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# Strong limit theorems for a simple random walk on the 2-dimensional comb

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**Abstract**

We study the path behaviour of a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ that is obtained from $\mathbb{Z}^2$ by removing all horizontal edges off the $x$-axis. In particular, we prove a strong approximation result for such a random walk which, in turn, enables us to establish strong limit theorems, like the joint Strassen type law of the iterated logarithm of its two components, as well as their marginal Hirsch type behaviour.

**Key words:** Random walk; 2-dimensional comb; Strong approximation; 2-dimensional Wiener

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process; Iterated Brownian motion; Laws of the iterated logarithm.

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# 1 Introdution and main results

The 2-dimensional comb lattice $\mathbb{C}^2$ is obtained from $\mathbb{Z}^2$ by removing all horizontal edges off the $x$-axis. As far as we know, the first paper that discusses the properties of a random walk on a particular tree that has the form of a comb is Weiss and Havlin [26].

A formal way of describing a simple random walk $\mathbf{C}(n)$ on the above 2-dimensional comb lattice $\mathbb{C}^2$ can be formulated via its transition probabilities as follows:

$$\mathbf{P}(\mathbf{C}(n+1)=(x,y\pm 1)\mid \mathbf{C}(n)=(x,y))=\frac{1}{2},\quad \text{if } y\neq 0, \tag{1.1}$$

$$\mathbf{P}(\mathbf{C}(n+1)=(x\pm 1,0)\mid \mathbf{C}(n)=(x,0))=\mathbf{P}(\mathbf{C}(n+1)=(x,\pm 1)\mid \mathbf{C}(n)=(x,0))=\frac{1}{4}. \tag{1.2}$$

Unless otherwise stated, we assume that $\mathbf{C}(0)=(0,0)$. The coordinates of the just defined vector valued simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ will be denoted by $C_1(n), C_2(n)$, i.e., $\mathbf{C}(n):=(C_1(n),C_2(n))$.

A compact way of describing the just introduced transition probabilities for this simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ is via defining

$$p(\mathbf{u},\mathbf{v}):=\mathbf{P}(\mathbf{C}(n+1)=\mathbf{v}\mid \mathbf{C}(n)=\mathbf{u})=\frac{1}{\deg(\mathbf{u})}, \tag{1.3}$$

for locations $\mathbf{u}$ and $\mathbf{v}$ that are neighbours on $\mathbb{C}^2$, where $\deg(\mathbf{u})$ is the number of neighbours of $\mathbf{u}$, otherwise $p(\mathbf{u},\mathbf{v}):=0$. Consequently, the non-zero transition probabilities are equal to $1/4$ if $\mathbf{u}$ is on the horizontal axis and they are equal to $1/2$ otherwise.

Weiss and Havlin [26] derived the asymptotic form for the probability that $\mathbf{C}(n)=(x,y)$ by appealing to a central limit argument. For further references along these lines we refer to Bertacchi [1]. Here we call attention to Bertacchi and Zucca [2], who obtained space-time asymptotic estimates for the $n$-step transition probabilities $p^{(n)}(\mathbf{u},\mathbf{v}):=\mathbf{P}(\mathbf{C}(n)=\mathbf{v}\mid \mathbf{C}(0)=\mathbf{u})$, $n\geq 0$, from $\mathbf{u}\in\mathbb{C}^2$ to $\mathbf{v}\in\mathbb{C}^2$, when $\mathbf{u}=(2k,0)$ or $(0,2k)$ and $\mathbf{v}=(0,0)$. Using their estimates, they concluded that, if $k/n$ goes to zero with a certain speed, then $p^{(2n)}((2k,0),(0,0))/p^{(2n)}((0,2k),(0,0))\to 0$, as $n\to\infty$, an indication that suggests that the particle in this random walk spends most of its time on some tooth of the comb. This provides also an insight into the remarkable result of Krishnapur and Peres [17], stating that 2 independent simple random walks on $\mathbb{C}^2$ meet only finitely often with probability 1, though the random walk itself is recurrent.

A further insight along these lines was provided by Bertacchi [1], where she analyzed the asymptotic behaviour of the horizontal and vertical components $C_1(n)$, $C_2(n)$ of $\mathbf{C}(n)$ on $\mathbb{C}^2$, and concluded that the expected values of various distances reached in $n$ steps are of order $n^{1/4}$ for $C_1(n)$ and of order $n^{1/2}$ for $C_2(n)$. Moreover, this conclusion, in turn, also led her to study the asymptotic law of the random walk $\mathbf{C}(n)=(C_1(n),C_2(n))$ on $\mathbb{C}^2$ via scaling the components $C_1(n),C_2(n)$ by $n^{1/4}$ and $n^{1/2}$, respectively. Namely, defining now the continuous time process $\mathbf{C}(nt)=(C_1(nt),C_2(nt))$ by linear interpolation, Bertacchi [1] established the following remarkable weak convergence result.

$$\left(\frac{C_1(nt)}{n^{1/4}},\frac{C_2(nt)}{n^{1/2}};\, t\geq 0\right)\xrightarrow{\text{Law}}(W_1(\eta_2(0,t)),W_2(t);\, t\geq 0),\quad n\to\infty, \tag{1.4}$$

where $W_1$, $W_2$ are two independent Wiener processes (Brownian motions) and $\eta_2(0,t)$ is the local time process of $W_2$ at zero, and $\xrightarrow{\text{Law}}$ denotes weak convergence on $C([0,\infty),\mathbb{R}^2)$ endowed with the topology of uniform convergence on compact intervals.

Recall that if $\{W(t),\ t \geq 0\}$ is a standard Wiener process (Brownian motion), then its two-parameter local time process $\{\eta(x,t),\ x \in \mathbb{R},\ t \geq 0\}$ can be defined via

$$\int_A \eta(x,t)\,dx = \lambda\{s : 0 \leq s \leq t,\ W(s) \in A\} \tag{1.5}$$

for any $t \geq 0$ and Borel set $A \subset \mathbb{R}$, where $\lambda(\cdot)$ is the Lebesgue measure, and $\eta(\cdot,\cdot)$ is frequently referred to as Wiener or Brownian local time.

The iterated stochastic process $\{W_1(\eta_2(0,t));\ t \geq 0\}$ provides an analogue of the equality in distribution $t^{-1/2}W(t) \stackrel{\text{Law}}{=} X$ for each fixed $t > 0$, where $W$ is a standard Wiener process and $X$ is a standard normal random variable. Namely, we have (cf., e.g., (1.7) and (1.8) in [7])

$$\frac{W_1(\eta_2(0,t))}{t^{1/4}} \stackrel{\text{Law}}{=} X|Y|^{1/2}, \quad t > 0 \text{ fixed}, \tag{1.6}$$

where $X$ and $Y$ are independent standard normal random variables.

