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# Intermittency on catalysts: three-dimensional simple symmetric exclusion*

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**Abstract**

We continue our study of intermittency for the parabolic Anderson model $\partial u/\partial t = \kappa\Delta u + \xi u$ in a space-time random medium $\xi$, where $\kappa$ is a positive diffusion constant, $\Delta$ is the lattice Laplacian on $\mathbb{Z}^d$, $d \geq 1$, and $\xi$ is a simple symmetric exclusion process on $\mathbb{Z}^d$ in Bernoulli equilibrium. This model describes the evolution of a *reactant* $u$ under the influence of a *catalyst* $\xi$.
In [3] we investigated the behavior of the annealed Lyapunov exponents, i.e., the exponential growth rates as $t \to \infty$ of the successive moments of the solution $u$. This led to an almost

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complete picture of intermittency as a function of $d$ and $\kappa$. In the present paper we finish our study by focussing on the asymptotics of the Lyaponov exponents as $\kappa \to \infty$ in the *critical* dimension $d = 3$, which was left open in [3] and which is the most challenging. We show that, interestingly, this asymptotics is characterized not only by a *Green* term, as in $d \geq 4$, but also by a *polaron* term. The presence of the latter implies intermittency of *all* orders above a finite threshold for $\kappa$.

**Key words:** Parabolic Anderson model, catalytic random medium, exclusion process, graphical representation, Lyapunov exponents, intermittency, large deviation.

**AMS 2000 Subject Classification:** Primary 60H25, 82C44; Secondary: 60F10, 35B40.

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# 1 Introduction and main result

## 1.1 Model

In this paper we consider the *parabolic Anderson model* (PAM) on $\mathbb{Z}^d$, $d \geq 1$,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \Delta u + \xi u & \text{on } \mathbb{Z}^d \times [0,\infty), \\ u(\cdot, 0) = 1 & \text{on } \mathbb{Z}^d, \end{cases} \tag{1.1}$$

where $\kappa$ is a positive diffusion constant, $\Delta$ is the lattice Laplacian acting on $u$ as

$$\Delta u(x,t) = \sum_{\substack{y \in \mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{1.2}$$

($\|\cdot\|$ is the Euclidian norm), and

$$\xi = (\xi_t)_{t \geq 0}, \qquad \xi_t = \{\xi_t(x) \colon x \in \mathbb{Z}^d\}, \tag{1.3}$$

is a space-time random field that drives the evolution. If $\xi$ is given by an infinite particle system dynamics, then the solution $u$ of the PAM may be interpreted as the concentration of a diffusing *reactant* under the influence of a *catalyst* performing such a dynamics.

In Gärtner, den Hollander and Maillard [3] we studied the PAM for $\xi$ Symmetric Exclusion (SE), and developed an almost complete qualitative picture. In the present paper we finish our study by focussing on the limiting behavior as $\kappa \to \infty$ in the *critical* dimension $d = 3$, which was left open in [3] and which is the most challenging. We restrict to *Simple Symmetric Exclusion* (SSE), i.e., $(\xi_t)_{t \geq 0}$ is the Markov dynamics on $\Omega = \{0,1\}^{\mathbb{Z}^3}$ (0 = vacancy, 1 = particle) with generator $L$ acting on cylinder functions $f : \Omega \to \mathbb{R}$ as

$$(Lf)(\eta) = \frac{1}{6} \sum_{\{a,b\}} \Big[ f\left(\eta^{a,b}\right) - f(\eta) \Big], \quad \eta \in \Omega, \tag{1.4}$$

where the sum is taken over all unoriented nearest-neighbor bonds $\{a,b\}$ of $\mathbb{Z}^3$, and $\eta^{a,b}$ denotes the configuration obtained from $\eta$ by interchanging the states at $a$ and $b$:

$$\eta^{a,b}(a) = \eta(b), \quad \eta^{a,b}(b) = \eta(a), \quad \eta^{a,b}(x) = \eta(x) \text{ for } x \notin \{a,b\}. \tag{1.5}$$

(See Liggett [7], Chapter VIII.) Let $\mathbb{P}_\eta$ and $\mathbb{E}_\eta$ denote probability and expectation for $\xi$ given $\xi_0 = \eta \in \Omega$. Let $\xi_0$ be drawn according to the Bernoulli product measure $\nu_\rho$ on $\Omega$ with density $\rho \in (0,1)$. The probability measures $\nu_\rho$, $\rho \in (0,1)$, are the only extremal equilibria of the SSE dynamics. (See Liggett [7], Chapter VIII, Theorem 1.44.) We write $\mathbb{P}_{\nu_\rho} = \int_\Omega \nu_\rho(d\eta)\mathbb{P}_\eta$ and $\mathbb{E}_{\nu_\rho} = \int_\Omega \nu_\rho(d\eta)\mathbb{E}_\eta$.

## 1.2 Lyapunov exponents

For $p \in \mathbb{N}$, define the $p$-th *annealed Lyapunov exponent* of the PAM by

$$\lambda_p(\kappa,\rho) = \lim_{t \to \infty} \frac{1}{pt} \log \mathbb{E}_{\nu_\rho}\left([u(0,t)]^p\right). \tag{1.6}$$

We are interested in the asymptotic behavior of $\lambda_p(\kappa,\rho)$ as $\kappa\to\infty$ for fixed $\rho$ and $p$. To this end, let $G$ denote the value at 0 of the *Green function* of simple random walk on $\mathbb{Z}^3$ with jump rate 1 (i.e., the Markov process with generator $\frac{1}{6}\Delta$), and let $\mathscr{P}_3$ be the value of the *polaron variational problem*

$$\mathscr{P}_3=\sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}}\left[\left\|\left(-\Delta_{\mathbb{R}^3}\right)^{-1/2}f^2\right\|_2^2-\left\|\nabla_{\mathbb{R}^3}f\right\|_2^2\right], \tag{1.7}$$

where $\nabla_{\mathbb{R}^3}$ and $\Delta_{\mathbb{R}^3}$ are the *continuous* gradient and Laplacian, $\|\cdot\|_2$ is the $L^2(\mathbb{R}^3)$-norm, $H^1(\mathbb{R}^3)=\{f\in L^2(\mathbb{R}^3)\colon\ \nabla_{\mathbb{R}^3}f\in L^2(\mathbb{R}^3)\}$, and

$$\left\|\left(-\Delta_{\mathbb{R}^3}\right)^{-1/2}f^2\right\|_2^2=\int_{\mathbb{R}^3}dx\,f^2(x)\int_{\mathbb{R}^3}dy\,f^2(y)\,\frac{1}{4\pi\|x-y\|}. \tag{1.8}$$

(See Donsker and Varadhan [1] for background on how $\mathscr{P}_3$ arises in the context of a self-attracting Brownian motion referred to as the polaron model. See also Gärtner and den Hollander [2], Section 1.5.)

We are now ready to formulate our main result (which was already announced in Gärtner, den Hollander and Maillard [4]).

**Theorem 1.1.** *Let $d=3$, $\rho\in(0,1)$ and $p\in\mathbb{N}$. Then*

$$\lim_{\kappa\to\infty}\kappa[\lambda_p(\kappa,\rho)-\rho]=\frac{1}{6}\rho(1-\rho)G+[6\rho(1-\rho)p]^2\mathscr{P}_3. \tag{1.9}$$

Note that the expression in the r.h.s. of (1.9) is the sum of a *Green* term and a *polaron* term. The existence, continuity, monotonicity and convexity of $\kappa\mapsto\lambda_p(\kappa,\rho)$ were proved in [3] for all $d\geq 1$ for all exclusion processes with an irreducible and symmetric random walk transition kernel. It was further proved that $\lambda_p(\kappa,\rho)=1$ when the random walk is recurrent and $\rho<\lambda_p(\kappa,\rho)<1$ when the random walk is transient. Moreover, it was shown that for simple random walk in $d\geq 4$ the asymptotics as $\kappa\to\infty$ of $\lambda_p(\kappa,\rho)$ is similar to (1.9), but *without* the polaron term. In fact, the subtlety in $d=3$ is caused by the appearance of this extra term which, as we will see in Section 5, is related to the large deviation behavior of the occupation time measure of a rescaled random walk that lies deeply hidden in the problem. For the heuristics behind Theorem 1.1 we refer the reader to [3], Section 1.5.

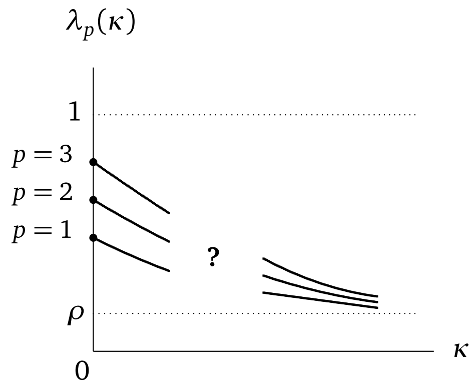
## 1.3 Intermittency

The presence of the polaron term in Theorem 1.1 implies that, for each $\rho\in(0,1)$, there exists a $\kappa_0(\rho)>0$ such that the *strict* inequality

$$\lambda_p(\kappa,\rho)>\lambda_{p-1}(\kappa,\rho)\quad\forall\,\kappa>\kappa_0(\rho) \tag{1.10}$$

holds for $p=2$ and, consequently, for all $p\geq 2$ by the convexity of $p\mapsto p\,\lambda_p(\kappa,\rho)$. This means that all moments of the solution $u$ are *intermittent* for $\kappa>\kappa_0(\rho)$, i.e., for large $t$ the random field $u(\cdot,t)$ develops sparse high spatial peaks dominating the moments in such a way that each moment is dominated by its own collection of peaks (see Gärtner and König [5], Section 1.3, and den Hollander [6], Chapter 8, for more explanation).

In [3] it was shown that for all $d \geq 3$ the PAM is intermittent for *small* $\kappa$. We conjecture that in $d = 3$ it is in fact intermittent for *all* $\kappa$. Unfortunately, our analysis does not allow us to treat intermediate values of $\kappa$ (see the figure).




Qualitative picture of $\kappa \mapsto \lambda_p(\kappa)$ for $p = 1, 2, 3$.

The formulation of Theorem 1.1 coincides with the corresponding result in Gärtner and den Hollander [2], where the random potential $\xi$ is given by independent simple random walks in a Poisson equilibrium in the so-called weakly catalytic regime. However, as we already pointed out in [3], the approach in [2] cannot be adapted to the exclusion process, since it relies on an explicit Feynman-Kac representation for the moments that is available only in the case of *independent* particle motion. We must therefore proceed in a totally different way. Only at the end of Section 5 will we be able to use some of the ideas in [2].

## 1.4 Outline

Each of Sections 2–5 is devoted to a major step in the proof of Theorem 1.1 for $p = 1$. The extension to $p \geq 2$ will be indicated in Section 6.

In Section 2 we start with the Feynman-Kac representation for the first moment of the solution $u$, which involves a random walk sampling the exclusion process. After rescaling time, we transform the representation w.r.t. the old measure to a representation w.r.t. a new measure via an appropriate absolutely continuous transformation. This allows us to separate the parts responsible for, respectively, the Green term and the polaron term in the r.h.s. of (1.9). Since the Green term has already been handled in [3], we need only concentrate on the polaron term. In Section 3 we show that, in the limit as $\kappa \to \infty$, the new measure may be replaced by the old measure. The resulting representation is used in Section 4 to prove the *lower bound* for the polaron term. This is done analytically with the help of a Rayleigh-Ritz formula. In Section 5, which is technical and takes up almost half of the paper, we prove the corresponding *upper bound*. This is done by freezing and defreezing the exclusion process over long time intervals, allowing us to approximate the representation in terms of the occupation time measures of the random walk over these time intervals. After applying spectral estimates and using a large deviation principle for these occupation time measures, we arrive at the polaron variational formula.

## 2 Separation of the Green term and the polaron term

In Section 2.1 we formulate the Feynman-Kac representation for the first moment of $u$ and show how to split this into two parts after an appropriate change of measure. In Section 2.2 we formulate two propositions for the asymptotics of these two parts, which lead to, respectively, the Green term and the polaron term in (1.9). These two propositions will be proved in Sections 3–5. In Section 2.3 we state and prove three elementary lemmas that will be needed along the way.

### 2.1 Key objects

The solution $u$ of the PAM in (1.1) admits the Feynman-Kac representation

$$u(x,t) = \mathrm{E}_x^X \left( \exp\left[ \int_0^t ds \; \xi_{t-s}\left(X_{\kappa s}\right) \right] \right), \tag{2.1}$$

where $X$ is simple random walk on $\mathbb{Z}^3$ with step rate 6 (i.e., with generator $\Delta$) and $\mathrm{P}_x^X$ and $\mathrm{E}_x^X$ denote probability and expectation with respect to $X$ given $X_0 = x$. Since $\xi$ is reversible w.r.t. $\nu_\rho$, we may reverse time in (2.1) to obtain

$$\mathbb{E}_{\nu_\rho}\big(u(0,t)\big) = \mathbb{E}_{\nu_\rho,0}\bigg( \exp\left[ \int_0^t ds \; \xi_s(X_{\kappa s}) \right] \bigg), \tag{2.2}$$

where $\mathbb{E}_{\nu_\rho,0}$ is expectation w.r.t. $\mathbb{P}_{\nu_\rho,0} = \mathbb{P}_{\nu_\rho} \otimes \mathrm{P}_0^X$.

As in [2] and [3], we rescale time and write

$$e^{-\rho(t/\kappa)}\mathbb{E}_{\nu_\rho}\big(u(0,t/\kappa)\big) = \mathbb{E}_{\nu_\rho,0}\bigg( \exp\left[ \frac{1}{\kappa}\int_0^t ds \, \phi(Z_s) \right] \bigg) \tag{2.3}$$

with

$$\phi(\eta,x) = \eta(x) - \rho \tag{2.4}$$

and

$$Z_t = (\xi_{t/\kappa}, X_t). \tag{2.5}$$

From (2.3) it is obvious that (1.9) in Theorem 1.1 (for $p = 1$) reduces to

$$\lim_{\kappa\to\infty} \kappa^2 \lambda^*(\kappa) = \frac{1}{6}\rho(1-\rho)G + [6\rho(1-\rho)]^2 \mathscr{P}_3, \tag{2.6}$$

where

$$\lambda^*(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\left( \exp\left[ \frac{1}{\kappa}\int_0^t ds \, \phi(Z_s) \right] \right). \tag{2.7}$$

Here and in the rest of the paper we suppress the dependence on $\rho \in (0,1)$ from the notation. Under $\mathbb{P}_{\eta,x} = \mathbb{P}_\eta \otimes \mathrm{P}_x^X$, $(Z_t)_{t\geq 0}$ is a Markov process with state space $\Omega \times \mathbb{Z}^3$ and generator

$$\mathscr{A} = \frac{1}{\kappa} L + \Delta \tag{2.8}$$

(acting on the Banach space of bounded continuous functions on $\Omega \times \mathbb{Z}^3$, equipped with the supremum norm). Let $(\mathscr{S}_t)_{t\geq 0}$ denote the semigroup generated by $\mathscr{A}$.

Our aim is to make an absolutely continuous transformation of the measure $\mathbb{P}_{\eta,x}$ with the help of an exponential martingale, in such a way that, under the new measure $\mathbb{P}^{\text{new}}_{\eta,x}$, $(Z_t)_{t\geq 0}$ is a Markov process with generator $\mathscr{A}^{\text{new}}$ of the form

$$\mathscr{A}^{\text{new}} f = e^{-\frac{1}{\kappa}\psi}\mathscr{A}\left(e^{\frac{1}{\kappa}\psi} f\right) - \left(e^{-\frac{1}{\kappa}\psi}\mathscr{A} e^{\frac{1}{\kappa}\psi}\right) f. \tag{2.9}$$

This transformation leads to an interaction between the exclusion process part and the random walk part of $(Z_t)_{t\geq 0}$, controlled by $\psi\colon \Omega \times \mathbb{Z}^3 \to \mathbb{R}$. As explained in [3], Section 4.2, it will be expedient to choose $\psi$ as

$$\psi = \int_0^T ds\, (\mathscr{S}_s \phi) \tag{2.10}$$

with $T$ a large constant (suppressed from the notation), implying that

$$-\mathscr{A}\psi = \phi - \mathscr{S}_T \phi. \tag{2.11}$$

It was shown in [3], Lemma 4.3.1, that

$$N_t = \exp\left[\frac{1}{\kappa}\left[\psi(Z_t) - \psi(Z_0)\right] - \int_0^t ds\, \left(e^{-\frac{1}{\kappa}\psi}\mathscr{A} e^{\frac{1}{\kappa}\psi}\right)(Z_s)\right] \tag{2.12}$$

is an exponential $\mathbb{P}_{\eta,x}$-martingale for all $(\eta, x) \in \Omega \times \mathbb{Z}^3$. Moreover, if we define $\mathbb{P}^{\text{new}}_{\eta,x}$ in such a way that

$$\mathbb{P}^{\text{new}}_{\eta,x}(A) = \mathbb{E}_{\eta,x}\left(N_t\, \mathbb{1}_A\right) \tag{2.13}$$

for all events $A$ in the $\sigma$-algebra generated by $(Z_s)_{s\in[0,t]}$, then under $\mathbb{P}^{\text{new}}_{\eta,x}$ indeed $(Z_s)_{s\geq 0}$ is a Markov process with generator $\mathscr{A}^{\text{new}}$. Using (2.11–2.13) and $\mathbb{E}^{\text{new}}_{\nu_\rho,0} = \int_\Omega \nu_\rho(d\eta)\mathbb{E}^{\text{new}}_{\eta,0}$, it then follows that the expectation in (2.7) can be written in the form

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{1}{\kappa}\int_0^t ds\, \phi(Z_s)\right]\right)\\
&= \mathbb{E}^{\text{new}}_{\nu_\rho,0}\Bigg(\exp\Bigg[\frac{1}{\kappa}\left[\psi(Z_0) - \psi(Z_t)\right] + \int_0^t ds\, \left[\left(e^{-\frac{1}{\kappa}\psi}\mathscr{A} e^{\frac{1}{\kappa}\psi}\right) - \mathscr{A}\left(\frac{1}{\kappa}\psi\right)\right](Z_s)\\
&\qquad + \frac{1}{\kappa}\int_0^t ds\, (\mathscr{S}_T \phi)(Z_s)\Bigg]\Bigg).
\end{aligned} \tag{2.14}$$

The first term in the exponent in the r.h.s. of (2.14) stays bounded as $t \to \infty$ and can therefore be discarded when computing $\lambda^*(\kappa)$ via (2.7). We will see later that the second term and the third term lead to the Green term and the polaron term in (2.6), respectively. These terms may be separated from each other with the help of Hölder's inequality, as stated in Proposition 2.1 below.

