) \mathbf{1}_{\{x < 0\}} + \xi(x) \mathbf{1}_{\{x \geq 0\}}$$

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We define $\nu^{\lambda,\rho}$ as the distribution of $T(\eta^\lambda, \eta^\rho) =: \eta^{\lambda,\rho}$, and $\bar\nu^{\lambda,\rho}$ as the coupling distribution of $(\eta^\lambda, \eta^\rho)$. Note that, by (20),

$$\bar\nu^{\lambda,\rho}\left\{(\eta,\xi) \in \mathbf{X}^2 : \eta \leq \xi\right\} = 1 \tag{26}$$

The measure $\nu^{\lambda,\rho}$ is non-explicit unless we are in the special case of [12] where the ν^ρ are product, one can use (21), and $\nu^{\lambda,\rho}$ is itself a product measure. In all cases, $\nu^{\lambda,\rho}$ enjoys the properties:

P1) Negative (nonnegative) sites are distributed as under ν^λ (ν^ρ);

P2) $\tau_1 \nu^{\lambda,\rho} \geq \nu^{\lambda,\rho}$ ($\tau_1 \nu^{\lambda,\rho} \leq \nu^{\lambda,\rho}$) if $\lambda \leq \rho$ ($\lambda \geq \rho$);

P3) $\nu^{\lambda,\rho}$ is stochastically increasing with respect to λ and ρ.

Let us also define an extended shift θ' on the compound probability space $\Omega' = \mathbf{X}^2 \times \Omega$. This is a particular case of $\widetilde\Omega$ when the "initial" probability space Ω_0 is the set $\mathbf{X}^2$ of coupled particle configurations (η, ξ). Let

$$\omega' = (\eta, \xi, \omega) \tag{27}$$

denote a generic element of this space. We set

$$\theta'_{x,t}\omega' = (\tau_x \eta_t(\eta, \omega), \tau_x \eta_t(\xi, \omega), \theta_{x,t}\omega) \tag{28}$$

We can now state and prove the main results of this section.

Proposition 3.2. *Set*

$$\mathcal{N}_t^{v,w}(\omega') := \sum_{[vt] < x \leq [wt]} \eta_t(T(\eta,\xi), \omega)(x) \tag{29}$$

Then, for every $t > 0$, $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$ and $v, w \in \mathbb{R} \setminus \Sigma_{low}$,

$$\lim_{t \to \infty} \frac{1}{t} \mathcal{N}_t^{v,w} \circ \theta'_{[\beta t], \alpha t}(\omega')$$
$$= [G(R_{\lambda,\rho}(v,1)) - v R_{\lambda,\rho}(v,1)] - [G(R_{\lambda,\rho}(w,1)) - w R_{\lambda,\rho}(w,1)] \tag{30}$$

$\bar\nu^{\lambda,\rho} \otimes \mathbf{P}$-*a.s.*

Remark 3.2. *This result is an almost sure version of [3, Lemma 3.2], where the limit of the corresponding expectation was derived.*

Corollary 3.1. *Set*

$$\beta_t^N(\omega')(dx) := \alpha^N(\eta_t(T(\eta,\xi), \omega))(dx) \tag{31}$$

(i) For every $t > 0$, $s_0 \geq 0$ and $x_0 \in \mathbb{R}$, we have the $\bar\nu^{\lambda,\rho} \otimes \mathbf{P}$-a.s. convergence

$$\lim_{N \to \infty} \beta_{Nt}^N(\theta'_{[Nx_0], Ns_0}\omega')(dx) = R_{\lambda,\rho}(.,t) dx$$

(ii) In particular, for an initial sequence $(\eta_0^N)_N$ such that $\eta_0^N = \eta^{\lambda,\rho}$ for every $N \in \mathbb{N}$, the conclusion of Theorem 2.1 holds without the finite-range assumption on $p(.)$.

For the asymmetric exclusion process, [1] proved a statement equivalent to the particular case $\alpha = \beta = 0$ of Proposition 3.2. Their argument, which is a correction of [6], is based on subadditivity. As will appear in Section 4 (in the proof of Lemma 4.4 for example, that is the main step of that section), we do need to consider nonzero α and β in order to prove a.s. hydrodynamics for general (non-Riemann) Cauchy datum. Using arguments similar to those in [1], it would be possible to prove Proposition 3.2 with $\alpha = \beta = 0$ for our model (1). However this no longer works for nonzero α and β (see Appendix B). Therefore we construct a new approach for a.s. Riemann hydrodynamics, that does not use subadditivity.

To prove Proposition 3.2, we first rewrite (in Subsection 3.1) the quantity $\mathcal{N}_t^{v,w}$ in terms of particle currents for which we then state (in Subsection 3.2) a series of lemmas (proven in Subsection 3.4), and finally obtain the desired limits for the currents, by deriving upper and lower bounds. For one bound (the upper bound if $\lambda < \rho$ or lower bound if $\lambda > \rho$), we derive an "almost-sure proof" inspired by the ideas of [3] and their extension in [8]. For the other bound we use new ideas based on large deviations of the empirical measure.

3.1 Currents

Let us define particle currents in a system $(\eta_t)_{t \geq 0}$ governed by (3)–(4). Let $x_\cdot = (x_t, t \geq 0)$ be a $\mathbb{Z}$-valued *cadlag* path with $\left| x_t - x_{t^-} \right| \leq 1$. In the sequel this path will be either deterministic, or a random path independent of the Poisson measure ω. We define the particle current as seen by an observer travelling along this path. We first consider a semi-infinite system, that is with $\sum_{x>0} \eta_0(x) < +\infty$: in this case, we set

$$\varphi_t^{x_\cdot}(\eta_0, \omega) := \sum_{y > x_t} \eta_t(\eta_0, \omega)(y) - \sum_{y > x_0} \eta_0(y) \tag{32}$$

where $\eta_t(\eta_0, \omega)$ is the mapping introduced in (2). In the sequel we shall most often omit ω and write η_t for $\eta_t(\eta_0, \omega)$ when this creates no ambiguity. We have

$$\varphi_t^{x_\cdot}(\eta_0, \omega) = \varphi_t^{x_\cdot,+}(\eta_0, \omega) - \varphi_t^{x_\cdot,-}(\eta_0, \omega) + \tilde{\varphi}_t^{x_\cdot}(\eta_0, \omega) \tag{33}$$

where

$$
\begin{aligned}
\varphi_t^{x_\cdot,+}(\eta_0, \omega) &= \omega \left\{ (s, y, z, u) : 0 \leq s \leq t,\ y \leq x_s < y + z, \right. \\
&\qquad\qquad\qquad\qquad \left. u \leq \frac{b(\eta_{s^-}(y), \eta_{s^-}(y+z))}{\|b\|_\infty} \right\} \\
\varphi_t^{x_\cdot,-}(\eta_0, \omega) &= \omega \left\{ (s, y, z, u) : 0 \leq s \leq t,\ y + z \leq x_s < y, \right. \\
&\qquad\qquad\qquad\qquad \left. u \leq \frac{b(\eta_{s^-}(y), \eta_{s^-}(y+z))}{\|b\|_\infty} \right\}
\end{aligned}
\tag{34}
$$

count the number of rightward/leftward crossings due to particle jumps, and

$$\tilde{\varphi}_t^{x_\cdot}(\eta_0, \omega) = - \int_{[0,t]} \eta_{s^-}(x_s \vee x_{s^-}) dx_s \tag{35}$$

is the current due to the self-motion of the observer. For an infinite system, we may still *define* $\varphi_t^{x_\cdot}(\eta_0, \omega)$ by equations (33) to (35). We shall use the notation φ_t^v in the particular case $x_t = [vt]$.

The following identities are immediate from (32) in the semi-infinite case, and extend to the infinite case:

$$\left| \varphi_t^x(\eta_0, \omega) - \varphi_t^y(\eta_0, \omega) \right| \leq K \left(\left| x_t - y_t \right| + \left| x_0 - y_0 \right| \right) \tag{36}$$

$$\sum_{x=1+[vt]}^{[wt]} \eta_t(\eta_0, \omega)(x) = \varphi_t^v(\eta_0, \omega) - \varphi_t^w(\eta_0, \omega) \tag{37}$$

Following (37), the quantity $\mathscr{N}_t^{v,w}(\omega')$ defined in (29) can be written as

$$\mathscr{N}_t^{v,w}(\omega') = \phi_t^v(\omega') - \phi_t^w(\omega') \tag{38}$$

where we define the current

$$\phi_t^v(\omega') := \varphi_t^v(T(\eta, \xi), \omega)$$

Notice that

$$\phi_t^v(\eta, \eta, \omega) = \varphi_t^v(\eta, \omega)$$

The proof of the existence of the limit in Proposition 3.2 is thus reduced to

$$\lim_{t \to \infty} t^{-1} \phi_t^v \left(\theta'_{[\beta t], \alpha t} \omega' \right) \quad \text{exists } \bar{v}^{\lambda, \rho} \otimes \mathbf{P}\text{-a.s.} \tag{39}$$

3.2 Lemmas

We fix $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$. The first lemma deals with equilibrium processes.

Lemma 3.1. *For all $r \in [\lambda, \rho] \cap \mathscr{R}$, $\varsigma \in \mathbf{X}$, $v \in \mathbb{R} \setminus \Sigma_{low}$,*

$$\lim_{t \to \infty} t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\varsigma, \varsigma, \omega) = G(r) - vr, \quad \bar{v}^{r,r} \otimes \mathbf{P}\text{-a.s.}$$

The second lemma relates the current of the process under study with equilibrium currents; here it plays the role of Lemma 3.3 in [3]. It is connected to the "finite propagation property" of the particle model (see Lemma 4.3):

Lemma 3.2. *There exist $\bar{v}$ and $\underline{v}$ (depending on b and p) such that we have, $\bar{v}^{\lambda, \rho} \otimes \mathbf{P}$-a.s.,*

$$\lim_{t \to \infty} \left[t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\eta, \xi, \omega) - t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\xi, \xi, \omega) \right] = 0, \tag{40}$$

for all $v > \bar{v}$, and

$$\lim_{t \to \infty} \left[t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\eta, \xi, \omega) - t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\eta, \eta, \omega) \right] = 0, \tag{41}$$

for all $v < \underline{v}$.

For the next lemmas we need some more notation and definitions. Let $v \in \mathbb{R}$. We consider a probability space Ω^+, whose generic element is denoted by ω^+, on which is defined a Poisson process $N_t = N_t(\omega^+)$ with intensity $|v|$. We denote by $\mathbf{P}^+$ the associated probability. We set

$$x_s^t(\omega^+) := (\text{sgn}(v)) \left[N_{\alpha t+s}(\omega^+) - N_{\alpha t}(\omega^+) \right] \tag{42}$$

$$\widetilde{\eta}_s^t(\eta_0, \omega, \omega^+) := \tau_{x_s^t(\omega^+)} \eta_s(\eta_0, \omega) \tag{43}$$

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Thus $(\widetilde{\eta}_s^t)_{s\geq 0}$ is a Feller process with generator

$$L_v = L + S_v, \quad S_v f(\zeta) = |v| \left[f(\tau_{\mathrm{sgn}(v)}\zeta) - f(\zeta) \right] \tag{44}$$

(for f local and $\zeta \in \mathbf{X}$) for which the set of local functions is a core, as it is known to be ([30]) for L. We denote by $\mathcal{I}_v$ the set of invariant measures for L_v. Since any translation invariant measure on $\mathbf{X}$ is stationary for the pure shift generator S_v, we have

$$\mathcal{I} \cap \mathcal{S} = \mathcal{I}_v \cap \mathcal{S} \tag{45}$$

It can be shown (see [16, Theorem 3.1] and [17, Corollary 9.6]) that $\mathcal{I}_v \subset \mathcal{S}$ for $v \neq 0$, which implies

$$\mathcal{I} \cap \mathcal{S} = \mathcal{I}_v \cap \mathcal{S} = \mathcal{I}_v$$

but we shall not use this fact here. Define the time empirical measure

$$m_t(\omega', \omega^+) := t^{-1} \int_0^t \delta_{\widetilde{\eta}_s^t(T(\eta,\xi),\omega,\omega^+)} ds \tag{46}$$

and space-time empirical measure (where $\varepsilon > 0$) by

$$m_{t,\varepsilon}(\omega', \omega^+) := |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{x \in \mathbb{Z}: |x| \leq \varepsilon t} \tau_x m_t(\omega', \omega^+) \tag{47}$$

We introduce the set

$$\mathcal{M}_{\lambda,\rho} := \left\{ \mu \in \mathcal{P}(\mathbf{X}) : v^\lambda \leq \mu \leq v^\rho \right\}$$

Notice that this is a closed (thus compact) subset of the compact space $\mathcal{P}(\mathbf{X})$.

Lemma 3.3. *(i) With $\bar{v}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$-probability one, every subsequential limit of $m_{t,\varepsilon}(\theta'_{[\beta t],\alpha t}\omega', \omega^+)$ as $t \to \infty$ lies in $\mathcal{I}_v \cap \mathcal{S} \cap \mathcal{M}_{\lambda,\rho} = \mathcal{I} \cap \mathcal{S} \cap \mathcal{M}_{\lambda,\rho}$.
(ii) $\mathcal{I} \cap \mathcal{S} \cap \mathcal{M}_{\lambda,\rho}$ is the set of probability measures v of the form $v = \int v^r \gamma(dr)$, where γ is a probability measure supported on $\mathcal{R} \cap [\lambda, \rho]$.*

The proof of Lemma 3.3 will be based on the following large deviation result in the spirit of [15].

Lemma 3.4. *(i) The functional $\mathcal{D}_v$ defined on $\mathcal{P}(\mathbf{X})$ by*

$$\mathcal{D}_v(\mu) := \sup_{f \text{ local}} - \int \frac{L_v e^f}{e^f}(\widetilde{\eta}) d\mu(\widetilde{\eta}) \tag{48}$$

is nonnegative, lower-semicontinuous, and $\mathcal{D}_v^{-1}(0) = \mathcal{I}_v$.