It is of interest to note that the iterated stochastic process $\{W_1(\eta_2(0,t));\ t \geq 0\}$ has first appeared in the context of studying the so-called second order limit law for additive functionals of a standard Wiener process $W$. See Dobrushin [13] for limiting distributions, Papanicolaou et al. [20], Ikeda and Watanabe [15], Kasahara [16] and Borodin [4] for weak convergence results.

For a related review of first and second order limit laws we refer to Csáki et al. [8], where the authors also established a strong approximation version of the just mentioned weak convergence result, and that of its simple symmetric random walk case as well, on the real line.

The investigations that are presented in this paper for the random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ were inspired by the above quoted weak limit law of Bertacchi [1] and the strong approximation methods and conclusions of Csáki et al. [7], [8], [9].

Bertacchi's method of proof for establishing the joint weak convergence statement (1.4) is based on showing that, on an appropriate probability space, each of the components converges in probability uniformly on compact intervals to the corresponding components of the conclusion of (1.4) (cf. Proposition 6.4 of Bertacchi [1]).

In this paper we extend this approach so that we provide joint strong invariance principles. Our main result is a strong approximation for the random walk $\mathbf{C}(n) = (C_1(n), C_2(n))$ on $\mathbb{C}^2$.

**Theorem 1.1.** *On an appropriate probability space for the random walk* $\{\mathbf{C}(n) = (C_1(n), C_2(n));$ $n = 0,1,2,\ldots\}$ *on* $\mathbb{C}^2$*, one can construct two independent standard Wiener processes* $\{W_1(t);\ t \geq 0\}$*,* $\{W_2(t);\ t \geq 0\}$ *so that, as* $n \to \infty$*, we have with any* $\varepsilon > 0$

$$n^{-1/4}|C_1(n) - W_1(\eta_2(0,n))| + n^{-1/2}|C_2(n) - W_2(n)| = O(n^{-1/8+\varepsilon}) \quad a.s.,$$

*where* $\eta_2(0,\cdot)$ *is the local time process at zero of* $W_2(\cdot)$*.*

In this section we will now present our further results and their corollaries.

Using Theorem 1.1, we first conclude the following strong invariance principle that will enable us to establish Strassen-type functional laws of the iterated logarithm for the continuous version of the random walk process $\{\mathbf{C}(xn) = (C_1(xn), C_2(xn));\ 0 \le x \le 1\}$ on the 2-dimensional comb lattice $\mathbb{C}^2$, that is defined by linear interpolation. We have almost surely, as $n \to \infty$,

$$\sup_{0\le x\le 1}\left\|\left(\frac{C_1(xn) - W_1(\eta_2(0,xn))}{n^{1/4}(\log\log n)^{3/4}}, \frac{C_2(xn) - W_2(xn)}{(n\log\log n)^{1/2}}\right)\right\| \to 0. \tag{1.7}$$

In view of this strong invariance principle we are now to study Strassen type laws of the iterated logarithm, in term of the set of limit points of the net of random vectors

$$\left(\frac{W_1(\eta_2(0,xt))}{2^{3/4}t^{1/4}(\log\log t)^{3/4}}, \frac{W_2(xt)}{(2t\log\log t)^{1/2}};\ 0 \le x \le 1\right)_{t\ge 3}, \tag{1.8}$$

as $t \to \infty$.

This will be accomplished in Theorem 1.2. In order to achieve this, we define the class $\widetilde{\mathscr{S}}^{(2)}$ as the set of $\mathbb{R}^2$ valued, absolutely continuous functions

$$\{(k(x), g(x));\ 0 \le x \le 1\} \tag{1.9}$$

for which $f(0) = g(0) = 0$, $\dot{k}(x)g(x) = 0$ a.e., and

$$\int_0^1 (|3^{3/4}2^{-1/2}\dot{k}(x)|^{4/3} + \dot{g}^2(x))\,dx \le 1. \tag{1.10}$$

**Theorem 1.2.** *Let $W_1(\cdot)$ and $W_2(\cdot)$ be two independent standard Wiener processes starting from 0, and let $\eta_2(0,\cdot)$ be the local time process at zero of $W_2(\cdot)$. Then the net of random vectors*

$$\left(\frac{W_1(\eta_2(0,xt))}{2^{3/4}t^{1/4}(\log\log t)^{3/4}}, \frac{W_2(xt)}{(2t\log\log t)^{1/2}};\ 0 \le x \le 1\right)_{t\ge 3}, \tag{1.11}$$

*as $t \to \infty$, is almost surely relatively compact in the space $C([0,1],\mathbb{R}^2)$ and its limit points is the set of functions $\widetilde{\mathscr{S}}^{(2)}$.*

**Corollary 1.1.** *For the random walk $\{\mathbf{C}(n) = (C_1(n), C_2(n));\ n = 1, 2, \ldots\}$ on the 2-dimensional comb lattice $\mathbb{C}^2$ we have that the sequence of random vector-valued functions*

$$\left(\frac{C_1(xn)}{2^{3/4}n^{1/4}(\log\log n)^{3/4}}, \frac{C_2(xn)}{(2n\log\log n)^{1/2}};\ 0 \le x \le 1\right)_{n\ge 3} \tag{1.12}$$

*is almost surely relatively compact in the space $C([0,1],\mathbb{R}^2)$ and its limit points is the set of functions $\widetilde{\mathscr{S}}^{(2)}$ as in* Theorem 1.2.

In order to illustrate the case of a joint functional law of the iterated logarithm for the two components of the random vectors in (1.12), we give the following example. Let

$$k(x,B,K_1) = k(x) = \begin{cases} \dfrac{Bx}{K_1} & \text{if} \quad 0 \le x \le K_1, \\ B & \text{if} \quad K_1 < x \le 1, \end{cases} \tag{1.13}$$

$$g(x,A,K_2)=g(x)=\begin{cases}0 & \text{if} \quad 0\le x\le K_2,\\ \dfrac{(x-K_2)A}{1-K_2} & \text{if} \quad K_2<x\le 1,\end{cases} \tag{1.14}$$

where $0\le K_1\le K_2\le 1$, and we see that $\dot{k}(x)g(x)=0$. Hence, provided that for $A,B,K_1$ and $K_2$

$$\frac{3\,B^{4/3}}{2^{2/3}K_1^{1/3}}+\frac{A^2}{(1-K_2)}\le 1,$$

we have $(k,g)\in\widetilde{\mathscr{S}}^{(2)}$. Consequently, in the two extreme cases,
(i) when $K_1=K_2=1$, then $|B|\le 2^{1/2}3^{-3/4}$ and on choosing $k(x)=\pm 2^{1/2}3^{-3/4}x$, $0\le x\le 1$,
then $g(x)=0$, $0\le x\le 1$, and
(ii) when $K_1=K_2=0$, then $|A|\le 1$ and on choosing $g(x)=\pm x$, $0\le x\le 1$,
then $k(x)=0$, $0\le x\le 1$.
Concerning now the joint limit points of $C_1(n)$ and $C_2(n)$, a consequence of Theorem 1.2 reads as follows.

