Lower estimates for random walks on a class of amenable $p$-adic groups

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Abstract
We give central lower estimates for the transition kernels corresponding to symmetric random walks on certain amenable $p$-adic groups.

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1 Introduction

Let $G$ be a locally compact amenable group. We shall denote by $d^r g$ (resp. $d^l g$) the right (resp. left) Haar measure on $G$ and by $\delta(g) = d^r g / d^l g$ the modular function on $G$ normalized by $\delta(e) = 1$ where $e$ denotes the identity of $G$. Let $d\mu(g) = \varphi(g) d^r g \in P(G)$ be a probability measure on $G$ where $\varphi(g) \in L^\infty(G)$ is assumed to have a compact support or a fast decay at infinity. Let us assume that $\mu$ is symmetric (i.e. the involution $g \mapsto g^{-1}$ stabilizes $\mu$) and consider the random walk on $G$ induced by $\mu$, i.e. the $G$-valued process that evolves as follows: if $X_n = g$ is the position at time $n$ then $X_{n+1} = gh$ where $h$ is chosen according to $\mu$. We shall denote by

$$d\mu^n(g) = \varphi_n(g) d^r g$$

the $n^{th}$ convolution power of $\mu$ and examine the behaviour of the decay of $\varphi_n(e)$ as $n \to \infty$.

If we restrict ourselves to unimodular real Lie groups then the answer lies in the behaviour of the volume growth of $G$. Let us recall that if $G$ is a locally compact group that is generated by some symmetric compact neighbourhood $\Omega \subset G$ of the identity in $G$, the volume growth function is (cf. [9])

$$\gamma(t) = \text{Vol}(B_t(e)), \quad t = 1, 2, ...$$

where the volume is taken with respect to $d^r g$ (or $d^l g$) and where $B_t(e)$ is the ball of radius $t$ centred on $e$ defined by

$$B_t(e) = \Omega \cdot \Omega, \quad t \text{ times.}$$

For $x \in G$ the distance from $e$ is defined by $|x| = \inf\{t, \ x \in B_t(e)\}$ and a left invariant distance can be defined on $G$ by setting $d(x, y) = |y^{-1}x|$, $x, y \in G$. If $\Omega_1, \Omega_2$ are two neighbourhoods of $e$ as above it is not difficult to check that there exists $C > 0$ such that $C^{-1} \leq |r|_2 / |l|_1 \leq C$ and that the corresponding growth functions satisfy the obvious equivalence $\gamma_1 \approx \gamma_2$, i.e.

$$\gamma_1(t) \leq C \gamma_2(Ct) + C \leq C' \gamma_1(C't) + C', \quad t \geq 1.$$  

For real Lie groups we have the following dichotomy (cf. [9], [13]): either

$$\gamma(t) \approx t^D$$

where $D = D(G) = 1, 2, ..., or$

$$\gamma(t) \approx e^t.$$

In the first case we say that $G$ is of polynomial growth and in the second case we say that $G$ is of exponential growth and the answer to our problem in the case of unimodular amenable real Lie groups was given by Varopoulos and is the following

$$\varphi_n(e) \approx n^{-D/2} \iff \gamma(t) \approx t^D$$

$$\varphi_n(e) \approx e^{-n^{1/3}} \iff \gamma(t) \approx e^t$$

(cf. [1], [6], [7], [10], [28]).

Varopoulos showed that the $e^{-n^{1/3}}$ versus polynomial behaviour extends to the non-unimodular amenable case depending on whether the Lie group $G$ is $(C)$ or $(NC)$. This classification introduced
in [24] can be expressed in terms of the roots of the $ad$-action of the radical of the Lie algebra of the group on its nilradical.

The discrete case is more complicated (cf. [1], [11], [16], [17], [22], [23]). First there is no dichotomy in the volume growth (cf. [8]). On the other hand if we suppose that the group $G$ is of exponential growth then one can claim only the upper bound (cf. [11])

$$\varphi_n(e) \leq C \exp(-cn^{1/3}), \quad n \geq 1.$$ 

In general, the matching lower bound fails. Ch. Pittet and L. Saloff-Coste showed (cf. [16]) that there are soluble groups with exponential volume growth for which the heat kernel decays as $\exp(-cn^\alpha)$ with $\alpha \in (0, 1)$ which can be taken arbitrarily close to 1. This, as mentioned above, can not happen in the case of real Lie groups.

In a recent paper (cf. [17]) Ch. Pittet and L. Saloff-Coste established the lower bound $\varphi_{2n}(e) \geq c \exp(-Cn^{1/3})$, $n = 1, 2, \ldots$ for the large times asymptotic behaviours of the probabilities of return to the origin at even times $2n$, for random walks associated with finite symmetric generating sets of solvable groups of finite Prüfer rank. They asked in this paper (cf. [17], §8) if a similar lower bound is available in the case of analytic $p$-adic groups. An answer to this problem was given in [15] (cf. also [14]). The aim of this paper is to show that the $e^{-n^{1/3}}$ lower bound obtained in [14] can be substantially improved for a large class of amenable $p$-adic groups.

## 2 Amenable $p$-adic groups

In this section $G$ will denote an algebraic connected amenable group over $\mathbb{Q}_p$, the field of $p$-adic numbers; $U \subset Q \subset G$ will denote the radical and the unipotent radical (cf. [3], [5]). Amenability of $G$ is equivalent to the fact that the semi-simple group $G/Q$ is compact (cf. [19]). Let $S$ denote a fixed levi subgroup $S$ of $G$ (cf. [3]). The group $G$ can then be written as a semi-direct product:

$$G = Q \ltimes S = (U \ltimes A) \ltimes S \cong U \ltimes (A \times S)$$

where $A$ is abelian and can be identified to the direct product of a finite group and a $d$-torus $T \cong (\mathbb{Q}_p^*)^d$ (cf. [3], [5]). Here $\mathbb{Q}_p^*$ denotes the multiplicative group of the field $\mathbb{Q}_p$. Since $(\mathbb{Q}_p^*)^d \cong \mathbb{Z}^d \times K$ (where $K$ is compact, cf. [2], [4]), by considering the projection

$$\pi : A \longrightarrow \mathbb{Z}^d,$$

we can fix $\pi_1, \ldots, \pi_d \in A$ so that each $z \in A$ admits a unique decomposition

$$z = \pi_1^{n_1} \cdot \pi_d^{n_d}, \quad n_1, \ldots, n_d \in \mathbb{Z}, \quad \tau \in \mathbb{K},$$

where $\mathbb{K}$ denotes a compact subgroup of $A$. Let $\mathcal{U} = \text{Lie}(U)$ denote the Lie algebra of $U$. Let $\mathcal{E}_p$ denotes a finite extension of $\mathbb{Q}_p$ which contains all the eigenvalues defined by

$$\det (\text{Ad}(\pi_j) - \lambda I) = 0, \quad j = 1, \ldots, d.$$ 

The $Ad$-action of $A$ on $\mathcal{U}$ extends to $\mathcal{U} \otimes_{\mathbb{Q}_p} \mathcal{E}_p$, and it follows from the proof of the Zassenhaus lemma (cf. [12]) that we have a decomposition of $\mathcal{U} \otimes_{\mathbb{Q}_p} \mathcal{E}_p$ into a direct sum

$$\mathcal{U} \otimes_{\mathbb{Q}_p} \mathcal{E}_p = W_1 \oplus \ldots \oplus W_r$$