(ii) Let $\widetilde{\xi}_.$ be a Markov process with generator L_v and distribution denoted by $\mathbf{P}$. Define the empirical measures

$$\pi_t(\widetilde{\xi}_.) := t^{-1} \int_0^t \delta_{\widetilde{\xi}_s} ds, \quad \pi_{t,\varepsilon} := |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{x \in \mathbb{Z} \cap [-\varepsilon t, \varepsilon t]} \tau_x \pi_t \tag{49}$$

where $\varepsilon > 0$. Then, for every closed subset F of $\mathcal{P}(\mathbf{X})$ and every $t \geq 0$,

$$\limsup_{t \to \infty} t^{-1} \log \mathbf{P} \left(\pi_{t,\varepsilon}(\widetilde{\xi}_.) \in F \right) \leq - \inf_{\mu \in F} \mathcal{D}_v(\mu) \tag{50}$$

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3.3 Proofs of Lemma 3.1, Proposition 3.2 and Corollary 3.1

Proof of Lemma 3.1. The result is a particular case, for $\lambda = \rho$, of (51) proven below. $\qquad\square$

Proof of Proposition 3.2. We denote by $\omega' = (\eta, \xi, \omega)$ a generic element of Ω'. We will establish the following limits. First,

$$
\begin{aligned}
\inf_{r \in [\lambda, \rho] \cap \mathscr{R}} [G(r) - vr] \quad &\leq \quad \liminf_{t \to \infty} t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\omega') \\
&\leq \quad \limsup_{t \to \infty} t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\omega') \\
&\leq \quad \sup_{r \in [\lambda, \rho] \cap \mathscr{R}} [G(r) - vr], \quad \bar{v}^{\lambda, \rho} \otimes \mathbf{P}\text{-a.s.}
\end{aligned}
\tag{51}
$$

and then

$$
\limsup_{t \to \infty} t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\omega') \leq \inf_{r \in [\lambda, \rho] \cap \mathscr{R}} [G(r) - vr], \quad \bar{v}^{\lambda, \rho} \otimes \mathbf{P}\text{-a.s.}
\tag{52}
$$

which will imply the result, when combined with Proposition 3.1 and the expression (38) of $\mathscr{N}_t^{v,w}(\omega')$. Though only the first inequality in (51) (together with (52)) seems relevant to Proposition 3.2, we will need the whole set of inequalities: Indeed, writing (51) for $\lambda = \rho = r \in \mathscr{R}$ proves the equilibrium result of Lemma 3.1.

To obtain the bounds in (51) we proceed as follows. First we replace the deterministic path vt in the current ϕ_t^v by $x_t^t(\omega^+)$. Then we consider a spatial average of $\varphi_t^{x^t(\omega^+)+x}$ for $x \in [-\varepsilon t, \varepsilon t]$, and we introduce, for $\zeta \in \mathbf{X}$, the martingale $M_t^{x,v}(\zeta, \omega, \omega^+)$ associated to $\varphi_t^{x^t(\omega^+)+x}(\zeta, \omega)$ (see below (57)). An exponential bound on the martingale reduces the derivation of (51) to bounds (deduced thanks to Lemma 3.3) on

$$
\int [f(\eta) - v\eta(1)] m_{t,\varepsilon}(\theta'_{[\beta t], \alpha t} \omega', \omega^+)(d\eta)
$$

(see below (62)), where $[f(\eta) - v\eta(1)]$ corresponds to the compensator of $\varphi_t^{x^t(\omega^+)+x}(\zeta, \omega)$ in $M_t^{x,v}(\zeta, \omega, \omega^+)$. The bound (52) relies on Lemmas 3.1 and 3.2 combined with the monotonicity of the process.

Step one: proof of (51). We have

$$
t^{-1} \phi_t^v \circ \theta'_{[\beta t], \alpha t}(\omega') = t^{-1} \varphi_t^v(\varpi_{\alpha t}, \theta_{[\beta t], \alpha t} \omega)
\tag{53}
$$

where the configuration

$$
\varpi_{\alpha t} = \varpi_{\alpha t}(\eta, \xi, \omega) := T\left(\tau_{[\beta t]} \eta_{\alpha t}(\eta, \omega), \tau_{[\beta t]} \eta_{\alpha t}(\xi, \omega)\right)
\tag{54}
$$

depends only on the restriction of ω to $[0, \alpha t] \times \mathbb{Z}$. Thus, since ω is a Poisson measure under $\mathbf{P}$, $\theta_{[\beta t], \alpha t} \omega$ is independent of $\varpi_{\alpha t}(\eta, \xi, \omega)$ under $\bar{v}^{\lambda, \rho} \otimes \mathbf{P}$, and

$$
\text{under } \bar{v}^{\lambda, \rho} \otimes \mathbf{P} \otimes \mathbf{P}^+, \ \varpi_{\alpha t} \text{ is independent of } (\theta_{[\beta t], \alpha t} \omega, \omega^+)
\tag{55}
$$

Define

$$\psi_t^{v,\varepsilon}(\zeta,\omega,\omega^+) := |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{y\in\mathbb{Z}:|y|\leq\varepsilon t} \varphi_t^{x_\cdot^t(\omega^+)+y}(\zeta,\omega)$$

for every $(\zeta,\omega,\omega^+) \in \mathbf{X} \times \Omega \times \Omega^+$, with $x_\cdot^t(\omega^+)$ given by (42). By (36),

$$\left|\varphi_t^v(\zeta,\omega) - \psi_t^{v,\varepsilon}(\zeta,\omega,\omega^+)\right| \leq K\left(2\varepsilon t + \left|x_t^t(\omega^+) - vt\right|\right)$$

Since $t^{-1}x_t^t(\omega^+) \to v$ with $\mathbf{P}^+$-probability one, the proof of (51) can be reduced to that of the same inequalities with the l.h.s. of (53) replaced by

$$t^{-1}\psi_t^{v,\varepsilon}(\varpi_{\alpha t}, \theta_{[\beta t],\alpha t}\omega, \omega^+) \tag{56}$$

and $\bar{v}^{\lambda,\rho} \otimes \mathbf{P}$ replaced by $\bar{v}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$. Let $f(\eta) := f^+(\eta) - f^-(\eta)$, with

$$\begin{align*}
f^+(\eta) &= \sum_{y,z\in\mathbb{Z}:\,y\leq 0<y+z} p(z)b(\eta(y),\eta(y+z))\\
f^-(\eta) &= \sum_{y,z\in\mathbb{Z}:\,y+z\leq 0<y} p(z)b(\eta(y),\eta(y+z))
\end{align*}$$

and $n(\eta) := \eta(\delta)$, where δ is defined to be 1 if $v > 0$, 0 if $v < 0$, and any integer if $v = 0$. By the definition of particle current (33)–(35), we have that, for any $\zeta \in \mathbf{X}$,

$$\begin{align*}
M_t^{x,v}(\zeta,\omega,\omega^+) &:= \varphi_t^{x_\cdot^t(\omega^+)+x}(\zeta,\omega)\\
&\quad - \int_0^t \tau_x[f - vn](\tilde{\eta}_{s^-}^t(\zeta,\omega,\omega^+)) \tag{57}\\
E_t^{x,v,\theta}(\zeta,\omega,\omega^+) &:= \exp\Big\{\theta\varphi_t^{x_\cdot^t(\omega^+)+x}(\zeta,\omega)\\
&\quad - \left(e^\theta - 1\right)\int_0^t \tau_x f^+(\tilde{\eta}_{s^-}^t(\zeta,\omega,\omega^+))ds\\
&\quad - \left(e^{-\theta} - 1\right)\int_0^t \tau_x f^-(\tilde{\eta}_{s^-}^t(\zeta,\omega,\omega^+))ds\\
&\quad - \int_0^t |v|\left(e^{-\mathrm{sgn}(v)\theta\tau_x n\left(\tilde{\eta}_{s^-}^t(\zeta,\omega,\omega^+)\right)} - 1\right)ds\Big\} \tag{58}\\
&= \exp\Big\{\theta M_t^{x,v}(\zeta,\omega,\omega^+)\\
&\quad - \left(e^\theta - 1 - \theta\right)\int_0^t \tau_x f^+(\tilde{\eta}_{s^-}^t(\zeta,\omega,\omega^+))ds\\
&\quad - \left(e^{-\theta} - 1 + \theta\right)\int_0^t \tau_x f^-(\tilde{\eta}_{s^-}^t(\zeta,\omega,\omega^+))ds\\
&\quad - \int_0^t |v|\left\{e^{-\mathrm{sgn}(v)\theta\tau_x n\left(\tilde{\eta}_{s^-}^t(\zeta,\omega,\omega^+)\right)} - 1\right.\\
&\quad \left. + \mathrm{sgn}(v)\theta\tau_x n(\tilde{\eta}_{s^-}^t(\zeta,\omega,\omega^+))\right)ds\Big\}
\end{align*}$$

are martingales under $\mathbf{P} \otimes \mathbf{P}^+$, with respective means 0 and 1. Notice that η_{s^-} and η_s can be replaced with each other in the above martingales, because, by the graphical construction of Section

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2.1, $s \mapsto \eta_s(x)$ is $\mathbf{P} \otimes \mathbf{P}^+$-a.s. locally piecewise constant for every $\zeta \in \mathbf{X}$ and $x \in \mathbb{Z}$. It follows from (58) that

$$\mathbf{E}\left(e^{\theta M_t^{x,\nu}}\right) \leq e^{Ct\left(e^{K|\theta|} - 1 - K|\theta|\right)} \tag{59}$$

for any $\zeta \in \mathbf{X}$, where expectation is w.r.t. $\mathbf{P} \otimes \mathbf{P}^+$, and the constant C depends only on $p(.)$, $b(.)$ and ν but not on ζ. Cramer's inequality and (59) imply the large deviation bound

$$\mathbf{P} \otimes \mathbf{P}^+(\{|M_t^{x,\nu}| \geq y\}) \leq 2e^{-t\mathscr{I}_C(y)} \tag{60}$$

for any $\zeta \in \mathbf{X}$ and $y \geq 0$, with the rate function

$$\mathscr{I}_C(y) = \frac{y}{K} \log\left(1 + \frac{y}{CK}\right) - C\left[\frac{y}{CK} - \log\left(1 + \frac{y}{CK}\right)\right]$$

Since(60) holds for every $\zeta \in \mathbf{X}$, the independence property (55) also implies

$$\bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+\left(\{|M_t^{x,\nu}(\varpi_{\alpha t}, \theta_{[\beta t],\alpha t}\omega, \omega^+))| \geq y\}\right) \leq 2e^{-t\mathscr{I}_C(y)}$$

This and Borel Cantelli's lemma imply that

$$\lim_{t \to \infty} t^{-1} |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{x \in \mathbb{Z}: |x| \leq \varepsilon t} M_t^{x,\nu}(\varpi_{\alpha t}, \theta_{[\beta t],\alpha t}\omega, \omega^+) = 0, \tag{61}$$

$\bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$-a.s. In view of (57) and (61), the proof of (51) is now reduced to that of the same set of inequalities with (56) replaced by

$$t^{-1} |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \int_0^t \sum_{x \in \mathbb{Z}: |x| \leq \varepsilon t} \tau_x[f - \nu n](\tilde{\eta}_s^t(\varpi_{\alpha t}, \theta_{[\beta t],\alpha t}\omega, \omega^+))ds$$

with δ as in (57), which is exactly, because of (54),

$$\int [f(\eta) - \nu\eta(\delta)]m_{t,\varepsilon}(\theta'_{[\beta t],\alpha t}\omega', \omega^+)(d\eta) \tag{62}$$

with the empirical measure $m_{t,\varepsilon}$ defined in (47). By Lemma 3.3, every subsequential limit ν as $t \to \infty$ of $m_{t,\varepsilon}(\theta'_{[\beta t],\alpha t}\omega', \omega^+)$ is of the form

$$\nu = \int \nu^r \gamma(dr)$$

for some measure γ supported on $\mathscr{R} \cap [\lambda, \rho]$. Then the corresponding subsequential limit as $t \to \infty$ of (62) is of the form

$$\int [G(r) - \nu r]\gamma(dr)$$

as one verifies, using shift invariance of ν^r, that

$$\int [f(\eta) - \nu\eta(\delta)]\nu^r(d\eta) = G(r) - \nu r$$

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This concludes the proof.

Step two: proof of (52). Let $u_1 \leq \underline{v} \leq v \leq \bar{v} \leq v_1$, $\lambda < \rho \in \mathcal{R}$, and $r \in [\lambda, \rho] \cap \mathcal{R}$. We set $\varsigma = \eta^r$, and we define $\bar{v}^{\lambda,r,\rho}$ as the coupling distribution of $(\eta^\lambda, \eta^r, \eta^\rho)$. Note that, by the stochastic ordering property (20),

$$\bar{v}^{\lambda,r,\rho}\left\{(\eta,\varsigma,\xi) \in \mathbf{X}^3 : \eta \leq \varsigma \leq \xi\right\} = 1 \tag{63}$$

The following limits are true for $\bar{v}^{\lambda,r,\rho}$-a.e. (η,ς,ξ). By the expression (38) of $\mathcal{N}_t^{\cdot,\cdot}$ and the equilibrium limit Lemma 3.1,

$$\lim_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{u_1,v} \circ \theta'_{[\beta t],\alpha t}(\varsigma,\varsigma,\omega) = r(v - u_1) \tag{64}$$

By attractiveness,

$$\liminf_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{u_1,v} \circ \theta'_{[\beta t],\alpha t}(\varsigma,\xi,\omega) \geq \lim_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{u_1,v} \circ \theta'_{[\beta t],\alpha t}(\varsigma,\varsigma,\omega) \tag{65}$$

Putting together (64) and (65),

$$\liminf_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{u_1,v} \circ \theta'_{[\beta t],\alpha t}(\varsigma,\xi,\omega) \geq r(v - u_1) \tag{66}$$

Now, by (38), Lemma 3.2 and Lemma 3.1 respectively for r and ρ,

$$\lim_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{u_1,v_1} \circ \theta'_{[\beta t],\alpha t}(\varsigma,\xi,\omega) = (G(r) - u_1 r) - (G(\rho) - v_1\rho) \tag{67}$$

Subtracting (66) from (67), we get

$$\limsup_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{v,v_1} \circ \theta'_{[\beta t],\alpha t}(\varsigma,\xi,\omega) \leq (G(r) - vr) - (G(\rho) - v_1\rho) \tag{68}$$

By attractiveness, (63) and (68), we have

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{v,v_1} \circ \theta'_{[\beta t],\alpha t}(\eta,\xi,\omega) &\leq \limsup_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{v,v_1} \circ \theta'_{[\beta t],\alpha t}(\varsigma,\xi,\omega) \\
&\leq (G(r) - vr) - (G(\rho) - v_1\rho)
\end{aligned} \tag{69}$$

Using (38), (69), Lemma 3.2, and Lemma 3.1 for ρ, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \phi_t^v \circ \theta'_{[\beta t],\alpha t}(\omega') \\
=\ &\limsup_{t\to\infty} \left(\frac{1}{t} \phi_t^v \circ \theta'_{[\beta t],\alpha t}(\omega') - \frac{1}{t} \phi_t^{v_1} \circ \theta'_{[\beta t],\alpha t}(\omega') + \frac{1}{t} \phi_t^{v_1} \circ \theta'_{[\beta t],\alpha t}(\omega') \right) \\
\leq\ &\limsup_{t\to\infty} \frac{1}{t} \mathcal{N}_t^{v,v_1} \circ \theta'_{[\beta t],\alpha t}(\omega') + \limsup_{t\to\infty} \frac{1}{t} \phi_t^{v_1} \circ \theta'_{[\beta t],\alpha t}(\omega') \\
\leq\ &((G(r) - vr) - (G(\rho) - v_1\rho)) + (G(\rho) - v_1\rho) \\
=\ &G(r) - vr
\end{aligned}$$

for every $r \in [\lambda, \rho] \cap \mathcal{R}$. Since ς is no longer involved in the above inequalities, we obtain a $\bar{v}^{\lambda,\rho}$-a.s. limit with respect to (η,ξ) for every $r \in [\lambda, \rho] \cap \mathcal{R}$. By continuity of G this holds outside

a common exceptional set of $\bar{v}^{\lambda,\rho}$-probability 0 for all $r \in [\lambda,\rho] \cap \mathscr{R}$. This proves (52). $\qquad\square$

Proof of Corollary 3.1.