**Corollary 1.2.** *The sequence*

$$\left(\frac{C_1(n)}{n^{1/4}(\log\log n)^{3/4}},\frac{C_2(n)}{(2n\log\log n)^{1/2}}\right)_{n\ge 3}$$

*is almost surely relatively compact in the rectangle*

$$R=\left[-\frac{2^{5/4}}{3^{3/4}},\frac{2^{5/4}}{3^{3/4}}\right]\times[-1,1]$$

*and the set of its limit points is the domain*

$$D=\{(u,v):\ k(1)=u,\ g(1)=v,\ (k(\cdot),g(\cdot))\in\widetilde{\mathscr{S}}^{(2)}\}. \tag{1.15}$$

It is of interest to find a more explicit description of $D$. In order to formulate the corresponding result for describing also the intrinsic nature of the domain $D$, we introduce the following notations:

$$F(B,A,K):=\frac{3B^{4/3}}{2^{2/3}K^{1/3}}+\frac{A^2}{1-K}\quad(0\le B,A,K\le 1), \tag{1.16}$$

$$D_1(K):=\{(u,v):\ F(|u|,|v|,K)\le 1\},$$

$$D_2:=\bigcup_{K\in(0,1)}D_1(K). \tag{1.17}$$

**Proposition 1.1.** *The two domains $D$, in* (1.15) *and $D_2$ in* (1.17) *are the same.*

**Remark 1.** Let

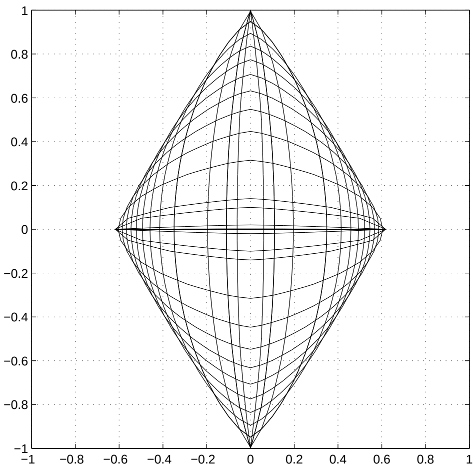


Figure 1: A picture of $D_2$.

(i) $A = A(B,K)$ be defined by the equation

$$F(B,A(B,K),K) = 1, \tag{1.18}$$

(ii) $K = K(B)$ be defined by the equation

$$A(B,K(B)) = \max_{0\le K\le 1} A(B,K). \tag{1.19}$$

Then clearly

$$D_2 = \{(B,A):\ |A| \le A(|B|,K(|B|)\}.$$

The explicit form of $A(B,K)$ can be easily obtained, and that of $K(B)$ can be obtained by the solution of a cubic equation. Hence, theoretically, we have the explicit form of $D_2$. However this explicit form is too complicated. A picture of $D_2$ can be given by numerical methods (Fig. 1.)

Now we turn to investigate the liminf behaviour of the components of our random walk on $\mathbb{C}^2$.

As to the liminf behaviour of the max functionals of the two components, from Bertoin [3], Nane [19] and Hirsch [14], we conclude the following Hirsch type behaviour of the respective components of the random walk process $\mathbf{C}(n)$ on the 2-dimensional comb lattice $\mathbb{C}^2$.

**Corollary 1.3.** *Let $\beta(n)$, $n = 1,2,\dots$, be a non-increasing sequence of positive numbers such that $n^{1/4}\beta(n)$ is non-decreasing. Then we have almost surely that*

$$\liminf_{n\to\infty} \frac{\max_{0\le k\le n} C_1(k)}{n^{1/4}\beta(n)} = 0 \quad or \quad \infty$$

*and*

$$\liminf_{n\to\infty} \frac{\max_{0\le k\le n} C_2(k)}{n^{1/2}\beta(n)} = 0 \quad or \quad \infty$$

*according as to whether the series $\sum_1^\infty \beta(n)/n$ diverges or converges.*

On account of Theorem 1.1, the so-called other law of the iterated logarithm due to Chung [6] obtains for $C_2(n)$.

**Corollary 1.4.**

$$\liminf_{n\to\infty}\left(\frac{8\log\log n}{\pi^2 n}\right)^{1/2}\max_{0\le k\le n}|C_2(k)|=1 \qquad a.s. \tag{1.20}$$

On the other hand, for the max functional of $|C_1(\cdot)|$ we do obtain a different result.

**Theorem 1.3.** *Let $\beta(t)>0$, $t\ge 0$, be a non-increasing function such that $t^{1/4}\beta(t)$ is non-decreasing. Then we have almost surely that*

$$\liminf_{t\to\infty}\frac{\sup_{0\le s\le t}|W_1(\eta_2(0,s))|}{t^{1/4}\beta(t)}=0 \quad or \quad \infty$$

*according as to whether the integral $\int_1^\infty \beta^2(t)/t\,dt$ diverges or converges.*

**Corollary 1.5.** *Let $\beta(n)$, $n=1,2,\ldots$, be a non-increasing sequence of positive numbers such that $n^{1/4}\beta(n)$ is non-decreasing. Then we have almost surely that*

$$\liminf_{n\to\infty}\frac{\max_{0\le k\le n}|C_1(k)|}{n^{1/4}\beta(n)}=0 \quad or \quad \infty$$

*according as to whether the series $\sum_1^\infty \beta^2(n)/n$ diverges or converges.*

## 2 Proof of Theorem 1.1

Let $X_i$, $i=1,2,\ldots$, be i.i.d. random variables with the distribution $P(X_i=1)=P(X_i=-1)=1/2$, and put $S(0):=0$, $S(n):=X_1+\ldots+X_n$, $n=1,2,\ldots$ The local time process of this simple symmetric random walk is defined by

$$\xi(k,n):=\#\{i:\,1\le i\le n,\,S(i)=k\},\quad k=0,\pm1,\pm2,\ldots,\,n=1,2,\ldots \tag{2.1}$$

Let $\rho(N)$ be the time of the $N$-th return to zero of the simple symmetric random walk on the integer lattice $\mathbb{Z}$, i.e., $\rho(0):=0$,

$$\rho(N):=\min\{i>\rho(N-1):\,S(i)=0\},\quad N=1,2,\ldots \tag{2.2}$$

We may construct the simple random walk on the 2-dimensional comb lattice as follows.