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where the subspaces \( W_j \) (1 \( \leq \) \( j \) \( \leq \) \( r \)) are invariant by \( \text{Ad}(\pi_i) \), \( i = 1, ..., d \), and such that the restriction of \( \text{Ad}(\pi_i) \) to \( W_j \) is the sum of the scalar \( \lambda_j(\pi_i) \) and a nilpotent endomorphism.

Let \( \chi_j : A \rightarrow \overline{Q}_p^* \) defined by

\[
\chi_j(z) = \lambda_j(\pi_{1})^{n_1}...\lambda_j(\pi_{d})^{n_d}, \quad z = \pi_{1}^{n_1}...\pi_{d}^{n_d} \tau, \quad j = 1, ..., r.
\]

Let \( |.| \) denote the standard \( p \)-adic norm. We shall denote by \( |.|' \) its extension to \( \overline{Q}_p \). We have \( |x|' \in \{ p^n \cup \{ 0 \}, x \in \overline{Q}_p \} \), where \( p \) denotes a rational power of the prime \( p \) (cf. [2]). Let \( \alpha_1, \alpha_2, ..., \alpha_s \) denote the different norms of the \( \chi_j \)'s, i.e. the different homomorphisms obtained by considering \( A \rightarrow \overline{p}^\mathbb{Z}, z \rightarrow |\chi_j(z)|_p' \), \( j = 1, ..., r \). Let \( \mathcal{L} = \{ \alpha_1, \alpha_2, ..., \alpha_s \} \). We shall assume that \( \mathcal{L} \) is nonempty and that

\[
1 \not\in \mathcal{L}.
\]

Here 1 denotes the homomorphism \( A \rightarrow \overline{p}^\mathbb{Z} \) identically equal to 1. This assumption guarantees the fact that the group \( G \) is compactly generated (cf. [3]). Observe that the group \( G \) is then of exponential volume growth (cf. [18]).

Let \( \gamma_{j,1}, ..., \gamma_{j,d} \in \mathbb{Z} \) the integers defined by

\[
\alpha_j(z) = p^{\gamma_{j,1}n_1+\gamma_{j,2}n_2+...+\gamma_{j,d}n_d}, \quad z = \pi_{1}^{n_1}...\pi_{d}^{n_d} \tau, \quad j = 1, ..., s.
\]

We shall denote by \( L_j, j = 1, ..., s \), the linear forms on \( \mathbb{Z}^d \) defined by

\[
L_j(n_1, ..., n_d) = \gamma_{j,1}n_1 + \gamma_{j,2}n_2 + ... + \gamma_{j,d}n_d,
\]

(\( n_1, ..., n_d \) \( \in \) \( \mathbb{Z}^d \)) and by \( \tilde{L}_j \) the linear forms on \( \mathbb{R}^d \) induced by the \( L_j \)'s.

Let now \( d\mu(g) = \varphi(g)d\gamma g \in \mathbb{P}(G) \) denote a symmetric probability measure on \( G \). The density \( \varphi(g) \) is assumed to be a continuous compactly supported function on \( G \). To avoid unnecessary complications we shall assume that there exists \( e \in \Omega = \Omega^{-1} \subset G \) such that

\[
\inf \{ \varphi(g), \quad g \in \Omega \} > 0, \quad G = \bigcup_{n \geq 0} \Omega^n;
\]

the last condition in (7) implies that \( \text{supp} (\mu) \) generates the group \( G \).

Let

\[
p : G \rightarrow G/U \rightarrow A
\]

denote the projection that we obtain from the identifications (1) and let

\[
\tilde{\mu} = (\pi \circ p)(\mu) \in \mathbb{P}(\mathbb{Z}^d) \subset \mathbb{P}(\mathbb{R}^d),
\]

where \( \pi \) denotes the canonical projection (2). It follows from (7) that there exists a choice of coordinates on \( \mathbb{R}^d \) for which the covariance matrix of \( \tilde{\mu} \) satisfies

\[
\int_{\mathbb{R}^d} x_i x_j d\tilde{\mu}(x) = \delta_{i,j}, \quad 1 \leq i, j \leq d.
\]

We shall assume that \( \mathbb{R}^d \) is equipped with the Euclidean structure associated to these coordinates.
From now on we shall assume that the $L_j$'s induce a "Weyl chamber". More precisely we suppose that

$$\Pi_{\mathcal{L}} = \{ x \in \mathbb{R}^d, \; \tilde{L}_j(x) > 0, \; j = 1, 2, ..., s \} \subset \mathbb{R}^d$$

define a nonempty convex cone in $\mathbb{R}^d$. This condition is the analogue of the $(NC)$-condition introduced by Varopoulos in [24] in the setting of real amenable Lie groups. We shall prove that, under this condition, we have a lower bound of the form $\varphi_n(e) \geq cn^{-\nu}$. The argument follows the approach introduced by Varopoulos in [24].

The exact value of $\nu$ is defined as in the real case and is expressed in terms a parameter $\lambda = \lambda(d, \mathcal{L})$ that is defined as follows. In the rank one case (i.e. $d = 1$) we shall set

$$(10a) \quad \lambda = 0.$$ 

In the case $d \geq 2$ let us denote by $\Sigma = \{ x \in \mathbb{R}^d, \; |x| = 1 \}$ the unit sphere in $\mathbb{R}^d$. Let $\Delta_\Sigma$ be the corresponding spherical Laplacian. Let $\Pi_\Sigma = \Sigma \cap \Pi_{\mathcal{L}} \subset \Sigma$. We then set

$$(10b) \quad \lambda = \inf\{-(\Delta_\Sigma f, f), \; \|f\|_2 = 1, \; f \in C_0^\infty(\Pi_\Sigma)\};$$

i.e. $\lambda$ is the first Dirichlet eigenvalue of the region $\Pi_\Sigma$. The scalar product and the $L^2$-norm in $(10b)$ are taken with respect to the Euclidean volume element on $\Sigma$.