## 2.2 Key propositions

**Proposition 2.1.** *For any $\kappa > 0$,*

$$\lambda^*(\kappa) \underset{\geq}{\leq} I_1^q(\kappa) + I_2^r(\kappa) \tag{2.15}$$

*with*

$$I_1^q(\kappa) = \frac{1}{q}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\rm new}_{\nu_\rho,0}\left(\exp\left[q\int_0^t ds\,\left[\left(e^{-\frac{1}{\kappa}\psi}\mathscr{A}e^{\frac{1}{\kappa}\psi}\right)-\mathscr{A}\left(\frac{1}{\kappa}\psi\right)\right](Z_s)\right]\right),$$
$$I_2^r(\kappa) = \frac{1}{r}\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\rm new}_{\nu_\rho,0}\left(\exp\left[\frac{r}{\kappa}\int_0^t ds\,(\mathscr{S}_T\phi)(Z_s)\right]\right), \tag{2.16}$$

*where* $1/q+1/r=1$*, with* $q>0$*,* $r>1$ *in the first inequality and* $q<0$*,* $0<r<1$ *in the second inequality.*

*Proof.* See [3], Proposition 4.4.1. The existence and finiteness of the limits in (2.16) follow from Lemma 3.1 below. ∎

By choosing $r$ arbitrarily close to 1, we see that the proof of our main statement in (2.6) reduces to the following two propositions, where we abbreviate

$$\limsup_{t,\kappa,T\to\infty} = \limsup_{T\to\infty}\limsup_{\kappa\to\infty}\limsup_{t\to\infty} \quad\text{and}\quad \lim_{t,\kappa,T\to\infty} = \lim_{T\to\infty}\lim_{\kappa\to\infty}\lim_{t\to\infty}. \tag{2.17}$$

In the next proposition we write $\psi_T$ instead of $\psi$ to indicate the dependence on the parameter $T$.

**Proposition 2.2.** *For any* $\alpha\in\mathbb{R}$*,*

$$\limsup_{t,\kappa,T\to\infty}\frac{\kappa^2}{t}\log\mathbb{E}^{\rm new}_{\nu_\rho,0}\left(\exp\left[\alpha\int_0^t ds\,\left[\left(e^{-\frac{1}{\kappa}\psi_T}\mathscr{A}e^{\frac{1}{\kappa}\psi_T}\right)-\mathscr{A}\left(\frac{1}{\kappa}\psi_T\right)\right](Z_s)\right]\right)\le\frac{\alpha}{6}\rho(1-\rho)G. \tag{2.18}$$

**Proposition 2.3.** *For any* $\alpha>0$*,*

$$\lim_{t,\kappa,T\to\infty}\frac{\kappa^2}{t}\log\mathbb{E}^{\rm new}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa}\int_0^t ds\,(\mathscr{S}_T\phi)(Z_s)\right]\right)=[6\alpha^2\rho(1-\rho)]^2\mathscr{P}_3. \tag{2.19}$$

These propositions will be proved in Sections 3–5.

## 2.3 Preparatory lemmas

This section contains three elementary lemmas that will be used frequently in Sections 3–5.

Let $p_t^{(1)}(x,y)$ and $p_t(x,y)=p_t^{(3)}(x,y)$ be the transition kernels of simple random walk in $d=1$ and $d=3$, respectively, with step rate 1.

**Lemma 2.4.** *There exists* $C>0$ *such that, for all* $t\ge0$ *and* $x,y,e\in\mathbb{Z}^3$ *with* $\|e\|=1$*,*

$$p_t^{(1)}(x,y)\le\frac{C}{(1+t)^{\frac{1}{2}}},\quad p_t(x,y)\le\frac{C}{(1+t)^{\frac{3}{2}}},\quad \left|p_t(x+e,y)-p_t(x,y)\right|\le\frac{C}{(1+t)^2}. \tag{2.20}$$

*Proof.* Standard. ∎

(In the sequel we will frequently write $p_t(x-y)$ instead of $p_t(x,y)$.)

From the graphical representation for SSE (Liggett [7], Chapter VIII, Theorem 1.1) it is immediate that

$$\mathbb{E}_\eta\big(\xi_t(x)\big) = \sum_{y\in\mathbb{Z}^d} p_t(x,y)\,\eta(y). \tag{2.21}$$

Recalling (2.4–2.5) and (2.10), we therefore have

$$\begin{aligned}\mathscr{S}_s\phi(\eta,x) = \mathbb{E}_{\eta,x}\Big(\phi(Z_s)\Big) &= \mathbb{E}_\eta\Bigg(\sum_{y\in\mathbb{Z}^3} p_{6s}(x,y)\big[\xi_{s/\kappa}(y)-\rho\big]\Bigg)\\ &= \sum_{z\in\mathbb{Z}^3} p_{6s1[\kappa]}(x,z)\big[\eta(z)-\rho\big]\end{aligned} \tag{2.22}$$

and

$$\psi(\eta,x) = \int_0^T ds \sum_{z\in\mathbb{Z}^3} p_{6s1[\kappa]}(x,z)\big[\eta(z)-\rho\big], \tag{2.23}$$

where we abbreviate

$$1[\kappa] = 1 + \frac{1}{6\kappa}. \tag{2.24}$$

**Lemma 2.5.** *For all $\kappa, T > 0$, $\eta\in\Omega$, $a,b\in\mathbb{Z}^3$ with $\|a-b\|=1$ and $x\in\mathbb{Z}^3$,*

$$|\psi(\eta,b)-\psi(\eta,a)| \le 2C\sqrt{T} \quad \textit{for } T\ge 1, \tag{2.25}$$

$$\Big|\psi\big(\eta^{a,b},x\big)-\psi(\eta,x)\Big| \le 2G, \tag{2.26}$$

$$\sum_{\{a,b\}}\Big(\psi\big(\eta^{a,b},x\big)-\psi(\eta,x)\Big)^2 \le \frac{1}{6}G, \tag{2.27}$$

*where $C>0$ is the same constant as in Lemma 2.4, and $G$ is the value at 0 of the Green function of simple random walk on $\mathbb{Z}^3$.*

*Proof.* For a proof of (2.26–2.27), see [3], Lemma 4.5.1. To prove (2.25), we may without loss of generality consider $b = a+e_1$ with $e_1=(1,0,0)$. Then, by (2.23), we have

$$\begin{aligned}|\psi(\eta,b)-\psi(\eta,a)| &\le \int_0^T ds \sum_{z\in\mathbb{Z}^3}\Big|p_{6s1[\kappa]}(z+e_1)-p_{6s1[\kappa]}(z)\Big|\\ &= \int_0^T ds \sum_{z\in\mathbb{Z}^3}\Big|p^{(1)}_{6s1[\kappa]}(z_1+e_1)-p^{(1)}_{6s1[\kappa]}(z_1)\Big|\,p^{(1)}_{6s1[\kappa]}(z_2)\,p^{(1)}_{6s1[\kappa]}(z_3)\\ &= \int_0^T ds \sum_{z_1\in\mathbb{Z}}\Big|p^{(1)}_{6s1[\kappa]}(z_1+e_1)-p^{(1)}_{6s1[\kappa]}(z_1)\Big|\\ &= 2\int_0^T ds\, p^{(1)}_{6s1[\kappa]}(0) \le 2C\sqrt{T}.\end{aligned} \tag{2.28}$$

In the last line we have used the first inequality in (2.20). ∎

Let $\mathcal{G}$ be the Green operator acting on functions $V : \mathbb{Z}^3 \to [0,\infty)$ as

$$\mathcal{G}V(x) = \sum_{y\in\mathbb{Z}^3} G(x-y)V(y), \quad x \in \mathbb{Z}^3, \tag{2.29}$$

with $G(z) = \int_0^\infty dt\, p_t(z)$. Let $\|\cdot\|_\infty$ denote the supremum norm.

**Lemma 2.6.** *For all $V : \mathbb{Z}^3 \to [0,\infty)$ and $x \in \mathbb{Z}^3$,*

$$\mathrm{E}_x^X\left(\exp\left[\int_0^\infty dt\, V(X_t)\right]\right) \le \left(1 - \|\mathcal{G}V\|_\infty\right)^{-1} \le \exp\left(\frac{\|\mathcal{G}V\|_\infty}{1-\|\mathcal{G}V\|_\infty}\right), \tag{2.30}$$

*provided that*

$$\|\mathcal{G}V\|_\infty < 1. \tag{2.31}$$

*Proof.* See [2], Lemma 8.1. ∎

# 3 Reduction to the original measure

In this section we show that the expectations in Propositions 2.2–2.3 w.r.t. the new measure $\mathbb{P}^{\text{new}}_{\nu_\rho,0}$ are asymptotically the same as the expectations w.r.t. the old measure $\mathbb{P}_{\nu_\rho,0}$. In Section 3.1 we state a Rayleigh-Ritz formula from which we draw the desired comparison. In Section 3.2 we state the analogues of Propositions 2.2–2.3 whose proof will be the subject of Sections 4–5.

## 3.1 Rayleigh-Ritz formula

Recall the definition of $\psi$ in (2.10). Let $m$ denote the counting measure on $\mathbb{Z}^3$. It is easily checked that both $\mu_\rho = \nu_\rho \otimes m$ and $\mu_\rho^{\text{new}}$ given by

$$d\mu_\rho^{\text{new}} = e^{\frac{2}{\kappa}\psi}\, d\mu_\rho \tag{3.1}$$

are reversible invariant measures of the Markov processes with generators $\mathcal{A}$ defined in (2.8), respectively, $\mathcal{A}^{\text{new}}$ defined in (2.9). In particular, $\mathcal{A}$ and $\mathcal{A}^{\text{new}}$ are self-adjoint operators in $L^2(\mu_\rho)$ and $L^2(\mu_\rho^{\text{new}})$. Let $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A}^{\text{new}})$ denote their domains.

**Lemma 3.1.** *For all bounded measurable $V : \Omega \times \mathbb{Z}^3 \to \mathbb{R}$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\text{new}}_{\nu_\rho,0}\left(\exp\left[\int_0^t ds\, V(Z_s)\right]\right) = \sup_{\substack{F\in\mathcal{D}(\mathcal{A}^{\text{new}})\\ \|F\|_{L^2(\mu_\rho^{\text{new}})}=1}} \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho^{\text{new}} \left(VF^2 + F\,\mathcal{A}^{\text{new}}F\right). \tag{3.2}$$

*The same is true when $\mathbb{E}^{\text{new}}_{\nu_\rho,0}$, $\mu_\rho^{\text{new}}$, $\mathcal{A}^{\text{new}}$ are replaced by $\mathbb{E}_{\nu_\rho,0}$, $\mu_\rho$, $\mathcal{A}$, respectively.*

*Proof.* The limit in the l.h.s. of (3.2) coincides with the upper boundary of the spectrum of the operator $\mathcal{A}^{\text{new}} + V$ on $L^2(\mu_\rho^{\text{new}})$, which may be represented by the Rayleigh-Ritz formula. The latter coincides with the expression in the r.h.s. of (3.2). The details are similar to [3], Section 2.2. ∎

Lemma 3.1 can be used to express the limits as $t \to \infty$ in Propositions 2.2–2.3 as variational expressions involving the new measure. Lemma 3.2 below says that, for large $\kappa$, these variational expressions are close to the corresponding variational expressions for the old measure. Using Lemma 3.1 for the original measure, we may therefore arrive at the corresponding limit for the old measure.

For later use, in the statement of Lemma 3.2 we do not assume that $\psi$ is given by (2.10). Instead, we only suppose that $\eta \mapsto \psi(\eta)$ is bounded and measurable and that there is a constant $K > 0$ such that for all $\eta \in \Omega$, $a, b \in \mathbb{Z}^3$ with $\|a-b\| = 1$ and $x \in \mathbb{Z}^3$,

$$|\psi(\eta,b)-\psi(\eta,a)| \le K \quad \text{and} \quad \left|\psi\left(\eta^{a,b},x\right)-\psi(\eta,x)\right| \le K, \tag{3.3}$$

but retain that $\mathscr{A}^{\mathrm{new}}$ and $\mu_\rho^{\mathrm{new}}$ are given by (2.9) and (3.1), respectively.

**Lemma 3.2.** *Assume* (3.3). *Then, for all bounded measurable* $V\colon \Omega\times\mathbb{Z}^3 \to \mathbb{R}$,

$$\begin{aligned}
&\sup_{\substack{F\in\mathscr{D}(\mathscr{A}^{\mathrm{new}})\\ \|F\|_{L^2(\mu_\rho^{\mathrm{new}})}=1}} \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho^{\mathrm{new}} \left(VF^2 + F\,\mathscr{A}^{\mathrm{new}}F\right)\\
&\quad \substack{\le\\ \ge}\; e^{\mp\frac{K}{\kappa}} \sup_{\substack{F\in\mathscr{D}(\mathscr{A})\\ \|F\|_{L^2(\mu_\rho)}=1}} \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho \left(e^{\pm\frac{K}{\kappa}}VF^2 + F\,\mathscr{A}F\right),
\end{aligned} \tag{3.4}$$

*where $\pm$ means $+$ in the first inequality and $-$ in the second inequality, and $\mp$ means the reverse.*

*Proof*. Combining (1.2), (1.4) and (2.8–2.9), we have for all $(\eta,x)\in\Omega\times\mathbb{Z}^3$ and all $F\in\mathscr{D}(\mathscr{A}^{\mathrm{new}})$,

$$\begin{aligned}
\left(VF^2 + F\,\mathscr{A}^{\mathrm{new}}F\right)(\eta,x) &= V(\eta,x)F^2(\eta,x)\\
&\quad + \frac{1}{6\kappa}\sum_{\{a,b\}} F(\eta,x)\, e^{\frac{1}{\kappa}[\psi(\eta^{a,b},x)-\psi(\eta,x)]}\left[F(\eta^{a,b},x)-F(\eta,x)\right]\\
&\quad + \sum_{y\colon \|y-x\|=1} F(\eta,x)\, e^{\frac{1}{\kappa}[\psi(\eta,y)-\psi(\eta,x)]}\left[F(\eta,y)-F(\eta,x)\right].
\end{aligned} \tag{3.5}$$

Therefore, taking into account (2.9), (3.1) and the exchangeability of $\nu_\rho$, we find that

$$\begin{aligned}
\iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho^{\mathrm{new}}\left(VF^2 + F\,\mathscr{A}^{\mathrm{new}}F\right) &= \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho^{\mathrm{new}}(\eta,x)\Bigg(V(\eta,x)F^2(\eta,x)\\
&\quad - \frac{1}{12\kappa}\sum_{\{a,b\}} e^{\frac{1}{\kappa}[\psi(\eta^{a,b},x)-\psi(\eta,x)]}\left[F(\eta^{a,b},x)-F(\eta,x)\right]^2\\
&\quad - \frac{1}{2}\sum_{y\colon \|y-x\|=1} e^{\frac{1}{\kappa}[\psi(\eta,y)-\psi(\eta,x)]}\left[F(\eta,y)-F(\eta,x)\right]^2\Bigg).
\end{aligned} \tag{3.6}$$

Let $\widetilde{F} = e^{\psi/\kappa} F$. Then, by (3.1) and (3.3),

$$
\begin{aligned}
(3.6)\ \begin{matrix}\le\\ \ge\end{matrix}\ & \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho^{\mathrm{new}}(\eta,x)\Bigg(V(\eta,x)F^2(\eta,x)\\
&\quad - \frac{e^{\mp\frac{K}{\kappa}}}{12\kappa}\sum_{\{a,b\}}\Big[F(\eta^{a,b},x)-F(\eta,x)\Big]^2 - \frac{e^{\mp\frac{K}{\kappa}}}{2}\sum_{y\colon \|y-x\|=1}\big[F(\eta,y)-F(\eta,x)\big]^2\Bigg)\\
=\ & \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho(\eta,x)\Bigg(V(\eta,x)\widetilde{F}^2(\eta,x) \qquad\qquad (3.7)\\
&\quad - \frac{e^{\mp\frac{K}{\kappa}}}{12\kappa}\sum_{\{a,b\}}\Big[\widetilde{F}(\eta^{a,b},x)-\widetilde{F}(\eta,x)\Big]^2 - \frac{e^{\mp\frac{K}{\kappa}}}{2}\sum_{y\colon \|y-x\|=1}\Big[\widetilde{F}(\eta,y)-\widetilde{F}(\eta,x)\Big]^2\Bigg)\\
=\ & e^{\mp\frac{K}{\kappa}}\iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho\left(e^{\pm\frac{K}{\kappa}}V\widetilde{F}^2 + \widetilde{F}\mathscr{A}\widetilde{F}\right).
\end{aligned}
$$

Taking further into account that

$$
\big\|\widetilde{F}\big\|^2_{L^2(\mu_\rho)} = \|F\|^2_{L^2(\mu_\rho^{\mathrm{new}})}, \tag{3.8}
$$

and that $\widetilde{F}\in\mathscr{D}(\mathscr{A})$ if and only if $F\in\mathscr{D}(\mathscr{A}^{\mathrm{new}})$, we get the claim. ∎

## 3.2 Reduced key propositions

At this point we may combine the assertions in Lemmas 3.1–3.2 for the potentials

$$
V = \alpha\left[\left(e^{-\frac{1}{\kappa}\psi}\mathscr{A}e^{\frac{1}{\kappa}\psi}\right) - \mathscr{A}\left(\frac{1}{\kappa}\psi\right)\right] \tag{3.9}
$$

and

$$
V = \frac{\alpha}{\kappa}(\mathscr{S}_T\phi) \tag{3.10}
$$

with $\psi$ given by (2.10). Because of (2.25–2.26), the constant $K$ in (3.3) may be chosen to be the maximum of $2G$ and $2C\sqrt{T}$, resulting in $K/\kappa\to 0$ as $\kappa\to\infty$. Moreover, from (2.27) and a Taylor expansion of the r.h.s. of (3.9) we see that the potential in (3.9) is bounded for each $\kappa$ and $T$, and the same is obviously true for the potential in (3.10) because of (2.4). In this way, using a moment inequality to replace the factor $e^{\pm K/\kappa}\alpha$ by a slightly larger, respectively, smaller factor $\alpha'$ independent of $T$ and $\kappa$, we see that the limits in Propositions 2.2–2.3 do not change when we replace $\mathbb{E}^{\mathrm{new}}_{\nu_\rho,0}$ by $\mathbb{E}_{\nu_\rho,0}$. Hence it will be enough to prove the following two propositions.