Assume that on a probability space we have two independent 1-dimensional simple symmetric random walks $\{S_j(n);\,n=0,1,\ldots\}$, with respective local times $\{\xi_j(k,n);\,k\in\mathbb{Z},\,n=0,1,\ldots\}$, return times $\{\rho_j(N);\,N=0,1,\ldots\}$, $j=1,2$, and an i.i.d. sequence $G_1,G_2,\ldots$ of geometric random variables with

$$\mathbf{P}(G_1=k)=\frac{1}{2^{k+1}},\quad k=0,1,2,\ldots,$$

independent of the random walks $S_j(\cdot)$, $j=1,2$. On this probability space we may construct a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ as follows. Put $T_N=G_1+G_2+\ldots G_N$,

$N = 1, 2, \ldots$ For $n = 0, \ldots, T_1$, let $C_1(n) = S_1(n)$ and $C_2(n) = 0$. For $n = T_1 + 1, \ldots, T_1 + \rho_2(1)$, let $C_1(n) = C_1(T_1)$, $C_2(n) = S_2(n - T_1)$. In general, for $T_N + \rho_2(N) < n \le T_{N+1} + \rho_2(N)$, let

$$C_1(n) = S_1(n - \rho_2(N)),$$

$$C_2(n) = 0,$$

and, for $T_{N+1} + \rho_2(N) < n \le T_{N+1} + \rho_2(N+1)$, let

$$C_1(n) = C_1(T_{N+1} + \rho_2(N)) = S_1(T_{N+1}),$$

$$C_2(n) = S_2(n - T_{N+1}).$$

Then it can be seen in terms of these definitions for $C_1(n)$ and $C_2(n)$ that $\mathbf{C}(n) = (C_1(n), C_2(n))$ is a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$.

**Lemma 2.1.** *If $T_N + \rho_2(N) \le n < T_{N+1} + \rho_2(N+1)$, then, as $n \to \infty$, we have for any $\varepsilon > 0$*

$$N = O(n^{1/2+\varepsilon}) \quad a.s.$$

*and*

$$\xi_2(0, n) = N + O(n^{1/4+\varepsilon}) \quad a.s.$$

**Proof.** First consider $\rho_2(N) + T_N \le n < T_{N+1} + \rho_2(N+1)$.

We use the following result (cf. Révész [22], Theorem 11.6).

For any $0 < \varepsilon < 1$ we have with probability 1 for all large enough $N$

$$(1-\varepsilon)\frac{N^2}{2\log\log N} \le \rho(N) \le N^2(\log N)^{2+\varepsilon}.$$

This and the law of large numbers for $\{T_N\}_{N\ge 1}$ imply

$$(1-\varepsilon)\left(\frac{N^2}{2\log\log N} + N\right) \le n \le (1+\varepsilon)(N+1) + N^2(\log N)^{2+\varepsilon}.$$

Hence,

$$n^{1/2-\varepsilon} \le N \le n^{1/2+\varepsilon}.$$

Also, $T_N = N + O(N^{1/2+\varepsilon})$ a.s., and

$$N = \xi_2(0, \rho_2(N)) \le \xi_2(0, T_N + \rho_2(N)) \le \xi_2(0, n) \le \xi_2(0, T_{N+1} + \rho_2(N+1)).$$

Consequently, with $\varepsilon > 0$, using increment results for the local time (cf. Csáki and Földes [10])

$$\xi_2(0, T_{N+1} + \rho_2(N+1)) = \xi_2(0, \rho_2(N+1)) + O(T_{N+1}^{1/2+\varepsilon}) = N + O(N^{1/2+\varepsilon}) = N + O(n^{1/4+\varepsilon}).$$

This completes the proof of Lemma 2.1. □

**Proof of Theorem 1.1.**

According to the joint strong invariance principle by Révész [21], we may also assume that on our probability space we have two independent Wiener processes $\{W_j(t); t \geq 0\}$ with their local time processes $\{\eta_j(x,t);\ x\in\mathbb{R},\ t\geq 0\}$, $j=1,2$, satisfying

$$S_j(n) - W_j(n) = O(n^{1/4+\varepsilon}) \quad \text{a.s.} \tag{2.3}$$

and

$$\sup_{x\in\mathbb{Z}} |\xi_j(x,n) - \eta_j(x,n)| = O(n^{1/4+\varepsilon}) \quad \text{a.s.}, \tag{2.4}$$

simultaneously as $n\to\infty$, $j=1,2$.

Now, using the above introduced definition for $C_1(n)$, in the case of $\rho_2(N)+T_N \leq n < T_{N+1}+\rho_2(N)$, Lemma 2.1, in combination with (2.3), (2.4) and increment results for the Wiener process (cf. Csörgő and Révész [12]) imply that, for any $\varepsilon>0$,

$$C_1(n) = S_1(n-\rho_2(N)) = W_1(n-\rho_2(N)) + O(T_N^{1/4+\varepsilon}) = W_1(T_N) + O(N^{1/4+\varepsilon}) = W_1(N) + O(N^{1/4+\varepsilon})$$

$$= W_1(\xi_2(0,n)) + O(n^{1/8+\varepsilon}) = W_1(\eta_2(0,n)) + O(n^{1/8+\varepsilon}) \quad \text{a.s.}$$

On the other hand, since $C_2(n)=0$ in the interval $\rho_2(N)+T_N \leq n \leq \rho_2(N)+T_{N+1}$ under consideration, we only have to estimate $W_2(n)$ in that domain. In this regard we have

$$|W_2(n)| \leq |W_2(\rho_2(N))| + |W_2(T_N+\rho_2(N)) - W_2(\rho_2(N))|$$

$$+ \sup_{T_N\leq t\leq T_{N+1}} |W_2(\rho_2(N)+t) - W_2(\rho_2(N))| = O(N^{1/2+\varepsilon}) = O(n^{1/4+\varepsilon}),$$

i.e.,

$$0 = C_2(n) = W_2(n) + O(n^{1/4+\varepsilon}).$$

In the case when $T_{N+1}+\rho_2(N) \leq n < T_{N+1}+\rho_2(N+1)$, by Lemma 2.1 and using again that $T_N = N + O(N^{1/2+\varepsilon})$, for any $\varepsilon>0$, we have almost surely

$$C_1(n) = S_1(T_{N+1}) = W_1(\xi_2(0,n)) + O(n^{1/8+\varepsilon}) = W_1(\eta_2(0,n)) + O(n^{1/8+\varepsilon}),$$

and

$$C_2(n) = S_2(n-T_{N+1}) = W_2(n-T_{N+1}) + O(N^{1/2+\varepsilon}) = W_2(n) + O(n^{1/4+\varepsilon}).$$

This completes the proof of Theorem 1.1. □

## 3 Proof of Theorem 1.2

The relative compactness follows from that of the components. In particular, for the first component we refer to Csáki et al. [11], while for the second one to Strassen [25]. So we only deal with the set of limit points as $t\to\infty$.