Theorem 1. Let $G$ and $d\mu^\ast_n(g) = \varphi_n(g)d^rg$, $n = 1, 2, ...$ be as above. Then there exists $C > 0$ such that

$$(11) \quad \varphi_n(e) \geq \frac{1}{Cn^{1+\sqrt{d^2-4}\lambda^2/2}}, \; n = 1, 2, ...$$

where $\lambda$ is defined by (10).

The following comments may be helpful in placing the above theorem in its proper perspective.

(i) It is enough to to prove the estimate (11) when $n$ is even since $\varphi$ satisfies (7).

(ii) Observe that the group $G$ is automatically non-unimodular, for otherwise we have (cf. [11]):

$$\varphi_n(e) \leq Ce^{-cn^{1/3}}, \; n = 1, 2, ...$$

(iii) The upper estimate

$$\varphi_n(e) \leq \frac{C}{n^{3/2}}, \; n = 1, 2, ...$$

is known to hold for general non-unimodular locally compact groups (cf. [26]). This shows that the index 3/2 cannot be improved in the rank one case.

(iv) We will show (cf. §4 below) that in the case of metabelian $p$-adic groups, the lower bound (11) can be complemented with a similar upper bound.

Throughout the remainder of the paper $C$ denotes a positive constant which is not always the same, even in a given line.
3 Proof of Theorem 1

Let $G$, $\mu (g) = \varphi (g) d^r g \in \mathbf{P}(G)$ and $\mu ^n (g) = \varphi _n (g) d^r g$ be as in Theorem 1. Let $\xi _1, \xi _2, \ldots \in G$ be a sequence of independent equidistributed random variables of law $\mu (g)$ and let $X_n = \xi _1 \xi _2 \ldots \xi _n$, $n = 1, 2, \ldots$ denote the corresponding random walk starting at $X_0 = e$. Let $B \subset G$ a borel subset, we have

\[(12) \quad \mathbb{P} [X_n \in B] = \int _B \varphi _n (g) d^r g, \quad n = 1, 2, \ldots \]

The symmetry of $\mu (g)$ implies that

$$d\mu (g) = d\mu (g^{-1}) = \varphi (g) d^r g = \varphi (g^{-1}) d^r (g^{-1}) = \varphi (g^{-1}) d^r (g^{-1}) d^r g,$$

hence

$$\varphi (g^{-1}) = \varphi (g) \delta (g), \quad g \in G.$$ 

We have also

$$\varphi _n (g^{-1}) = \varphi _n (g) \delta (g), \quad g \in G, \quad n = 1, 2, \ldots$$

On the other hand we have

$$\varphi _{2n}(e) = \int _G \varphi _n (g^{-1}) \varphi _n (g) d g = \int _G \varphi _n (g) \delta (g) \varphi _n (g) d g = \int _G \varphi _n (g)^2 d^r g.$$ 

Schwarz inequality applied to (12) gives then

\[(13) \quad \varphi _{2n}(e) \geq \left( \frac{\mathbb{P} [X_n \in B]}{|B|} \right)^2, \quad B \subset G, \quad n = 1, 2, \ldots \]

where $|B|$ denotes the right Haar measure of $B$.

Let us further observe that the group $G$ (resp. $G/U$) decomposes as a semi-direct (resp. direct) product

$$G = Q \ltimes S \cong U \ltimes (\mathbb{Z}^d \times \tilde{S})$$

$$G/U \cong A \times S \cong \mathbb{Z}^d \times \tilde{S}$$

where $\tilde{S}$ is compact. This follows from (1).

Let us write

\[(14) \quad X_n = \xi _1 \xi _2 \ldots \xi _n = u_1 z_1 u_2 z_2 \ldots u_n z_n, \quad n = 1, 2, \ldots \]

where $\xi _j = u_j z_j$ with $u_j \in U$ and $z_j \in A \times S$, $j = 1, 2, \ldots$. Using the interior automorphisms $x^y = y x y^{-1}$, $x, y \in G$, we rewrite (14)

\[(15) \quad X_n = u_1 u_2 z_1 z_2 \ldots u_n z_1 z_2 \ldots z_n = \Gamma _n z_n, \quad \Gamma _n \in U, \quad z_n \in A \times S, \quad n = 1, 2, \ldots \]
We shall use the exponential map and identify \( U \) to its Lie algebra \( \mathcal{U} \) (cf. [5], [20]) and write each \( u_j \) in the above expression

\[
(16) \quad u_i = \exp(v_i), \quad v_i \in \mathcal{U}, \quad i = 1, 2, ...
\]

We have therefore

\[
(17) \quad u_j^{z_1 z_2 ... z_{j-1}} = \exp \left( \text{Ad}(z_1 ... z_{j-1})v_j \right), \quad j \geq 2.
\]

Let us fix \( e_1, e_2, ..., e_m \) a basis of \( \mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \) adapted to the decomposition (4). For \( x = x_1 e_1 + x_2 e_2 + ... + x_m e_m \in \mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \), we set

\[
\|x\| = \max_{1 \leq i \leq m} |x_i|.
\]

Since \( \mu \) is compactly supported we can suppose that the \( v_i \)'s in (16) satisfy

\[
(18) \quad \|v_j\| \leq C, \quad j = 1, 2, ...
\]

where \( C > 0 \) is an appropriate positive constant.