**Proposition 3.3.** *For all $\alpha\in\mathbb{R}$,*

$$
\limsup_{t,\kappa,T\to\infty}\frac{\kappa^2}{t}\log\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\alpha\int_0^t ds\left[\left(e^{-\frac{1}{\kappa}\psi}\mathscr{A}e^{\frac{1}{\kappa}\psi}\right)-\mathscr{A}\left(\frac{1}{\kappa}\psi\right)\right](Z_s)\right]\right) \le \frac{\alpha}{6}\rho(1-\rho)G. \tag{3.11}
$$

**Proposition 3.4.** *For all $\alpha>0$,*

$$
\lim_{t,\kappa,T\to\infty}\frac{\kappa^2}{t}\log\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa}\int_0^t ds\,(\mathscr{S}_T\phi)(Z_s)\right]\right) = \left[6\alpha^2\rho(1-\rho)\right]^2\mathscr{P}_3. \tag{3.12}
$$

Proposition 3.3 has already been proven in [3], Proposition 4.4.2. Sections 4–5 are dedicated to the proof of the lower, respectively, upper bound in Proposition 3.4.

# 4 Proof of Proposition 3.4: lower bound

In this section we derive the lower bound in Proposition 3.4. We fix $\alpha, \kappa, T > 0$ and use Lemma 3.1, to obtain

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho,0}\left( \exp\left[ \frac{\alpha}{\kappa} \int_0^t ds\, (\mathscr{S}_T\phi)(Z_s) \right] \right) = \sup_{\substack{F\in\mathscr{D}(\mathscr{A}) \\ \|F\|_{L^2(\mu_\rho)}=1}} \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho \left( \frac{\alpha}{\kappa} (\mathscr{S}_T\phi) F^2 + F\mathscr{A}F \right). \tag{4.1}$$

In Section 4.1 we choose a test function. In Section 4.2 we compute and estimate the resulting expression. In Section 4.3 we take the limit $\kappa, T \to \infty$ and show that this gives the desired lower bound.

## 4.1 Choice of test function

To get the desired lower bound, we use test functions $F$ of the form

$$F(\eta, x) = F_1(\eta) F_2(x). \tag{4.2}$$

Before specifying $F_1$ and $F_2$, we introduce some further notation. In addition to the counting measure $m$ on $\mathbb{Z}^3$, consider the discrete Lebesgue measure $m_\kappa$ on $\mathbb{Z}^3_\kappa = \kappa^{-1}\mathbb{Z}^3$ giving weight $\kappa^{-3}$ to each site in $\mathbb{Z}^3_\kappa$. Let $l^2(\mathbb{Z}^3)$ and $l^2(\mathbb{Z}^3_\kappa)$ denote the corresponding $l^2$-spaces. Let $\Delta_\kappa$ denote the lattice Laplacian on $\mathbb{Z}^3_\kappa$ defined by

$$(\Delta_\kappa f)(x) = \kappa^2 \sum_{\substack{y\in\mathbb{Z}^3_\kappa \\ \|y-x\|=\kappa^{-1}}} [f(y) - f(x)]. \tag{4.3}$$

Choose $f \in \mathscr{C}_c^\infty(\mathbb{R}^3)$ with $\|f\|_{L^2(\mathbb{R}^3)} = 1$ arbitrarily, where $\mathscr{C}_c^\infty(\mathbb{R}^3)$ is the set of infinitely differentiable functions on $\mathbb{R}^3$ with compact support. Define

$$f_\kappa(x) = \kappa^{-3/2} f\left(\kappa^{-1}x\right), \quad x \in \mathbb{Z}^3, \tag{4.4}$$

and note that

$$\|f_\kappa\|_{l^2(\mathbb{Z}^3)} = \|f\|_{l^2(\mathbb{Z}^3_\kappa)} \to 1 \quad \text{as } \kappa \to \infty. \tag{4.5}$$

For $F_2$ choose

$$F_2 = \|f_\kappa\|^{-1}_{l^2(\mathbb{Z}^3)} f_\kappa. \tag{4.6}$$

To choose $F_1$, introduce the function

$$\widetilde{\phi}(\eta) = \frac{\alpha}{\|f_\kappa\|^2_{l^2(\mathbb{Z}^3)}} \sum_{x\in\mathbb{Z}^3} (\mathscr{S}_T\phi)(\eta, x) f_\kappa^2(x). \tag{4.7}$$

Given $K > 0$, abbreviate

$$S = 6T1[\kappa] \quad \text{and} \quad U = 6K\kappa^2 1[\kappa] \tag{4.8}$$

(recall (2.24)). For $\kappa > \sqrt{T/K}$, define $\widetilde{\psi}\colon \Omega \to \mathbb{R}$ by

$$\widetilde{\psi} = \int_0^{U-S} ds\, \mathscr{T}_s \widetilde{\phi}, \tag{4.9}$$

where $(\mathscr{T}_t)_{t\geq 0}$ is the semigroup generated by the operator $L$ in (1.4). Note that the construction of $\widetilde{\psi}$ from $\widetilde{\phi}$ in (4.9) is similar to the construction of $\psi$ from $\phi$ in (2.10). In particular,

$$-L\widetilde{\psi} = \widetilde{\phi} - \mathscr{T}_{U-S}\widetilde{\phi}. \tag{4.10}$$

Combining the probabilistic representations of the semigroups $(\mathscr{S}_t)_{t\geq 0}$ (generated by $\mathscr{A}$ in (2.8)) and $(\mathscr{T}_t)_{t\geq 0}$ (generated by $L$ in (1.4)) with the graphical representation formulas (2.21–2.22), and using (4.4–4.5), we find that

$$\widetilde{\phi}(\eta) = \frac{\alpha}{\|f\|^2_{l^2(\mathbb{Z}^3_\kappa)}} \int_{\mathbb{Z}^3_\kappa} m_\kappa(dx) f^2(x) \sum_{z\in\mathbb{Z}^3} p_S(\kappa x, z)[\eta(z)-\rho] \tag{4.11}$$

and

$$\widetilde{\psi}(\eta) = \sum_{z\in\mathbb{Z}^3} h(z)[\eta(z)-\rho] \tag{4.12}$$

with

$$h(z) = \frac{\alpha}{\|f\|^2_{l^2(\mathbb{Z}^3_\kappa)}} \int_{\mathbb{Z}^3_\kappa} m_\kappa(dx) f^2(x) \int_S^U ds\, p_s(\kappa x, z). \tag{4.13}$$

Using the second inequality in (2.20), we have

$$0 \leq h(z) \leq \frac{C\alpha}{\sqrt{T}}, \quad z \in \mathbb{Z}^3. \tag{4.14}$$

Now choose $F_1$ as

$$F_1 = \left\|e^{\widetilde{\psi}}\right\|^{-1}_{L^2(\nu_\rho)} e^{\widetilde{\psi}}. \tag{4.15}$$

For the above choice of $F_1$ and $F_2$, we have $\|F_1\|_{L^2(\nu_\rho)} = \|F_2\|_{l^2(\mathbb{Z}^3)} = 1$ and, consequently, $\|F\|_{L^2(\mu_\rho)} = 1$. With $F_1$, $F_2$ and $\widetilde{\phi}$ as above, and $\mathscr{A}$ as in (2.8), after scaling space by $\kappa$ we arrive at the following lemma.

**Lemma 4.1.** *For $F$ as in* (4.2), (4.6) *and* (4.15)*, all $\alpha, T, K > 0$ and $\kappa > \sqrt{T/K}$,*

$$\begin{aligned}\kappa^2 \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho \left(\frac{\alpha}{\kappa}(\mathscr{S}_T\phi)F^2 + F\mathscr{A}F\right) \\ = \frac{1}{\|f\|^2_{l^2(\mathbb{Z}^3_\kappa)}} \int_{\mathbb{Z}^3_\kappa} dm_\kappa\, f\Delta_\kappa f + \frac{\kappa}{\|e^{\widetilde{\psi}}\|^2_{L^2(\nu_\rho)}} \int_\Omega d\nu_\rho \left(\widetilde{\phi} e^{2\widetilde{\psi}} + e^{\widetilde{\psi}} L e^{\widetilde{\psi}}\right),\end{aligned} \tag{4.16}$$

*where $\widetilde{\phi}$ and $\widetilde{\psi}$ are as in* (4.7) *and* (4.9).

## 4.2 Computation of the r.h.s. of (4.16)

Clearly, as $\kappa \to \infty$ the first summand in the r.h.s. of (4.16) converges to

$$\int_{\mathbb{R}^3} dx\, f(x)\Delta f(x) = -\left\|\nabla_{\mathbb{R}^3} f\right\|^2_{L^2(\mathbb{R}^3)}. \tag{4.17}$$

The computation of the second summand in the r.h.s. of (4.16) is more delicate:

**Lemma 4.2.** *For all $\alpha > 0$ and $0 < \epsilon < K$,*

$$\liminf_{\kappa,T\to\infty} \frac{\kappa}{\|e^{\widetilde{\psi}}\|^2_{L^2(\nu_\rho)}} \int_\Omega d\nu_\rho \left( \widetilde{\phi} e^{2\widetilde{\psi}} + e^{\widetilde{\psi}} L e^{\widetilde{\psi}} \right)$$
$$\geq 6\alpha^2 \rho(1-\rho) \int_{\mathbb{R}^3} dx\, f^2(x) \int_{\mathbb{R}^3} dy\, f^2(y) \left( \int_{6\epsilon}^{6K} dt\, p_t^{(G)}(x,y) - \int_{6K}^{12K} dt\, p_t^{(G)}(x,y) \right), \tag{4.18}$$

*where*

$$p_t^{(G)}(x,y) = (4\pi t)^{-3/2} \exp[-\|x-y\|^2/4t] \tag{4.19}$$

*denotes the Gaussian transition kernel associated with $\Delta_{\mathbb{R}^3}$, the continuous Laplacian on $\mathbb{R}^3$.*

*Proof.* Using the probability measure

$$d\nu_\rho^{\text{new}} = \left\|e^{\widetilde{\psi}}\right\|^{-2}_{L^2(\nu_\rho)} e^{2\widetilde{\psi}}\, d\nu_\rho \tag{4.20}$$

in combination with (4.10), we may write the term under the lim inf in (4.18) in the form

$$\kappa \int_\Omega d\nu_\rho^{\text{new}} \left( e^{-\widetilde{\psi}} L e^{\widetilde{\psi}} - L\widetilde{\psi} + \mathcal{T}_{U-S}\widetilde{\phi} \right). \tag{4.21}$$

This expression can be handled by making a Taylor expansion of the $L$-terms and showing that the $\mathcal{T}_{U-S}$-term is nonnegative. Indeed, by the definition of $L$ in (1.4), we have

$$\left( e^{-\widetilde{\psi}} L e^{\widetilde{\psi}} - L\widetilde{\psi} \right)(\eta) = \frac{1}{6} \sum_{\{a,b\}} \left( e^{[\widetilde{\psi}(\eta^{a,b}) - \widetilde{\psi}(\eta)]} - 1 - \left[ \widetilde{\psi}(\eta^{a,b}) - \widetilde{\psi}(\eta) \right] \right). \tag{4.22}$$

Recalling the expressions for $\widetilde{\psi}$ in (4.12–4.13) and using (4.14), we get for $a, b \in \mathbb{Z}^3$ with $\|a-b\| = 1$,

$$\left| \widetilde{\psi}(\eta^{a,b}) - \widetilde{\psi}(\eta) \right| = |h(a) - h(b)|\, |\eta(b) - \eta(a)| \leq \frac{C\alpha}{\sqrt{T}}. \tag{4.23}$$

Hence, a Taylor expansion of the exponent in the r.h.s. of (4.22) gives

$$\int_\Omega d\nu_\rho^{\text{new}} \left( e^{-\widetilde{\psi}} L e^{\widetilde{\psi}} - L\widetilde{\psi} \right) \geq \frac{e^{-C\alpha/\sqrt{T}}}{12} \int_\Omega d\nu_\rho^{\text{new}} \sum_{\{a,b\}} \left[ \widetilde{\psi}(\eta^{a,b}) - \widetilde{\psi}(\eta) \right]^2. \tag{4.24}$$

Using (4.12), we obtain

$$\int_\Omega \nu_\rho^{\text{new}}(d\eta) \sum_{\{a,b\}} \left[ \widetilde{\psi}(\eta^{a,b}) - \widetilde{\psi}(\eta) \right]^2 = \sum_{\{a,b\}} [h(a) - h(b)]^2 \int_\Omega \nu_\rho^{\text{new}}(d\eta) [\eta(b) - \eta(a)]^2. \tag{4.25}$$

Using (4.20), we have (after cancellation of factors not depending on $a$ or $b$)

$$\int_\Omega \nu_\rho^{\text{new}}(d\eta) [\eta(b) - \eta(a)]^2 = \frac{\displaystyle\int_\Omega \nu_\rho(d\eta)\, e^{2\chi_{a,b}(\eta)} [\eta(b) - \eta(a)]^2}{\displaystyle\int_\Omega \nu_\rho(d\eta)\, e^{2\chi_{a,b}(\eta)}} \tag{4.26}$$

with

$$\chi_{a,b}(\eta) = h(a)\eta(a) + h(b)\eta(b). \tag{4.27}$$

Using (4.14), we obtain that

$$\int_\Omega \nu_\rho^{\mathrm{new}}(d\eta)\big[\eta(b)-\eta(a)\big]^2 \ge e^{-4C\alpha/\sqrt{T}} \int_\Omega \nu_\rho(d\eta)\big[\eta(b)-\eta(a)\big]^2 = e^{-4C\alpha/\sqrt{T}} 2\rho(1-\rho). \tag{4.28}$$

On the other hand, by (4.13),

$$\begin{aligned}\sum_{\{a,b\}} \big[h(a)-h(b)\big]^2 = \frac{\alpha^2}{\|f\|^4_{l^2(\mathbb{Z}^3_\kappa)}} \int_S^U dt \int_S^U ds \int_{\mathbb{Z}^3_\kappa} m_\kappa(dx) f^2(x) \int_{\mathbb{Z}^3_\kappa} m_\kappa(dy) f^2(y) \\ \times \sum_{\{a,b\}} \big[p_t(\kappa x,a)-p_t(\kappa x,b)\big]\big[p_s(\kappa y,a)-p_s(\kappa y,b)\big]\end{aligned} \tag{4.29}$$

with

$$\begin{aligned}\sum_{\{a,b\}} \big[p_t(\kappa x,a)-p_t(\kappa x,b)\big]\big[p_s(\kappa y,a)-p_s(\kappa y,b)\big] &= -\sum_{a\in\mathbb{Z}^3} p_t(\kappa x,a)\Delta p_s(\kappa x,a) \\ &= -6\sum_{a\in\mathbb{Z}^3} p_t(\kappa x,a)\Big(\frac{\partial}{\partial s} p_s(\kappa y,a)\Big),\end{aligned} \tag{4.30}$$

where $\Delta$ acts on the first spatial variable of $p_s(\cdot,\cdot)$ and $\Delta p_s = 6(\partial p_s/\partial s)$. Therefore,

$$\begin{aligned}(4.29) &= 6\int_S^U dt \int_{\mathbb{Z}^3_\kappa} m_\kappa(dx) f^2(x) \int_{\mathbb{Z}^3_\kappa} m_\kappa(dy) f^2(y) \sum_{a\in\mathbb{Z}^3} p_t(\kappa x,a)\big[p_S(\kappa y,a)-p_U(\kappa y,a)\big] \\ &= 6\int_{\mathbb{Z}^3_\kappa} m_\kappa(dx) f^2(x) \int_{\mathbb{Z}^3_\kappa} m_\kappa(dy) f^2(y) \Big(\Big(\int_{2S}^{S+U} dt\, p_t(\kappa x,\kappa y) - \int_{U+S}^{2U} dt\, p_t(\kappa x,\kappa y)\Big).\end{aligned} \tag{4.31}$$

Combining (4.24–4.25) and (4.28–4.29) and (4.31), we arrive at

$$\begin{aligned}\int_\Omega d\nu_\rho^{\mathrm{new}}\Big(e^{-\widetilde{\psi}} L e^{\widetilde{\psi}} - L\widetilde{\psi}\Big) \ge \frac{e^{-5C\alpha/\sqrt{T}}\alpha^2}{\|f\|^4_{l^2(\mathbb{Z}^3_\kappa)}} \rho(1-\rho) \int_{\mathbb{Z}^3_\kappa} m_\kappa(dx) f^2(x) \int_{\mathbb{Z}^3_\kappa} m_\kappa(dy) f^2(y) \\ \times \Big(\int_{2S}^{S+U} dt\, p_t(\kappa x,\kappa y) - \int_{U+S}^{2U} dt\, p_t(\kappa x,\kappa y)\Big).\end{aligned} \tag{4.32}$$

After replacing $2S$ in the first integral by $6\epsilon\kappa^2 1[\kappa]$, using a Gaussian approximation of the transition kernel $p_t(x,y)$ and recalling the definitions of $S$ and $U$ in (4.8), we get that, for any $\epsilon > 0$,

$$\begin{aligned}&\liminf_{\kappa,T\to\infty} \kappa \int_\Omega d\nu_\rho^{\mathrm{new}}\Big(e^{-\widetilde{\psi}} L e^{\widetilde{\psi}} - L\widetilde{\psi}\Big) \\ &\quad\ge 6\alpha^2\rho(1-\rho) \int_{\mathbb{R}^3} dx\, f^2(x) \int_{\mathbb{R}^3} dy\, f^2(y) \Big(\int_{6\epsilon}^{6K} dt\, p_t^{(G)}(x,y) - \int_{6K}^{12K} dt\, p_t^{(G)}(x,y)\Big).\end{aligned} \tag{4.33}$$

At this point it only remains to check that the $\mathcal{T}_{U-S}$-term in (4.21) is nonnegative. By (4.11) and the probabilistic representation of the semigroup $(\mathcal{T}_t)_{t\geq 0}$, we have

$$\int_\Omega d\nu_\rho^{\text{new}} \mathcal{T}_{U-S}\widetilde{\phi} = \frac{\alpha}{\|f\|^2_{l^2(\mathbb{Z}^3_\kappa)}} \int_{\mathbb{Z}^3_\kappa} m_\kappa(dx) f^2(x) \sum_{z\in\mathbb{Z}^3} p_U(\kappa x, z) \int_\Omega \nu_\rho^{\text{new}}(d\eta)[\eta(z)-\rho] \tag{4.34}$$

and, by (4.20),

$$\begin{aligned}\int_\Omega \nu_\rho^{\text{new}}(d\eta)[\eta(z)-\rho] &= -\rho + \frac{\rho e^{2h(z)}}{\rho e^{2h(z)}+1-\rho} = -\rho + \frac{\rho}{1-(1-\rho)(1-e^{-2h(z)})} \\ &\geq -\rho + \rho\Big[1+(1-\rho)\Big(1-e^{-2h(z)}\Big)\Big] = \rho(1-\rho)\Big(1-e^{-2h(z)}\Big),\end{aligned} \tag{4.35}$$

which proves the claim. ∎

### 4.3 Proof of the lower bound in Proposition 3.4

We finish by using Lemma 4.2 to prove the lower bound in Proposition 3.4.