(i). By Proposition 3.1 (cf. [8, (28)]),

$$\frac{d}{dv}[G(h_c(v)) - vh_c(v)] = -h_c(v)$$

weakly with respect to v. Thus, setting $u = R_{\lambda,\rho}$, we have

$$[G(u(v,1)) - vu(v,1)] - [G(u(w,1)) - wu(w,1)] = \int_v^w u(x,1)dx \qquad (70)$$

for all $v, w \in \mathbb{R}$. Let $a < b$ in $\mathbb{R}$. Setting

$$\varpi = T\left(\tau_{[Nx_0]}\eta_{Ns_0}(\eta,\omega), \tau_{[Nx_0]}\eta_{Ns_0}(\xi,\omega)\right)$$

we have

$$
\begin{aligned}
\beta_{Nt}^N(\theta'_{[Nx_0],Ns_0}(\eta,\xi,\omega))((a,b)) &= N^{-1} \sum_{[Na] < x \le [Nb]} \eta_{Nt}(\varpi, \theta_{[Nx_0],Ns_0}\omega)(x) \\
&= t(Nt)^{-1} \mathscr{N}_{Nt}^{a/t,b/t}(\theta'_{[Nx_0],Ns_0}\omega')
\end{aligned}
$$

Thus, by Proposition 3.2 and (70),

$$\lim_{N\to\infty} \beta_{Nt}^N(\theta'_{[Nx_0],Ns_0}\omega')((a,b)) = t\int_{a/t}^{b/t} u(x,1)dx = \int_a^b u(x,t)dx \qquad (71)$$

$\bar{v}^{\lambda,\rho} \otimes \mathbf{P}$-a.s., where the last equality follows from Proposition 3.1. Now (70) implies that the r.h.s. of (30) is a continuous function of (v,w), while the l.h.s. is a uniformly Lipschitz function of (v,w), since the number of particles per site is bounded. It follows that one can find a single exceptional set of $\bar{v}^{\lambda,\rho} \otimes \mathbf{P}$-probability 0 outside which (71) holds simultaneously for all a, b, which proves the claim.

(ii). Since $\eta^{\lambda,\rho}$ has distribution $\bar{v}^{\lambda,\rho}$ under $\mathbf{P}_0$, the statement follows from *(i)* with $x_0 = s_0 = 0$. $\quad\square$

3.4 Proofs of remaining lemmas

Proof of Lemma 3.2. Let $\varepsilon > 0$. We consider the probability space $\Omega' \times (\mathbb{Z}^+)^{\mathbb{Z}}$ equipped with the product measure

$$\mathbf{P}'_\varepsilon := \bar{v}^{\lambda,\rho} \otimes \mathbf{P} \otimes P_\varepsilon$$

where P_ε is the product measure on $\mathbb{Z}$ whose marginal at each site is Poisson with mean $K(1 + \varepsilon)$. A generic element of this space is denoted by (ω', χ), with $\omega' = (\eta, \xi, \omega)$ and $\chi \in (\mathbb{Z}^+)^{\mathbb{Z}}$. We first prove (40). Because of (26) (that is, coupled configurations are ordered under $\bar{v}^{\lambda,\rho}$), by the attractiveness property (15), we may define

$$\gamma_s(\omega') := \eta_s(\xi,\omega) - \eta_s(T(\eta,\xi),\omega) \qquad (72)$$

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for $s \geq 0$. Therefore γ-particles represent the discrepancies between the system starting from ξ and the system starting from $T(\eta, \xi)$. We set

$$\overline{p}(z) = (p(z) + p(-z))\|b\|_\infty \tag{73}$$

and fix $\bar{v}$ such that

$$\bar{v} > \sum_{z \in \mathbb{Z}} z \overline{p}(z) \tag{74}$$

Let $v > \bar{v}$. Because $\gamma_0(x) = 0$ for all $x > 0$, by the definition of current (32),

$$\phi_t^v(\xi, \xi, \omega) = \phi_t^v(\omega') + \sum_{y > [vt]} \gamma_t(y)$$

Therefore, to prove (40), we want to obtain

$$\lim_{t \to \infty} t^{-1} \sum_{y > \bar{v}t} \gamma_t \circ \theta'_{[\beta t], \alpha t}(\omega')(y) = 0, \quad \bar{v}^{\lambda, \rho} \otimes \mathbf{P}\text{-a.s.} \tag{75}$$

To this end, we follow the proof of [1, Proposition 5], with minor modifications. We emphasize that even if the latter proof corresponds to $\alpha = \beta = 0$, we will see that the arguments extend to $(\alpha, \beta) \neq (0,0)$. Each γ_0-particle and χ-particle is given an integer label $j \leq 0$, and we denote by R_0^j (resp. Z_0^j) the position of the γ_0-particle (resp. χ-particle) with label j. That is, $R_0^\cdot$ and $Z_0^\cdot$ must satisfy

$$\sum_{j \leq 0} \delta_{R_0^j} = \sum_{x \in \mathbb{Z}} \gamma_0(x) \delta_x, \quad \sum_{j \leq 0} \delta_{Z_0^j} = \sum_{x \in \mathbb{Z}} \chi(x) \delta_x \tag{76}$$

There is a unique way to choose such $R_0^\cdot$ (resp. $Z_0^\cdot$) if we impose moreover that $R_0^j \leq R_0^{j+1}$ (resp. $Z_0^j \leq Z_0^{j+1}$) for all $j < 0$, which we will assume. By the definition of χ, the number of χ-particles between $-n$ and 0 will be eventually larger than nK. Let

$$W(\chi) := \inf\left\{n \in \mathbb{Z}^+ : Z_0^j \geq -[|j|/K] \text{ for every } j \leq -n\right\}$$

By Poisson large deviation bounds, the random variable W is $\mathbf{P}'_\varepsilon$-a.s. finite with exponentially decaying distribution. Since $\gamma_0(x) \leq K$ for every $x \in \mathbb{Z}$, we have $R_0^j \leq -[|j|/K]$ for all $j \leq 0$, hence $Z_0^j \geq R_0^j$ for every $j \leq -W(\chi)$. The dynamics of $(R_t^\cdot)_{t \geq 0}$ is defined as follows. By (72) and the graphical construction, at most one discrepancy can jump at each event time of ω. If a γ-particle jumps at time t from x to y, we set $R_t^j = y$ and $R_t^k = R_{t^-}^k$ for all $k \neq j$, where j is the largest label l such that $R_{t^-}^l = x$. The choice of the largest label is arbitrary and could be replaced by any deterministic or random choice procedure, provided the corresponding source of randomness is independent of (ω', χ). Note that we do not in general have $R_t^j \leq R_t^{j+1}$ for $t > 0$, but this does not matter for our purpose. The dynamics of Z^j is defined as follows: if $R_{t^-}^j = x$ and, for some $z > 0$ and $u \in [0,1]$,

$$\{(t, x, z, u), (t, x+z, -z, u)\} \cap \omega \neq \emptyset \tag{77}$$

then $Z_t^j = Z_{t^-}^j + z$. In other words, χ-particles evolve as mutually independent (given their initial positions) random walks, that jump from y to $y+z$ at rate $\overline{p}(z)$ given by (73) for all $y \in \mathbb{Z}, z \geq 0$. Since a jump for R^j from $R_{t^-}^j = x$ to $R_t^j = x+z$ is possible only under (77),

$$R_0^j \leq Z_0^j \Rightarrow \forall t > 0, R_t^j \leq Z_t^j \tag{78}$$

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In view of (75), (78), we estimate

$$
\begin{aligned}
\sum_{x>\bar{v}t} \gamma_t(x) = \sum_{j\le 0} \mathbf{1}_{\{R_t^j(\omega')>\bar{v}t\}} &= \sum_{j\le -W(\chi)} \mathbf{1}_{\{R_t^j(\omega')>\bar{v}t\}} + \sum_{-W(\chi)<j\le 0} \mathbf{1}_{\{R_t^j(\omega')>\bar{v}t\}} \\
&\le \sum_{j\le -W(\chi)} \mathbf{1}_{\{Z_t^j(\omega')>\bar{v}t\}} + W(\chi) \\
&\le \bar{Z}_t(\omega,\chi) + W(\chi)
\end{aligned}
\tag{79}
$$

where $\bar{Z}_t := \sum_{i\le 0} \mathbf{1}_{\{Z_t^i>\bar{v}t\}}$ is a Poisson random variable with mean

$$
\mathbf{E}'_\varepsilon \bar{Z}_t = K(1+\varepsilon) \sum_{j\ge \bar{v}t} \mathbf{P}'_\varepsilon(Y_t > j)
\tag{80}
$$

for Y_t a random walk starting at 0 that jumps from y to $y+z$ with rate $\overline{p}(z)$. Since $\bar{v}$ satisfies (74), repeating $[1, (43)$–$(45)]$ gives that

$$
\lim_{t\to\infty} \mathbf{E}'_\varepsilon \bar{Z}_t / t = 0
\tag{81}
$$

Let $\delta > 0$. Since $\bar{Z}_t$ is a Poisson variable,

$$
\begin{aligned}
\mathbf{P}'_\varepsilon(|\bar{Z}_t - \mathbf{E}'_\varepsilon \bar{Z}_t| > \delta t) &\le \frac{\mathbf{E}'_\varepsilon(\bar{Z}_t - \mathbf{E}'_\varepsilon \bar{Z}_t)^4}{(\delta t)^4} \\
&\le \frac{[\mathbf{E}'_\varepsilon(\bar{Z}_t - \mathbf{E}'_\varepsilon \bar{Z}_t)^2]^2}{(\delta t)^4} \\
&\le \frac{(\mathbf{E}'_\varepsilon \bar{Z}_t / t)^2 t^2}{(\delta t)^4}
\end{aligned}
\tag{82}
$$

Therefore, by Borel Cantelli lemma

$$
\lim_{t\to\infty} (\bar{Z}_t - \mathbf{E}'_\varepsilon \bar{Z}_t)/t = 0, \quad \mathbf{P}'_\varepsilon\text{-a.s.}
\tag{83}
$$

Since $\mathbf{P}$ is invariant by $\theta_{[\beta t],\alpha t}$, $\bar{Z}_t(\theta_{[\beta t],\alpha t}\omega,\chi)$ has the same distribution as $\bar{Z}_t(\omega,\chi)$ under $\mathbf{P}'_\varepsilon$. Thus (80)–(83) still hold with $\bar{Z}_t(\theta_{[\beta t],\alpha t}\omega,\chi)$ instead of $\bar{Z}_t(\omega,\chi)$, and

$$
\lim_{t\to\infty} t^{-1}\bar{Z}_t(\theta_{[\beta t],\alpha t}\omega,\chi) = 0, \quad \mathbf{P}'_\varepsilon\text{-a.s.}
\tag{84}
$$

Because the random variable W in (79) does not depend on ω', (84) and (79) imply (75). This concludes the proof of (40).

If we now define γ_s as

$$
\gamma_s(\omega') := \eta_s(T(\eta,\xi),\omega) - \eta_s(\eta,\omega)
$$

and replace $\phi_t^\gamma(\omega')$ by $-\phi_t^\gamma(\omega')$ (so that the current, which was rightwards, becomes leftwards), then the proof of (41) can be obtained by repeating the same steps as in the previous argument. $\square$

Proof of Lemma 3.3.

(i). Since

$$m_t(\omega', \omega^+) - m_{[t]}(\omega', \omega^+) = \frac{[t] - t}{t} m_{[t]}(\omega', \omega^+) + t^{-1} \int_{[t]}^t \delta_{\widetilde{\eta}_s^t(T(\eta, \xi), \omega, \omega^+)} ds$$

has total variation bounded by $2/t$, it is enough to prove the result for every subsequential limit of the sequence $m_{n,\varepsilon}(\theta'_{[\beta n], \alpha n} \omega', \omega^+)$ as $n \to \infty$, $n \in \mathbb{N}$.

Step one. We prove that every subsequential limit lies in $\mathscr{I}_\nu$. It is enough to show that, for every open neighborhood O of $\mathscr{I}_\nu$, with $\bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$-probability one, $m_{n,\varepsilon}$ lies in O for sufficiently large n. One can see from (47) and (49) that

$$m_{t,\varepsilon}(\theta'_{[\beta t], \alpha t} \omega', \omega^+) = \pi_{t,\varepsilon}(\widetilde{\xi}_.^t)$$

where, for fixed t, $\widetilde{\xi}_.^t$ is the process defined by

$$\widetilde{\xi}_s^t := \widetilde{\eta}_s^t(\varpi_{\alpha t}, \theta_{[\beta t], \alpha t} \omega, \omega^+)$$

with the configuration $\varpi_{\alpha t}$ defined in (54) and satisfying (55). Hence under $\bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$, $\widetilde{\xi}_.^t$ is a Markov process with generator L_ν and initial distribution $\nu^{\lambda,\rho}$ independent of t. By Lemma 3.4,

$$\limsup_{n \to \infty} n^{-1} \log \bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+ \left(m_{n,\varepsilon}(\theta'_{[\beta n], \alpha n} \omega', \omega^+) \notin O \right) < 0$$

Now Borel-Cantelli's lemma implies that, a.s. with respect to $\bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$, $m_{n,\varepsilon}(\theta'_{[\beta n], \alpha n} \omega', \omega^+) \in O$ for large n.