First consider the a.s. limit points of

$$\left(\frac{W_1(xt)}{(2t\log\log t)^{1/2}}, \frac{|W_2(xt)|}{(2t\log\log t)^{1/2}}, \frac{\eta_2(0,xt)}{(2t\log\log t)^{1/2}};\ 0\leq x\leq 1\right)_{t\geq 3} \tag{3.1}$$

and

$$\left(\frac{W_1(\eta_2(0,xt))}{2^{3/4}t^{1/4}(\log\log t)^{3/4}}, \frac{|W_2(xt)|}{(2t\log\log t)^{1/2}}; 0 \le x \le 1\right)_{t\ge 3}. \tag{3.2}$$

Clearly we need only the limit points of (3.2) but we will use the limit points of (3.1) to facilitate our argument. According to a theorem of Lévy [18], the following equality in distribution holds:

$$\{(\eta_2(0,t), |W_2(t)|),\ t \ge 0\} \overset{\text{Law}}{=} \{(M_2(t), M_2(t) - W_2(t),\ t \ge 0\},$$

$M_2(t) := \sup_{0\le s\le t} W_2(s)$. Hence the set of a.s. limit points of (3.1) is the same as that of

$$\left(\frac{W_1(xt)}{(2t\log\log t)^{1/2}}, \frac{M_2(xt) - W_2(xt)}{(2t\log\log t)^{1/2}}, \frac{M_2(xt)}{(2t\log\log t)^{1/2}}; 0 \le x \le 1\right)_{t\ge 3}, \tag{3.3}$$

and the set of a.s. limit points of (3.2) is the same as that of

$$\left(\frac{W_1(M_2(xt))}{2^{3/4}t^{1/4}(\log\log t)^{3/4}}, \frac{M_2(xt) - W_2(xt)}{(2t\log\log t)^{1/2}}; 0 \le x \le 1\right)_{t\ge 3}. \tag{3.4}$$

By the well-known theorem of Strassen [25] the set of a.s. limit points of

$$\left(\frac{W_1(xt)}{(2t\log\log t)^{1/2}}, \frac{W_2(xt)}{(2t\log\log t)^{1/2}}; 0 \le x \le 1\right)_{t\ge 3} \tag{3.5}$$

is $\mathscr{S}^2$, the set of absolutely continuous $\mathbb{R}^2$-valued functions $\{(f(x), \ell(x)); x \in [0,1]\}$ for which $f(0) = \ell(0) = 0$ and

$$\int_0^1 (\dot{f}^2(x) + \dot{\ell}^2(x))dx \le 1. \tag{3.6}$$

It follows that the set of a.s. limit points of (3.3), and hence also that of (3.1), is

$$\{(f(x), h(x) - \ell(x), h(x)) : (f,\ell) \in \mathscr{S}^2\}, \tag{3.7}$$

where

$$h(x) = \max_{0\le u\le x} \ell(u).$$

Moreover, applying Theorem 3.1 of [9], we get that the set of a.s. limit points of (3.4), hence also that of (3.2), is

$$\{(f(h(x)), h(x) - \ell(x)) : (f,\ell) \in \mathscr{S}^2\}.$$

It is easy to see that $\dot{h}(x)(h(x) - \ell(x)) = \dot{h}(x)(\dot{h}(x) - \dot{\ell}(x)) = 0$ and

$$\int_0^1 ((\dot{h}(x) - \dot{\ell}(x))^2 + \dot{h}^2(x))\,dx = \int_0^1 \dot{\ell}^2(x)\,dx + 2\int_0^1 \dot{h}(x)(\dot{h}(x) - \dot{\ell}(x))\,dx = \int_0^1 \dot{\ell}^2(x)\,dx.$$

Since $(f,\ell) \in \mathscr{S}^2$, we have

$$\int_0^1 (\dot{f}^2(x) + (\dot{h}(x) - \dot{\ell}(x))^2 + \dot{h}^2(x))\,dx \le 1.$$

On denoting the function $h(\cdot)-\ell(\cdot)$ in (3.7) by $g_1(\cdot)$, we can now conclude that the set of a.s. limit points of the net in (3.1) is the set of functions $(f, g_1, h)$, where $(f, g_1, h)$ are absolutely continuous functions on $[0,1]$ such that $g_1 \geq 0$, $h$ is non-decreasing, $f(0)=g_1(0)=h(0)=0$, $g_1\dot{h}=0$ a.e. and

$$\int_0^1 (\dot{f}^2(x)+\dot{g_1}^2(x)+\dot{h}^2(x))\,dx \leq 1. \tag{3.8}$$

Also, the set of a.s. limit points of the net (3.2) consists of functions $(f(h), g_1)$, where $(f, g_1, h)$ are as above.

Now put $k(x) = f(h(x))$. Obviously $k, g_1$ are absolutely continuous, $k(0) = g_1(0) = 0$ and $\dot{k}(x)g_1(x) = 0$ a.e. Using Hölder's inequality, the simple inequality $A^{2/3}B^{1/3} \leq 2^{2/3}3^{-1}(A+B)$ and $h(1)\leq 1$ (cf. the proof of Lemma 2.1 in [11]), we get

$$\int_0^1 (3^{3/4}2^{-1/2}|\dot{k}(x)|)^{4/3}\,dx \leq 3/2^{2/3}\left(\int_0^1 \dot{f}^2(x)\,dx\right)^{2/3}\left(\int_0^1 \dot{h}^2(x)\,dx\right)^{1/3} \leq \int_0^1 (\dot{f}^2(x)+\dot{h}^2(x))\,dx,$$

showing that the limit points of (3.2) are the functions $(k, g_1) \in \widetilde{\mathscr{S}}^{(2)}$ with $g_1 \geq 0$.

On the other hand, assume that $(k, g_1) \in \widetilde{\mathscr{S}}^{(2)}$, $g_1 \geq 0$. Define

$$h(x) = \frac{1}{2^{1/3}}\int_0^x |\dot{k}(u)|^{2/3}\,du$$

and

$$f(u) = \begin{cases} k(h^{-1}(u)) & \text{for} \quad 0 \leq u \leq h(1), \\ k(1) & \text{for} \quad h(1) \leq u \leq 1. \end{cases}$$

Then (cf. [11])

$$\int_0^1 \dot{f}^2(u)\,du + \int_0^1 \dot{h}^2(x)\,dx = \int_0^1 |\dot{f}(h(x))|^2\dot{h}(x)\,dx + \int_0^1 \frac{1}{2^{2/3}}|\dot{k}(x)|^{4/3}\,dx$$

$$= \frac{3}{2^{2/3}}\int_0^1 |\dot{k}(x)|^{4/3}\,dx,$$

i.e. $(k, g_1)$ is a limit point of (3.2) by (1.10) and (3.8).

Now assume that $(k, g)$ is a limit point of (1.11). Then obviously $(k, |g|)$ is a limit point of (3.2), i.e. $(k, |g|) \in \widetilde{\mathscr{S}}^{(2)}$, and as easily seen, also $(k, g) \in \widetilde{\mathscr{S}}^{(2)}$, where the absolute continuity comes from the fact, that $(k, g)$ is a limit point of (1.11).

It remains to show that any $(k, g) \in \widetilde{\mathscr{S}}^{(2)}$ is a limit point of (1.11).