*Proof.* Combining (4.16–4.18), we get

$$\begin{aligned}&\liminf_{\kappa,T\to\infty} \kappa^2 \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho \left(\frac{\alpha}{\kappa}(\mathcal{S}_T\phi)F^2 - F\mathcal{A}F\right) \\ &\quad\geq 6\alpha^2\rho(1-\rho)\int_{\mathbb{R}^3} dx\, f^2(x) \int_{\mathbb{R}^3} dy\, f^2(y) \left(\int_{6\epsilon}^{6K} dt\, p_t^{(G)}(x,y) - \int_{6K}^{12K} dt\, p_t^{(G)}(x,y)\right) \\ &\qquad - \left\|\nabla_{\mathbb{R}^3} f\right\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{4.36}$$

Letting $\epsilon \downarrow 0$, $K \to \infty$, replacing $f(x)$ by $\gamma^{3/2} f(\gamma x)$ with $\gamma = 6\alpha^2\rho(1-\rho)$, taking the supremum over all $f \in C_c^\infty(\mathbb{R}^3)$ such that $\|f\|_{L^2(\mathbb{R}^3)} = 1$ and recalling (4.1), we arrive at

$$\liminf_{t,\kappa,T\to\infty} \frac{\kappa^2}{t} \log \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa}\int_0^t ds\,(\mathcal{S}_T\phi)(Z_s)\right]\right) \geq \left[6\alpha^2\rho(1-\rho)\right]^2 \mathcal{P}_3, \tag{4.37}$$

which is the desired inequality. ∎

## 5 Proof of Proposition 3.4: upper bound

In this section we prove the upper bound in Proposition 3.4. The proof is long and technical. In Sections 5.1 we "freeze" and "defreeze" the exclusion dynamics on long time intervals. This allows us to approximate the relevant functionals of the random walk in terms of its occupation time measures on those intervals. In Section 5.2 we use a spectral bound to reduce the study of the long-time asymptotics for the resulting time-dependent potentials to the investigation of time-independent potentials. In Section 5.3 we make a cut-off for small times, showing that these times are negligible in the limit as $\kappa \to \infty$, perform a space-time scaling and compactification of the underlying random walk, and apply a large deviation principle for the occupation time measures, culminating in the appearance of the variational expression for the polaron term $\mathcal{P}_3$.

## 5.1 Freezing, defreezing and reduction to two key lemmas

### 5.1.1 Freezing

We begin by deriving a preliminary upper bound for the expectation in Proposition 3.4 given by

$$\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^t ds\, V(Z_s)\right]\right) \tag{5.1}$$

with

$$V(\eta,x) = \frac{\alpha}{\kappa}(\mathscr{S}_T\phi)(\eta,x) = \frac{\alpha}{\kappa}\sum_{y\in\mathbb{Z}^3} p_{6T1[\kappa]}(x,y)(\eta(y)-\rho), \tag{5.2}$$

where, as before, $T$ is a large constant. To this end, we divide the time interval $[0,t]$ into $\lfloor t/R_\kappa\rfloor$ intervals of length

$$R_\kappa = R\kappa^2 \tag{5.3}$$

with $R$ a large constant, and "freeze" the exclusion dynamics $(\xi_{t/\kappa})_{t\geq 0}$ on each of these intervals. As will become clear later on, this procedure allows us to express the dependence of (5.1) on the random walk $X$ in terms of objects that are close to integrals over occupation time measures of $X$ on time intervals of length $R_\kappa$. We will see that the resulting expression can be estimated from above by "defreezing" the exclusion dynamics. We will subsequently see that, after we have taken the limits $t\to\infty$, $\kappa\to\infty$ and $T\to\infty$, the resulting estimate can be handled by applying a large deviation principle for the space-time rescaled occupation time measures in the limit as $R\to\infty$. The latter will lead us to the polaron term.

Ignoring the negligible final time interval $[\lfloor t/R_\kappa\rfloor R_\kappa, t]$, using Hölder's inequality with $p,q>1$ and $1/p+1/q=1$, and inserting (5.2), we see that (5.1) may be estimated from above as

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\int_0^{\lfloor t/R_\kappa\rfloor R_\kappa} ds\, V(Z_s)\right]\right)\\
&\quad = \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa} ds\sum_{y\in\mathbb{Z}^3} p_{6T1[\kappa]}(X_s,y)(\xi_{s/\kappa}(y)-\rho)\right]\right)\\
&\quad \leq \left(\mathscr{E}^{(1)}_{R,\alpha q}(t)\right)^{1/q}\left(\mathscr{E}^{(2)}_{R,\alpha p}(t)\right)^{1/p}
\end{aligned} \tag{5.4}$$

with

$$\begin{aligned}
\mathscr{E}^{(1)}_{R,\alpha}(t) = \mathscr{E}^{(1)}_{R,\alpha}(\kappa,T;t) = \mathbb{E}_{\nu_\rho,0}\Bigg(\exp\Bigg[\frac{\alpha}{\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa} ds\sum_{y\in\mathbb{Z}^3}\Big(& p_{6T1[\kappa]}(X_s,y)\,\xi_{\frac{s}{\kappa}}(y)\\
&- p_{6T1[\kappa]+\frac{s-(k-1)R_\kappa}{\kappa}}(X_s,y)\,\xi_{\frac{(k-1)R_\kappa}{\kappa}}(y)\Big)\Bigg]\Bigg)
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
&\mathscr{E}^{(2)}_{R,\alpha}(t) = \mathscr{E}^{(2)}_{R,\alpha}(\kappa,T;t)\\
&\quad = \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa} ds\sum_{y\in\mathbb{Z}^3} p_{6T1[\kappa]+\frac{s-(k-1)R_\kappa}{\kappa}}(X_s,y)\left(\xi_{\frac{(k-1)R_\kappa}{\kappa}}(y)-\rho\right)\right]\right).
\end{aligned} \tag{5.6}$$

Therefore, by choosing $p$ close to 1, the proof of the upper bound in Proposition 3.4 reduces to the proof of the following two lemmas.

**Lemma 5.1.** *For all $R, \alpha > 0$,*

$$\limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{t} \log \mathscr{E}^{(1)}_{R,\alpha}(\kappa, T; t) \le 0. \tag{5.7}$$

**Lemma 5.2.** *For all $\alpha > 0$,*

$$\limsup_{R\to\infty} \limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{t} \log \mathscr{E}^{(2)}_{R,\alpha}(\kappa, T; t) \le \left[6\alpha^2\rho(1-\rho)\right]^2 \mathscr{P}_3. \tag{5.8}$$

Lemma 5.1 will be proved in Section 5.1.2, Lemma 5.2 in Sections 5.1.3–5.3.3.

### 5.1.2 Proof of Lemma 5.1

*Proof.* Fix $R, \alpha > 0$ arbitrarily. Given a path $X$, an initial configuration $\eta \in \Omega$ and $k \in \mathbb{N}$, we first derive an upper bound for

$$\mathbb{E}_\eta\left( \exp\left[ \frac{\alpha}{\kappa} \int_0^{R_\kappa} ds \sum_{y\in\mathbb{Z}^3} \left( p_{6T1[\kappa]}\left(X_s^{(k,\kappa)}, y\right) \xi_{\frac{s}{\kappa}}(y) - p_{6T1[\kappa]+\frac{s}{\kappa}}\left(X_s^{(k,\kappa)}, y\right) \eta(y) \right) \right] \right), \tag{5.9}$$

where

$$X_s^{(k,\kappa)} = X_{(k-1)R_\kappa+s}. \tag{5.10}$$

To this end, we use the independent random walk approximation $\widetilde{\xi}$ of $\xi$ (cf. [3], Proposition 1.2.1), to obtain

$$(5.9) \le \prod_{y\in A_\eta} \mathrm{E}_0^Y \left( \exp\left[ \frac{\alpha}{\kappa} \int_0^{R_\kappa} ds \left( p_{6T1[\kappa]}\left(X_s^{(k,\kappa)}, y + Y_{\frac{s}{\kappa}}\right) - p_{6T1[\kappa]+\frac{s}{\kappa}}\left(X_s^{(k,\kappa)}, y\right) \right) \right] \right), \tag{5.11}$$

where $Y$ is simple random walk on $\mathbb{Z}^3$ with jump rate 1 (i.e., with generator $\frac{1}{6}\Delta$), $\mathrm{E}_0^Y$ is expectation w.r.t. $Y$ starting from 0, and

$$A_\eta = \{x \in \mathbb{Z}^3 \colon \eta(x) = 1\}. \tag{5.12}$$

Observe that the expectation w.r.t. $Y$ of the expression in the exponent is zero. Therefore, a Taylor expansion of the exponential function yields the bound

$$\begin{aligned} &\mathrm{E}_0^Y \left( \exp\left[ \frac{\alpha}{\kappa} \int_0^{R_\kappa} ds \left( p_{6T1[\kappa]}\left(X_s^{(k,\kappa)}, y + Y_{\frac{s}{\kappa}}\right) - p_{6T1[\kappa]+\frac{s}{\kappa}}\left(X_s^{(k,\kappa)}, y\right) \right) \right] \right) \\ &\quad \le 1 + \sum_{n=2}^{\infty} \prod_{l=1}^{n} \left( \frac{\alpha}{\kappa} \int_{s_{l-1}}^{R_\kappa} ds_l \sum_{y_l\in\mathbb{Z}^3} p_{\frac{s_l - s_{l-1}}{\kappa}}(y_{l-1}, y_l) \right. \\ &\qquad\qquad \left. \times \left[ p_{6T1[\kappa]}\left(X_{s_l}^{(k,\kappa)}, y + y_l\right) + p_{6T1[\kappa]+\frac{s_l}{\kappa}}\left(X_{s_l}^{(k,\kappa)}, y\right) \right] \right), \end{aligned} \tag{5.13}$$

where $s_0 = 0$, $y_0 = 0$, and the product has to be understood in a noncommutative way. Using the Chapman-Kolmogorov equation and the inequality $p_t(z) \le p_t(0)$, $z \in \mathbb{Z}^3$, we find that

$$\begin{aligned}\int_{s_{l-1}}^{R_\kappa} ds_l \sum_{y_l \in \mathbb{Z}^3} p_{\frac{s_l - s_{l-1}}{\kappa}}(y_{l-1}, y_l) \left[ p_{6T1[\kappa]}\left(X_{s_l}^{(k,\kappa)}, y + y_l\right) + p_{6T1[\kappa]+\frac{s_l}{\kappa}}\left(X_{s_l}^{(k,\kappa)}, y\right)\right]\Bigg) \\ \le 2\int_0^\infty ds\, p_{T+\frac{s}{\kappa}}(0) = 2\kappa G_T(0)\end{aligned} \tag{5.14}$$

with

$$G_T(0) = \int_T^\infty ds\, p_s(0) \tag{5.15}$$

the cut-off Green function of simple random walk at 0 at time $T$. Substituting this into the above bound for $l = n, n-1, \cdots, 3$, computing the resulting geometric series, and using the inequality $1 + x \le e^x$, we obtain

$$\begin{aligned}(5.13) \le \exp\Bigg[\frac{C_T\alpha^2}{\kappa^2} \prod_{l=1}^{2} \int_{s_{l-1}}^{R_\kappa} ds_l \sum_{y_l \in \mathbb{Z}^3} p_{\frac{s_l - s_{l-1}}{\kappa}}(y_{l-1}, y_l) \\ \times \left(p_{6T1[\kappa]}\left(X_{s_l}^{(k,\kappa)}, y + y_l\right) + p_{6T1[\kappa]+\frac{s_l}{\kappa}}\left(X_{s_l}^{(k,\kappa)}, y\right)\right)\Bigg]\end{aligned} \tag{5.16}$$

with

$$C_T = \frac{1}{1 - 2\alpha G_T(0)}, \tag{5.17}$$

provided that $2\alpha G_T(0) < 1$, which is true for $T$ large enough. Note that $C_T \to 1$ as $T \to \infty$. Substituting (5.16) into (5.11), we find that

$$\begin{aligned}(5.9) \le \exp\Bigg[\frac{C_T\alpha^2}{\kappa^2} \sum_{y \in \mathbb{Z}^3} \prod_{l=1}^{2} \int_{s_{l-1}}^{R_\kappa} ds_l \sum_{y_l \in \mathbb{Z}^3} p_{\frac{s_l - s_{l-1}}{\kappa}}(y_{l-1}, y_l) \\ \times \left(p_{6T1[\kappa]}\left(X_{s_l}^{(k,\kappa)}, y + y_l\right) + p_{6T1[\kappa]+\frac{s_l}{\kappa}}\left(X_{s_l}^{(k,\kappa)}, y\right)\right)\Bigg].\end{aligned} \tag{5.18}$$

Using once more the Chapman-Kolmogorov equation and $p_t(x, y) = p_t(x - y)$, we may compute the sums in the exponent, to arrive at

$$\begin{aligned}(5.9) \le \exp\Bigg[\frac{C_T\alpha^2}{\kappa^2} \int_0^{R_\kappa} ds_1 \int_{s_1}^{R_\kappa} ds_2 \Bigg(p_{12T1[\kappa]+\frac{s_2 - s_1}{\kappa}}\left(X_{s_2}^{(k,\kappa)} - X_{s_1}^{(k,\kappa)}\right) \\ + 3p_{12T1[\kappa]+\frac{s_2+s_1}{\kappa}}\left(X_{s_2}^{(k,\kappa)} - X_{s_1}^{(k,\kappa)}\right)\Bigg)\Bigg].\end{aligned} \tag{5.19}$$

Note that this bound does not depend on the initial configuration $\eta$ and depends on the process $X$ only via its increments on the time interval $[(k-1)R_\kappa, kR_\kappa]$. By (5.10), the increments over the time intervals labelled $k = 1, 2, \cdots, \lfloor t/R_\kappa \rfloor$ are independent and identically distributed. Using $\mathbb{E}_{\nu_\rho,0} = \int \nu_\rho(d\eta) \mathrm{E}_0^X \mathbb{E}_\eta$, we can therefore apply the Markov property of the exclusion dynamics

$(\xi_{t/\kappa})_{t\geq 0}$ at times $R_\kappa$, $2R_\kappa$, $\cdots$, $(\lfloor t/R_\kappa \rfloor - 1)R_\kappa$ to the expectation in the r.h.s. of (5.5), insert the bound (5.19) and afterwards use that $(X_t)_{t\geq 0}$ has independent increments, to arrive at

$$\log \mathscr{E}^{(1)}_{R,\alpha}(t) \leq \frac{t}{R_\kappa} \log \mathrm{E}^X_0 \Bigg( \exp\Bigg[ \frac{C_T\alpha^2}{\kappa^2} \int_0^{R_\kappa} ds_1 \int_{s_1}^{R_\kappa} ds_2 \Big( p_{12T1[\kappa]+\frac{s_2-s_1}{\kappa}}(X_{s_2}-X_{s_1}) + 3p_{12T1[\kappa]+\frac{s_2+s_1}{\kappa}}(X_{s_2}-X_{s_1}) \Big) \Bigg] \Bigg). \tag{5.20}$$

Hence, recalling the definition of $R_\kappa$ in (5.3), we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \frac{\kappa^2}{t} \log \mathscr{E}^{(1)}_{R,\alpha}(t) \\ &\leq \frac{1}{R} \log \mathrm{E}^X_0 \Bigg( \exp\Bigg[ \frac{C_T\alpha^2 R}{R_\kappa} \int_0^{R_\kappa} ds_1 \int_{s_1}^{R_\kappa} ds_2 \Big( p_{12T1[\kappa]+\frac{s_2-s_1}{\kappa}}(X_{s_2}-X_{s_1}) \\ &\qquad + 3p_{12T1[\kappa]+\frac{s_2+s_1}{\kappa}}(X_{s_2}-X_{s_1}) \Big) \Bigg] \Bigg). \end{aligned} \tag{5.21}$$

Let

$$\widehat{X}_t = X_t + Y_{t/\kappa}, \tag{5.22}$$

and let $\mathrm{E}^{\widehat{X}}_0 = \mathrm{E}^X_0 \mathrm{E}^Y_0$ be the expectation w.r.t. $\widehat{X}$ starting at 0. Observe that

$$p_{t+s/\kappa}(z) = \mathrm{E}^Y_0 \Big( p_t(z + Y_{s/\kappa}) \Big). \tag{5.23}$$

We next apply Jensen's inequality w.r.t. the first integral in the r.h.s. of (5.21), substitute $s_2 = s_1 + s$, take into account that $X$ has independent increments, and afterwards apply Jensen's inequality w.r.t. $\mathrm{E}^Y_0$, to arrive at the following upper bound for the expectation in (5.21):

$$\begin{aligned} &\mathrm{E}^X_0 \Bigg( \exp\Bigg[ \frac{C_T\alpha^2 R}{R_\kappa} \int_0^{R_\kappa} ds_1 \int_{s_1}^{R_\kappa} ds_2 \Big( p_{12T1[\kappa]+\frac{s_2-s_1}{\kappa}}(X_{s_2}-X_{s_1}) \\ &\qquad + 3p_{12T1[\kappa]+\frac{s_2+s_1}{\kappa}}(X_{s_2}-X_{s_1}) \Big) \Bigg] \\ &\leq \frac{1}{R_\kappa} \int_0^{R_\kappa} ds_1\, \mathrm{E}^X_0 \Bigg( \exp\Bigg[ C_T\alpha^2 R \int_0^\infty ds\, \mathrm{E}^Y_0 \Big( p_{12T1[\kappa]}\Big(X_s + Y_{\frac{s}{\kappa}}\Big) \\ &\qquad + 3p_{12T1[\kappa]+\frac{2s_1}{\kappa}}\Big(X_s + Y_{\frac{s}{\kappa}}\Big) \Big) \Bigg] \Bigg) \\ &\leq \frac{1}{R_\kappa} \int_0^{R_\kappa} ds_1\, \mathrm{E}^{\widehat{X}}_0 \Bigg( \exp\Bigg[ C_T\alpha^2 R \int_0^\infty ds \Big( p_{12T1[\kappa]}(\widehat{X}_s) + 3p_{12T1[\kappa]+\frac{2s_1}{\kappa}}(\widehat{X}_s) \Big) \Bigg] \Bigg). \end{aligned} \tag{5.24}$$