Step two. We prove that every subsequential limit lies in $\mathscr{M}_{\lambda,\rho}$. Since

$$\eta \leq T(\eta, \xi) \leq \xi$$

for $\bar{\nu}^{\lambda,\rho}$- a.e. (η, ξ), (54) and the monotonicity property (15) imply

$$\tau_{[\beta t]} \eta_{\alpha t}(\eta, \omega) \leq \varpi_{\alpha t}(\eta, \xi, \omega) \leq \tau_{[\beta t]} \eta_{\alpha t}(\xi, \omega)$$

for all $t \geq 0$. By (46) and (54),

$$m_t(\theta'_{[\beta t], \alpha t} \omega', \omega^+) = t^{-1} \int_0^t \delta_{\widetilde{\eta}_s^t(\varpi_{\alpha t}, \theta_{[\beta t], \alpha t} \omega, \omega^+)} ds$$

By (5)–(6) and (42)–(43),

$$\widetilde{\eta}_s^t(\tau_{[\beta t]} \eta_{\alpha t}(\eta, \omega), \theta_{[\beta t], \alpha t} \omega, \omega^+) = \tau_{-\mathrm{sgn}(v)N_{\alpha t}(\omega^+) + [\beta t]} \widetilde{\eta}_{\alpha t + s}^0(\eta, \omega, \omega^+)$$

Thus

$$l_t(\eta, \omega, \omega^+) := t^{-1} \int_{\alpha t}^{\alpha t + t} |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{x \in \mathscr{X}_t(\omega^+)} \tau_x \delta_{\widetilde{\eta}_s^0(\eta, \omega, \omega^+)} ds$$

$$\leq m_{t,\varepsilon}(\theta'_{[\beta t], \alpha t} \omega', \omega^+)$$

$$\leq t^{-1} \int_{\alpha t}^{\alpha t + t} |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{x \in \mathscr{X}_t(\omega^+)} \tau_x \delta_{\widetilde{\eta}_s^0(\xi, \omega, \omega^+)} ds =: r_t(\xi, \omega, \omega^+) \qquad (85)$$

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$\bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$- a.s. for all $t \geq 0$, where

$$\mathscr{X}_t(\omega^+) := \mathbb{Z} \cap \left[[\beta t] - \operatorname{sgn}(v) N_{\alpha t}(\omega^+) - \varepsilon t, [\beta t] - \operatorname{sgn}(v) N_{\alpha t}(\omega^+) + \varepsilon t \right]$$

We now argue that l_t and r_t respectively converge a.s. to ν^λ and ν^ρ with respect to $\bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$. Let us consider for instance l_t. Let L_t denote the measure defined as l_t but with $\mathscr{X}_t(\omega^+)$ replaced by

$$\mathscr{Y}_t := \mathbb{Z} \cap [(\beta - \alpha v - \varepsilon)t, (\beta - \alpha v + \varepsilon)t]$$

By the strong law of large numbers for Poisson processes, there exists a subset $\mathscr{C} \subset \Omega^+$, with $\mathbf{P}^+$-probability one, on which $\operatorname{sgn}(v) N_t(\omega^+)/t \to v$ as $t \to \infty$. The total variation of $l_t - L_t$ is bounded by $(2\varepsilon t + 1)^{-1} |\mathscr{X}_t \Delta \mathscr{Y}_t|$, where Δ denotes the symmetric difference of two sets. Hence, for $\omega^+ \in \mathscr{C}$, $l_t - L_t$ converges to 0 in total variation. We are thus reduced to proving a.s. convergence of L_t to ν^λ. Under $\bar{\nu}^{\lambda,\rho} \otimes \mathbf{P} \otimes \mathbf{P}^+$, $\tilde{\eta}^0(\eta, \omega, \omega^+)$ is a Feller process with generator L_v and initial distribution ν^λ. Thanks to (17) and (45), we can apply Proposition 2.3, or more precisely, its extended form (24). This implies convergence of L_t.

Step three. We prove that every subsequential limit lies in $\mathscr{S}$. To this end we simply note that the measure

$$\tau_1 m_{n,\varepsilon} - m_{n,\varepsilon} \;=\; |[-\varepsilon n, \varepsilon n] \cap \mathbb{Z}|^{-1} \left(\sum_{x \in \mathbb{Z} \cap (-\varepsilon n + 1, \varepsilon n + 1]} \tau_x m_n \right. \\ \left. - \sum_{x \in \mathbb{Z} \cap [-\varepsilon n, \varepsilon n)} \tau_x m_n \right) \tag{86}$$

has total variation bounded by $2|[-\varepsilon n, \varepsilon n] \cap \mathbb{Z}|^{-1} = O(1/n)$. Letting $n \to \infty$ in (86) shows that $\tau_1 m = m$ for any subsequential limit m of $m_{n,\varepsilon}$.

(ii). Proposition 2.1 implies $\nu = \int \nu^r \gamma(dr)$ with γ supported on $\mathscr{R}$. Let $\lambda' \in \mathscr{R}$ and $\rho' \in \mathscr{R}$ respectively denote the infimum and supremum of the support of γ. Assume for instance that $\lambda' < \lambda$. Choose some $\lambda'' \in (\lambda', \lambda)$. By Proposition 2.2, the random variable

$$M(\eta) := \lim_{l \to \infty} (2l + 1)^{-1} \sum_{x=-l}^{l} \eta(x)$$

is defined ν^r-a.s. for every $r \in \mathscr{R}$, and thus also ν-a.s. It is a nondecreasing function of η. Thus, $\nu^\lambda \leq \nu$ implies

$$\nu(M < \lambda'') \leq \nu^\lambda(M < \lambda'') = 0$$

where the last equality follows from Proposition 2.2, hence a contradiction. Similarly $\rho' > \rho$ implies a contradiction. Thus γ is supported on $\mathscr{R} \cap [\lambda, \rho]$. □

Proof of Lemma 3.4.

(i). Nonnegativity follows from taking $f = 0$ in (48). As a supremum of continuous functions, $\mathscr{D}_v$ is lower semicontinuous : indeed, because the number of particles per site is bounded,

each local function e^f is continuous and bounded and so is $L_\nu(e^f)/e^f$, hence the functional defined on $\mathscr{P}(\mathbf{X})$ by

$$\phi_f(\mu) := -\int \frac{L_\nu(e^f)}{e^f} d\mu \tag{87}$$

is continuous. The inclusion $\mathscr{I}_\nu \subset \mathscr{D}_\nu^{-1}(0)$ holds because of

$$L_\nu(\log g) \leq L_\nu g / g$$

which follows from the elementary inequality $\log b - \log a \leq (b-a)/a$, and the fact that $\int L_\nu(\log g) d\mu = 0$ if $\mu \in \mathscr{I}_\nu$. We eventually prove the reverse inclusion $\mathscr{D}_\nu^{-1}(0) \subset \mathscr{I}_\nu$. Fix a local test function f. If $\mu \in \mathscr{D}_\nu^{-1}(0)$, we must have

$$I(t) := \int \frac{L_\nu(e^{tf})}{e^{tf}} d\mu \geq 0, \quad \forall t \in \mathbb{R} \tag{88}$$

As f is local and the space $\{0,\dots,K\}$ is finite, integrability conditions are satisfied to differentiate $I(t)$ in (88) under the integral. Since $I(0) = 0$, equation (88) implies that $I(t)$ has a minimum at $t = 0$, hence

$$0 = \frac{dI(t)}{dt}\bigg|_{t=0} = \int \frac{d}{dt} \frac{L_\nu(e^{tf})}{e^{tf}}\bigg|_{t=0} d\mu = \int L_\nu f \, d\mu$$

and thus $\mu \in \mathscr{I}_\nu$, since this is true for any local function.

(ii). Since ϕ_f is continuous, by [25, Appendix 2, Lemma 3.3], it is enough to prove that

$$\limsup_{t\to\infty} t^{-1} \log \mathbf{P}\left(\pi_{t,\varepsilon}(\tilde{\xi}_\cdot) \in O\right) \leq \inf_{f \text{ local}} \sup_{\mu \in O} -\phi_f(\mu) \tag{89}$$

for every open subset $O \subset \mathscr{P}(\mathbf{X})$. Let f be a local test function on $\mathbf{X}$, and set

$$\bar{f}(t,\eta) := |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{x \in \mathbb{Z} \cap [-\varepsilon t, \varepsilon t]} \tau_x f(\eta) = \sum_{n=0}^{\infty} 1_{[n\varepsilon^{-1},(n+1)\varepsilon^{-1})}(t) \bar{f}_n(\eta)$$

where

$$\bar{f}_n(\eta) := (2n+1)^{-1} \sum_{x=-n}^{n} \tau_x f(\eta)$$

For each $n \in \mathbb{Z}^+$,

$$M_t^{f,n} := \exp\left\{ \bar{f}_n(\tilde{\xi}_t) - \bar{f}_n(\tilde{\xi}_{n\varepsilon^{-1}}) - \int_{n\varepsilon^{-1}}^{t} e^{-\bar{f}_n} L_\nu[e^{\bar{f}_n}](\tilde{\xi}_{s^-}) ds \right\}, \quad t \geq n\varepsilon^{-1}$$

is a mean 1 martingale under $\mathbf{P}$ with respect to the σ-field $\mathscr{G}_t$ generated by $(\tilde{\xi}_s, s \leq t)$ (cf. [25, Section 7 of Appendix 1]). It follows that M_t^f defined for $t \geq 0$ by

$$M_t^f := \prod_{k=1}^{n} M_{(k\varepsilon^{-1})^-}^{f,k-1} M_t^{f,n}, \quad t \in [n\varepsilon^{-1}, (n+1)\varepsilon^{-1})$$

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(where the product is 1 for $n = 0$) is a mean 1 $\mathcal{G}_t$-martingale under $\mathbf{P}$. Thus we can define a probability measure $\mathbf{P}^f$ on $\mathcal{G}_t$ by $d\mathbf{P}^f/d\mathbf{P} = M_t^f$. A simple computation shows that

$$M_t^f = \exp\left\{\bar{f}(t, \widetilde{\xi}_t) - \bar{f}(0, \widetilde{\xi}_0) - \int_0^t e^{-\bar{f}} L_\nu[e^{\bar{f}}](s, \widetilde{\xi}_{s^-}) ds + R_t^f\right\} \tag{90}$$

where

$$R_t^f = \sum_{n=1}^{[\varepsilon t]} \left[\bar{f}_{n-1}(\widetilde{\xi}_{(n\varepsilon^{-1})^-}) - \bar{f}_n(\widetilde{\xi}_{n\varepsilon^{-1}})\right] \tag{91}$$

Notice that, by the graphical construction of Section 2.1, $s \mapsto \eta_s(x)$ is for each $x \in \mathbb{Z}$ a piecewise constant function whose jumps occur at (random) times which are a subset of some Poisson process. Thus s^- in (90) can be replaced by s, and $(n\varepsilon^{-1})^-$ in (91) by $n\varepsilon^{-1}$. The latter implies that the summand in (91) is bounded in modulus by $4(2n+1)^{-1} \sup|f|$. Hence

$$\left|R_t^f\right| \le 2[1 + \log(\varepsilon t)] \sup|f| \tag{92}$$

We claim (this will be established below) that, for every probability measure μ on $\mathbf{X}$, the mapping $f \mapsto \int e^{-f} L_\nu[e^f] d\mu$ (defined on the set of local functions $f : \mathbf{X} \to \mathbb{R}$) is convex. Since L_ν commutes with the space shift, this implies

$$\frac{d\mathbf{P}}{d\mathbf{P}^f} \le \exp\left\{-R_t^f + \bar{f}(0, \widetilde{\xi}_0) - \bar{f}(t, \widetilde{\xi}_t) + t \int \frac{L_\nu[e^f]}{e^f}(\eta) \pi_{t,\varepsilon}(\widetilde{\xi}_\cdot)(d\eta)\right\}$$

Thus, for any open subset O of $\mathcal{P}(\mathbf{X})$, we have (cf. (87))

$$\mathbf{P}(\pi_{t,\varepsilon}(\widetilde{\xi}_\cdot) \in O) \ \le \ e^{-R_t^f + 2\sup|f|} \int e^{-t\phi_f[\pi_{t,\varepsilon}(\widetilde{\xi}_\cdot)]} \mathbf{1}_O[\pi_{t,\varepsilon}(\widetilde{\xi}_\cdot)] d\mathbf{P}^f(\widetilde{\xi}_\cdot)$$

$$\le \ \exp\left\{-R_t^f + 2\sup|f| - t \inf_{\mu \in O} \phi_f(\mu)\right\} \tag{93}$$

Using (92) and minimizing the r.h.s. of (93) over local functions f, we obtain (89).

Proof of claim. We prove that $f \mapsto \int e^{-f} L_\nu[e^f] d\mu = -\phi_f(\mu)$ is convex for any $\mu \in \mathcal{P}(\mathbf{X})$. Equivalently we show that, for any local functions f, g on $\mathbf{X}$, $t \mapsto -\phi_{tf+g}(\mu)$ is convex on $\mathbb{R}$. We have

$$\frac{d^2}{dt^2} \int \frac{L_\nu e^{tf+g}}{e^{tf+g}} d\mu = \int \frac{L_\nu(f^2 e^{tf+g}) - 2(tf+g) L_\nu(f e^{tf+g}) + f^2 L_\nu(e^{tf+g})}{e^{tf+g}} d\mu$$

The above integrand is nonnegative. Indeed, for local functions φ and ψ,

$$L_\nu(\varphi^2 \psi) - 2\varphi L_\nu(\varphi \psi) + \varphi^2 L_\nu \psi = L_\nu^\psi(\varphi^2) - 2\varphi L_\nu^\psi \varphi \tag{94}$$

where $L_\nu^\psi \varphi := L_\nu(\varphi \psi) - \varphi L_\nu \psi$. For $\psi \ge 0$, L_ν^ψ is a Markov generator, and thus the r.h.s. of (94) is nonnegative. $\qquad\square$

4 The Cauchy problem

In Corollary 3.1, we established an almost sure hydrodynamic limit for initial measures corresponding to the Riemann problem with $\mathscr{R}$-valued initial densities. In this section we prove that this implies Theorem 2.1, that is the almost sure hydrodynamic limit for *any* initial sequence associated with *any* measurable initial density profile (thus we add in this Section the hypothesis $p(.)$ finite range, which was not necessary for Riemann a.s. hydrodynamics). This passage is inspired by Glimm's scheme, a well-known procedure in the theory of hyperbolic conservation laws, by which one constructs general entropy solutions using only Riemann solutions (see *e.g.* [39, Chapter 5]). In [8, Section 5], we undertook such a derivation for convergence in probability. In the present context of almost sure convergence, new error analysis is necessary. In particular, we have to do an explicit time discretization (vs. the "instantaneous limit" of [7, Section 3, Theorem 3.2] or [8, Section 5] for the analogue of (105) below), we need estimates uniform in time (Lemma 4.2), and each approximation step requires a control with exponential bounds (Proposition 4.2 and Lemma 4.3).

4.1 Preliminary results

For two measures $\alpha, \beta \in \mathscr{M}^+(\mathbb{R})$ with compact support, we define

$$\Delta(\alpha, \beta) := \sup_{x \in \mathbb{R}} \left| \alpha((-\infty, x]) - \beta((-\infty, x]) \right| \tag{95}$$

When α or β is of the form $u(.)dx$ for $u(.) \in L^\infty(\mathbb{R})$ with compact support, we simply write u in (95) instead of $u(.)dx$. A connection between Δ and vague convergence is given by the following technical lemma, whose proof is left to the reader.

Lemma 4.1. *(i) Let $(\alpha^N)_N$ be a $\mathscr{M}^+(\mathbb{R})$-valued sequence supported on a common compact subset of $\mathbb{R}$, and $u(.) \in L^\infty(\mathbb{R})$. The following statements are equivalent: (a) $\alpha^N \to u(.)dx$ as $N \to \infty$; (b) $\Delta(\alpha^N, u(.)) \to 0$ as $N \to \infty$.*

(ii) Let $(\alpha^N(.))_N$ be a sequence of $\mathscr{M}^+(\mathbb{R})$-valued functions $\alpha^N : \mathscr{T} \to \mathscr{M}^+(\mathbb{R})$, where $\mathscr{T}$ is an arbitrary set, such that the measures $\alpha^N(t)$ are supported on a common compact subset of $\mathbb{R}$. Assume that, for some $\alpha : [0, +\infty) \to \mathscr{M}^+(\mathbb{R})$, $\Delta(\alpha^N(t), \alpha(t))$ converges to 0 uniformly on $\mathscr{T}$. Then $\alpha^N(.)$ converges to $\alpha(.)$ uniformly on $\mathscr{T}$.