Let us equip \( \text{End}_{\mathbb{Q}_p} \left( \mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \right) \) with the norm

\[
\|T\| = \sup_{\|v\| \leq 1} \|Tv\|, \quad T \in \text{End}_{\mathbb{Q}_p} \left( \mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \right).
\]

It is clear that \( \|\cdot\| \) satisfies the ultrametric property

\[
(19) \quad \|T + T'\| \leq \max \left( \|T\|, \|T'\| \right).
\]

Let \( 1 \leq l \leq r \). For \( z = \pi_1^{n_1} ... \pi_d^{n_d} \tilde{z} \in A \times S \cong \mathbb{Z}_d \times \tilde{S} \) we have

\[
(20) \quad \text{Ad}(z)|_{W_l} = \text{Ad}(\tilde{z})|_{W_l} \circ \left( \tilde{x}_l(z)I + \mathcal{T}(z) \right),
\]

where \( \mathcal{T}(z) \) denotes an upper triangular matrix and where

\[
\tilde{x}_l(z) = \chi_l(\pi_1^{n_1} ... \pi_d^{n_d}) = \chi_l(\pi_1)^{n_1} ... \chi_l(\pi_d)^{n_d}.
\]

On the other hand it is clear that

\[
(21) \quad |\tilde{x}_l(z)|_p', \quad |\tilde{x}_l(z)|_p^{-1} \leq p^{C|\tilde{z}|_{\mathbb{Z}_d}}, \quad z \in A \times S,
\]

where \( |\cdot|_{\mathbb{Z}_d} \) denotes the Euclidean norm on \( \mathbb{Z}_d \). We have also

\[
(22) \quad \|\text{Ad}(z)\| \leq C\|\text{Ad}(\pi_1)\|^{n_1} ... \|\text{Ad}(\pi_d)\|^{n_d} \leq C p^{C|\tilde{z}|_{\mathbb{Z}_d}}, \quad z \in A \times S.
\]

Combining (20), (21), (22) we deduce that the triangular matrix \( \mathcal{T}(z) \) that appears in (20) satisfies

\[
(23) \quad \|\mathcal{T}(z)\| \leq C p^{C|\tilde{z}|_{\mathbb{Z}_d}}, \quad z \in A \times S.
\]
Let now \( z_1, z_2, \ldots, z_k \in A \times S \) and \( 1 \leq l \leq r \). We have

\[
\text{Ad}(z_1 z_2 \ldots z_k)|_{W_l} = \text{Ad}(\tilde{\tau}_1 \tilde{\tau}_2 \ldots \tilde{\tau}_k)|_{W_l} \circ \prod_{j=1}^{k} (\tilde{\chi}_l(z_j)|I + \mathcal{F}(z_j)) \\
= \tilde{\chi}_l(z_1 z_2 \ldots z_k) \text{Ad}(\tilde{\tau}_1 \tilde{\tau}_2 \ldots \tilde{\tau}_k)|_{W_l} \\
\circ \sum_{i,j} (\tilde{\chi}_l(z_i) \tilde{\chi}_l(z_{i,j}))^{-1} \mathcal{F}(z_{i,j}) \ldots \mathcal{F}(z_{i_{l_a}}).
\]

Using the fact that in the last sum all the terms corresponding to indexes \( \alpha > n \) vanish and combining this with (21), (23) and the ultrametric property (19) we deduce that

\[
||\text{Ad}(z_1 z_2 \ldots z_k)|_{W_l}|| \leq C \left| \tilde{\chi}_l(z_1 z_2 \ldots z_k) \right|_{L^p} \rho^{C \max_{1 \leq l \leq k} ||\mathcal{F}||_{L^d}}.
\]

If we use the linear forms \( L_l \) defined by (5) and (6) we then deduce

\[
||\text{Ad}(z_1 z_2 \ldots z_k)|_{W_l}|| \leq C p^{\max_{1 \leq l \leq k} L_l(\zeta_1 + \ldots + \zeta_k) + C \max_{1 \leq l \leq k} ||\mathcal{F}||_{L^d}}.
\]

By the above considerations we have finally proved that

\[
||\text{Ad}(z_1 z_2 \ldots z_k)|| \leq C p^{\max_{1 \leq l \leq k} L_l(\zeta_1 + \ldots + \zeta_k) + C \max_{1 \leq l \leq k} ||\mathcal{F}||_{L^d}}
\]

\[
\zeta_j = p(z_j), \quad z_j \in A \times S, \quad j = 1, \ldots, k, \quad k = 1, 2, \ldots
\]

If we apply this estimate to \( u_j^{z_1 z_2 \ldots z_j-1} = \exp(\text{Ad}(z_1 \ldots z_j-1)\nu_j) \) where the \( u_j \)'s, \( \nu_j \)'s and \( z_j \)'s are as in (16), (17), (18) we then deduce

\[
||\text{Ad}(z_1 \ldots z_{j-1})\nu_j|| \leq C p^{\max_{1 \leq l \leq k} L_l(\zeta_1 + \ldots + \zeta_{j-1}) + C \max_{1 \leq l \leq k} ||\mathcal{F}||_{L^d}}
\]

\[
\zeta_j = p(z_j), \quad j \geq 2.
\]

Observe that the measure that controls the random walk \( (S_j)_{j \in \mathbb{N}} \) defined by \( S_j = \zeta_0 + \zeta_1 + \ldots + \zeta_j \) (\( \zeta_0 = 0 \)) is the symmetric measure \( \tilde{\mu} \) defined by (8). The random variables \( \zeta_j \) are in particular compactly supported and we have

(24) \[ ||\text{Ad}(z_1 \ldots z_{j-1})\nu_j|| \leq C p^{\max_{1 \leq l \leq k} L_l(\zeta_1 + \ldots + \zeta_{j-1})}, \quad \zeta_j = p(z_j), \quad j \geq 2. \]

Let us consider, for \( n = 1, 2, \ldots \), the event \( E_n \) defined by

(25) \[
E_n = \left( L_l(\zeta_1 + \ldots + \zeta_j) \leq C, \quad j = 1, \ldots, n, \quad l = 1, \ldots, s; \right)
\]

\[
\left| \zeta_1 + \ldots + \zeta_n \right|_{L^d} \leq C \sqrt{n},
\]

where \( C > 0 \) denotes an appropriate large constant. Using (15), Campbell-Hausdorff, (24) and the ultrametric property (19) we see that the event \( E_n \) verifies

\[
E_n \subset \left[ X_n \in B_n \right]
\]

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where $B_n$ is defined by

$$B_n = \exp \{ \{ u \in \mathcal{U}, \ |u| \leq C \} . \{ z \in A \times S, \ |p(z)|_{\mathbb{R}^d} \leq An^{1/2} \} \}.$$