Applying Lemma 2.6, we can bound the last expression from above by

$$\exp\left[ \frac{4C_T\alpha^2 R \widehat{G}_{2T}(0)}{1 - 4C_T\alpha^2 R\widehat{G}_{2T}(0)} \right], \tag{5.25}$$

where $\widehat{G}_{2T}(0)$ is the cut-off at time $2T$ of the Green function $\widehat{G}$ at 0 for $\widehat{X}$ (which has generator $1[\kappa]\Delta$). Since $\widehat{G}_{2T}(0) \to \frac{1}{6}G_{12T}(0)$ as $\kappa \to \infty$, and since the latter converges to zero as $T \to \infty$, a combination of the above estimates with (5.21) gives the claim. ∎

#### 5.1.3 Defreezing

To prove Lemma 5.2, we next "defreeze" the exclusion dynamics in $\mathcal{E}^{(2)}_{R,\alpha}(t)$. This can be done in a similar way as the "freezing" we did in Section 5.1.1, by taking into account the following remarks. In (5.6), each single summand is asymptotically negligible as $t \to \infty$. Hence, we can safely remove a summand at the beginning and add a summand at the end. After that we can bound the resulting expression from above with the help of Hölder's inequality with weights $p, q > 1$, $1/p + 1/q = 1$, namely,

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{kR_\kappa}^{(k+1)R_\kappa} ds \sum_{y\in\mathbb{Z}^3} p_{6T1[\kappa]+\frac{s-kR_\kappa}{\kappa}}(X_s,y)\Big(\xi_{\frac{kR_\kappa}{\kappa}}(y)-\rho\Big)\right]\right) \\
&\quad\le \Big(\mathcal{E}^{(3)}_{R,\alpha q}(t)\Big)^{1/q}\Big(\mathcal{E}^{(4)}_{R,\alpha p}(t)\Big)^{1/p}
\end{aligned} \tag{5.26}$$

with

$$\begin{aligned}
&\mathcal{E}^{(3)}_{R,\alpha}(t) = \mathcal{E}^{(3)}_{R,\alpha}(\kappa, T; t) \\
&= \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa R_\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa} du \int_{kR_\kappa}^{(k+1)R_\kappa} ds \sum_{y\in\mathbb{Z}^3}\Big(p_{6T1[\kappa]+\frac{s-kR_\kappa}{\kappa}}(X_s,y)\,\xi_{\frac{kR_\kappa}{\kappa}}(y)\right.\right. \\
&\qquad\qquad\left.\left. - p_{6T1[\kappa]+\frac{s-u}{\kappa}}(X_s,y)\,\xi_{\frac{u}{\kappa}}(y)\Big)\right]\right)
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
&\mathcal{E}^{(4)}_{R,\alpha}(t) = \mathcal{E}^{(4)}_{R,\alpha}(\kappa, T; t) \\
&= \mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa R_\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa} du \int_{kR_\kappa}^{(k+1)R_\kappa} ds \sum_{y\in\mathbb{Z}^3} p_{6T1[\kappa]+\frac{s-u}{\kappa}}(X_s,y)\Big(\xi_{\frac{u}{\kappa}}(y)-\rho\Big)\right]\right).
\end{aligned} \tag{5.28}$$

In this way, choosing $p$ close to 1, we see that the proof of Lemma 5.2 reduces to the proof of the following two lemmas.

**Lemma 5.3.** *For all $R, \alpha > 0$,*

$$\limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{t}\log \mathcal{E}^{(3)}_{R,\alpha}(\kappa, T; t) \le 0. \tag{5.29}$$

**Lemma 5.4.** *For all $\alpha > 0$,*

$$\limsup_{R\to\infty}\limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{t}\log \mathcal{E}^{(4)}_{R,\alpha}(\kappa, T; t) \le \big[6\alpha^2\rho(1-\rho)\big]^2\mathcal{P}_3. \tag{5.30}$$

In the remaining sections we prove Lemmas 5.3–5.4 and thereby complete the proof of the upper bound in Proposition 3.4.

### 5.1.4 Proof of Lemma 5.3

*Proof.* The proof goes along the same lines as the proof of Lemma 5.1. Instead of (5.9), we consider

$$\begin{aligned}\mathbb{E}_\eta\Bigg(\exp\Bigg[\frac{\alpha}{\kappa R_\kappa}\int_0^{R_\kappa}du\int_{R_\kappa}^{2R_\kappa}ds\sum_{y\in\mathbb{Z}^3}\Big(p_{6T1[\kappa]+\frac{s-R_\kappa}{\kappa}}\Big(X_s^{(k,\kappa)},y\Big)\xi_{\frac{R_\kappa}{\kappa}}(y)\\ -p_{6T1[\kappa]+\frac{s-u}{\kappa}}\Big(X_s^{(k,\kappa)},y\Big)\xi_{\frac{u}{\kappa}}(y)\Big)\Bigg]\Bigg).\end{aligned}\tag{5.31}$$

Applying Jensen's inequality w.r.t. the first integral and the Markov property of the exclusion dynamics $(\xi_{t/\kappa})_{t\geq 0}$ at time $u/\kappa$, we see that it is enough to derive an appropriate upper bound for

$$\begin{aligned}\mathbb{E}_\zeta\Bigg(\exp\Bigg[\frac{\alpha}{\kappa}\int_{R_\kappa}^{2R_\kappa}ds\sum_{y\in\mathbb{Z}^3}\Big(p_{6T1[\kappa]+\frac{s-R_\kappa}{\kappa}}\Big(X_s^{(k,\kappa)},y\Big)\xi_{\frac{R_\kappa-u}{\kappa}}(y)\\ -p_{6T1[\kappa]+\frac{s-u}{\kappa}}\Big(X_s^{(k,\kappa)},y\Big)\zeta(y)\Big)\Bigg]\Bigg)\end{aligned}\tag{5.32}$$

uniformly in $\zeta\in\Omega$ and $u\in[0,R_\kappa]$. The main steps are the same as in the proof of Lemma 5.1. Instead of (5.19), we obtain

$$\begin{aligned}(5.32)\leq\exp\Bigg[\frac{C_T\alpha^2}{\kappa^2}\int_{R_\kappa}^{2R_\kappa}ds_1\int_{s_1}^{2R_\kappa}ds_2\Big(p_{12T1[\kappa]+\frac{s_2-s_1}{\kappa}+\frac{2(s_1-R_\kappa)}{\kappa}}\Big(X_{s_2}^{(k,\kappa)}-X_{s_1}^{(k,\kappa)}\Big)\\ +3p_{12T1[\kappa]+\frac{s_2-s_1}{\kappa}+\frac{2(s_1-u)}{\kappa}}\Big(X_{s_2}^{(k,\kappa)}-X_{s_1}^{(k,\kappa)}\Big)\Big)\Bigg],\end{aligned}\tag{5.33}$$

and this expression may be bounded from above by (5.25). ∎

## 5.2 Spectral bound

The advantage of Lemma 5.4 compared to the original upper bound in Proposition 3.4 is that, modulo a small time correction of the form $(s-u)/\kappa$, the expression under the expectation in (5.28) depends on $X$ only via its occupation time measures on the time intervals $[kR_\kappa,(k+1)R_\kappa]$, $k=1,2,\cdots,\lfloor t/R_\kappa\rfloor$. This will allow us in Section 5.3 to use a large deviation principle for these occupation time measures. The present section consists of five steps, organized in Sections 5.2.1–5.2.5, leading up to a final lemma that will be proved in Section 5.3.

We abbreviate

$$V_{k,u}(\eta)=V_{k,u}^{\kappa,X}(\eta)=\frac{1}{R_\kappa}\int_{kR_\kappa}^{(k+1)R_\kappa}ds\sum_{y\in\mathbb{Z}^3}p_{6T1[\kappa]+\frac{s-u}{\kappa}}(X_s,y)(\eta(y)-\rho)\tag{5.34}$$

and rewrite the expression for $\mathscr{E}_{R,\alpha}^{(4)}(t)$ in (5.28) in the form

$$\mathscr{E}_{R,\alpha}^{(4)}(t)=\mathbb{E}_{\nu_\rho,0}\Bigg(\exp\Bigg[\frac{\alpha}{\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa}du\,V_{k,u}(\xi_{u/\kappa})\Bigg]\Bigg).\tag{5.35}$$

In (5.34) and subsequent expressions we suppress the dependence on $T$ and $R$.

### 5.2.1 Reduction to a spectral bound

Let $B(\Omega)$ denote the Banach space of bounded measurable functions on $\Omega$ equipped with the supremum norm $\|\cdot\|_\infty$. Given $V \in B(\Omega)$, let

$$\lambda(V) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho}\left( \exp\left[ \int_0^t V(\xi_s)\,ds \right]\right) \tag{5.36}$$

denote the associated Lyapunov exponent. The limit in (5.36) exists and coincides with the upper boundary of the spectrum of the self-adjoint operator $L+V$ on $L^2(\nu_\rho)$, written

$$\lambda(V) = \sup \mathrm{Sp}(L+V). \tag{5.37}$$

**Lemma 5.5.** *For all $t>0$ and all bounded and piecewise continuous $V\colon [0,t] \to B(\Omega)$,*

$$\mathbb{E}_{\nu_\rho}\left( \exp\left[ \int_0^t V_u(\xi_u)\,du \right]\right) \le \exp\left[ \int_0^t \lambda(V_s)\,ds \right]. \tag{5.38}$$

*Proof.* In the proof we will assume that $s \mapsto V_s$ is continuous. The extension to piecewise continuous $s \mapsto V_s$ will be straightforward. Let $0 = t_0 < t_1 < \cdots < t_r = t$ be a partition of the interval $[0,t]$. Then

$$\begin{aligned} \int_0^t V_u(\xi_u)\,du &\le \sum_{k=1}^r \int_{t_{k-1}}^{t_k} V_{t_{k-1}}(\xi_s)\,ds + \sum_{k=1}^r \max_{s\in[t_{k-1},t_k]} \|V_s - V_{t_{k-1}}\|_\infty (t_k - t_{k-1}) \\ &\le \sum_{k=1}^r \int_{t_{k-1}}^{t_k} V_{t_{k-1}}(\xi_s)\,ds + t \max_{k=1,\cdots,r} \max_{s\in[t_{k-1},t_k]} \|V_s - V_{t_{k-1}}\|_\infty. \end{aligned} \tag{5.39}$$

Let $(\mathscr{S}_t^V)_{t\ge 0}$ denote the semigroup generated by $L+V$ on $L^2(\nu_\rho)$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Then

$$\left\| \mathscr{S}_t^V \right\| = e^{t\lambda(V)}. \tag{5.40}$$

Using the Markov property, we find that

$$\begin{aligned} \mathbb{E}_{\nu_\rho}\left( \exp\left[ \sum_{k=1}^r \int_{t_{k-1}}^{t_k} V_{t_{k-1}}(\xi_s)\,ds \right]\right) &= \left( \mathscr{S}_{t_1}^{V_{t_0}} \mathscr{S}_{t_2-t_1}^{V_{t_1}} \cdots \mathscr{S}_{t_r-t_{r-1}}^{V_{t_{r-1}}} \mathbb{1}, \mathbb{1} \right) \\ &\le \left\| \mathscr{S}_{t_1}^{V_{t_0}} \right\| \left\| \mathscr{S}_{t_2-t_1}^{V_{t_1}} \right\| \cdots \left\| \mathscr{S}_{t_r-t_{r-1}}^{V_{t_{r-1}}} \right\| \\ &= \exp\left[ \sum_{k=1}^r \lambda(V_{t_{k-1}})(t_k - t_{k-1}) \right]. \end{aligned} \tag{5.41}$$

Combining (5.39) and (5.41), we arrive at

$$\log \mathbb{E}_{\nu_\rho}\left( \int_0^t V_s(\xi_s)\,ds \right) \le \sum_{k=1}^r \lambda(V_{t_{k-1}})(t_k - t_{k-1}) + t \max_{k=1,\cdots,r} \max_{s\in[t_{k-1},t_k]} \left\| V_s - V_{t_{k-1}} \right\|_\infty. \tag{5.42}$$

Since the map $V \mapsto \lambda(V)$ from $B(\Omega)$ to $\mathbb{R}$ is continuous (which can be seen e.g. from (5.40) and the Feynman-Kac representation of $\mathscr{S}_t^V$), the claim follows by letting the mesh of the partition tend to zero. ∎

**Lemma 5.6.** *For all $\alpha, T, R, t, \kappa > 0$,*

$$\mathbb{E}_{\nu_\rho,0}\left(\exp\left[\frac{\alpha}{\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa} du\, V_{k,u}(\xi_{u/\kappa})\right]\right) \leq \mathrm{E}_0^X\left(\exp\left[\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa} du\,\lambda_{k,u}\right]\right) \tag{5.43}$$

*with*

$$\lambda_{k,u} = \lambda_{k,u}^{\kappa,X} = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha}{\kappa}\int_0^t ds\, V_{k,u}^{\kappa,X}(\xi_{s/\kappa})\right]\right), \tag{5.44}$$

*where $u \in [(k-1)R_\kappa, kR_\kappa]$, $k = 1, 2, \cdots, \lfloor t/R_\kappa\rfloor$.*

*Proof.* Apply Lemma 5.5 to the potential $V_u(\eta) = (\alpha/\kappa)V_{k,u}(\eta)$ for $u \in [(k-1)R_\kappa, kR_\kappa]$ with $(\xi_u)_{u\geq 0}$ replaced by $(\xi_{u/\kappa})_{u\geq 0}$, and take the expectation w.r.t. $\mathrm{E}_0^X$. ∎

The spectral bound in Lemma 5.6 enables us to estimate the expression in (5.35) from above by finding upper bounds for the expectation in (5.44) with a *time-independent* potential $V_{k,u}$. This goes as follows. Fix $\kappa$, $X$, $k$ and $u$, and abbreviate

$$\widehat{\phi} = \alpha V_{k,u}^{\kappa,X}. \tag{5.45}$$

Let $(\mathcal{Q}_t)_{t\geq 0}$ be the semigroup generated by $(1/\kappa)L$, and define

$$\widehat{\psi} = \int_0^M dr\,(\mathcal{Q}_r\widehat{\phi}) \tag{5.46}$$

with

$$M = 3K1[\kappa]\kappa^3 \tag{5.47}$$

for a large constant $K > 0$. Then

$$-\frac{1}{\kappa}L\widehat{\psi} = \widehat{\phi} - \mathcal{Q}_M\widehat{\phi} \tag{5.48}$$

with

$$\begin{aligned}(\mathcal{Q}_r\widehat{\phi})(\eta) &= \frac{\alpha}{R_\kappa}\int_{kR_\kappa}^{(k+1)R_\kappa} ds \sum_{y\in\mathbb{Z}^3} p_{6T1[\kappa]+\frac{s-u+r}{\kappa}}(X_s, y)[\eta(y)-\rho] \\ &= \alpha\sum_{y\in\mathbb{Z}^3}\Xi_r(y)[\eta(y)-\rho]\end{aligned} \tag{5.49}$$

and

$$\Xi_r(x) = \Xi_{k,u,r}^{\kappa,X}(x) = \frac{1}{R_\kappa}\int_{kR_\kappa}^{(k+1)R_\kappa} ds\, p_{6T1[\kappa]+\frac{s-u+r}{\kappa}}(X_s, x). \tag{5.50}$$

As in Section 2, we introduce new probability measures $\mathbb{P}_\eta^{\mathrm{new}}$ by an absolute continuous transformation of the probability measures $\mathbb{P}_\eta$, in the same way as in (2.12–2.13) with $\psi$ and $\mathscr{A}$ replaced by $\widehat{\psi}$ and $(1/\kappa)L$, respectively. Under $\mathbb{P}_\eta^{\mathrm{new}}$, $(\xi_{t/\kappa})_{t\geq 0}$ is a Markov process with generator

$$\frac{1}{\kappa}L^{\mathrm{new}}f = e^{-\frac{1}{\kappa}\widehat{\psi}}\frac{1}{\kappa}L\left(e^{\frac{1}{\kappa}\widehat{\psi}}f\right) - \left(e^{-\frac{1}{\kappa}\widehat{\psi}}\frac{1}{\kappa}Le^{\frac{1}{\kappa}\widehat{\psi}}\right)f. \tag{5.51}$$

Since $\eta \mapsto \widehat{\psi}(\eta)$ is bounded, we have, similarly as in Proposition 2.1 with $q = r = 2$,

$$\lambda_{k,u}^{\kappa,X} \leq \limsup_{t\to\infty} \frac{1}{2t} \log\left(\mathscr{E}_{k,u}^{(5)}(t)\right) + \limsup_{t\to\infty} \frac{1}{2t} \log\left(\mathscr{E}_{k,u}^{(6)}(t)\right) \tag{5.52}$$

with

$$\mathscr{E}_{k,u}^{(5)}(t) = \mathscr{E}_{k,u}^{(5)}(\kappa, X; t) = \mathbb{E}_{\nu_\rho}^{\text{new}}\left(\exp\left[\frac{2}{\kappa}\int_0^t dr \left[\left(e^{-\frac{1}{\kappa}\widehat{\psi}} L e^{\frac{1}{\kappa}\widehat{\psi}}\right) - L\left(\frac{1}{\kappa}\widehat{\psi}\right)\right](\xi_{r/\kappa})\right]\right) \tag{5.53}$$

and

$$\mathscr{E}_{k,u}^{(6)}(t) = \mathscr{E}_{k,u}^{(6)}(\kappa, X; t) = \mathbb{E}_{\nu_\rho}^{\text{new}}\left(\exp\left[\frac{2}{\kappa}\int_0^t dr\, (\mathscr{Q}_M \widehat{\phi})(\xi_{r/\kappa})\right]\right), \tag{5.54}$$

where $\mathbb{E}_{\nu_\rho}^{\text{new}} = \int_\Omega \nu_\rho(d\eta)\mathbb{E}_\eta^{\text{new}}$, and we suppress the dependence on the constants $T$, $K$, $R$.