Consider the stochastic process $\{V(t,\omega)\, t \geq 0\}$, $\omega \in \Omega_1$, that is living on a probability space $\{\Omega_1, \mathscr{A}_1, P_1\}$ and is equal in distribution to the absolute value of a standard Wiener process. Assume also that on this probability space there is a standard Wiener process, independent of $V(\cdot)$. Our aim is now to extend this probability space so that it would carry a Wiener process $W_2(\cdot)$, constructed from the just introduced stochastic process $V(\cdot)$. During this construction $\eta_2(0,\cdot)$ and $W_1(\cdot)$ as well as the function $k$, remain unchanged, and we only have to deal with $W_2$ and $g$.

The construction will be accomplished by assigning random signs to the excursions of $V(\cdot)$. In order to realize this construction, we start with introducing an appropriate set of tools.

Let $r(u)$, $u \geq 0$, be a nonnegative continuous function with $r(0) = 0$. We introduce the following notations.

$$\begin{aligned}
R_0 &:= R_0(r) = \{u \geq 0 : r(u) = 0,\, r(u+v) > 0 \;\forall\; 0 < v \leq 1\},\\
R_1 &:= R_1(r) = \{u \geq 0 : u \notin R_0,\, r(u) = 0,\, r(u+v) > 0 \;\forall\; 0 < v \leq 1/2\},\\
&\dots\\
R_k &:= R_k(r) = \{u \geq 0 : u \notin R_j, j = 0,1,\dots,k-1,\, r(u) = 0,\, r(u+v) > 0,\;\forall\; 0 < v \leq 1/2^k\},
\end{aligned}$$

$k = 1,2,\dots$

$$\begin{aligned}
u_{\ell 1} &:= u_{\ell 1}(r) = \min\{u : u \in R_\ell\},\\
&\dots\\
u_{\ell j} &:= u_{\ell j}(r) = \min\{u : u > u_{\ell,j-1},\, u \in R_\ell\}, \quad j = 2,3,\dots\\
v_{\ell j} &:= v_{\ell j}(r) = \min\{u : u > u_{\ell j},\, r(u) = 0\}, \quad j = 1,2,\dots
\end{aligned}$$

$\ell = 0,1,2,\dots$

Let $\{\delta_{\ell j},\, \ell = 0,1,2,\dots,\, j = 1,2,\dots\}$ be a double sequence of i.i.d. random variables with distribution

$$P_2(\delta_{\ell j} = 1) = P_2(\delta_{\ell j} = -1) = \frac{1}{2},$$

that is assumed to be independent of $V(\cdot)$, and lives on the probability space $(\Omega_2, \mathscr{A}_2, P_2)$.

Now, replace the function $r(\cdot)$ by $V(\cdot)$ in the above construction of $u_{\ell j}$ and $v_{\ell j}$ and define the stochastic process

$$W_2(u) = W_2(u,\omega) = \sum_{\ell=0}^{\infty}\sum_{j=1}^{\infty} \delta_{\ell j} V(u) 1_{(u_{\ell j}, v_{\ell j}]}(u), \quad u \geq 0,\, \omega \in \Omega, \tag{3.9}$$

that lives on the probability space

$$(\Omega, \mathscr{A}, \mathbf{P}) := (\Omega_1, \mathscr{A}_1, P_1) \times (\Omega_2, \mathscr{A}_2, P_2).$$

Clearly, $W_2(\cdot)$ as defined in (3.9) is a standard Wiener process on the latter probability space, independent of $W_1$. Moreover, $V(u) = |W_2(u)|$ and $\eta_2(0,\cdot)$ is the local time at zero of both $V$ and $W_2$. We show that $(k,g)$ is a limit point of (1.11) in terms of $W_2$ that we have just defined in (3.9).

In order to accomplish the just announced goal, we first note that it suffices to consider only those $g$ for which there are finitely many zero-free intervals $(\alpha_i, \beta_i), i = 1,2,\dots,m$, in their support $[0,1]$, since the set of the latter functions is dense. Clearly then, such a function $g(\cdot)$ can be written as

$$g(x) = \sum_{i=1}^{m} \varepsilon_i |g(x)| 1_{(\alpha_i,\beta_i]}(x),$$

where $\varepsilon_i \in \{-1,1\}$, $i = 1,\dots,m$.

On account of what we just established for $W_2(\cdot)$ of (3.9) above, for $P_1$-almost all $\omega \in \Omega_1$ there exists a sequence $\{t_K = t_K(\omega)\}_{K=1}^{\infty}$ with $\lim_{K\to\infty} t_K = \infty$, such that

$$\lim_{K\to\infty} \sup_{0\leq x\leq 1} \left| \frac{|W_2(x t_K)|}{(2 t_K \log\log t_K)^{1/2}} - |g(x)| \right| = 0, \tag{3.10}$$

with $W_2(\cdot)$ as in (3.9).

On recalling the construction of the latter $W_2(\cdot)$ via the excursion intervals $(u_{\ell j}, v_{\ell j}]$, we conclude that, for $K$ large enough, there exists a finite number of excursion intervals $(u(K,i), v(K,i)]$, $i = 1,2,\dots,m$, such that

$$\lim_{K\to\infty}\frac{u(K,i)}{t_K} = \alpha_i, \qquad \lim_{K\to\infty}\frac{v(K,i)}{t_K} = \beta_i,$$

for each $\omega \in \Omega_1$ for which (3.10) and the construction of the excursion intervals $(u_{\ell j}, v_{\ell j}]$ hold true.

The finite set of the just defined intervals $(u(K,i), v(K,i)]$ is a subset of the excursion intervals $(u_{\ell j}, v_{\ell j}]$ that are paired with the double sequence of i.i.d. random variables $\delta_{\ell j}$ in the construction of $W_2(\cdot)$ as in (3.9). Let $\delta(K,i)$ denote the $\delta_{\ell j}$ that belongs to $(u(K,i), v(K,i))$. Since these random variables are independent, there exists a subsequence $\delta(K_N, i)$, $N = 1,2,\dots$ such that we have

$$\delta(K_N, i) = \varepsilon_i, \qquad i = 1,\dots,m, \quad N = 1,2,\dots. \tag{3.11}$$

$P_2$-almost surely.

Hence on account of (3.10) and (3.11), we have

$$\lim_{N\to\infty}\sup_{\alpha_i\le x\le\beta_i}\left|\frac{\delta(K_N,i)|W_2(xt_{K_N})|}{(2t_{K_N}\log\log t_{K_N})^{1/2}} - \varepsilon_i|g(x)|\right| = 0, \quad i = 1,\dots,m. \tag{3.12}$$

for $\mathbf{P}$-almost all $\omega \in \Omega$.

Also, as a consequence of (3.10), we have

$$\lim_{N\to\infty}\sup_{x:g(x)=0}\left|\frac{W_2(xt_{K_N})}{(2t_{K_N}\log\log t_{K_N})^{1/2}}\right| = 0 \tag{3.13}$$

$\mathbf{P}$-almost surely.

Consequently, on account of (3.12) and (3.13), we conclude

$$\lim_{N\to\infty}\sup_{0\le x\le 1}\left|\frac{W_2(xt_{K_N})}{(2t_{K_N}\log\log t_{K_N})^{1/2}} - g(x)\right| = 0.$$

$\mathbf{P}$-almost surely, i.e. $(k,g)$ is a limit point of (1.11).