It is easy to see that $d^r g$ shows that

$$|B_n| \leq C n^d, \quad n = 1, 2, \ldots$$

It remains to estimate the probability of the event (25). Let $\Pi$ denote the polyhedral region in $\mathbb{Z}^d$ defined by

$$\Pi = \{ z \in \mathbb{Z}^d, \ L_j(z) \leq C, \ j = 1, \ldots, s \},$$

where $C$ denotes the same constant as in (25). Let $h_n(x, y)$, $n = 1, 2, \ldots, x, y \in \Pi$, denote the transition kernel corresponding to the random walk $\{ S_j \}_{j \in \mathbb{N}}$ with killing outside of $\Pi$. Precise lower estimates for the kernel $h_n(x, y)$ can be obtained thanks to the results of [26] and [27]. To write down these estimates we need the following notations. Let us assume that $d \geq 2$ and let $0 < u_0(\sigma)$, $\sigma \in \Pi$, denote the eigenfunction corresponding to the first Dirichlet eigenvalue of the region $\Pi$ defined by (10). Let $u(x)$ be the function defined on $\Pi$ by

$$u(x) = u(r, \sigma) = r^a u_0(\sigma), \quad x = (r, \sigma) \in \mathbb{R}_+^* \times \Pi$$

where

$$a = \frac{\sqrt{(d-2)^2 + 4\lambda} - (d-2)}{2},$$

and where $(r, \sigma)$ denote the polar coordinates on $\mathbb{R}_+^* \times \Sigma$. The function $u$ defined in this way is a positive function harmonic in $\Pi$ which vanishes on $\partial \Pi$ (cf. [24]). In the case where $d = 1$, the function $u$ is defined by $u(x) \equiv x$, $x \in \mathbb{R}_+^*$.

It follows easily from [27] (cf. estimate (2) p. 359) and [26] (cf. Theorem 1 and estimate (0.3.4)) that there exists $C > 0$ such that

$$h_n(0, y) \geq \frac{u(-y)}{C n^{a+d/2}}, \quad |y| \leq C \sqrt{n}, \quad n \geq C.$$

We have therefore

$$P(E_n) \geq \frac{1}{C n^{a+d/2}} \sum_{y \in \Pi, |y| \leq C \sqrt{n}} u(-y).$$

Using the homogeneity of $u$ we deduce then that

$$P(E_n) \geq \frac{1}{C n^{a+d/2}} \int_{C \sqrt{n}}^{C \sqrt{n}} r^{a+d-1} dr = \frac{1}{C n^{a/2}}.$$

The lower estimate (11) is an immediate consequence of (13), (26), (27) and (28). This completes the proof of Theorem 1.
4 Metabelian $p$-adic groups

Our aim in this section is to show that in the case of metabelian groups the lower estimate (11) can be complemented with a similar upper bound. We keep the notation of §2. We shall denote by $Z_p^* = \{x \in \mathbb{Q}_p^*, |x|_p = 1\}$ and by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p, |x|_p \leq 1\}$. Let $dx$ denote the Haar measure on $\mathbb{Q}_p$ normalized by $dx(\mathbb{Z}_p) = 1$ and let $d^*x$ denote the Haar measure on $\mathbb{Q}_p^*$ normalized by $d^*(Z_p^*) = 1$.

Let us fix $k, l \geq 2$ and consider

\begin{equation}
G = \mathbb{Q}_p^k \rtimes_{\sigma} \left( \mathbb{Q}_p^* \right)^l
\end{equation}

the semi-direct product of $\left( \mathbb{Q}_p^* \right)^l$ with the vector space $\mathbb{Q}_p^k$ where $\left( \mathbb{Q}_p^* \right)^l$ acts on $\mathbb{Q}_p^k$ by

\[
x = (x_1, ..., x_k) \mapsto \sigma(y)x = (\chi_1(y)x_1, ..., \chi_k(y)x_k), \quad y \in \left( \mathbb{Q}_p^* \right)^l,
\]

where $\chi_1, ..., \chi_k$ denote $k$ morphisms

\begin{equation}
\chi_1, ..., \chi_k : \left( \mathbb{Q}_p^* \right)^l \longrightarrow \mathbb{Q}_p^*.
\end{equation}

More precisely, we assume that the multiplication in $G$ is given by

\[
g \cdot g' = (x; y)(x'; y') = (x + \sigma(y)x'; y.y')
\]

\[
= \left( x_1 + \chi_1(y)x'_1, x_2 + \chi_2(y)x'_2, ..., x_k + \chi_k(y)x'_k; y_1, y_2, ..., y_l \right)
\]

\[
g = (x, y), \quad g' = (x', y') \in G; \quad x = (x_1, ..., x_k), \quad x' = (x'_1, ..., x'_k) \in \mathbb{Q}_p^k;
\]

\[
y = (y_1, ..., y_l), \quad y = (y'_1, ..., y'_l) \in \left( \mathbb{Q}_p^* \right)^l.
\]

We shall denote by

\[
d^*g = dxd^*y = dx_1...dx_kd^*y_1...d^*y_l; \quad d^l g = dg = \delta(g)^{-1}d^rg
\]

the right and the left invariant Haar measure on $G$. The modular function $\delta(g)$ is given by

\begin{equation}
\delta(g) = \delta(y) = |\chi_1(y)||\chi_2(y)|...|\chi_k(y)|, \quad g = (x, y) \in G.
\end{equation}

Let $d\mu(g) = \varphi(g)d^*r(g)$ denote the symmetric, compactly supported, probability measure on $G$ defined by

\begin{equation}
\varphi(g) = \alpha \delta^{-1/2}(g)I_{\mathbb{Z}_p}(x_1)...I_{\mathbb{Z}_p}(x_k)I_{\mathbb{Z}_p}(\chi_1(y)^{-1}x_1)...I_{\mathbb{Z}_p}(\chi_k(y)^{-1}x_k)
\]

\[
\times I_{\left( p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^* \right)}(y_1)...I_{\left( p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^* \right)}(y_l),
\]

\[
g = (x_1, ..., x_k; y_1, ..., y_l) \in G.
\]

The notation $I_A$ is used here to denote the indicator of a subset $A$ and $\alpha > 0$ denotes an appropriate positive constant.
We shall assume that the \( \chi_j \)'s in (30) verify the condition

\[
|\chi_j| \neq 1, \quad j = 1, \ldots, k.
\]

This guarantees (cf. §2) that the group \( G \) defined by (29) is compactly generated and it is easy to see that \( \text{supp}(\mu) \) is a generating set of the group \( G \). We shall denote, as in §2, by \( L_j, j = 1, \ldots, k \), the \( k \) linear forms on \( \mathbb{Z}^l \) induced by the \( \chi_j \)'s and we shall assume that these linear forms induce a "Weyl chamber" \( \Pi = \{ x \in \mathbb{R}^l, \quad L_j(x) > 0, \quad j = 1, 2, \ldots, k \} \) as in §2. We shall denote by \( \Sigma = \{ x \in \mathbb{R}^l, \quad |x| = 1 \} \) the unit sphere in \( \mathbb{R}^l \), by \( \Pi_\Sigma = \Sigma \cap \Pi \) and by \( \lambda \) be the first Dirichlet eigenvalue of the region \( \Pi_\Sigma \). We have then the following:

**Theorem 2.** Let \( G = \mathbb{Q}_p^k \ltimes_{\sigma} \left( \mathbb{Q}_p^* \right)^l \) and let \( \mu \in \mathcal{P}(G) \) be as above. Let \( d\mu^n(g) = d(\mu * \cdots * \mu)(g) = \varphi_n(g)d^*g \) denote the \( n \)th convolution power of \( \mu \). Then there exists \( C > 0 \) such that

\[
\varphi_n(e) \leq \frac{C}{n^{1+\sqrt{(i-2)^2+4\lambda}}}, \quad n = 1, 2, \ldots
\]

The first step to prove Theorem 2 consists in establishing an explicit formula for \( \varphi_n(g), \quad g \in G \).

In what follows we shall use the notation \( x_i = (x_{i,1}, \ldots, x_{i,k}) \) (resp. \( y_i = (y_{i,1}, \ldots, y_{i,l}) \)) for \( x_i \in \mathbb{Q}_p^k, \quad i = 1, 2, \ldots \) (resp. \( y_i \in \left( \mathbb{Q}_p^* \right)^l, \quad i = 1, 2, \ldots \)). Let us fix \( g = (\xi, \zeta) = (\xi_1, \ldots, \xi_k, \zeta_1, \ldots, \zeta_l) \in G \) and \( n = 1, 2, \ldots \). By definition of convolution product we have:

\[
\varphi_{n+1}(g) = \int_G \varphi_n(h)\varphi_1(h^{-1}g)dh = \int_G \varphi_1(h^{-1}g)\delta(h^{-1})\varphi_n(h)d^*h = \int_G \varphi_1(h^{-1}g)\delta(h^{-1})d\mu^n(h)
\]

\[
= \int_G \cdots \int_G \varphi_1 g_1 \cdots g_n^{-1} g \delta(g_1 \cdots g_n)^{-1} d\mu(g_1) \cdots d\mu(g_n)
\]

\[
= \int_G \cdots \int_G \varphi_1 \left( -\sigma(y_1 \cdots y_n)^{-1} x_1 + \sigma(y_2 \cdots y_n)^{-1} x_2 + \cdots + \sigma(y_n)^{-1} x_n \right) \delta(y_1 \cdots y_n)^{-1} dx_1 dx_2 \cdots dx_n d^*y_1 \cdots d^*y_n.
\]
An obvious change of variables combined with Fubini gives

\[
\varphi_{n+1}(g) = \int_{(Q_p^*)^n} \ldots \int_{(Q_p^*)^y} \left[ \int_{Q_p^x} \ldots \int_{Q_p^z} \left[ \varphi_1 \left( -\left( x_1 + x_2 + \ldots + x_n \right) 
\right. \right. \\
+ \sigma(y_1 \ldots y_n)^{-1} \xi, (y_1 \ldots y_n)^{-1} \xi \right] \varphi_1(\sigma(y_1 \ldots y_n)x_1, y_1) \\
+ \varphi_1(\sigma(y_2 \ldots y_n)x_2, y_2) \ldots \varphi_1(\sigma(y_n)x_n, y_n) \right] \, dx_1 \ldots dx_n \\
\prod_{i=2}^n |\chi_1(y_1 \ldots y_n)| \ldots |\chi_k(y_1 \ldots y_n)| \, d^*y_1 \ldots d^*y_n,
\]

where we used (31). Let us consider the product

\[
\prod_{i=1}^n \varphi_1(\sigma(y_1 \ldots y_n)x_i, y_i)
\]

that appears in equation (33). By (32), this product is equal to

\[
a^n \prod_{i=1}^n \delta^{-1/2}(y_i) \prod_{i=1}^n \left( I_{Z_p}(\chi_1(y_1 \ldots y_n)x_{i,1}) \ldots I_{Z_p}(\chi_k(y_1 \ldots y_n)x_{i,k}) \right) \\
\times \prod_{i=1}^{n-1} \left( I_{Z_p}(\chi_1(y_{i+1} \ldots y_n)x_{i,1}) \ldots I_{Z_p}(\chi_k(y_{i+1} \ldots y_n)x_{i,k}) \right) \\
\times I_{Z_p}(x_{n,1}) \ldots I_{Z_p}(x_{n,k}) \prod_{i=1}^n I_{\left( p^{-1}Z_p^1 \cup Z_p^2 \cup pZ_p^3 \right)}(y_{i,1}) \ldots I_{\left( p^{-1}Z_p^1 \cup Z_p^2 \cup pZ_p^3 \right)}(y_{i,l}) \\
= a^n \prod_{i=1}^n \delta^{-1/2}(y_i) \prod_{i=1}^{n-1} \left( I_{|\chi_1(y_{i+1} \ldots y_n)|\max(1,|\chi_1(y_i)|)Z_p}(x_{i,1}) \\
\ldots I_{\max(1,|\chi_k(y_i)|)Z_p}(x_{i,k}) \right) \\
I_{\max(1,|\chi_1(y_n)|)Z_p}(x_{n,1}) \ldots I_{\max(1,|\chi_k(y_n)|)Z_p}(x_{n,k}) \\
\prod_{i=1}^n I_{\left( p^{-1}Z_p^1 \cup Z_p^2 \cup pZ_p^3 \right)}(y_{i,1}) \ldots I_{\left( p^{-1}Z_p^1 \cup Z_p^2 \cup pZ_p^3 \right)}(y_{i,l}).
\]