### 5.2.2 Two further lemmas

For $a, b \in \mathbb{Z}^3$ with $\|a - b\| = 1$, define

$$\mathscr{K}_{k,u}(a,b) = \mathscr{K}_{k,u}^{\kappa,X}(a,b) = e^{2C\alpha/T}\frac{\alpha^2}{3\kappa^3}\int_0^M dr \int_r^M d\widetilde{r}\, [\Xi_r(a) - \Xi_r(b)]\,[\Xi_{\widetilde{r}}(a) - \Xi_{\widetilde{r}}(b)] \tag{5.55}$$

with $\Xi_r$ given by (5.50) and $C$ the constant from Lemma 2.4. Abbreviate

$$\left\|\mathscr{K}_{k,u}\right\|_1 = \sum_{\{a,b\}} \mathscr{K}_{k,u}^{\kappa,X}(a,b). \tag{5.56}$$

**Lemma 5.7.** *For all $\alpha, T, K, R, \kappa, t > 0$, $u \in [(k-1)R_\kappa, kR_\kappa]$, $k = 1, 2, \cdots, \lfloor t/R_\kappa \rfloor$, and all paths $X$,*

$$\mathscr{E}_{k,u}^{(5)}(t) \leq \mathbb{E}_{\nu_\rho}\left(\exp\left[\kappa\left\|\mathscr{K}_{k,u}\right\|_1 \int_0^{t/\kappa} dr\, [\xi_r(e_1) - \xi_r(0)]^2\right]\right) \tag{5.57}$$

*with*

$$\left\|\mathscr{K}_{k,u}\right\|_1 \leq e^{2C\alpha/T}\frac{2\alpha^2}{\kappa^2 R_\kappa^2}\int_{kR_\kappa}^{(k+1)R_\kappa} ds \int_{kR_\kappa}^{(k+1)R_\kappa} d\widetilde{s} \int_0^M dr\, p_{12T1[\kappa] + \frac{s+\widetilde{s}-2u+2r}{\kappa}}(X_{\widetilde{s}} - X_s). \tag{5.58}$$

**Lemma 5.8.** *There exists $\kappa_0 > 0$ such that for all $\kappa > \kappa_0$, $K > 1$, $\alpha, T, R, \kappa, t > 0$, $u \in [(k-1)R_\kappa, kR_\kappa]$, $k = 1, 2, \cdots, \lfloor t/R_\kappa \rfloor$, and all paths $X$,*

$$\mathscr{E}_{k,u}^{(6)}(t) \leq \exp\left(\frac{D_{\alpha,T,K}}{\kappa^2}\rho t\right), \tag{5.59}$$

*where the constant $D_{\alpha,T,K}$ does not depend on $R$, $t$, $\kappa$, $u$ or $k$ and satisfies*

$$\lim_{K\to\infty} D_{\alpha,T,K} = 0, \quad \textit{uniformly in } T \geq 1. \tag{5.60}$$

### 5.2.3 Proof of Lemma 5.7

*Proof.* We want to replace $\mathbb{E}^{\text{new}}_{\nu_\rho}$ by $\mathbb{E}_{\nu_\rho}$ in formula (5.53) by applying the analogues of Lemmas 3.1 and 3.2. To this end, we need to compute the constant $K$ in (3.3) for $\psi$ replaced by $\widehat{\psi}$. Recalling (5.46) and (5.49), we have, for $\eta \in \Omega$ and $a, b \in \mathbb{Z}^3$ with $\|a-b\| = 1$,

$$\widehat{\psi}(\eta^{a,b}) - \widehat{\psi}(\eta) = \alpha \int_0^M dr\, \big[\Xi_r(a) - \Xi_r(b)\big]\,[\eta(b) - \eta(a)]. \tag{5.61}$$

Hence,

$$\Big|\widehat{\psi}(\eta^{a,b}) - \widehat{\psi}(\eta)\Big| \le \alpha \int_0^M dr\, \Big|\Xi_r(a) - \Xi_r(b)\Big| \le C\alpha \int_0^\infty dr \left(1 + 6T + \frac{r}{\kappa}\right)^{-2} \le \frac{C\alpha}{T}\kappa. \tag{5.62}$$

Here we have used (5.50) and the right-most inequality in (2.20). This yields

$$\mathscr{E}^{(5)}_{k,u}(t) \le \mathbb{E}_{\nu_\rho}\left( \exp\left[ \frac{2}{\kappa} e^{C\alpha/T} \int_0^t dr \left[ \left( e^{-\frac{1}{\kappa}\widehat{\psi}} L e^{\frac{1}{\kappa}\widehat{\psi}} \right) - L\left(\frac{1}{\kappa}\widehat{\psi}\right) \right] (\xi_{r/\kappa}) \right] \right). \tag{5.63}$$

By (1.4), we have

$$\frac{1}{\kappa}\left[ e^{-\frac{1}{\kappa}\widehat{\psi}} L e^{\frac{1}{\kappa}\widehat{\psi}} - L\left(\frac{1}{\kappa}\widehat{\psi}\right) \right](\eta) = \frac{1}{6\kappa} \sum_{\{a,b\}} \left( e^{\frac{1}{\kappa}[\widehat{\psi}(\eta^{a,b}) - \widehat{\psi}(\eta)]} - 1 - \frac{1}{\kappa}\Big[\widehat{\psi}(\eta^{a,b}) - \widehat{\psi}(\eta)\Big] \right). \tag{5.64}$$

In view of (5.62), a Taylor expansion of the r.h.s. of (5.64) gives

$$\frac{1}{\kappa}\left[ e^{-\frac{1}{\kappa}\widehat{\psi}} L e^{\frac{1}{\kappa}\widehat{\psi}} - L\left(\frac{1}{\kappa}\widehat{\psi}\right) \right](\eta) \le \frac{e^{C\alpha/T}}{12\kappa^3} \sum_{\{a,b\}} \Big( \widehat{\psi}(\eta^{a,b}) - \widehat{\psi}(\eta) \Big)^2. \tag{5.65}$$

Hence, recalling (5.55) and (5.61), we get

$$\begin{aligned} &\mathbb{E}_{\nu_\rho}\left( \exp\left[ \frac{2}{\kappa} e^{C\alpha/T} \int_0^t dr \left[ \left( e^{-\frac{1}{\kappa}\widehat{\psi}} L e^{\frac{1}{\kappa}\widehat{\psi}} \right) - L\left(\frac{1}{\kappa}\widehat{\psi}\right) \right] (\xi_{r/\kappa}) \right] \right) \\ &\qquad \le \mathbb{E}_{\nu_\rho}\left( \exp\left[ \int_0^t dr \sum_{\{a,b\}} \mathscr{K}_{k,u}(a,b) \Big[ \xi_{\frac{r}{\kappa}}(b) - \xi_{\frac{r}{\kappa}}(a) \Big]^2 \right] \right). \end{aligned} \tag{5.66}$$

Using Jensen's inequality w.r.t. the probability kernel $\mathscr{K}_{k,u}/\|\mathscr{K}_{k,u}\|_1$, together with the translation invariance of $\xi$ under $\mathbb{P}_{\nu_\rho}$, we arrive at (5.57). To derive (5.58), observe that for arbitrary $h, \widetilde{h}, r, \widetilde{r} > 0$ and $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} &\sum_{\{a,b\}} \Big[ p_{h+\frac{r}{\kappa}}(x,a) - p_{h+\frac{r}{\kappa}}(x,b) \Big] \Big[ p_{\widetilde{h}+\frac{\widetilde{r}}{\kappa}}(y,a) - p_{\widetilde{h}+\frac{\widetilde{r}}{\kappa}}(y,b) \Big] \\ &\quad = -\sum_{a\in\mathbb{Z}^3} p_{h+\frac{r}{\kappa}}(x,a) \Delta p_{\widetilde{h}+\frac{\widetilde{r}}{\kappa}}(y,a) = -6\kappa \sum_{a\in\mathbb{Z}^3} p_{h+\frac{r}{\kappa}}(x,a) \frac{\partial}{\partial \widetilde{r}} p_{\widetilde{h}+\frac{\widetilde{r}}{\kappa}}(y,a), \end{aligned} \tag{5.67}$$

where $\Delta$ acts on the first spatial variable of $p_t(\cdot,\cdot)$ and $\frac{1}{6}\Delta p_{t/\kappa} = \kappa(\partial/\partial t)p_{t/\kappa}$. Recalling (5.50), it follows that

$$\sum_{\{a,b\}} \big[\Xi_r(a) - \Xi_r(b)\big] \big[\Xi_{\widetilde{r}}(a) - \Xi_{\widetilde{r}}(b)\big] = -6\kappa \sum_{a\in\mathbb{Z}^3} \Xi_r(a) \frac{\partial}{\partial \widetilde{r}} \Xi_{\widetilde{r}}(a) \tag{5.68}$$

and, consequently,

$$\begin{aligned}\left\|\mathscr{K}_{k,u}\right\|_1 &= e^{2C\alpha/T}\frac{2\alpha^2}{\kappa^2}\int_0^M dr \sum_{a\in\mathbb{Z}^3}\Xi_r(a)\left[\Xi_r(a)-\Xi_M(a)\right] \\ &\le e^{2C\alpha/T}\frac{2\alpha^2}{\kappa^2}\int_0^M dr \sum_{a\in\mathbb{Z}^3}\Xi_r(a)^2.\end{aligned} \tag{5.69}$$

Hence, taking into account (5.50), we arrive at (5.58). ∎

### 5.2.4 Proof of Lemma 5.8

*Proof.* Using the same arguments as in (5.62–5.63), we can replace $\mathbb{E}^{\text{new}}_{\nu_\rho}$ by $\mathbb{E}_{\nu_\rho}$ in formula (5.54), to obtain

$$\mathscr{E}^{(6)}_{k,u}(t) \le \mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{2}{\kappa}e^{C\alpha/T}\int_0^t dr\,\left(\mathscr{Q}_M\widehat{\phi}\,\right)\left(\xi_{r/\kappa}\right)\right]\right). \tag{5.70}$$

Because of (5.49), this yields

$$\exp\left[\frac{2\alpha}{\kappa}e^{C\alpha/T}\rho t\right]\mathscr{E}^{(6)}_{k,u}(t) \le \mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{2\alpha}{\kappa}e^{C\alpha/T}\int_0^t dr \sum_{y\in\mathbb{Z}^3}\Xi_M(y)\xi_{r/\kappa}(y)\right]\right). \tag{5.71}$$

Now, using the independent random walk approximation $\widetilde{\xi}$ of $\xi$ (see [3], Proposition 1.2.1), we find that

$$\begin{aligned}&\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{2\alpha}{\kappa}e^{C\alpha/T}\int_0^t dr \sum_{y\in\mathbb{Z}^3}\Xi_M(y)\xi_{r/\kappa}(y)\right]\right) \\ &\qquad \le \int \nu_\rho(d\eta)\prod_{x\in A_\eta}\mathrm{E}^Y_x\left(\exp\left[\frac{2\alpha}{\kappa}e^{C\alpha/T}\int_0^t dr\,\Xi_M\left(Y_{r/\kappa}\right)\right]\right),\end{aligned} \tag{5.72}$$

where $A_\eta$ is given by (5.12) and $Y$ is simple random walk with step rate 1. Define

$$v(x,t) = \mathrm{E}^Y_x\left(\exp\left[\frac{2\alpha}{\kappa}e^{C\alpha/T}\int_0^t dr\,\Xi_M\left(Y_{r/\kappa}\right)\right]\right), \quad (x,t)\in\mathbb{Z}^3\times[0,\infty), \tag{5.73}$$

and write

$$w(x,t) = v(x,t)-1. \tag{5.74}$$

Then we may bound (5.71) from above as follows:

$$\begin{aligned}\text{r.h.s. (5.71)} &\le \int \nu_\rho(d\eta)\prod_{x\in\mathbb{Z}^3}\left[1+\eta(x)w(x,t)\right] \\ &= \prod_{x\in\mathbb{Z}^3}\left[1+\rho w(x,t)\right] \\ &\le \exp\left(\rho\sum_{x\in\mathbb{Z}^3}w(x,t)\right).\end{aligned} \tag{5.75}$$

By the Feynman-Kac formula, $w$ is the solution of the Cauchy problem

$$\frac{\partial}{\partial t}w(x,t)=\frac{1}{6\kappa}\Delta w(x,t)+\frac{2\alpha}{\kappa}e^{C\alpha/T}\Xi_M(x)\big[1+w(x,t)\big],\quad w(\cdot,0)\equiv 0. \tag{5.76}$$

Therefore

$$\frac{\partial}{\partial r}\sum_{x\in\mathbb{Z}^3}w(x,r)=\frac{2\alpha}{\kappa}e^{C\alpha/T}\sum_{x\in\mathbb{Z}^3}\Xi_M(x)\big[1+w(x,r)\big]. \tag{5.77}$$

Integrating (5.77) w.r.t. $r$ over the time interval $[0,t]$ and substituting the resulting expression into (5.75), we get

$$\text{r.h.s. (5.71)}\le\exp\left[\frac{2\alpha}{\kappa}e^{C\alpha/T}\rho\int_0^t dr\sum_{x\in\mathbb{Z}^3}\Xi_M(x)\big(1+w(x,r)\big)\right]. \tag{5.78}$$

Since $\sum_{x\in\mathbb{Z}^3}\Xi_M(x)=1$, this leads to

$$\mathscr{E}^{(6)}_{k,u}(t)\le\exp\left[\frac{2\alpha}{\kappa}e^{C\alpha/T}\rho\int_0^t dr\sum_{x\in\mathbb{Z}^3}\Xi_M(x)w(x,r)\right]. \tag{5.79}$$

An application of Lemma 2.6 to the expectation in the r.h.s. of (5.73) gives

$$v(x,t)\le\Big(1-2\alpha e^{C\alpha/T}\big\|\mathscr{G}\Xi_M\big\|_\infty\Big)^{-1}. \tag{5.80}$$

Next, using (5.47) and (5.50), we find that

$$\big\|\mathscr{G}\Xi_M\big\|_\infty\le G_{6T+M/\kappa}(0)\le G_{3K\kappa^2}(0), \tag{5.81}$$

where the r.h.s. tends to zero as $\kappa\to\infty$. Thus, if $K>1$ and $\kappa>\kappa_0$ with $\kappa_0$ large enough (not depending on the other parameters), then $v(x,t)\le 2$, and hence $w(x,t)\le 1$, for all $x\in\mathbb{Z}^3$ and $t\ge 0$, so that (5.76) implies that $w\le\widehat{w}$, where $\widehat{w}$ solves

$$\frac{\partial}{\partial t}\widehat{w}(x,t)=\frac{1}{6\kappa}\Delta\widehat{w}(x,t)+\frac{4\alpha}{\kappa}e^{C\alpha/T}\Xi_M(x),\qquad \widehat{w}(\cdot,0)\equiv 0. \tag{5.82}$$

The solution of this Cauchy problem has the representation

$$\widehat{w}(x,t)=\frac{4\alpha}{\kappa}e^{C\alpha/T}\int_0^t dr\sum_{y\in\mathbb{Z}^3}p_{\frac{r}{\kappa}}(x,y)\Xi_M(y)=\frac{4\alpha}{\kappa}e^{C\alpha/T}\int_0^t dr\,\Xi_{M+r}(x). \tag{5.83}$$

Hence

$$\begin{aligned}\sum_{x\in\mathbb{Z}^3}\Xi_M(x)w(x,r)&\le\frac{4\alpha}{\kappa}e^{C\alpha/T}\int_0^r d\widetilde{r}\sum_{x\in\mathbb{Z}^3}\Xi_M(x)\Xi_{M+\widetilde{r}}(x)\\&\le\frac{4\alpha}{\kappa}e^{C\alpha/T}\frac{1}{R_\kappa^2}\int_{kR_\kappa}^{(k+1)R_\kappa}ds\int_{kR_\kappa}^{(k+1)R_\kappa}d\widetilde{s}\int_0^r d\widetilde{r}\,p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2M+\widetilde{r}}{\kappa}}(0)\\&\le\frac{4\alpha}{\kappa}e^{C\alpha/T}\int_0^\infty d\widetilde{r}\,p_{\frac{2M+\widetilde{r}}{\kappa}}(0)\\&\le\frac{4C\alpha}{\sqrt{K}\kappa}e^{C\alpha/T},\end{aligned} \tag{5.84}$$

where we again use the second inequality of Lemma 2.4. Substituting (5.84) into (5.79), we arrive at the claim with $D_{\alpha,T,K}=8\alpha^2Ce^{2C\alpha/T}/\sqrt{K}$. ∎

### 5.2.5 Further reduction of Lemma 5.4

To further estimate the expectation in Lemma 5.7 from above, we use the following two lemmas.