This concludes the proof of Theorem 1.2. □

# 4 Proof of Proposition 1.1

Recall the definitions (1.13)-(1.19), and put

$$k(x,K) := k(x,B,K), \qquad g(x,K) := g(x,A,K).$$

It is easy to see that

$$\int_0^1 (|3^{3/4}2^{-1/2}\dot{k}(x,K)|^{4/3} + (\dot{g}(x,K))^2)dx = \frac{3B^{4/3}}{2^{2/3}K^{1/3}} + \frac{A^2}{1-K} = F(|B|,|A|,K).$$

Hence, if

$$F(|B|,|A|,K) \le 1,$$

then

$$(k(x,K),g(x,K)) \in \mathscr{F}^{(2)}$$

and

$$D_2 \subseteq D.$$

Now we have to show that $D \subseteq D_2$. On assuming that $(k_0(\cdot),g_0(\cdot)) \in \mathscr{F}^{(2)}$, we show that $(k_0(1),g_0(1)) \in D_2$. Let

$$\begin{aligned} L &= \{x: \ \dot{k}_0(x)=0\}, \qquad \lambda(L)=\kappa, \\ M &= \{x: \ g_0(x)=0\}, \qquad \lambda(M)=\mu, \end{aligned}$$

where $\lambda$ is the Lebesgue measure. Clearly $\mu+\kappa \ge 1$ and there exist monotone, measure preserving, one to one transformations $m(x)$ resp. $n(x)$ defined on the complements of the above sets $\overline{L}$ resp. $\overline{M}$ such that $m(x)$ maps $\overline{L}$ onto $[0,1-\kappa]$ and $n(x)$ maps $\overline{M}$ onto $[\mu,1]$ :

$$\begin{aligned} m(x) &\in [0,1-\kappa] \quad (x \in \overline{L}), \\ n(x) &\in [\mu,1] \quad (x \in \overline{M}). \end{aligned}$$

Define the function $k_1(y)$ resp. $g_1(y)$ by

$$k_1(y) = \begin{cases} k_0(m^{-1}(y)) \text{ for } y \in [0,1-\kappa] \\ k_1(1-\kappa) \text{ for } y \in (1-\kappa,1], \end{cases}$$

$$g_1(y) = \begin{cases} 0 & \text{for } y \in [0,\mu] \\ g_0(n^{-1}(y)) & \text{for } y \in (\mu,1]. \end{cases}$$

Note that

$$\begin{aligned} \int_0^1 |\dot{k}_1(y)|^{4/3} dy &= \int_0^1 |\dot{k}_0(x)|^{4/3} dx, \\ \int_0^1 (\dot{g}_1(y))^2 dy &= \int_0^1 (\dot{g}_0(x))^2 dx, \\ (k_1(y),g_1(y)) &\in \mathscr{F}^{(2)}. \end{aligned}$$

Taking into account that $1-\kappa \le \mu$, we define the following linear approximations $k_2$ resp. $g_2$ of $k_1$ resp. $g_1$ :

$$k_2(x) = k(x, k_1(1), 1-\kappa) = \begin{cases} \dfrac{x}{\mu} k_1(1) & \text{if} \quad 0 \le x \le \mu, \\ k_1(1) & \text{if} \quad \mu \le x \le 1, \end{cases}$$

$$g_2(x) = g(x, g_1(1), 1-\mu) = \begin{cases} 0 & \text{if} \quad 0 \le x \le \mu, \\ \dfrac{x-\mu}{1-\mu} g_1(1) & \text{if} \quad \mu \le x \le 1. \end{cases}$$

It follows from Hölder's inequality (cf., e.g. Riesz and Sz.-Nagy [23] p. 75) that

$$F(|k_1(1)|, |g_1(1)|, \mu) = F(|k_2(1)|, |g_2(1)|, \mu) = \int_0^1 (\, |3^{3/4} 2^{-1/2} \dot{k}_2(x)|^{4/3} + (\dot{g}_2(x))^2 \,)\, dx$$
$$\le \int_0^1 (\, |3^{3/4} 2^{-1/2} \dot{k}_1(x)|^{4/3} + (\dot{g}_1(x))^2 \,)\, dx \le 1,$$

implying that $(k_1(1), g_1(1)) \in D_2$. Taking into account that $|k_0(1)| \le |k_1(1)|$ and $|g_0(1)| \le |g_1(1)|$ by our construction, $(k_0(1), g_0(1)) \in D_2$ as well, which implies that $D \subseteq D_2$. □

## 5 Proof of Theorem 1.3

It is proved by Nane [19] that as $u \downarrow 0$,

$$\mathbf{P}(\sup_{0 \le t \le 1} |W_1(\eta_2(0,t))| \le u) \sim c u^2 \tag{5.1}$$

with some positive constant $c$. Consequently, for small $u$ we have

$$c_1 u^2 \le \mathbf{P}(\sup_{0 \le t \le 1} |W_1(\eta_2(0,t))| \le u) \le c_2 u^2 \tag{5.2}$$

with some positive constants $c_1$ and $c_2$.

First assume that $\int_1^\infty \beta^2(t)/t \, dt < \infty$. Put $t_n = e^n$. Then we also have $\sum_n \beta^2(t_n) < \infty$. Indeed, it is well known that the integral and series in hand are equiconvergent. For arbitrary $\varepsilon > 0$ consider the events

$$A_n = \left\{ \sup_{0 \le s \le t_n} |W_1(\eta_2(0,s))| \le \frac{1}{\varepsilon} t_{n+1}^{1/4} \beta(t_n) \right\},$$

$n = 1, 2, \ldots$ It follows from (5.2) that

$$\mathbf{P}(A_n) \le \frac{c_2}{\varepsilon^2} \left( \frac{t_{n+1}}{t_n} \right)^{1/2} \beta^2(t_n) = c_3 \beta^2(t_n),$$

which is summable, hence $\mathbf{P}(A_n \text{ i.o.}) = 0$. Consequently, for large $n$, we have

$$\sup_{0\le s\le t_n} |W_1(\eta_2(0,s))| \ge \frac{1}{\varepsilon} t_{n+1}^{1/4}\beta(t_n),$$

and for $t_n \le t < t_{n+1}$, we have as well

$$\sup_{0\le s\le t} |W_1(\eta_2(0,s))| \ge \frac{1}{\varepsilon} t^{1/4}\beta(t) \quad \text{a.s.}$$

Since the latter inequality is true for $t$ large enough and $\varepsilon > 0$ is arbitrary, we arrive at

$$\liminf_{t\to\infty} \frac{\sup_{0\le s\le t} |W_1(\eta_2(0,s))|}{t^{1/4}\beta(t)} = \infty \quad \text{a.s.}$$

Now assume that $\int_1^\infty \beta^2(t)/t\,dt = \infty$. Put $t_n = e^n$. Hence we have also $\sum_n \beta^2(t_n) = \infty$. Let $W^*(t) = \sup_{0\le s\le t} |W_1(\eta_2(0,s))|$. Consider the events

$$A_n = \left\{W^*(t_n) \le t_n^{1/4}\beta(t_n)\right\},$$

$n = 1, 2, \ldots$ It follows from (5.2) that

$$\mathbf{P}(A_n) \ge c\beta^2(t_n),$$

consequently $\sum_n \mathbf{P}(A_n) = \infty$.