The following proposition is a collection of results on entropy solutions. We first recall two definitions. A sequence $(u_n, n \in \mathbb{N})$ of Borel measurable functions on $\mathbb{R}$ is said to converge to u in $L^1_{\text{loc}}(\mathbb{R})$ if and only if

$$\lim_{n \to \infty} \int_I \left| u_n(x) - u(x) \right| dx = 0$$

for every bounded interval $I \subset \mathbb{R}$. The variation of a function $u(.)$ on an interval $I \subset \mathbb{R}$ is defined by

$$\text{TV}_I[u(.)] = \sup \left\{ \sum_{i=0}^{n-1} \left| u(x_{i+1}) - u(x_i) \right| : n \in \mathbb{N}, x_0, \ldots, x_n \in I, x_0 < \cdots < x_n \right\}$$

We shall simply write TV for $\text{TV}_{\mathbb{R}}$. We say that $u = u(., .)$ defined on $\mathbb{R} \times \mathbb{R}^{+*}$ has locally bounded space variation if for every bounded space interval $I \subset \mathbb{R}$ and bounded time interval $J \subset \mathbb{R}^{+*}$

$$\sup_{t \in J} \text{TV}_I[u(., t)] < +\infty$$

Proposition 4.1.

o) Let $u(.,.)$ be the entropy solution to (10) with Cauchy datum $u_0 \in L^\infty(\mathbb{R})$. Then the mapping $t \mapsto u(.,t)$ lies in $C^0([0,+\infty), L^1_{\text{loc}}(\mathbb{R}))$.

i) If $u_0(.)$ is a.e. $\mathscr{R}$-valued, then so is the corresponding entropy solution $u(.,t)$ to (10) at later times.

ii) If $u_0^i(.)$ has finite variation, that is $\text{TV}u_0^i(.) < +\infty$, then so does $u^i(.,t)$ for every $t > 0$, and $\text{TV}u^i(.,t) \leq \text{TV}u_0^i(.)$.

iii) Finite propagation property: Assume $u^i(.,.)$ ($i \in \{1,2\}$) is the entropy solution to (10) with Cauchy data $u_0^i(.)$. Let $V = ||G'||_\infty := \sup_\rho |G'(\rho)|$. Then: (a) for every $x < y$ and $0 \leq t < (y-x)/2V$,

$$\int_{x+Vt}^{y-Vt} \left|u^1(z,t) - u^2(z,t)\right| dz \leq \int_x^y \left|u_0^1(z) - u_0^2(z)\right| dz \tag{96}$$

In particular, if u_0^1 is supported (resp. coincides with u_0^2) in $[-R,R]$ for some $R > 0$, $u^1(.,t)$ is supported (resp. coincides with $u^2(.,t)$) in $[-R-Vt, R+Vt]$. (b) If $\int_\mathbb{R} u_0^i(z)dz < +\infty$,

$$\Delta(u^1(.,t), u^2(.,t)) \leq \Delta(u_0^1(.), u_0^2(.)) \tag{97}$$

iv) Let $x_0 = -\infty < x_1 < \cdots < x_n < x_{n+1} = +\infty$ and $\varepsilon := \min_{0 \leq k \leq n}(x_{k+1} - x_k)$. Denote by $u_0(.)$ the piecewise constant profile with value r_k on $I_k := (x_k, x_{k+1})$. Then, for $t < \varepsilon/(2V)$, the entropy solution $u(.,t)$ to (10) with Cauchy datum $u_0(.)$ is given by

$$u(x,t) = R_{r_{k-1}, r_k}(x - x_k, t), \quad \forall x \in \left(x_{k-1} + Vt, x_{k+1} - Vt\right)$$

Properties o), ii) and iii) are standard, see [28; 39; 29]. Properties i) and iv) are respectively [8, Lemma 5.3] and [7, Lemma 3.4]. The latter states that the entropy solution starting from a piecewise constant profile can be constructed at small times as a superposition of successive non-interacting Riemann waves. This is a consequence of iii). Note that the whole space is indeed covered by the definition of $u(x,t)$ in iv), since we have $x_{k+1} - Vt \geq x_k + Vt$ for $t \leq \varepsilon/(2V)$.

The next lemma improves [8, Lemma 5.5] by deriving an approximation uniform in time.

Lemma 4.2. *Assume $u_0(.)$ is a.e. $\mathscr{R}$-valued, has bounded support and finite variation, and let $(x,t) \mapsto u(x,t)$ be the entropy solution to (10) with Cauchy datum $u_0(.)$. For every $\varepsilon > 0$, let $\mathscr{P}_\varepsilon$ be the set of piecewise constant $\mathscr{R}$-valued functions on $\mathbb{R}$ with compact support and step lengths at least ε, and set*

$$\delta_\varepsilon(t) := \varepsilon^{-1} \inf\{\Delta(u(.), u(.,t)) : u(.) \in \mathscr{P}_\varepsilon\}$$

Then there is a sequence $\varepsilon_n \downarrow 0$ as $n \to \infty$ such that δ_{ε_n} converges to 0 uniformly on any bounded subset of $\mathbb{R}^+$.

Proof of Lemma 4.2. We first argue that, for every $\varepsilon > 0$, δ_ε is a continuous function. Indeed, by

Proposition 4.1, iii),a) for every $T > 0$, there exists a bounded set $K_T \subset \mathbb{R}$ such that the support of $u(.,t)$ is contained in K_T for every $t \in [0, T]$. Since

$$\Delta(v, w) \le \int_{\mathbb{R}} |v(x) - w(x)| \, dx \tag{98}$$

for $v, w \in L^\infty(\mathbb{R})$ with compact support, it follows by Proposition 4.1, o) and Lemma 4.1, (i) that

$$\lim_{s \to t} \Delta(u(.,s), u(.,t)) = 0 \tag{99}$$

for every $t \ge 0$. This and the inequality

$$\left| \delta_\varepsilon(t) - \delta_\varepsilon(s) \right| \le \varepsilon^{-1} \Delta(u(.,s), u(.,t))$$

imply continuity of δ_ε. By Proposition 4.1, i) and ii), $u(.,t)$ has bounded, finite space variation, and is $\mathcal{R}$-valued. Hence, by [8, Lemma 5.5], for any given $\delta > 0$, for $\varepsilon > 0$ small enough, there exists an approximation $u^{\varepsilon,\delta}(.,t) \in \mathscr{P}_\varepsilon$ of $u(.,t)$ with $\Delta\left(u^{\varepsilon,\delta}(.,t), u(.,t)\right) \le \varepsilon\delta$. This implies $\delta_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ for every $t > 0$. Let $\mathcal{T}$ be some countable dense subset of $[0, +\infty)$. By the diagonal extraction procedure, we can find a sequence $\varepsilon_n \downarrow 0$ such that $\delta_{\varepsilon_n}(t) \downarrow 0$ for each $t \in \mathcal{T}$. By continuity of δ_ε we also have that $\delta_{\varepsilon_n}(t) \downarrow 0$ for every $t \in [0, +\infty)$. Dini's theorem implies that δ_{ε_n} converges uniformly to 0 on every bounded subset of $[0, +\infty)$. $\square$

We now quote [10, Proposition 3.1], which yields that Δ is an "almost" nonincreasing functional for two coupled particle systems:

Proposition 4.2. *Assume $p(.)$ is finite range. Then there exist constants $C > 0$ and $c > 0$, depending only on $b(.,.)$ and $p(.)$, such that the following holds. For every $N \in \mathbb{N}$, $(\eta_0, \xi_0) \in \mathbf{X}^2$ with $|\eta_0| + |\xi_0| := \sum_{x \in \mathbb{Z}} [\eta_0(x) + \xi_0(x)] < +\infty$, and every $\gamma > 0$, the event*

$$\forall t > 0 : \Delta(\alpha^N(\eta_t(\eta_0, \omega)), \alpha^N(\eta_t(\xi_0, \omega))) \le \Delta(\alpha^N(\eta_0), \alpha^N(\xi_0)) + \gamma \tag{100}$$

has $\mathbf{P}$-probability at least $1 - C(|\eta_0| + |\xi_0|)e^{-cN\gamma}$.

We finally recall the *finite propagation* property at particle level (see [8, Lemma 5.2]), which is a microscopic analogue of Proposition 4.1, iii).

Lemma 4.3. *There exist constants v and C, depending only on $b(.,.)$ and $p(.)$, such that the following holds. For any $x, y \in \mathbb{Z}$, any $(\eta_0, \xi_0) \in \mathbf{X}^2$, and any $0 < t < (y - x)/(2v)$: if η_0 and ξ_0 coincide on the site interval $[x, y]$, then with $\mathbf{P}$-probability at least $1 - e^{-Ct}$, $\eta_s(\eta_0, \omega)$ and $\eta_s(\xi_0, \omega)$ coincide on the site interval $[x + vt, y - vt] \cap \mathbb{Z}$ for every $s \in [0, t]$.*

Remark 4.1. *The time uniformity in Proposition 4.2 and Lemma 4.3 does not appear in the original statements (repectively, [10, Proposition 3.1] and [8, Lemma 5.2]), but follows in each case from the proof.*

4.2 Proof of Theorem 2.1

4.2.1 Simplified initial conditions

We will first prove Theorem 2.1 under the simplifying assumptions:

$$u_0 \text{ is a.e. } \mathcal{R}\text{-valued} \tag{101}$$

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$$\mathrm{TV} u_0 < +\infty \tag{102}$$

and there exists $R > 0$ (independent of N) such that

$$\operatorname{supp} u_0 \subset [-R, R] \tag{103}$$

$$\forall N \in \mathbb{N}, \quad \mathbf{P}_0 \left(\eta_0^N(x) = 0 \text{ whenever } x \in \mathbb{Z}, |x| \geq RN \right) = 1 \tag{104}$$

The essential part of the work (that is, the approximation scheme) is contained here, and the proof under general assumptions will follow in Subsection 4.2.2 by approximation arguments.

Assumption (101) implies by Proposition 4.1, i), that $u(., t)$ is $\mathscr{R}$-valued, (102) by Proposition 4.1, ii), that $u(., t)$ has finite variation for $t > 0$, and (103) by Proposition 4.1, iii), that $u(., t)$ is supported on $[-(R + V t), R + V t]$. In the sequel we abbreviate, for $(\omega_0, \omega) \in \widetilde{\Omega}$

$$\eta_t^N = \eta_t(\eta_0^N(\omega_0), \omega)$$

We consider the random process on $\widetilde{\Omega}$

$$\Delta^N(t) := \Delta(\alpha^N(\eta_{Nt}^N), u(., t))$$

By initial assumption (8) and (i) of Lemma 4.1, $\Delta^N(0)$ converges to 0, $\mathbf{P}_0$-a.s. Fix an arbitrary time $T > 0$. We are going to prove that

$$\lim_{N \to \infty} \sup_{t \in [0,T]} \Delta^N(t) = 0, \quad \widetilde{\mathbf{P}}\text{-a.s.} \tag{105}$$

Then one can find a set of probability one on which this holds simultaneously for all $T > 0$. Theorem 2.1 follows from (ii) of Lemma 4.1.

Let $\varepsilon = \varepsilon_n$ be given by Lemma 4.2, and $\delta = \delta_n = 2 \sup_{t \in [0,T]} \delta_{\varepsilon_n}(t)$, so that $\delta_n \to 0$ as $n \to \infty$. In the sequel, for notational simplicity, we omit mention of n. We fix a time discretization step

$$\varepsilon' = \varepsilon \min((2v)^{-1}, (2V)^{-1}) \tag{106}$$

where V and v are the constants defined in Proposition 4.1 and Lemma 4.3. Let $t_k = k\varepsilon'$, for $k \leq \mathscr{K} := [T/\varepsilon']$, $t_{\mathscr{K}+1} = T$. The main step to derive (105) is to obtain a time discretized version of it, namely

Lemma 4.4.
$$\limsup_{N \to \infty} \sup_{k=0,\dots,\mathscr{K}-1} \left[\Delta^N(t_{k+1}) - \Delta^N(t_k) \right] \leq 3\delta\varepsilon, \quad \widetilde{\mathbf{P}}\text{-a.s.}$$

The second, more technical step, will be to fill in the gaps between discretized times by a uniform estimate for the time modulus of continuity, that is

Lemma 4.5.
$$\lim_{\varepsilon = \varepsilon_n \to 0} \limsup_{N \to \infty} \sup_{k=0,\dots,\mathscr{K}} \sup_{t \in [t_k, t_{k+1}]} \Delta \left[\alpha^N(\eta_{Nt}^N), \alpha^N(\eta_{Nt_k}^N) \right] = 0, \quad \widetilde{\mathbf{P}}\text{-a.s.}$$

By o) of Proposition 4.1, $t \mapsto u(t,.)$ is uniformly continuous from $[0, T]$ to $L^1_{\text{loc}}(\mathbb{R})$. This and (98) imply an analogue of Lemma 4.5 at the level of entropy solutions, namely

$$\lim_{\varepsilon = \varepsilon_n \to 0} \sup_{k=0,\ldots,\mathcal{K}} \sup_{t \in [t_k, t_{k+1}]} \Delta(u(.,t), u(.,t_k)) = 0 \tag{107}$$

Then (105) follows from Lemma 4.4, Lemma 4.5 and (107).

Proof of Lemma 4.4. The method is to approximate $u(., t_k)$ by an $\mathcal{R}$-valued step function $v_k(.)$, to associate to the profile $v_k(.)$ a sequence of configurations $(\xi^{N,k})_N$ (in the sense (118) below), to use Riemann hydrodynamics for these approximations from time t_k up to time t_{k+1}, and to show that approximated systems at time t_{k+1} are close enough to the original ones, both at microscopic and macroscopic levels.