**Lemma 5.9.** *Let*

$$\Gamma(\beta) = \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{\nu_\rho}\left( \exp\left[ \beta \int_0^t du\, \left[ \xi_u(e_1) - \xi_u(0) \right]^2 \right] \right). \tag{5.85}$$

*Then*

$$\lim_{\beta\to 0} \frac{\Gamma(\beta)}{\beta} = 2\rho(1-\rho). \tag{5.86}$$

*Proof*. The proof is a straightforward adaptation of what is done in Gärtner, den Hollander and Maillard [3], Lemmas 4.6.8 and 4.6.10. ∎

**Lemma 5.10.** *For all $\alpha, T, K, R, \kappa > 0$, $u \in [(k-1)R_\kappa, kR_\kappa]$, $k = 1, 2, \cdots, \lfloor t/R_\kappa \rfloor$, and all paths $X$,*

$$\limsup_{t\to\infty} \frac{1}{2t} \log \mathscr{E}^{(5)}_{k,u}(t) \le \vartheta_{\alpha,T}\, \rho(1-\rho) \left\| \mathscr{K}_{k,u} \right\|_1, \tag{5.87}$$

*where $\vartheta_{\alpha,T}$ does not depend on $K, R, \kappa, u, k$ or $X$, and $\vartheta_{\alpha,T} \to 1$ as $T \to \infty$.*

*Proof*. Using the bound in (5.58) for $\|\mathscr{K}_{k,u}\|_1$, we find that

$$\kappa \left\| \mathscr{K}_{k,u} \right\|_1 \le e^{2C\alpha/T} 2\alpha^2 \int_0^\infty dr\, p_{12T+2r}(0) \le \frac{C\alpha^2 e^{2C\alpha/T}}{\sqrt{T}}, \tag{5.88}$$

which tends to zero as $T \to \infty$. Hence, we may apply Lemma 5.9 to (5.57) to get the claim. ∎

At this point we may combine Lemmas 5.10 and 5.8 with (5.52), to get

$$\lambda^{\kappa,X}_{k,u} \le \vartheta_{\alpha,T}\, \rho(1-\rho) \left\| \mathscr{K}_{k,u} \right\|_1 + \frac{D_{\alpha,T,K}}{2\kappa^2} \rho. \tag{5.89}$$

Note that the upper bound in (5.58) for $\|\mathscr{K}_{k,u}\|_1$ depends on $X$ only via its increments on the times interval $[(k-1)R_\kappa, kR_\kappa]$ and that these increments are i.i.d. for $k = 1, 2, \cdots, \lfloor t/R_\kappa \rfloor$. Hence, combining (5.35) and Lemma 5.6 with (5.89) and splitting the resulting expectation w.r.t. $\mathrm{E}^X_0$ into $\lfloor t/R_\kappa \rfloor$ equal factors with the help of the Markov property at times $kR_\kappa$, $k = 1, 2, \cdots, \lfloor t/R_\kappa \rfloor$, we obtain, after also substituting (5.58),

$$\limsup_{t\to\infty} \frac{\kappa^2}{t} \log \mathscr{E}^{(4)}_{R,\alpha}(t) \le \frac{1}{R} \log \mathscr{E}^{(7)}_{R,\alpha}(\kappa) + \frac{D_{\alpha,T,K}}{2} \rho \tag{5.90}$$

with

$$\begin{aligned} &\mathscr{E}^{(7)}_{R,\alpha}(\kappa) = \mathscr{E}^{(7)}_{R,\alpha}(T, K; \kappa) \\ &= \mathrm{E}^X_0\left( \exp\left[ \frac{\Theta_{\alpha,T,\rho}}{\kappa^2} \frac{1}{R_\kappa^2} \int_0^{R_\kappa} ds \int_s^{R_\kappa} d\widetilde{s} \int_{-R_\kappa}^0 du \int_0^M dr\, p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2r}{\kappa}}\left( X_{\widetilde{s}} - X_s \right) \right] \right), \end{aligned} \tag{5.91}$$

where

$$\Theta_{\alpha,T,\rho} = 4\vartheta_{\alpha,T}\, \alpha^2 e^{2C\alpha/T} \rho(1-\rho) \to 4\alpha^2 \rho(1-\rho) \quad \text{as } T \to \infty. \tag{5.92}$$

Because of (5.60), we therefore conclude that the proof of Lemma 5.4 reduces to the following lemma.

**Lemma 5.11.** *For all $\alpha, K > 0$,*

$$\limsup_{\kappa,T,R\to\infty} \frac{1}{R} \log \mathscr{E}^{(7)}_{R,\alpha}(T,K;\kappa) \le \left[6\alpha^2\rho(1-\rho)\right]^2 \mathscr{P}_3. \tag{5.93}$$

## 5.3 Small-time cut out, scaling and large deviations

### 5.3.1 Small-time cut out

The proof of Lemma 5.11 will be reduced to two further lemmas in which we cut out small times. These lemmas will be proved in Sections 5.3.2–5.3.3.

For $\epsilon > 0$ small, let

$$m = 3\epsilon\kappa^3 1[\kappa] \tag{5.94}$$

and define

$$\begin{aligned} &\mathscr{E}^{(8)}_{R,\alpha}(\kappa) = \mathscr{E}^{(8)}_{R,\alpha}(T,\epsilon;\kappa) \\ &= \mathrm{E}^X_0\left(\exp\left[\frac{\Theta_{\alpha,T,\rho}}{\kappa^2 R^2_\kappa}\int_0^{R_\kappa} ds \int_s^{R_\kappa} d\widetilde{s} \int_{-R_\kappa}^0 du \int_0^m dr\, p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2r}{\kappa}}\left(X_{\widetilde{s}} - X_s\right)\right]\right) \end{aligned} \tag{5.95}$$

and

$$\begin{aligned} &\mathscr{E}^{(9)}_{R,\alpha}(\kappa) = \mathscr{E}^{(9)}_{R,\alpha}(T,\epsilon,K;\kappa) \\ &= \mathrm{E}^X_0\left(\exp\left[\frac{\Theta_{\alpha,T,\rho}}{\kappa^2 R^2_\kappa}\int_0^{R_\kappa} ds \int_s^{R_\kappa} d\widetilde{s} \int_{-R_\kappa}^0 du \int_m^M dr\, p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2r}{\kappa}}\left(X_{\widetilde{s}} - X_s\right)\right]\right). \end{aligned} \tag{5.96}$$

By Hölder's inequality with weights $p,q > 1$, $1/p + 1/q = 1$, we have

$$\mathscr{E}^{(7)}_{R,\alpha}(\kappa) = \left(\mathscr{E}^{(8)}_{R,\sqrt{q}\alpha}(\kappa)\right)^{1/q}\left(\mathscr{E}^{(9)}_{R,\sqrt{p}\alpha}(\kappa)\right)^{1/p}. \tag{5.97}$$

Hence, by choosing $p$ close to 1, we see that the proof of Lemma 5.11 reduces to the following lemmas.

**Lemma 5.12.** *For all $\alpha > 0$ and $\epsilon > 0$ small enough,*

$$\limsup_{\kappa,T,R\to\infty} \frac{1}{R} \log \mathscr{E}^{(8)}_{R,\alpha}(T,\epsilon;\kappa) = 0. \tag{5.98}$$

**Lemma 5.13.** *For all $\alpha, \epsilon, K > 0$ with $0 < \epsilon < K$,*

$$\limsup_{\kappa,T,R\to\infty} \frac{1}{R} \log \mathscr{E}^{(9)}_{R,\alpha}(T,\epsilon,K;\kappa) \le \left[6\alpha^2\rho(1-\rho)\right]^2 \mathscr{P}_3. \tag{5.99}$$

Note that in $\mathscr{E}^{(8)}_{R,\alpha}(\kappa)$ we integrate the transition kernel over "small" times $r \in [0,m]$. What Lemma 5.12 shows is that the integral is asymptotically negligible.

### 5.3.2 Proof of Lemma 5.12

*Proof.* We need only prove the upper bound in (5.98). An application of Jensen's inequality yields

$$\mathscr{E}^{(8)}_{R,\alpha}(\kappa) \le \frac{1}{R_\kappa}\int_0^{R_\kappa} ds\, \mathrm{E}_0^X\left(\exp\left[\frac{\Theta_{\alpha,T,\rho}}{\kappa^2 R_\kappa}\int_0^\infty d\widetilde{s}\int_{-R_\kappa}^0 du\int_0^m dr\, p_{12T1[\kappa]+2\frac{s-u+r}{\kappa}+\frac{\widetilde{s}}{\kappa}}(X_{\widetilde{s}})\right]\right). \tag{5.100}$$

Observe that

$$p_{12T1[\kappa]+2\frac{s-u+r}{\kappa}+\frac{\widetilde{s}}{\kappa}}(X_{\widetilde{s}}) = \mathrm{E}_0^Y\Big(p_{12T1[\kappa]+2\frac{s-u+r}{\kappa}}(X_{\widetilde{s}}+Y_{\widetilde{s}/\kappa})\Big). \tag{5.101}$$

As in (5.22), let $\widehat{X}_t = X_t + Y_{t/\kappa}$ and let $\mathrm{E}_0^{\widehat{X}}$ denote expectation w.r.t. $\widehat{X}$ starting at 0. Then, using Jensen's inequality w.r.t. $\mathrm{E}_0^Y$, we find that

$$\mathscr{E}^{(8)}_{R,\alpha}(\kappa) \le \frac{1}{R_\kappa}\int_0^{R_\kappa} ds\, \mathrm{E}_0^{\widehat{X}}\left(\exp\left[\frac{\Theta_{\alpha,T,\rho}}{\kappa^2 R_\kappa}\int_0^\infty d\widetilde{s}\int_{-R_\kappa}^0 du\int_0^m dr\, p_{12T1[\kappa]+2\frac{s-u+r}{\kappa}}\left(\widehat{X}_{\widetilde{s}}\right)\right]\right). \tag{5.102}$$

For the potential

$$V_s^\kappa(x) = \frac{1}{\kappa^2 R_\kappa}\int_{-R_\kappa}^0 du\int_0^m dr\, p_{12T1[\kappa]+2\frac{s-u+r}{\kappa}}(x), \tag{5.103}$$

we obtain

$$\left\|\widehat{\mathscr{G}}V_s^\kappa\right\|_\infty \le \frac{1}{\kappa^2}\int_0^m dr\, \widehat{G}_{2T+\frac{r}{3\kappa 1[\kappa]}}(0) \le \frac{3}{\kappa}1[\kappa]\int_0^{\epsilon\kappa^2} dr\, \widehat{G}_r(0) \le C\sqrt{\epsilon}, \tag{5.104}$$

where $\widehat{\mathscr{G}}$ and $\widehat{G}$ are the Green operator, respectively, the Green function corresponding to $1[\kappa]\Delta$. Hence, an application of Lemma 2.6 to (5.102) yields

$$\mathscr{E}^{(8)}_{R,\alpha}(\kappa) \le \left(1 - C\Theta_{\alpha,T,\rho}\sqrt{\epsilon}\right)^{-1}, \tag{5.105}$$

which, together with (5.92), leads to the claim for $0 < \epsilon < (4C\rho(1-\rho)\alpha^2)^{-2}$. ∎

For further comments on Lemma 5.12, see the remark at the end of Section 5.3.3.

### 5.3.3 Scaling, compactification and large deviations

In this section we prove Lemma 5.13 with the help of scaling, compactification and large deviations.

*Proof.* Recalling the definition of $m$ in (5.94) and $M$ in (5.47), we obtain from (5.96), after appropriate time scaling ($s \to \kappa^2 s$, $\widetilde{s} \to \kappa^2\widetilde{s}$, $u \to \kappa^2 u$ and $r \to 3\kappa^3 1[\kappa] r$),

$$\begin{aligned}&\mathscr{E}^{(9)}_{R,\alpha}(\kappa)\\ &= \mathrm{E}_0^X\left(\exp\left[3\Theta_{\alpha,T,\rho}1[\kappa]\frac{1}{R^2}\int_0^R ds\int_s^R d\widetilde{s}\int_{-R}^0 du\int_\epsilon^K dr\, p^{(\kappa)}_{\frac{2T1[\kappa]}{\kappa^2}+\frac{s+\widetilde{s}-2u}{6\kappa}+1[\kappa]r}\left(X_s^{(\kappa)}, X_{\widetilde{s}}^{(\kappa)}\right)\right]\right)\end{aligned} \tag{5.106}$$

with the rescaled transition kernel

$$p_t^{(\kappa)}(x,y) = \kappa^3 p_{6\kappa^2 t}(\kappa x, \kappa y), \quad x,y \in \mathbb{Z}^3_\kappa = \frac{1}{\kappa}\mathbb{Z}^3, \tag{5.107}$$

and the rescaled random walk

$$X_t^{(\kappa)} = \kappa^{-1} X_{\kappa^2 t}, \quad t \in [0,\infty). \tag{5.108}$$

Let $Q$ be a large centered cube in $\mathbb{R}^3$, viewed as a torus, and let $Q^{(\kappa)} = Q \cap \mathbb{Z}_\kappa^3$. Let $l(Q)$, $l(Q^{(\kappa)})$ denote the side lengths of $Q$ and $Q^{(\kappa)}$, respectively. Define the periodized objects

$$p_t^{(\kappa,Q)}(x,y) = \sum_{k\in\mathbb{Z}^3} p_t^{(\kappa)}\Big(x, y + \frac{k}{\kappa}\, l\big(Q^{(\kappa)}\big)\Big) \tag{5.109}$$

and

$$X_t^{(\kappa,Q)} = X_t^{(\kappa)} \mod\big(Q^{(\kappa)}\big). \tag{5.110}$$

Clearly,

$$p_t^{(\kappa)}\Big(X_s^{(\kappa)}, X_{\widetilde{s}}^{(\kappa)}\Big) \le p_t^{(\kappa,Q)}\Big(X_s^{(\kappa,Q)}, X_{\widetilde{s}}^{(\kappa,Q)}\Big). \tag{5.111}$$

Let $\beta = (\beta_t)_{t\ge 0}$ be Brownian motion on the torus $Q$ with generator $\Delta_{\mathbb{R}^3}$ and transition kernel

$$p_t^{(G,Q)}(x,y) = \sum_{k\in\mathbb{Z}^3} p_t^{(G)}\Big(x, y + k\, l(Q)\Big) \tag{5.112}$$

obtained by periodization of the Gaussian kernel $p_t^{(G)}(x,y)$ defined in (4.19). Fix $\theta > 1$ (arbitrarily close to 1). Then there exists $\kappa_0 = \kappa_0(\theta;\epsilon,K,Q) > 0$ such that

$$p_t^{(\kappa,Q)}(x,y) \le \theta p_t^{(G,Q)}(x,y), \quad \text{for all } \kappa > \kappa_0 \text{ and } (t,x,y) \in [\epsilon/2, 2K] \times Q \times Q. \tag{5.113}$$

Hence, it follows from (5.106) that there exists $\kappa_1 = \kappa_1(\theta;T,\epsilon,K,R,Q) > 0$ such that

$$\mathscr{E}_{R,\alpha}^{(9)}(\kappa) \le \mathrm{E}_0^X\left( \exp\left[ \frac{3}{2}\theta^2 \Theta_{\alpha,T,\rho} \frac{1}{R}\int_0^R ds \int_0^R d\widetilde{s} \int_\epsilon^K dr\, p_r^{(G,Q)}\Big(X_s^{(\kappa,Q)}, X_{\widetilde{s}}^{(\kappa,Q)}\Big)\right]\right). \tag{5.114}$$

Applying Donsker's invariance principle and recalling (5.92), we find that

$$\begin{aligned} &\limsup_{\kappa,T\to\infty} \frac{1}{R}\log \mathscr{E}_{R,\alpha}^{(9)}(\kappa) \\ &\quad \le \frac{1}{R}\log \mathrm{E}_0^\beta\left( \exp\left[ 6\theta^2\alpha^2\rho(1-\rho)\frac{1}{R}\int_0^R ds\int_0^R d\widetilde{s}\int_\epsilon^K dr\, p_r^{(G,Q)}(\beta_s, \beta_{\widetilde{s}})\right]\right). \end{aligned} \tag{5.115}$$

Applying the large deviation principle for the occupation time measures of $\beta$, we get

$$\limsup_{\kappa,T,R\to\infty} \frac{1}{R}\log \mathscr{E}_{R,\alpha}^{(9)}(T,\epsilon;\kappa) \le \mathscr{P}_3^{(Q)}(\theta;\epsilon,K), \tag{5.116}$$

where

$$\mathscr{P}_3^{(Q)}(\theta;\epsilon,K) = \sup_{\nu\in\mathscr{M}_1(Q)} \left[ 6\theta^2\alpha^2\rho(1-\rho)\int_Q \nu(dx)\int_Q \nu(dy)\int_\epsilon^K dr\, p_r^{(G,Q)}(x,y) - S^Q(\nu)\right] \tag{5.117}$$

with large deviation rate function $S^Q\colon \mathscr{M}_1(Q) \to [0,\infty]$ defined by

$$S^Q(\mu) = \begin{cases} \|\nabla_{\mathbb{R}^3} f\|_2^2 & \text{if } \mu \ll dx \text{ and } \sqrt{\frac{d\mu}{dx}} = f(x) \text{ with } f \in H^1_{\mathrm{per}}(Q), \\ \infty & \text{otherwise,} \end{cases} \tag{5.118}$$

where $\mathcal{M}_1(Q)$ is the space of probability measures on $Q$, and $H^1_{\rm per}(Q)$ denotes the space of functions in $H^1(Q)$ with periodic boundary conditions. By [2], Lemma 7.4, we have

$$\limsup_{Q\uparrow\mathbb{R}^3} \mathscr{P}_3^{(Q)}(\theta;\epsilon,K) \le \mathscr{P}_3(\theta;\epsilon,K) \tag{5.119}$$

with

$$\begin{aligned}
&\mathscr{P}_3(\theta;\epsilon,K)\\
&= \sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}} \left[6\,\theta^2\alpha^2\rho(1-\rho)\int_{\mathbb{R}^3} dx\, f^2(x)\int_{\mathbb{R}^3} dy\, f^2(y)\int_\epsilon^K dr\, p_r^{(G)}(x,y) - \left\|\nabla_{\mathbb{R}^3} f\right\|^2_{L^2(\mathbb{R}^3)}\right]\\
&\le \sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}} \left[6\,\theta^2\alpha^2\rho(1-\rho)\int_{\mathbb{R}^3} dx\, f^2(x)\int_{\mathbb{R}^3} dy\, f^2(y)\int_0^\infty dr\, p_r^{(G)}(x,y) - \left\|\nabla_{\mathbb{R}^3} f\right\|^2_{L^2(\mathbb{R}^3)}\right]\\
&= \left[6\,\theta^2\alpha^2\rho(1-\rho)\right]^2\mathscr{P}_3.
\end{aligned} \tag{5.120}$$

Combining (5.116) and (5.120), and letting $\theta\downarrow 1$, we arrive at the claim of Lemma 5.13. ∎

This, after a long struggle by the authors and considerable patience on the side of the reader, completes the proof of the upper bound in Proposition 3.4.