We also have

\[
\varphi_1 \left( -\left( x_1 + x_2 + \ldots + x_n \right) + \sigma(y_1 \ldots y_n)^{-1} \xi, (y_1 \ldots y_n)^{-1} \xi \right) \\
= a \delta^{-1/2}((y_1 \ldots y_n)^{-1} \xi) \\
\times I_{\max(1,|\chi_1(y_1 \ldots y_n)|\xi^{-1})Z_p}(x_{1,1} + x_{2,1} + \ldots + x_{n,1} - \chi_1(y_1 \ldots y_n)^{-1} \xi_1) \\
\ldots I_{\max(1,|\chi_k(y_1 \ldots y_n)|\xi^{-1})Z_p}(x_{1,k} + x_{2,k} + \ldots + x_{n,k} - \chi_k(y_1 \ldots y_n)^{-1} \xi_k) \\
\times I_{\left( p^{-1}Z_p^1 \cup Z_p^2 \cup pZ_p^3 \right)}((y_{1,1} \ldots y_{n,1})^{-1} \xi_1) \ldots I_{\left( p^{-1}Z_p^1 \cup Z_p^2 \cup pZ_p^3 \right)}((y_{1,l} \ldots y_{n,l})^{-1} \xi_l).
\]
On the other hand we shall set

\[ m(y) = \alpha \delta^{-1/2}(y) \prod_{j=1}^{k} \min \left( 1, |\chi_j(y)| \right) I(\rho^{-1} z_p^* \cup z_p) (y_1) \cdots I(\rho^{-1} z_p^* \cup z_p) (y_l), \]

\[ y = (y_1, \ldots, y_l) \in \left( \mathbb{Q}_p^* \right)^l. \]

What motivates this notation is the fact that

\[ \int_{\mathbb{Q}_p^*} \varphi(x, y) dx = m(y), \quad y \in \left( \mathbb{Q}_p^* \right)^l. \]

Setting

\[ A(y_1, y_2, \ldots, y_n) = m \left( (y_1 \ldots y_n)^{-1} \zeta \right) \prod_{j=1}^{k} \max \left( 1, |\chi_j(y_1 \ldots y_n)\zeta^{-1}| \right) \]

\[ \times \prod_{i=1}^{n} \max \left( 1, |\chi_1(y_i)|^{-1} \right) \cdots \max \left( 1, |\chi_k(y_i)|^{-1} \right) m(y_i), \]

\[ B_j(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = B_j(x_1, x_2, \ldots, x_n) \]

\[ = I_{\max(1, |\chi_j(y_1 \ldots y_n)\zeta^{-1}|)} \left( x_{1,j} + x_{2,j} + \ldots + x_{n,j} - x_j(y_1 \ldots y_n)^{-1} \xi_j \right) I_{\max(1, |\chi_j(y_1)|)} (x_{i,j}), \]

\[ j = 1, 2, \ldots, k, \]

we rewrite (33) as

\[ \varphi_{n+1}(g) = \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \left[ \int_{\mathbb{Q}_p^k} \cdots \int_{\mathbb{Q}_p^k} B_j(x_1, x_2, \ldots, x_n) \, dx_1 \cdots dx_n \right] \]

\[ A(y_1, y_2, \ldots, y_n) \prod_{i=2}^{n} \left| \chi_1(y_i \ldots y_n) \right| \cdots \left| \chi_k(y_i \ldots y_n) \right| \]

\[ d^* y_1 \ldots d^* y_n. \]

The next step is to calculate

\[ \int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} B_j(x_1, x_2, \ldots, x_n) dx_{1,j} dx_{2,j} \ldots dx_{n,j}, \quad j = 1, \ldots, k. \]
Towards this we observe that:

\[
\int_{\mathbb{Q}_p} \ldots \int_{\mathbb{Q}_p} I_{\max(1,|\chi(1)_j(y_1 \ldots y_n)\xi^{-1}|)}(x) dx_1, \ldots dx_n
\]

\[
= \int_{\mathbb{Q}_p} \ldots \int_{\mathbb{Q}_p} B_j(x_1, x_2, \ldots, x_n)dx_1 dx_2, \ldots dx_n
\]

\[
= \text{Min} \left[ \min_{1 \leq i \leq n-1} \left( |\chi(1)_j(y_{i+1} \ldots y_n)| \max(1, |\chi(1)_j(y_i)|) \right) \right]
\]

\[
\times \left( I_{\min(1,|\chi(1)_j(y_{i+1} \ldots y_n)}(x) \right) \max(1, |\chi(1)_j(y_{i+1} \ldots y_n)|)
\]

\[
\times \left( I_{\max(1,|\chi(1)_j(y_{i+1} \ldots y_n)}(x) \right) \max(1, |\chi(1)_j(y_{i+1} \ldots y_n)|)
\]

where \( * \) denotes the usual convolution in \( \mathbb{Q}_p \) (cf. [21]). Taking Fourier transform (cf. [21]) we obtain

\[
\int_{\mathbb{Q}_p} \ldots \int_{\mathbb{Q}_p} \ldots \int_{\mathbb{Q}_p} B_j(x_1, x_2, \ldots, x_n)dx_1 dx_2, \ldots dx_n
\]

Let us denote by \( S^\xi_j(y_1, \ldots, y_n) \subset \mathbb{Q}_p, y_1, \ldots, y_n \in \mathbb{Q}_p^\ast \) the subset defined by

\[
S^\xi_j(y_1, \ldots, y_n) = \text{Min} \left[ \min_{1 \leq i \leq n-1} \left( |\chi(1)_j(y_{i+1} \ldots y_n)| \max(1, |\chi(1)_j(y_i)|) \right) \right]
\]

\[
\times \left( (1, \text{Min}(1, |\chi(1)_j(y_{i+1} \ldots y_n)|) \max(1, |\chi(1)_j(y_i)|)) \right)
\]

\[
\times \left( \max(1, |\chi(1)_j(y_{i+1} \ldots y_n)|) \right)
\]

With this notation we have

\[
(39) \int_{\mathbb{Q}_p} \ldots \int_{\mathbb{Q}_p} B_j(x_1, x_2, \ldots, x_n)dx_1 dx_2, \ldots dx_n
\]
Putting together (33), (36), (37), (38) and (39) and taking into account the fact that

\[
\prod_{j=1}^{k} \min\left(1, |\chi_j(y_n)|^{-1}\right) \prod_{i=1}^{n-1} \left(\prod_{j=1}^{k} \min\left(1, |\chi_j(y_i)|^{-1}\right)\right)
\]

\[
\times \prod_{i=1}^{n} \max\left(1, |\chi_1(y_i)|^{-1}\right) \ldots \max\left(1, |\chi_k(y_i)|^{-1}\right)
\]

\[
= \prod_{j=1}^{k} |\chi_j(y_1 \ldots y_n)|^{-1} = \delta^{-1}(y_1 \ldots y_n),
\]

and that

\[
\delta^{-1}(y_1 \ldots y_n) \prod_{j=1}^{k} \max\left(1, |\chi_j(y_1 \ldots y_n)\xi^{-1}|\right) m\left((y_1 \ldots y_n)^{-1}\xi\right)
\]