Let $v_k(.)$ be an approximation of $u(., t_k)$ given by Lemma 4.2, so that

$$\Delta(u(., t_k), v_k(.)) \le \delta \varepsilon, \quad k = 0, \ldots, \mathcal{K} - 1 \tag{108}$$

We write $v_k(.)$ as

$$v_k = \sum_{l=0}^{l_k} r_{k,l} \mathbf{1}_{[x_{k,l}, x_{k,l+1})} \tag{109}$$

where $-\infty = x_{k,0} < x_{k,1} < \ldots < x_{k,l_k} < x_{k,l_k+1} = +\infty$, $r_{k,l} \in \mathcal{R}$, $r_{k,0} = r_{k,l_k} = 0$, and, for $1 < l \le l_k$,

$$x_{k,l} - x_{k,l-1} \ge \varepsilon \tag{110}$$

For $t_k \le t < t_{k+1}$, we denote by $v_k(., t)$ the entropy solution to (10) at time t with Cauchy datum $v_k(.)$. For $l = 1, \ldots, l_k$, define on $\widetilde{\Omega}$ the configurations $\xi^{N,k,l}(\omega_0, \omega)$ and $\xi^{N,k}(\omega_0, \omega)$ by

$$\xi^{N,k,l}(\omega_0, \omega)(x) := \begin{cases} \eta_{Nt_k}(\eta^{r_{k,l-1}}(\omega_0), \omega)(x) & \text{if } x < [Nx_{k,l}] \\ \eta_{Nt_k}(\eta^{r_{k,l}}(\omega_0), \omega)(x) & \text{if } x \ge [Nx_{k,l}] \end{cases} \tag{111}$$

$$\xi^{N,k}(\omega_0, \omega)(x) := \eta_{Nt_k}(\eta^{r_{k,l}}(\omega_0), \omega)(x), \text{ if } [Nx_{k,l}] \le x < [Nx_{k,l+1}] \tag{112}$$

so that $\xi^{N,k}(\omega_0, \omega)$ has finitely many particles, and

$$\xi^{N,k}(x) = \xi^{N,k,l}(x), \text{ if } [Nx_{k,l-1}] \le x < [Nx_{k,l+1}] \tag{113}$$

Moreover by the commutation property (6),

$$\tau_{[Nx_{k,l}]} \xi^{N,k,l}(\omega_0, \omega) \tag{114}$$
$$= T(\eta_{Nt_k}(\tau_{[Nx_{k,l}]} \eta^{r_{k,l-1}}(\omega_0), \theta_{[Nx_{k,l}],0}\omega), \eta_{Nt_k}(\tau_{[Nx_{k,l}]} \eta^{r_{k,l}}(\omega_0), \theta_{[Nx_{k,l}],0}\omega))$$

Evolutions from (111)–(112) are denoted by

$$\begin{aligned} \xi_t^{N,k}(\omega_0, \omega) &= \eta_t(\xi^{N,k}(\omega_0, \omega), \theta_{0,Nt_k}\omega) \\ \xi_t^{N,k,l}(\omega_0, \omega) &= \eta_t(\xi^{N,k,l}(\omega_0, \omega), \theta_{0,Nt_k}\omega) \end{aligned} \tag{115}$$

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so that by the Markov property (5) and (114), we have

$$
\begin{aligned}
&\tau_{[Nx_{k,l}]}\xi_t^{N,k,l}(\omega_0,\omega) \\
={}& T(\eta_t(\tau_{[Nx_{k,l}]}\eta^{r_{k,l-1}}(\omega_0),\theta_{[Nx_{k,l}],Nt_k}\omega),\eta_t(\tau_{[Nx_{k,l}]}\eta^{r_{k,l}}(\omega_0),\theta_{[Nx_{k,l}],Nt_k}\omega))
\end{aligned}
\tag{116}
$$

We claim that

$$
\lim_{N\to\infty}\alpha^N(\eta_{Nt_k}(\eta^{r_{k,l}}(\omega_0),\omega))(dx)=r_{k,l}dx,\quad \widetilde{\mathbf{P}}\text{-a.s.}
\tag{117}
$$

For $k=0$, this follows from Proposition 2.2, since $\eta^{r_{k,l}}(\omega_0)\sim\nu^{r_{k,l}}$. For $k=1,\dots,\mathcal{K}-1$, this follows from Corollary 3.1 with $\lambda=\rho=r_{k,l}$ and $s_0=x_0=0$. Indeed, on the one hand we have $R_{r_{k,l},r_{k,l}}(.,.)\equiv r_{k,l}$; on the other hand, if $\omega'=(\eta,\xi,\omega)\sim\bar\nu^{r_{k,l},r_{k,l}}\otimes\mathbf{P}$, we have $\eta=\xi$ a.s., so that $\beta_{Nt_k}^N(\omega')=\alpha^N(\eta_{Nt_k}(\eta,\omega))$ a.s., with $(\eta,\omega)\sim\nu^{r_{k,l}}\otimes\mathbf{P}$.

By (112), for every continuous function ψ on $\mathbb{R}$ with compact support,

$$
\begin{aligned}
&\int_{\mathbb{R}}\psi(x)\alpha^N(\xi^{N,k}(\omega_0,\omega))(dx) \\
={}& \sum_{l=1}^{l_k}\int_{\mathbb{R}}\psi(x)\mathbf{1}_{[x_{k,l},x_{k,l+1})}(x)\alpha^N(\eta_{Nt_k}(\eta^{r_{k,l}}(\omega_0),\omega))(dx)+O(1/N) \\
&\overset{N\to\infty}{\longrightarrow}\int_{\mathbb{R}}\psi(x)\nu_k(x)dx,\quad \widetilde{\mathbf{P}}\text{-a.s.}
\end{aligned}
$$

where the convergence follows from (117) and (109). Hence,

$$
\lim_{N\to\infty}\alpha^N(\xi^{N,k}(\omega_0,\omega))(dx)=\nu_k(.)dx,\quad \widetilde{\mathbf{P}}\text{-a.s.}
\tag{118}
$$

that is, $\xi^{N,k}$ is a microscopic version of $\nu_k(.)$. For $k=0,\dots,\mathcal{K}-1$, we write (remember that $\varepsilon'=t_{k+1}-t_k$)

$$
\begin{aligned}
\Delta^N(t_{k+1})-\Delta^N(t_k) \le{}& \Delta\left[\alpha^N(\eta_{Nt_{k+1}}^N),\alpha^N(\xi_{N\varepsilon'}^{N,k})\right]-\Delta^N(t_k) \\
&+\Delta\left[\alpha^N(\xi_{N\varepsilon'}^{N,k}),\nu_k(.,\varepsilon')\right] \\
&+\Delta(\nu_k(.,\varepsilon'),u(.,t_{k+1}))
\end{aligned}
\tag{119}
$$

By (108) and iii), b) of Proposition 4.1,

$$
\Delta(\nu_k(.,t_{k+1}-t_k),u(.,t_{k+1}))\le\Delta(\nu_k(.),u(.,t_k))\le\delta\varepsilon
\tag{120}
$$

is a bound for the third term on the r.h.s. of (119). For the first term, we define the event

$$
E^{N,k}:=\left\{\Delta\left[\alpha^N(\eta_{Nt_{k+1}}^N),\alpha^N(\xi_{N\varepsilon'}^{N,k})\right]\le\Delta\left[\alpha^N(\eta_{Nt_k}^N),\alpha^N(\xi^{N,k})\right]+\delta\varepsilon\right\}
$$

By assumption (104) and Proposition 4.2,

$$
\widetilde{\mathbf{P}}(\widetilde\Omega-E^{N,k})\le C'Ne^{-cN\delta\varepsilon}
$$

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for some constant C' independent of k. Thus, by Borel-Cantelli's lemma, there exists a random $N_1(\omega_0, \omega)$ such that $\widetilde{\mathbf{P}}$-a.s., $E^{N,k}$ holds for all $N \geq N_1$ and $k = 0, \ldots, \mathscr{K} - 1$. On the other hand,

$$\Delta\left[\alpha^N(\eta^N_{Nt_k}), \alpha^N(\xi^{N,k})\right] \leq \Delta^N(t_k) + \Delta(u(., t_k), v_k(.)) + \Delta\left[v_k(.), \alpha^N(\xi^{N,k})\right]$$

Thus, by (108), (118) and (i) of Lemma 4.1,

$$\limsup_{N \to \infty}\left\{\Delta\left[\alpha^N(\eta^N_{Nt_k}), \alpha^N(\xi^{N,k})\right] - \Delta^N(t_k)\right\} \leq \delta\varepsilon$$

with $\widetilde{\mathbf{P}}$-probability 1. Therefore, $\widetilde{\mathbf{P}}$-a.s., for $k = 0, \ldots, \mathscr{K} - 1$,

$$\limsup_{N \to \infty}\left\{\Delta\left[\alpha^N(\eta^N_{Nt_{k+1}}), \alpha^N(\xi^{N,k}_{N\varepsilon'})\right] - \Delta^N(t_k)\right\} \leq 2\delta\varepsilon \tag{121}$$

is a bound for the first term on the r.h.s. of (119). We finally bound the second term on the r.h.s. of (119). For $k = 0, \ldots, \mathscr{K} - 1$ and $l = 1, \ldots, l_k$, by the respective definitions (28), (31) of θ', $\beta^N_.$, and (111), (115), (116) we have

$$(\tau_{-[Nx_{k,l}]/N})\alpha^N\left(\xi^{N,k,l}_{Nt}(\omega_0, \omega)\right) = \alpha^N\left((\tau_{[Nx_{k,l}]})\xi^{N,k,l}_{Nt}(\omega_0, \omega)\right)$$
$$= \beta^N_{Nt} \circ \theta'_{[Nx_{k,l}], Nt_k}(\eta^{r_{k,l-1}}(\omega_0), \eta^{r_{k,l}}(\omega_0), \omega) \tag{122}$$

for any $t \geq 0$, where the first line follows from definition of the shift operator τ on measures and particle configurations (see notation at the beginning of Section 2), and definition (46) of the empirical measure α^N. This and Corollary 3.1 imply

$$\lim_{N \to \infty} \alpha^N\left(\xi^{N,k,l}_{Nt}\right) = R_{r_{k,l-1}, r_{k,l}}(. - x_{k,l}, t)dx, \quad \widetilde{\mathbf{P}}\text{-a.s.} \tag{123}$$

Let us consider the events

$$F^{N,k,l} := \left\{\xi^{N,k}_{N\varepsilon'}(x) = \xi^{N,k,l}_{N\varepsilon'}(x), \forall x \in \mathbb{Z} \cap \left[N(x_{k,l-1} + v\varepsilon'), N(x_{k,l+1} - v\varepsilon')\right)\right\}$$

By (113), the definition (106) of ε', (110) and Lemma 4.3, we have

$$\widetilde{\mathbf{P}}\left(\widetilde{\Omega} - F^{N,k,l}\right) \leq e^{-CN\varepsilon'}$$

Thus there exists a random $N_2(\omega_0, \omega)$ such that $\widetilde{\mathbf{P}}$-a.s., $F^{N,l,k}$ holds for every $N \geq N_2$, $k = 0, \ldots, \mathscr{K} - 1$ and $l = 1, \ldots, l_k$. This combined with (123) implies that $\widetilde{\mathbf{P}}$-a.s., the restriction of $\alpha^N(\xi^{N,k}_{N\varepsilon'})$ to $(x_{k,l-1} + v\varepsilon', x_{k,l+1} - v\varepsilon')$ converges as $N \to \infty$ to the restriction of $R_{r_{k,l-1}, r_{k,l}}(. - x_{k,l}, \varepsilon')dx$. By (106) and iv) of Proposition 4.1, this induces

$$\lim_{N \to \infty} \alpha^N(\xi^{N,k}_{N\varepsilon'}) = v_k(., \varepsilon')dx, \quad \widetilde{\mathbf{P}}\text{-a.s.}$$

which, by Lemma 4.1, implies that the second term on the r.h.s. of (119) converges $\widetilde{\mathbf{P}}$-a.s. to 0 as $N \to \infty$. Together with (120) and (121), this yields Lemma 4.4. $\qquad\square$

Proof of Lemma 4.5. We label η-particles increasingly from left to right at each time Nt_k, denoting their positions by $(R^{k,i})_{i \in I}$, where I is a finite set whose cardinal $|I|$, of order $O(N)$ by assumption

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(104), is the number of particles in the system. For simplicity we omit the dependence of the labelling on N in the notation. The position of particle i at time $\theta \in [Nt_k, Nt_{k+1}]$ is denoted by $R_\theta^{k,i}$. Let for any $s, t \in [t_k, t_{k+1}]$,

$$\Delta_{s,t} := \Delta(\alpha^N(\eta_{Ns}^N), \alpha^N(\eta_{Nt}^N)) = N^{-1} \left| \sup_{x \in \mathbb{Z}} \sum_{y \leq x} \left[\eta_{Nt}^N(y) - \eta_{Ns}^N(y) \right] \right|$$

Let $z \in \mathbb{Z}$ be a point at which the supremum above is attained. We can suppose without loss of generality that

$$N\Delta_{s,t} = \sum_{y \leq z} \eta_{Ns}^N(y) - \sum_{y \leq z} \eta_{Nt}^N(y).$$

Therefore to the left of z at time Ns there are at least $N\Delta_{s,t}$ more particles than at time Nt. Let I_s and I_t be the subsets of I which label the particles to the left of or at z at times Ns and Nt respectively. Then we have $|I_s| - |I_t| \geq N\Delta_{s,t}$ which implies $|I_s \setminus I_t| \geq N\Delta_{s,t}$. Now if $i \in I_s \setminus I_t$,

$$R_{Nt}^{k,i} > z \text{ since } i \notin I_t \tag{124}$$
$$R_s^{k,i} \leq z \text{ since } i \in I_s \tag{125}$$

By (124), since we have at most K particles per site, $\max_{i \in I_s \setminus I_t} R_{Nt}^{k,i} \geq z + K^{-1}N\Delta_{s,t}$. This implies $\max_{i \in I_s \setminus I_t}(R_{Nt}^{k,i} - R_{Ns}^{k,i}) \geq K^{-1}N\Delta_{s,t}$ by (125), thus

$$K \sup_{i \in I}(R_{Ns}^{k,i} - R_{Nt}^{k,i}) \geq N\Delta_{s,t}$$

and we conclude that

$$\Delta(\alpha^N(\eta_{Ns}^N), \alpha^N(\eta_{Nt}^N)) \leq KN^{-1} \sup_{i \in I} \left| R_{Ns}^{k,i} - R_{Nt}^{k,i} \right|$$

Proceeding as in the proof of Lemma 3.2 it is possible to construct processes $Q^{k,i}$ and $S^{k,i}$ on the time interval $[Nt_k, Nt_{k+1}]$ such that

$$Q_{Nt}^{k,i} \leq R_{Nt}^{k,i} - R_{Nt_k}^{k,i} \leq S_{Nt}^{k,i}$$

for $t \in [t_k, t_{k+1}]$, and $S^{k,i}$ (resp. $Q^{k,i}$) is a Markov process on $\mathbb{Z}$ starting from 0 at time Nt_k, that jumps from x to $x+z$ at rate $p(z)\|b\|_\infty$ only for $z > 0$ (resp. only for $z < 0$). Therefore,

$$\mathbf{P} \left(\sup_k \sup_{t \in [t_k, t_{k+1}]} \Delta\left(\alpha^N(\eta_{Nt}^N), \alpha^N(\eta_{Nt_k}^N)\right) \geq C\varepsilon \right)$$

$$\leq \sum_k \sum_{i \in I} \mathbf{P} \left(\sup_{t \in [t_k, t_{k+1}]} \left| R_{Nt}^{k,i} - R_{Nt_k}^{k,i} \right| \geq CN\varepsilon \right)$$

$$\leq \sum_k \sum_{i \in I} \mathbf{P} \left(-Q_{Nt_{k+1}}^{k,i} \geq CN\varepsilon \right) \tag{126}$$

$$+ \sum_k \sum_{i \in I} \mathbf{P} \left(S_{Nt_{k+1}}^{k,i} \geq CN\varepsilon \right) \tag{127}$$

Since $p(.)$ has finite first moment, by large deviation bounds for random walks, the constant C can be chosen large enough such that the probabilities in (126)–(127) are smaller than $e^{-C'\varepsilon N}$ for some

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constant C' (recall that $t_{k+1} - t_k = \varepsilon'$ is a multiple of ε). By Borel Cantelli's lemma we conclude that

$$\limsup_{N\to\infty} \sup_{k=0,\ldots,\mathscr{K}} \sup_{t\in[t_k,t_{k+1}]} \Delta\left(\alpha^N(\eta^N_{Nt}), \alpha^N(\eta^N_{Nt_k})\right) \le C\varepsilon$$

for $\varepsilon = \varepsilon_n$ on a set of probability one, which can be chosen common to all (the countably many) values of $n \in \mathbb{N}$. On this set we thus have Lemma 4.5. $\qquad\square$

4.2.2   General case

We will relax assumptions (101)–(104) in two steps.