Now we are to estimate $\mathbf{P}(A_m A_n)$, $(m < n)$. In fact, we have to estimate the probability $\mathbf{P}(W^*(s) < a,\, W^*(t) < b)$ for $s = t_m$, $t = t_n$, with $a = t_m^{1/4}\beta(t_m)$, $b = t_n^{1/4}\beta(t_n)$. Applying Lemma 1 of Shi [24], we have for $0 < s < t$, $0 < a \le b$,

$$\mathbf{P}(W^*(s) < a,\, W^*(t) < b) \le \frac{16}{\pi^2}\mathbf{E}\left(\exp\left(-\frac{\pi^2}{8a^2}\eta_2(0,s) - \frac{\pi^2}{8b^2}(\eta_2(0,t) - \eta_2(0,s))\right)\right).$$

Next we wish to estimate the expected value on the right-hand side of the latter inequality. For the sake of our calculations, we write $\eta(0,s)$ instead of $\eta_2(0,s)$ to stand for the local time at zero of a standard Wiener process $W(\cdot)$, i.e., we also write $W$ instead of $W_2$. With this convenient notation, we now let

$$\alpha(s) = \max\{u < s : W(u) = 0\} \qquad \gamma(s) = \min\{v > s : W(v) = 0\},$$

and let $g(u,v)$, $0 < u < s < v$ denote the joint density function of these two random variables. Recall that the marginal distribution of $\alpha(s)$ is the arcsine law with density function

$$g_1(u) = \frac{1}{\pi\sqrt{u(s-u)}}, \quad 0 < u < s.$$

Putting $\lambda_1 = \pi^2/(8a^2)$, $\lambda_2 = \pi^2/(8b^2)$, a straightforward calculation yields

$$\mathbf{E}\left(\exp\left(-\lambda_1\eta(0,s) - \lambda_2(\eta(0,t) - \eta(0,s))\right)\right)$$

$$= \iint_{0<u<s<v} \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u) = 0) g(u,v) \mathbf{E}(e^{-\lambda_2(\eta(0,t)-\eta(0,v))} \mid W(v) = 0)\,du\,dv = I_1 + I_2,$$

where $I_1 = \iint_{0<u<s<v<t/2}$ and $I_2 = \iint_{0<u<s,\, t/2<v}$. The first part is not void if $s = e^m$, $t = e^n$, $m < n$, since obviously $e^m < e^n/2$. Estimating them now, in the first case we use the inequality

$$\mathbf{E}(e^{-\lambda_2(\eta(0,t)-\eta(0,v))} \mid W(v) = 0) \le \mathbf{E}(e^{-\lambda_2\eta(0,t/2)}),$$

while in the second case we simply estimate this expectation by 1. Thus

$$\begin{aligned} I_1 &= \iint_{0<u<s<v<t/2} \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u) = 0) g(u,v) \mathbf{E}(e^{-\lambda_2(\eta(0,t)-\eta(0,v))} \mid W(v) = 0)\, du\, dv \\ &\le \mathbf{E}(e^{-\lambda_2\eta(0,t/2)}) \iint_{0<u<s<v} \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u) = 0) g(u,v)\, du\, dv \\ &= \mathbf{E}(e^{-\lambda_1\eta(0,s)}) \mathbf{E}(e^{-\lambda_2\eta(0,t/2)}). \end{aligned}$$

In the second case we have

$$\left( \int_{t/2}^{\infty} g(u,v)\, dv \right) du = \mathbf{P}(\alpha(t/2) \in du).$$

But

$$\frac{\mathbf{P}(\alpha(t/2) \in du)}{\mathbf{P}(\alpha(s) \in du)} \le c \frac{\sqrt{s-u}}{\sqrt{t/2-u}} \le c\sqrt{\frac{2s}{t}}.$$

Hence

$$\begin{aligned} I_2 &= \iint_{0<u<s,\, v>t/2} \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u) = 0) g(u,v) \mathbf{E}(e^{-\lambda_2(\eta(0,t)-\eta(0,v))} \mid W(v) = 0)\, du\, dv \\ &\le c\sqrt{\frac{s}{t}} \int_0^s \mathbf{E}(e^{-\lambda_1\eta(0,u)} \mid W(u) = 0) g_1(u)\, du = c\sqrt{\frac{s}{t}} \mathbf{E}\left(e^{-\lambda_1\eta(0,s)}\right). \end{aligned}$$

From Borodin-Salminen [5], 1.3.3 on p. 127, we obtain for $\theta > 0$ that

$$\mathbf{E}\left(e^{-\theta\eta(0,t)}\right) = 2e^{\theta^2 t/2}(1 - \Phi(\theta\sqrt{t})),$$

where $\Phi$ is the standard normal distribution function. From this and the well-known asymptotic formula

$$(1 - \Phi(z)) \sim \frac{c}{z} e^{-z^2/2}, \quad z \to \infty$$

we get for $\theta\sqrt{t} \to \infty$

$$\mathbf{E}\left(e^{-\theta\eta(0,t)}\right) \sim \frac{c}{\theta\sqrt{t}} \tag{5.3}$$

with some positive constant $c$.

On using (5.3) now, we arrive at

$$I_1 + I_2 \le \frac{c}{\lambda_1\lambda_2\sqrt{st}} + \frac{c}{\lambda_1\sqrt{t}},$$

with some positive constant $c$. To estimate $\mathbf{P}(A_m A_n)$, put $s = t_m = e^m$, $t = t_n = e^n$. Then, on recalling the definitions of $a$ and $b$, respectively in $\lambda_1$ and $\lambda_2$, we get

$$\lambda_1 = \frac{\pi^2}{8t_m^{1/2}\beta^2(t_m)}, \quad \lambda_2 = \frac{\pi^2}{8t_n^{1/2}\beta^2(t_n)},$$

which in turn implies

$$\mathbf{P}(A_m A_n) \le c\beta^2(t_m)\beta^2(t_n) + c\frac{t_m^{1/2}}{t_n^{1/2}}\beta^2(t_m) \le c\mathbf{P}(A_m)\mathbf{P}(A_n) + ce^{(m-n)/2}\mathbf{P}(A_m).$$

Since $e^{(m-n)/2}$ is summable for fixed $m$, by the Borel-Cantelli lemma we get $\mathbf{P}(A_n \text{ i.o.}) > 0$. Also, by 0-1 law, this probability is equal to 1. This completes the proof of Theorem 1.3. □

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