\[
= \alpha \delta^{-1/2}(y_1 \ldots y_n) \delta^{-1/2}(\zeta) \prod_{p}^{(z_p^*)^l} ((y_{1,1} \ldots y_{n,1})^{-1}\xi_1)
\]

\[
\ldots \prod_{j=1}^{l} ((y_{1,j} \ldots y_{n,j})^{-1}\xi_j),
\]

we finally deduce the formula

\[
\varphi_{n+1}(g) = \alpha \delta^{-1/2}(\zeta) \int_{(Q_p^*)^l} \ldots \int_{(Q_p^*)^l} \prod_{j=1}^{k} \text{Min} \left(1; \min\left(\prod_{i=1}^{n} \max\left(1, |\chi_j(y_i)|\right)\right); \max(1, |\chi_j(y_n)|) \min(1, |\chi_j(y_1 \ldots y_n)^{-1}\xi|)\right)
\]

\[
\delta^{-1/2}(y_1 \ldots y_n) \prod_{j=1}^{k} I_{\zeta_j}(y_{1,1} \ldots y_{n,1}) (\chi_j(y_1 \ldots y_n)^{-1}\xi_j)
\]

\[
\prod_{j=1}^{l} I_{\chi_j}(y_{1,j} \ldots y_{n,j})^{-1}\xi_j)
\]

\[
m(y_1) \ldots m(y_n) d^* y_1 \ldots d^* y_n.
\]

The next step is to give a probabilistic interpretation of (40). Towards that let us consider a sequence of independent identically \((Q_p^*)^l\)-valued random variables \(Y_1, Y_2, \ldots\) with distribution on \((Q_p^*)^l\) given by

\[
P[Y_j \in dy] = m(y) d^* y
\]
where \(m(y)\) is as in (34). Observe that by (35) we have \(\int m(y)d^*y = 1\). The formula (40) can be rewritten

\[
\varphi_{n+1}(g) = a\delta^{-1/2}(\xi)E\left[\prod_{j=1}^{k}\min\left(1; \min_{1\leq i \leq n} \left|\chi_j(Y_{i+1}...Y_n)\right|\max(1, |\chi_j(Y_i)|)\right)\right.
\]

\[
\left.\max(1, |\chi_j(Y_n)|) \min(1, |\chi_j(Y_1...Y_n)^{-1}\xi_j|)\right]
\]

\[
\delta^{-1/2}(Y_1...Y_n)\prod_{j=1}^{k}I_{\delta_j(Y_1,...,y_n)}(\chi_j(Y_1...Y_n)^{-1}\xi_j)
\]

\[
\prod_{j=1}^{l}I_{(p^{-1}Z_p^* \cup Z_p^* \cup pZ_p^*)}(Y_{i,j}...Y_{n,j})^{-1}\xi_j \right] .
\]

In the case \(g = e\) the above formula simplifies considerably and we obtain

\[
\varphi_{n+1}(e) \leq CE\left[\prod_{j=1}^{k}\min\left(1, \min_{1\leq i \leq n} |\chi_j(Y_i...Y_n)|\right)\right]
\]

\[
\times \prod_{j=1}^{l}I_{(p^{-1}Z_p^* \cup Z_p^* \cup pZ_p^*)}(Y_{i,j}...Y_{n,j})^{-1}) .
\]

Now we project the random walk \(Y_1...Y_n\) controlled by (41) on \(Z^l \subset \mathbb{R}^l\) and we assume that \(\mathbb{R}^l\) is equipped with the Euclidean structure ensuring (9). We shall denote by \(S_n = X_1 + X_2 + ... + X_n\) \((n \geq 1)\) the random walk obtained via this projection. We have

\[
\varphi_{n+1}(e) \leq CE\left[\prod_{j=1}^{k}\min_{1\leq i \leq n} p^{L_j(S_n-S_i)}, S_n \in K\right], \quad n = 1, 2, ...
\]

where \(K\) denotes a neighbourhood of the origin in \(Z^l\). It follows then that

\[
\varphi_{n+1}(e) \leq CE\left[\prod_{j=1}^{k}p^{\max_{1\leq i \leq n} L_j(S_i)}, S_n \in K\right], \quad n = 1, 2, ...
\]

and therefore

\[
(42) \quad \varphi_{n+1}(e) \leq C \sum_{\lambda_1 \geq 0,...,\lambda_k \geq 0} p^{-\sum_{i=1}^{k}\lambda_i} \mathbb{P}\left(\max_{0 \leq i \leq n} L_1(S_i) = \lambda_1, ..., \right.
\]

\[
\left.\max_{0 \leq i \leq n} L_k(S_i) = \lambda_k; \quad S_n \in K\right), \quad n = 1, 2, ...
\]

Let us now fix \(x_0 \in \Pi\) such that

\[
L_j(x_0) > 0, \quad j = 1, ..., k.
\]

Let

\[
\lambda_0 = \min_{1 \leq j \leq k} L_j(x_0).
\]
We have
\[ L_j(u_0) = L_j(x_0/\lambda_0) \geq 1. \]

It follows then form (42) that
\[
\varphi_{n+1}(e) \leq C \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P}\left( L_1(S_i) \leq \lambda_1 L_1(u_0), \ldots, L_k(S_i) \leq \lambda_k L_k(u_0), \right)
\]

\[
0 \leq i \leq n; \ S_n \in K)
\]

\[
\leq C \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P}\left( L_1(S_i - (\max_{1 \leq i \leq k} \lambda_i + C)u_0) \leq 0, \ldots, \right)
\]

\[
L_k(S_i - (\max_{1 \leq i \leq k} \lambda_i + C)u_0) \leq 0, \ 0 \leq i \leq n; \ S_n \in K).
\]

Choosing \( C > 0 \) large enough, we can apply Theorem 4 of [26] and we deduce
\[
\varphi_{n+1}(e) \leq \frac{C}{n^{1+\sqrt{(l-2)^2+4k}}} \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} \left( \max_{1 \leq i \leq k} \lambda_i + 1 \right)^{C} p^{-\sum_{i=1}^k \lambda_i}
\]

\[
\leq \frac{C}{n^{1+\sqrt{(l-2)^2+4k}}} \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} \left( \sum_{i=1}^k \lambda_i + 1 \right)^{C} p^{-\sum_{i=1}^k \lambda_i}
\]

\[
\leq \frac{C}{n^{1+\sqrt{(l-2)^2+4k}}}.
\]

This completes the proof of Theorem 2.

**References**