Step one: compact support only.   We prove Theorem 2.1 when the additional assumptions (103)–(104) are maintained, but (101)–(102) are relaxed. Let $T > 0$. By approximating the initial profile by $\mathscr{R}$-valued ones, we define a sequence $(u_0^n)_{n\in\mathbb{N}}$ of $[0,K]$-valued functions satisfying (103)–(104) and (101)–(102) for fixed n, such that

$$\lim_{n\to\infty} \Delta(u_0^n, u_0) = 0 \tag{128}$$

and a family of (deterministic) particle configurations $(\eta_0^{n,N})_{n\in\mathbb{N},N\in\mathbb{N}}$ satisfying (104) for fixed n, such that

$$\lim_{N\to\infty} \Delta\left(\alpha^N(\eta_0^{n,N}), u_0^n\right) = 0 \tag{129}$$

for each $n \in \mathbb{N}$. Indeed, let us partition $[-R, R]$ into finitely many intervals $I_{n,k}$ of length at most $\delta_n \to 0$, and set

$$u_0^n = \sum_k K \mathbf{1}_{(x_{n,k}, x_{n,k} + \rho_{n,k} l_{n,k}/K)}$$

where $l_{n,k}$ denotes the length of $I_{n,k}$, $x_{n,k}$ its left extremity, and $\rho_{n,k}$ the mean value of u_0 on $I_{n,k}$. Then u_0^n has the same mean value as u_0 on $I_{n,k}$, hence $\Delta(u_0^n, u_0) \le K\delta_n$. Then we define a sequence of particle configurations associated to u_0^n by

$$\eta_0^{n,N}(x) = u_0^n\left(\frac{x}{N}\right), \quad \forall x \in \mathbb{Z}$$

We denote by $u^n(x, t)$ the entropy solution to (10) at time t starting from Cauchy datum u_0^n, and by $\eta_t^{n,N} := \eta_t(\eta_0^{n,N}, \omega)$ the evolved particle configuration starting from $\eta_0^{n,N}$. By triangle inequality for Δ,

$$\begin{aligned}
\Delta\left(\alpha^N(\eta_{Nt}^N), u(., t)\right) &\le \Delta\left(\alpha^N(\eta_{Nt}^N), \alpha^N(\eta_{Nt}^{n,N})\right) \\
&+ \Delta\left(\alpha^N(\eta_{Nt}^{n,N}), u^n(., t)\right) \\
&+ \Delta(u^n(., t), u(., t))
\end{aligned} \tag{130}$$

We have by iii), b) of Proposition 4.1,

$$\Delta(u^n(., t), u(., t)) \le \Delta(u_0^n, u_0) \tag{131}$$

Further, by Subsection 4.2.1, (129) implies the analogue of (105) for $\eta_{\cdot}^{n,N}$, that is

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \Delta\left(\alpha^N(\eta_{Nt}^{n,N}), u^n(., t)\right) = 0 \qquad \widetilde{\mathbf{P}}\text{-a.s.} \tag{132}$$

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On the other hand,

$$\Delta\left(\alpha^N(\eta_{Nt}^N), \alpha^N(\eta_{Nt}^{n,N})\right) = \Delta\left(\alpha^N(\eta_0^N), \alpha^N(\eta_0^{n,N})\right) + \Gamma_{Nt}^{N,n} \tag{133}$$

where, by Proposition 4.2, $\Gamma_{Nt}^{N,n}$ is a random variable which satisfies

$$\widetilde{\mathbf{P}}\left(\sup_{t\geq 0}\Gamma_{Nt}^{n,N} \geq \gamma\right) \leq C'Ne^{-cN\gamma}, \quad \forall \gamma > 0$$

for some constant $C' > 0$ independent of n. Applying Borel-Cantelli's lemma to a vanishing sequence of values of γ,

$$\lim_{N\to\infty}\sup_{t\geq 0}\Gamma_{Nt}^{n,N} = 0 \tag{134}$$

$\widetilde{\mathbf{P}}$-a.s.. Furthermore,

$$\begin{aligned}\Delta\left(\alpha^N(\eta_0^N), \alpha^N(\eta_0^{n,N})\right) &\leq \Delta\left(\alpha^N(\eta_0^N), u_0\right) + \Delta\left(u_0, u_0^n\right)\\ &\quad + \Delta\left(u_0^n, \alpha^N(\eta_0^{n,N})\right)\end{aligned} \tag{135}$$

By (8), (i) of Lemma 4.1 and (129)–(135),

$$\limsup_{N\to\infty}\sup_{t\in[0,T]}\Delta\left(\alpha^N(\eta_{Nt}^N), u(.,t)\right) \leq 2\Delta(u_0^n, u_0)$$

on a subset of $\widetilde{\Omega}$ with $\widetilde{\mathbf{P}}$-probability one, which can be chosen to be the same for all (countably many) values of $n \in \mathbb{N}$ and $T > 0$. The conclusion of Theorem 2.1 then follows from (128) and (ii) of Lemma 4.1.

Step two: general case. We now finally relax assumptions (103)–(104), thanks to the finite propagation property (both at microscopic and macroscopic levels). Consider u_0 and η_0^N as in the statement of Theorem 2.1, without any restriction. Let $w = \max(V, v)$, where V and v are the constants given respectively in Proposition 4.1 and Lemma 4.3. For $n \in \mathbb{N}$, we set

$$u_0^n := u_0\mathbf{1}_{[-n,n]}, \quad \eta_0^{n,N}(x) = \eta_0^N(x)\mathbf{1}_{\mathbb{Z}\cap[-Nn,Nn]}(x)$$

By Lemma 4.3 and Borel-Cantelli's lemma, $\widetilde{\mathbf{P}}$-a.s. for large enough N,

$$\eta_{Nt}^N(x) = \eta_{Nt}^{n,N}(x), \forall t \leq n/(2w), \forall x \in [-Nn/2, Nn/2] \cap \mathbb{Z}$$

By the previous step, for each $n \in \mathbb{N}$, $\widetilde{\mathbf{P}}$-a.s., $\alpha^N(\eta_{Nt}^{n,N})$ converges to $u^n(.,t)dx$ as $N \to \infty$, uniformly on bounded times intervals. By iii), (a) of Proposition 4.1, for every $t \leq n/(2w)$, $u^n(.,t) = u(.,t)$ on $[-Nn/2, Nn/2]$. Thus, for every continuous functions $\psi : \mathbb{R} \to \mathbb{R}$ supported on $[-Nn/2, Nn/2]$, there is an event of $\widetilde{\mathbf{P}}$-probability one on which

$$\int_{\mathbb{R}}\psi(x)\alpha^N(\eta_{Nt}^N)(dx) \to \int_{\mathbb{R}}\psi(x)u(x,t)dx$$

uniformly on the time interval $[0, n/(2w)]$. This event can be chosen to be the same for all values of n and for a countable set of continuous functions with compact support that is convergence determining for the vague topology. This establishes the result.

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A Proof of Corollary 2.1

Let μ_t^N denote the distribution at time t of a Markov process with generator (1). Assume $\alpha^N(\eta)(dx)$ converges in μ_0^N-probability to $u_0(.)dx$, that is, for all $\varepsilon > 0$ and every continuous function ψ on $\mathbb{R}$ with compact support,

$$\lim_{N\to\infty} \mu_0^N\left(\left\{\eta : \left|\int_{\mathbb{R}} \psi(x)\alpha^N(\eta)(dx) - \int \psi(x)u_0(x)dx\right| > \varepsilon\right\}\right) = 0$$

Then for every $t > 0$, $\alpha^N(\eta)(dx)$ converges in μ_{Nt}^N-probability to $u(.,t)dx$. This weak law follows from the strong law in Theorem 2.1. Indeed, by Skorokhod's representation theorem, we can find a probability space $(\Omega_0, \mathscr{F}_0, \mathbf{P}_0)$ and a sequence $(\eta_0^N)_N$ of $\mathbf{X}$-valued random variables on Ω_0 such that η_0^N has distribution μ_0^N, and $\alpha^N(\eta_0^N)(dx)$ converges $\mathbf{P}_0$-a.s. to $u_0(.)dx$.

B Remarks on subadditivity

As outlined below, it would be possible to establish (39) in the particular case $\beta = \alpha = 0$ by using the subadditive ergodic theorem as in [1, Proposition 3]. However we cannot use this approach when $(\beta, \alpha) \neq (0,0)$.

Let us introduce

$$\begin{aligned}
X_{0,n}(\omega') &:= \phi_{n/v}^v(\omega') - \varphi_{n/v}^v(\eta, \omega) \\
X_{m,n}(\omega') &:= X_{0,n-m}(\theta'_{m,m/v}\omega')
\end{aligned}$$ (136)

then $X_{m,n}$ is the same as defined in equation [1, (27)] and, by [1, p. 226], it satisfies the *superadditivity* property

$$X_{0,n} \geq X_{0,m} + X_{m,n}$$ (137)

(superadditivity is obtained here rather than subadditivity in [1], because we have $\lambda < \rho$ instead of $\lambda > \rho$). We point out that the proof of (137) in [1] uses only attractiveness and the fact that we start with $\eta \leq \xi$, but not the choice of the distribution of (η, ξ). It can thus be generalized from the asymmetric exclusion process to our setting. Let us now assume that the probability measure on Ω' is $\bar{v}^{\lambda,\rho} \otimes \mathbf{P}$. We can proceed as in [1]. Indeed, because $\bar{v}^{\lambda,\rho}$ is invariant for the coupled process, (28) implies that $\bar{v}^{\lambda,\rho} \otimes \mathbf{P}$ is invariant by the shift $\theta'_{x,t}$. By (26), (137) is true $\bar{v}^{\lambda,\rho} \otimes \mathbf{P}$-a.s. This and Poisson bounds on the expectation of $X_{0,n}$ imply, by Kingman's subadditive ergodic theorem, that $n^{-1}X_{0,n}(\omega')$ converges $\bar{v}^{\lambda,\rho} \otimes \mathbf{P}$-a.s. On the other hand, $n^{-1}\varphi_{n/v}^v(\eta, \omega)$ converges $\bar{v}^{\lambda,\rho} \otimes \mathbf{P}$-a.s. by (3.1) below. Hence,

$$\bar{v}^{\lambda,\rho} \otimes \mathbf{P} \text{ a.s., } \exists \lim_{n\to\infty} n^{-1}\phi_{n/v}^v(\omega')$$ (138)

The limit in (138) can then be identified using the hydrodynamic limit of [8], in the same way as [3] is used in [1]. We thus obtain a particular case of (39) for $\beta = \alpha = 0$. However, the case $(\beta, \alpha) \neq (0,0)$ would require

$$\bar{v}^{\lambda,\rho} \otimes \mathbf{P} \text{ a.s., } \exists \lim_{n\to\infty} n^{-1}\phi_{n/v}^v(\theta'_{[\beta n],\alpha n}\omega')$$ (139)

for every $\beta \in \mathbb{R}$ and $\alpha \neq 0$. The a.s. limit (138) only implies a limit in probability for the shifted current in (139), as the distribution of a single current is unchanged by the shift. In contrast the *joint* distribution of the *sequence* of currents may change from (138) to (139): Thus we cannot simply derive (139) from (138). On the other hand, the shifted currents $Y_{0,n} := X_{0,n} \circ \theta'_{[\beta n], \alpha n}$ no longer enjoy a super-additivity property like (137), so we cannot use the subadditive ergodic theorem to obtain (139). Our approach to obtain (139) overcomes this difficulty by avoiding the use of subadditivity.

C   Proof of Proposition 2.3

The main ingredient is a two-dimensional extension of Birkhoff's ergodic theorem:

Proposition C.1. *Let $(\mathcal{X}, \mathcal{F}, P)$ be a probability space and $T, S : \mathcal{X} \to \mathcal{X}$ two measurable mappings such that $P \circ T^{-1} = P \circ S^{-1} = P$. Then, for every bounded $\mathcal{F}$-measurable $f : \mathcal{X} \to \mathcal{X}$, the limit*

$$f_{**}(x) := \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{n} \sum_{i=1}^{n} f(S^i T^j x) \tag{140}$$

exists for almost every $x \in \mathcal{X}$ with respect to P.

This follows from more general results established for instance in [45] or [27, Chapter 6]. However we include a simpler proof adapted to our case.

Proof of proposition C.1. By ergodic theorem there exists bounded functions f_* and f_{**} such that

$$\frac{1}{n} \sum_{i=1}^{n} f(S^i x) \quad := \quad f_*^n(x) \to f_*(x), \ P - a.s. \tag{141}$$

$$\frac{1}{n} \sum_{j=1}^{n} f_*(T^j x) \quad := \quad f_{**}^n(x) \to f_{**}(x), \ P - a.s. \tag{142}$$

By (141) and Egorov's theorem, there is a sequence of subsets $A_k \in \mathcal{F}$, $k \in \mathbb{N}$, such that $\nu(A_k) \to 0$ as $k \to \infty$, and $f_*^n \to f_*$ uniformly on $\mathcal{X} \backslash A_k$. Let

$$F_{**}^n(x) := \frac{1}{n} \sum_{j=1}^{n} \frac{1}{n} \sum_{i=1}^{n} f(S^i T^j x) = \frac{1}{n} \sum_{j=1}^{n} f_*^n(T^j x)$$

Then,

$$\left| F_{**}^n(x) - f_{**}(x) \right| \leq \frac{1}{n} \left| \sum_{j=1}^n \left[f_*^n(T^j x) - f_*(T^j x) \right] 1_{\mathscr{X} \setminus A_k}(T^j x) \right|$$

$$+ \left| \frac{1}{n} \sum_{j=1}^n \left[f_*^n(T^j x) - f_*(T^j x) \right] 1_{A_k}(T^j x) \right|$$

$$+ \left| \frac{1}{n} \sum_{j=1}^n f_*(T^j x) - f_{**}(x) \right|$$

$$\leq \sup_{\mathscr{X} \setminus A_k} \left| f_*^n - f_* \right| + 2M g_{A_k}^n(x) + \left| f_{**}^n(x) - f_{**}(x) \right| =: B_{n,k}(x)$$

where $M := \sup_{\mathscr{X}} |f|$ and, by ergodic theorem,

$$g_{A_k}^n(x) := \frac{1}{n} \sum_{j=1}^n 1_{A_k}(T^j x) \overset{n \to \infty}{\to} g_{A_k}(x), \quad P\text{-a.s.} \tag{143}$$

for some bounded, nonnegative, $\mathscr{F}$-measurable g_{A_k} such that

$$\int g_{A_k} d\nu = \nu(A_k) \overset{k \to \infty}{\to} 0 \tag{144}$$

By uniform convergence of f_*^n on $\mathscr{X} \setminus A_k$, a.s. convergence (142), and (143), $\limsup_{n \to \infty} B_{n,k}(x) \leq 2M g_{A_k}(x)$ holds P-a.s. By (144), g_{A_k} goes to 0 in $L^1(P)$ as $k \to \infty$. Thus it has a subsequence converging to 0 P-a.s. Letting $k \to \infty$ along this subsequence concludes the proof. $\qquad \square$

Proof of proposition 2.3.