*Remark.* The reader might be surprised that the expression in the l.h.s. of (5.98) does not only vanish in the limit as $\epsilon\downarrow 0$ but vanishes for *all* $\epsilon>0$ sufficiently small. This fact is closely related to the observation that

$$\mathscr{P}_3(\pi^3)=0 \quad \text{whereas} \quad \mathscr{P}_3(\infty)=\mathscr{P}_3>0 \tag{5.121}$$

with

$$\mathscr{P}_3(\epsilon) = \sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}} \left[\int_{\mathbb{R}^3} dx\, f^2(x)\int_{\mathbb{R}^3} dy\, f^2(y)\int_0^\epsilon dr\, p_r^{(G)}(x-y) - \left\|\nabla_{\mathbb{R}^3} f\right\|_2^2\right]. \tag{5.122}$$

Indeed, given a potential $V\ge 0$ with $\|\mathscr{G}_{\mathbb{R}^3}V\|_\infty<1/2$, where $\mathscr{G}_{\mathbb{R}^3}$ denotes the Green operator associated with $\Delta_{\mathbb{R}^3}$, the method used in the proof of Lemma 5.12 leads to

$$\lim_{R\to\infty}\frac{1}{R}\log \mathrm{E}_0^\beta\left(\exp\left[\frac{1}{R}\int_0^R ds\int_0^R d\widetilde{s}\, V(\beta_{\widetilde{s}}-\beta_s)\right]\right)=0. \tag{5.123}$$

On the other hand, the large deviation principle for the occupation time measures of $\beta$ shows that this limit coincides with

$$\sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}} \left[\int_{\mathbb{R}^3} dx\, f^2(x)\int_{\mathbb{R}^3} dy\, f^2(y)\, V(x-y) - \left\|\nabla_{\mathbb{R}^3} f\right\|_2^2\right]. \tag{5.124}$$

But, for $0<\epsilon<\pi^3$ the potential

$$V_\epsilon(x)=\int_0^\epsilon dr\, p_r^{(G)}(x) \tag{5.125}$$

satisfies the assumption $\|\mathscr{G}_{\mathbb{R}^3}V_\epsilon\|_\infty<1/2$, implying $\mathscr{P}_3(\pi^3)=0$.

## 6 Higher moments

In this last section we explain how to extend the proof of Theorem 1.1 to higher moments $p \geq 2$. Sections 6.1–6.3 parallel Sections 2.1, 3.2, 4 and 5.

### 6.1 Two key propositions

Our starting point is the Feynman-Kac representation for the $p$-th moment,

$$\mathbb{E}_{\nu_\rho}\Big(u(0,t)^p\Big) = \mathbb{E}^{(p)}_{\nu_\rho;0}\left(\exp\left[\int_0^t ds \sum_{j=1}^p \xi_s(X^j_{\kappa s})\right]\right), \tag{6.1}$$

where $X^1,\cdots,X^p$ are independent simple random walks on $\mathbb{Z}^3$ starting at 0 and $\mathbb{E}^{(p)}_{\nu_\rho;x}$ denotes expectation w.r.t. $\mathbb{P}^{(p)}_{\nu_\rho;x} = \mathbb{P}_{\nu_\rho} \otimes \mathrm{P}^{X^1}_{x_1} \otimes \cdots \otimes \mathrm{P}^{X^p}_{x_p}$, $x = (x_1,\cdots,x_p) \in (\mathbb{Z}^3)^p$.

The arguments in Sections 2 and 3 easily extend to this more general case by replacing $Z$, $\mathscr{A}$, $(\mathscr{S}_t)_{t\geq 0}$, $\phi$ and $\psi$ by their $p$-dimensional analogues $Z^{(p)}$, $\mathscr{A}^{(p)}$, $(\mathscr{S}^{(p)}_t)_{t\geq 0}$, $\phi^{(p)}$ and $\psi^{(p)}$. To be precise, consider the Markov process

$$Z^{(p)}_t = \big(\xi_{t/\kappa}, X^1_t, \cdots, X^p_t\big) \quad \text{on } \Omega \times (\mathbb{Z}^3)^p \tag{6.2}$$

with generator

$$\mathscr{A}^{(p)} = \frac{1}{\kappa} L + \sum_{j=1}^p \Delta_j, \tag{6.3}$$

where the lattice Laplacian $\Delta_j$ acts on the $j$-th spatial variable. Denote by $(\mathscr{S}^{(p)}_t)_{t\geq 0}$ the associated semigroup. We define

$$\phi^{(p)}(\eta; x_1,\cdots,x_p) = \sum_{j=1}^p \phi(\eta, x_j) = \sum_{j=1}^p (\eta(x_j) - \rho) \tag{6.4}$$

and

$$\psi^{(p)} = \int_0^T ds\, \mathscr{S}^{(p)}_s \phi^{(p)}. \tag{6.5}$$

Then

$$\psi^{(p)}(\eta; x_1,\cdots,x_p) = \sum_{j=1}^p \psi(\eta, x_j). \tag{6.6}$$

In this way the proof of Theorem 1.1 for $p \geq 2$ reduces to the proof of the following extension of Propositions 3.3 and 3.4.

**Proposition 6.1.** *For all $p \in \mathbb{N}$ and $\alpha \in \mathbb{R}$,*

$$\begin{aligned}\limsup_{t,\kappa,T\to\infty} \frac{\kappa^2}{pt} \log \mathbb{E}^{(p)}_{\nu_\rho;0}\left(\exp\left[\alpha \int_0^t ds \left[\left(e^{-\frac{1}{\kappa}\psi^{(p)}} \mathscr{A}^{(p)} e^{\frac{1}{\kappa}\psi^{(p)}}\right) - \mathscr{A}^{(p)}\left(\frac{1}{\kappa}\psi^{(p)}\right)\right](Z^{(p)}_s)\right]\right) \\ \leq \frac{\alpha}{6}\rho(1-\rho)G.\end{aligned} \tag{6.7}$$

**Proposition 6.2.** *For all $p \in \mathbb{N}$ and $\alpha > 0$,*

$$\lim_{t,\kappa,T\to\infty} \frac{\kappa^2}{pt} \log \mathbb{E}^{(p)}_{\nu_\rho;0}\left( \exp\left[ \frac{\alpha}{\kappa} \int_0^t ds \left( \mathscr{S}_T^{(p)} \phi^{(p)} \right)(Z_s^{(p)}) \right] \right) = \left[ 6\alpha^2 \rho(1-\rho) p \right]^2 \mathscr{P}_3. \tag{6.8}$$

Proposition 6.1 has already been proven for all $p \in \mathbb{N}$ in [3], Proposition 4.4.2 and Section 4.8.

## 6.2 Lower bound in Proposition 6.2

We use the following analogue of the variational representation (4.1)

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{(p)}_{\nu_\rho;0}\left( \exp\left[ \frac{\alpha}{\kappa} \int_0^t ds \left( \mathscr{S}_T^{(p)} \phi^{(p)} \right)(Z_s^{(p)}) \right] \right) \\ &\quad = \sup_{\substack{F^{(p)} \in \mathscr{D}(\mathscr{A}^{(p)}) \\ \|F^{(p)}\|_{L^2(\mu_\rho^p)}=1}} \iint_{\Omega\times(\mathbb{Z}^3)^p} d\mu_\rho^p \left[ \frac{\alpha}{\kappa} \left( \mathscr{S}_T^{(p)} \phi^{(p)} \right) (F^{(p)})^2 + F^{(p)} \mathscr{A}^{(p)} F^{(p)} \right]. \end{aligned} \tag{6.9}$$

To obtain the appropriate lower bound, we use test functions $F^{(p)}$ of the form

$$F^{(p)}(\eta; x_1, \cdots, x_p) = F_1(\eta) F_2(x_1) \cdots F_2(x_p), \tag{6.10}$$

where $F_1$, $F_2$ and $F = F^{(1)}$ are the same as in (4.15), (4.6) and (4.2), respectively. One easily checks that

$$\begin{aligned} &\frac{\kappa^2}{p} \iint_{\Omega\times(\mathbb{Z}^3)^p} d\mu_\rho^p \left[ \frac{\alpha}{\kappa} \left( \mathscr{S}_T^{(p)} \phi^{(p)} \right) (F^{(p)})^2 + F^{(p)} \mathscr{A}^{(p)} F^{(p)} \right] \\ &\quad = \frac{(p\kappa)^2}{p^2} \iint_{\Omega\times\mathbb{Z}^3} d\mu_\rho \left[ \frac{p\alpha}{p\kappa} \left( \mathscr{S}_T \phi \right) F^2 + F \left( \frac{1}{p\kappa} L + \Delta \right) F \right]. \end{aligned} \tag{6.11}$$

But this is $1/p^2$ times the expression in Lemma 4.1 with $\alpha$ and $\kappa$ replaced by $p\alpha$ and $p\kappa$, respectively. Hence, we may use the lower bounds for this expression in Section 4 to arrive at the lower bound in Proposition 6.2.

## 6.3 Upper bound in Proposition 6.2

1. Freezing and defreezing can be done in the same way as in Section 5.1, but with $V(\eta, x)$ in (5.2) replaced by

$$V^{(p)}(\eta, x) = \frac{\alpha}{\kappa} \sum_{y\in\mathbb{Z}^3} \left( \sum_{j=1}^p p_{6T1[\kappa]}(x_j, y) \right) (\eta(y) - \rho). \tag{6.12}$$

This leads to the analogues of Lemmas 5.1 and 5.3 along the lines of Sections 5.1.2 and 5.1.4.

2. Considering

$$V_{k,u}^{(p)}(\eta) = \frac{1}{R_\kappa} \int_{kR_\kappa}^{(k+1)R_\kappa} ds \sum_{y\in\mathbb{Z}^3} \left( \sum_{j=1}^p p_{6T1[\kappa]+\frac{s-u}{\kappa}}(X_s^j, y) \right) (\eta(y) - \rho) \tag{6.13}$$

and

$$\mathscr{E}^{(4,p)}_{R,\alpha}(t)=\mathbb{E}_{\nu_\rho;0}\left(\exp\left[\frac{\alpha}{\kappa}\sum_{k=1}^{\lfloor t/R_\kappa\rfloor}\int_{(k-1)R_\kappa}^{kR_\kappa}du\,V^{(p)}_{k,u}(\xi_{u/\kappa})\right]\right) \tag{6.14}$$

instead of (5.34–5.35), the proof reduces to the following analogue of Lemma 5.4.

**Lemma 6.3.** *For each $\alpha>0$,*

$$\limsup_{R\to\infty}\limsup_{t,\kappa,T\to\infty}\frac{\kappa^2}{pt}\log\mathscr{E}^{(4,p)}_{R,\alpha}(t)\le\left[6\alpha^2\rho(1-\rho)p\right]^2\mathscr{P}_3. \tag{6.15}$$

3. The proof of Lemma 6.3 follows the lines of Sections 5.2–5.3. The spectral bound is essentially the same as in Section 5.2.1. In Lemma 5.6 we have to replace $V_{k,u}$ by $V^{(p)}_{k,u}$ and $\lambda_{k,u}$ by

$$\lambda^{(p)}_{k,u}=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_{\nu_\rho}\left(\exp\left[\frac{\alpha}{\kappa}\int_0^t ds\,V^{(p)}_{k,u}(\xi_{s/\kappa})\right]\right). \tag{6.16}$$

Subsequently, we replace $V_{k,u}$ by $V^{(p)}_{k,u}$ in (5.45), to obtain functions $\widehat{\phi}^{(p)}$, $\widehat{\psi}^{(p)}$ replacing (5.45–5.46), and

$$\Xi^{(p)}_r(x)=\frac{1}{R_\kappa}\int_{kR_\kappa}^{(k+1)R_\kappa}ds\sum_{j=1}^{p}p_{6T1[\kappa]+\frac{s-u+r}{\kappa}}\left(X^j_s,x\right) \tag{6.17}$$

replacing (5.50). Then, in the analogue of Lemma 5.7, instead of (5.58) we get the bound

$$\left\|\mathscr{K}^{(p)}_{k,u}\right\|_1\le e^{2C\alpha/T}\frac{2\alpha^2}{\kappa^2R_\kappa^2}\int_{kR_\kappa}^{(k+1)R_\kappa}ds\int_{kR_\kappa}^{(k+1)R_\kappa}d\widetilde{s}\int_0^M dr\sum_{i,j=1}^{p}p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2r}{\kappa}}\left(X^j_{\widetilde{s}}-X^i_s\right) \tag{6.18}$$

along the line of Section 5.2.3. Similarly, the proof of the analogue of Lemma 5.8 follows the argument in Section 5.2.4, leading to a reduction of Lemma 6.3 to the analogue of Lemma 5.11, as in Section 5.2.5.

4. To make the small-time cut-off, instead of (5.95) we consider

$$\begin{aligned}&\mathscr{E}^{(8,p)}_{R,\alpha}(\kappa)\\&=\mathrm{E}^X_0\left(\exp\left[\frac{\Theta_{\alpha,T,\rho}}{\kappa^2R_\kappa^2}\int_0^{R_\kappa}ds\int_s^{R_\kappa}d\widetilde{s}\int_{-R_\kappa}^0du\int_0^m dr\sum_{i,j=1}^{p}p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2r}{\kappa}}\left(X^j_{\widetilde{s}}-X^i_s\right)\right]\right).\end{aligned} \tag{6.19}$$

Using the Chapman-Kolmogorov equation, we see that

$$\begin{aligned}&\int_0^{R_\kappa}ds\int_0^{R_\kappa}d\widetilde{s}\sum_{i,j=1}^{p}p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2r}{\kappa}}\left(X^j_{\widetilde{s}}-X^i_s\right)\\&\quad=\sum_{z\in\mathbb{Z}^3}\left(\sum_{i=1}^{p}\int_0^{R_\kappa}ds\,p_{6T1[\kappa]+\frac{s-u+r}{\kappa}}\left(X^i_s,z\right)\right)^2\\&\quad\le p\sum_{z\in\mathbb{Z}^3}\sum_{i=1}^{p}\left(\int_0^{R_\kappa}ds\,p_{6T1[\kappa]+\frac{s-u+r}{\kappa}}\left(X^i_s,z\right)\right)^2\\&\quad=p\sum_{i=1}^{p}\int_0^{R_\kappa}ds\int_0^{R_\kappa}d\widetilde{s}\,p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2r}{\kappa}}\left(X^i_{\widetilde{s}}-X^i_s\right).\end{aligned} \tag{6.20}$$

Substituting this into the r.h.s. of (6.19) and applying Hölder's inequality for the $p$ exponential factors, we reduce the problem to the consideration of a single random walk and can proceed as in Section 5.3, leading to the analogues of Lemmas 5.12–5.13.

5. The proof of the analogue of Lemma 5.12 runs along the line of Section 5.3.2. To prove the analogue of Lemma 5.13, instead of (5.96) we consider

$$\begin{aligned}&\mathscr{E}_{R,\alpha}^{(9,p)}(\kappa)\\ &= \mathrm{E}_0^X\left(\exp\left[\frac{\Theta_{\alpha,T,\rho}}{\kappa^2 R_\kappa^2}\int_0^{R_\kappa} ds \int_s^{R_\kappa} d\widetilde{s} \int_{-R_\kappa}^0 du \int_m^M dr \sum_{i,j=1}^p p_{12T1[\kappa]+\frac{s+\widetilde{s}-2u+2r}{\kappa}}\left(X_{\widetilde{s}}^j - X_s^i\right)\right]\right).\end{aligned} \tag{6.21}$$

As in Section 5.3.3, this leads to

$$\begin{aligned}&\limsup_{\kappa,T,R\to\infty}\frac{1}{pR}\log\mathscr{E}_{R,\alpha}^{(9,p)}(\kappa)\\ &\le \frac{1}{p}\sup_{\substack{\nu_i\in\mathscr{M}_1(Q)\\ 1\le i\le p}}\left[6\theta^2\alpha^2\rho(1-\rho)\sum_{i,j=1}^p\int_Q \nu_i(dx)\int_Q \nu_j(dy)\int_\epsilon^K dr\, p_r^{(G,Q)}(x,y) - \sum_{i=1}^p S^Q(\nu_i)\right]\end{aligned} \tag{6.22}$$

instead of (5.116–5.117). Now we can proceed similarly as in [2], Lemma 7.3. Consider the Fourier transforms $\widehat{\nu}_i$ of the measures $\nu_i \in \mathscr{M}_1(Q)$ defined by

$$\widehat{\nu}_j(k) = \int_Q e^{-i(2\pi/l(Q))k\cdot x}\nu_j(dx), \quad k\in\mathbb{Z}^3, j=1,\cdots,p. \tag{6.23}$$

The transition kernel $p^{(G,Q)}$ admits the Fourier representation

$$p_r^{(G,Q)}(x) = \frac{1}{l(Q)^3}\sum_{k\in\mathbb{Z}^3} e^{-(2\pi/l(Q))^2|k|^2 r} e^{-i(2\pi/l(Q))k\cdot x}, \quad (x,t)\in Q\times(0,\infty). \tag{6.24}$$

Therefore we may write

$$\int_Q \nu_i(dx)\int_Q \nu_j(dy)\, p_r^{(G,Q)}(x,y) = \frac{1}{l(Q)^3}\sum_{k\in\mathbb{Z}^3} e^{-(2\pi/l(Q))^2|k|^2 r}\widehat{\nu}_i(k)\overline{\widehat{\nu}_j(k)}. \tag{6.25}$$

Using that

$$\mathrm{Re}\left(\widehat{\nu}_i(k)\overline{\widehat{\nu}_j(k)}\right) \le \frac{1}{2}\left|\widehat{\nu}_i(k)\right|^2 + \frac{1}{2}\left|\widehat{\nu}_j(k)\right|^2, \tag{6.26}$$

we obtain

$$\begin{aligned}&\int_Q \nu_i(dx)\int_Q \nu_j(dy)\, p_r^{(G,Q)}(x,y)\\ &\le \frac{1}{2}\int_Q \nu_i(dx)\int_Q \nu_i(dy)\, p_r^{(G,Q)}(x,y) + \frac{1}{2}\int_Q \nu_j(dx)\int_Q \nu_j(dy)\, p_r^{(G,Q)}(x,y).\end{aligned} \tag{6.27}$$

Therefore the term inside the square brackets in the r.h.s. of (6.22) does not exceed

$$\sum_{i=1}^p\left[6\theta^2\alpha^2\rho(1-\rho)p\int_Q \nu_i(dx)\int_Q \nu_i(dy)\int_\epsilon^K dr\, p_r^{(G,Q)}(x,y) - S^Q(\nu_i)\right], \tag{6.28}$$

and we arrive at

$$\limsup_{\kappa,T,R\to\infty} \frac{p}{R} \log \mathscr{E}_{R,\alpha}^{(9,p)}(\kappa) \le \mathscr{P}_3^{(Q,p)}(\theta;\epsilon,K) \tag{6.29}$$

with

$$\mathscr{P}_3^{(Q,p)}(\theta;\epsilon,K) = \sup_{\nu\in\mathscr{M}_1(Q)} \left[ 6\theta^2\alpha^2\rho(1-\rho)p \int_Q \nu(dx) \int_Q \nu(dy) \int_\epsilon^K dr\, p_r^{(G,Q)}(x,y) - S^Q(\nu) \right], \tag{6.30}$$

which is the analogue of (5.116–5.117) for $p \ge 2$. The rest of the proof can be easily obtained from the analogues of (5.119–5.120).

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