Existence of the limit. Define the random variables $X_{i,j} := \int_{(j-1)a}^{ja} f(\tau^{i-1} \eta_s) ds$, where $i, j \in \mathbb{N}$ and $(\eta_s)_{s \geq 0}$ is the stationary Markov process with generator L and initial distribution μ. Take $\mathscr{X} = \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$, $\mathscr{F}$ the product Borel σ-field, P the distribution of the $\mathscr{X}$-valued random variable $(X_{i,j})_{i,j \in \mathbb{N}}$, $T\left[(x_{i,j})_{i,j \in \mathbb{N}}\right] = (x_{i,j+1})_{i,j \in \mathbb{N}}$, $S\left[(x_{i,j})_{i,j \in \mathbb{N}}\right] = (x_{i+1,j})_{i,j \in \mathbb{N}}$. We have $P \circ T^{-1} = P$ because $(\eta_s)_{s \geq 0}$ is stationary, and $P \circ S^{-1} = S$ because μ and L are invariant by τ. Then the existence of the limit follows from Proposition C.1.

Identification of the limit. Let now $\mathscr{X}$ be the Skorokhod space of **X**-valued paths, and $P = P_\mu$ the law of the Markov process with generator L and initial distribution μ. We consider on $\mathscr{X}$ the space shifts $(\tau_x)_{x \in \mathbb{Z}}$ and time shifts $(T_t)_{t \geq 0}$ defined as follows: if $\eta_. = (\eta_s)_{s \geq 0} \in \mathscr{X}$, then $\tau_x \eta_. := (\tau_x \eta_s)_{s \geq 0}$, where τ_x on the r.h.s. is the spatial shift on particle configurations defined in Section 2, and $T_t \eta_. := (\eta_{t+s})_{s \geq 0}$. What follows is a generalization of a standard result for one-parameter Markov processes (see e.g. [11, Chapter 7]). By the above existence step, we can define

$$f_{**}(\eta_.) := \lim_{n \to \infty} F_{**}^n(\eta_.) \tag{145}$$

P_μ-a.s., where

$$F_{**}^n(\eta_.) = \frac{1}{an} \int_0^{an} \frac{1}{n} \sum_{i=1}^n \tau^i f(\eta_t) dt$$

As a limit of measurable functions, f_{**} is measurable. For every $t > 0$, $T_t F_{**}^n - F_{**}^n$ and $\tau F_{**}^n - F_{**}^n$ consist of space-time sums over boundary domains of order $O(n) = o(n^2)$, hence in the limit $n \to \infty$, f_{**} is invariant by $(T_t)_{t \geq 0}$ and $\tau := \tau_1$. To show that this implies f_{**} is a P-a.s. constant function, we will prove that any measurable subset F of $\mathcal{X}$ which is invariant by $(T_t)_{t \geq 0}$ and τ has P_μ-probability 0 or 1. Taking expectations in (145), the constant value of f_{**} must be $\int f \, d\mu$, and Proposition 2.3 is thus established.

Let $F \subset \mathcal{X}$ be measurable, and invariant by $(T_t)_{t \geq 0}$ and τ. Set

$$g(\eta) = P_\mu(F|\eta_0 = \eta) =: P_\eta(F) \tag{146}$$

which is defined for μ-a.e. η. Here, P_η denotes the law of the Markov process starting from deterministic state $\eta \in \mathbf{X}$. We are going to prove that

$$g \equiv 0 \text{ or } g \equiv 1 \tag{147}$$

μ-a.s., which will imply $P(F) = \int_{\mathbf{X}} g(\eta) \mu(d\eta) \in \{0, 1\}$.

We have

$$g(\tau\eta) = P_{\tau\eta}(F) = P_\eta(\tau^{-1}F) = P_\eta(F) = g(\eta)$$

where the second equality follows from translation invariance (22) of L (which implies $P_{\tau\eta} = \tau P_\eta$), and the third from τ-invariance of F. Therefore g is μ-a.s. invariant by the spatial shift τ. We claim that $g = \mathbf{1}_G$ μ-a.s. for some $G \subset \mathbf{X}$. Indeed, let $\mathcal{F}_t$ denote the σ-field of $\mathcal{X}$ generated by the mappings $\eta_. \mapsto \eta_s$ for $s \leq t$. With P_μ-probability one,

$$g(\eta_t) = P_{\eta_t}(F) = P_\mu(T_t^{-1}F|\mathcal{F}_t) = P_\mu(F|\mathcal{F}_t)$$

where the second equality follows from Markov property, and the third from the T_t invariance of F. By the martingale convergence theorem, we have the P_μ-a.s. limit

$$\lim_{t \to \infty} g(\eta_t) = \mathbf{1}_F(\eta_.) \tag{148}$$

Since $\eta_t \sim \mu$ for all $t \geq 0$, for every $\varepsilon > 0$,

$$P_\mu(\eta_. : \varepsilon \leq g(\eta_t) \leq 1 - \varepsilon) = \mu(\eta : \varepsilon \leq g(\eta) \leq 1 - \varepsilon) \tag{149}$$

we conclude from (148)–(149) that the law of $g(\eta_t)$ is Bernoulli, hence $g = \mathbf{1}_G$ μ-a.s. for some $G \subset \mathbf{X}$. The desired conclusion (147) is thus equivalent to $\mu(G) \in \{0, 1\}$, which we now establish.

First we claim that, with P_μ-probability one, we have $g(\eta_t) = g(\eta_0)$ for all $t > 0$. Indeed,

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by T_t invariance of F, Markov property and definition of G,

$$
\begin{aligned}
P_\mu\left(\{\eta_t \notin G\} \cap F\right) &= P\left(\{\eta_t \notin G\} \cap T_t^{-1}F\right) \\
&= \int_{\{\eta_t \notin G\} \subset \mathscr{X}} P_{\eta_t}(F) P_\mu(d\eta_.) \\
&= \int_{\{\eta_t \notin G\} \subset \mathscr{X}} g(\eta_t) P_\mu(d\eta_.) \\
&= 0
\end{aligned}
\tag{150}
$$

Similarly we have

$$
P_\mu\left(\{\eta_t \in G\} \cap (\mathscr{X} \setminus F)\right) = 0
\tag{151}
$$

It follows from (150)–(151) that $\{\eta_t \in G\} = F$ up to a set of P_μ-probability 0. In other words, for every $t > 0$, we have $g(\eta_t) = g(\eta_0) = \mathbf{1}_F(\eta_.)$ with P_μ-probability one.

Now, assume $\mu(G) \notin \{0,1\}$, and define the conditioned measures $\mu_G(d\eta) := \mu(d\eta | \eta \in G)$ and $\mu_{\mathscr{X}\setminus G}(d\eta) := \mu(d\eta | \eta \in \mathscr{X}\setminus G)$, so that $\mu = \mu(G)\mu_G + \mu(\mathscr{X}\setminus G)\mu_{\mathscr{X}\setminus G}$. The invariance of L and μ with respect to space shift imply the same for μ_G and $\mu_{\mathscr{X}\setminus G}$. A simple computation, using invariance of μ for L, and the fact that $g(\eta_t)$ is P_μ a.s. constant, shows that μ_G and $\mu_{\mathscr{X}\setminus G}$ are also invariant for L. This contradicts the fact that $\mu \in (\mathscr{I} \cap \mathscr{S})_e$. $\square$

References

[1] Andjel, E., Ferrari, P.A., Siqueira, A. Law of large numbers for the asymmetric exclusion process. *Stoch. Process. Appl.* **132** (2004) no. 2, 217–233.

[2] Andjel, E.D., Kipnis, C. Derivation of the hydrodynamical equation for the zero range interaction process. *Ann. Probab.* **12** (1984), 325–334. MR0735841

[3] Andjel, E.D., Vares, M.E. Hydrodynamic equations for attractive particle systems on $\mathbb{Z}$. *J. Stat. Phys.* **47** (1987) no. 1/2, 265–288. MR0892931

[4] Andjel, E.D., Vares, M.E. Correction to : "Hydrodynamic equations for attractive particle systems on $\mathbb{Z}$". *J. Stat. Phys.* **113** (2003) 379-380.

[5] Belletini, G., Bertini, L., Mariani, M., Novaga, M. Γ-entropy cost for scalar conservation laws. *Archive Rational Mechanics Anal.* **195** (2010), 261–309.

[6] Benassi, A., Fouque, J.P. Hydrodynamical limit for the asymmetric exclusion process. *Ann. Probab.* **15** (1987), 546–560. MR0885130

[7] Bahadoran, C., Guiol, H., Ravishankhar, K., Saada, E. A constructive approach to Euler hydro-dynamics for attractive particle systems. Application to k-step exclusion. *Stoch. Process. Appl.* **99** (2002) no. 1, 1–30. MR1894249

[8] Bahadoran, C., Guiol, H., Ravishankhar, K., Saada, E. Euler hydrodynamics of one-dimensional attractive particle systems. *Ann. Probab.* **34** (2006), 1339–1369. MR2257649

[9] Biane, P., Durrett, R. *Lectures on Probability Theory, Ecole d'Eté de Probabilités de Saint-Flour XXIII-1993.* LN in Math. 1608, Springer, 1995. MR1383120

[10] Bramson, M., Mountford, T. Existence of blocking measures for one-dimensional nonzero mean exclusion processes. *Ann. Probab.* **30** (2002), 1082–1130. MR1920102

[11] Breiman, L. *Probability.* Addison-Wesley, 1968. MR0229267

[12] Cocozza-Thivent, C. Processus des misanthropes. *Z. Wahrsch. Verw. Gebiete* **70** (1985) no. 4, 509–523. MR0807334

[13] De Lellis, C., Otto, F., Westdickenberg, M. Minimal entropy conditions for Burgers Equation.*Quarterly of Applied Math.* **62** (2004), 687–700. MR2104269

[14] De Masi A., Presutti E. *Mathematical Methods for Hydrodynamic Limits.* LN in Math. 1501, Springer, 1991. MR1175626

[15] Donsker, M.D., Varadhan, S.R.S. Asymptotic evaluation of certain Markov process expectations for large time I. *Commun. Pure Appl. Math.* **28** (1975), 1–47. MR0386024

[16] Ferrari, P.A. Shock fluctuations in asymmetric simple exclusion. *Probab. Theory Rel. Fields* **91** (1992), 81–101. MR1142763

[17] Ferrari, P.A. Shocks in the Burgers equation and the asymmetric simple exclusion process; in *Automata Networks, Dynamical Systems and Statistical Physics*, edited by E. Goles, S. Martinez, Kluwer Math. and its Appl. (1992), 25–64. MR1263704

[18] Guiol, H. Some properties of k-step exclusion processes. *J. Stat. Phys.* **94** (1999) no. (3/4), 495–511. MR1675362

[19] Harris, T.E. Nearest neighbor Markov interaction processes on multidimensional lattices. *Adv. in Math.* **9** (1972), 66–89. MR0307392

[20] Harris, T.E. Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** (1978), 355–378. MR0488377

[21] Hayes, B.T., le Floch, P.G. Non-Classical Shocks and Kinetic Relations: Scalar Conservation Laws. *Archive for Rational Mech. Anal.* **139** (1997), 1–56. MR1475777

[22] Jensen, L. Large deviations of the asymmetric simple exclusion process in one dimension. *PhD dissertation, New York University* (2000).

[23] Kamae, T., Krengel, U. Stochastic partial ordering. *Ann. Probab.* **6** (1978), 1044–1049. MR0512419

[24] Kamae, T., Krengel, U., O'Brien, L.G. Stochastic inequalities on partially ordered spaces. *Ann. Probab.* **5** (1977), 899–912. MR0494447

[25] Kipnis, C., Landim, C. *Scaling limits of interacting particle systems*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **320**. Springer-Verlag, Berlin, 1999. MR1707314

[26] Kipnis, C., Olla, S., Varadhan, S.R.S. Hydrodynamics and large deviations for simple exclusion process. *Comm. Pure. Appl. Math.* **XLII** (1989), 115–137. MR0978701

[27] Krengel, U. *Ergodic theory.* De Gruyter Studies in Mathematics **6**, 1985. MR0797411

[28] Kružkov, N. First order quasilinear equations in several independent variables. *Math. USSR Sb.* **10** (1970), 217–243.

[29] Lax, P.D. Hyperbolic systems of conservation laws II. *Comm. Pure. Appl. Math.* **10** (1957), 537–566. MR0093653

[30] Liggett, T.M. *Interacting particle systems.* Classics in Mathematics (Reprint of first edition), Springer-Verlag, New York, 2005. MR2108619

[31] Liggett, T.M. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **324**, Springer-Verlag, New York, 1999. MR1717346

[32] Mariani, M. Large deviations principle for stochastic conservation laws. M. Large deviations principle for stochastic scalar conservation laws. *Probab. Theory Related Fields* (2009) (Published online)

[33] Mountford, T.S., Ravishankar, K., Saada, E. Macroscopic stability for nonfinite range kernels. http://fr.arxiv.org/abs/0903.0498, *Braz. J. Probab. Stat.* (2009).

[34] Rezakhanlou, F. Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. *Comm. Math. Phys.* **140** (1991), no. 3, 417–448. MR1130693

[35] Rezakhanlou, F. Continuum limit for some growth models. II. *Ann. Probab.* **29** (2001) no. 3, 1329–1372. MR1872745

[36] Rost, H. Nonequilibrium behaviour of a many particle process: density profile and local equilibria, *Z. Wahrsch. Verw. Gebiete* **58** (1981) no. 1, 41–43.

[37] Seppäläinen, T. Existence of hydrodynamics for the totally asymmetric simple *K*-exclusion process. *Ann. Probab.* **27** (1999) no. 1, 361–415. MR1681094

[38] Seppäläinen, T. *Translation Invariant Exclusion Processes*. In preparation. `http://www.math. wisc.edu/~seppalai/excl-book/etusivu.html`.

[39] Serre, D. *Systèmes de lois de conservation*. Diderot Ed., 1996.

[40] Spohn, H. *Large Scale Dynamics of Interacting Particles.* Springer, 1991.

[41] Strassen, V. The existence of probability measures with given marginals. *Ann. Math. Statist.* **36** (1965), 423–439. MR0177430

[42] Varadhan, S.R.S. Lectures on Hydrodynamic Scaling; in *Hydrodynamic Limits and Related Topics;* edited by S. Feng, A. Lawniczak, S.R.S. Varadhan, Fields Institute Communications, AMS, 2000. MR1798641

[43] Varadhan, S.R.S. Large deviations for the asymmetric simple exclusion process. *Stochastic analysis on large scale interacting systems*, 1–27, *Adv. Stud. Pure Math.* **39**, Math. Soc. Japan, Tokyo, 2004. MR2073328

[44] Vol'pert, A.I. The spaces BV and quasilinear equations. *Math. USSR Sbornik* **2** (1967) no. 2, 225–267. *Transl. from* (Russian) *Mat. Sb. (N.S.)* **73** (1967) no. 115, 255–302. MR0216338

[45] Wiener, N. The ergodic theorem. *Duke Math.* **15** (1939) no. 1, 1–18.