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# Glauber dynamics on nonamenable graphs: boundary conditions and mixing time

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**Abstract**

We study the stochastic Ising model on finite graphs with $n$ vertices and bounded degree and analyze the effect of boundary conditions on the mixing time. We show that for all low enough temperatures, the spectral gap of the dynamics with $(+)$-boundary condition on a class of nonamenable graphs, is strictly positive uniformly in $n$. This implies that the mixing time grows at most linearly in $n$. The class of graphs we consider includes hyperbolic graphs with sufficiently high degree, where the best upper bound on the mixing time of the free boundary dynamics is polynomial in $n$, with exponent growing with the inverse temperature. In addition, we construct a graph in this class, for which the mixing time in the free boundary case is exponentially large in $n$. This provides a first example where the mixing time jumps from exponential to linear in $n$ while passing from free to $(+)$-boundary condition. These results extend the analysis of Martinelli, Sinclair and Weitz to a wider class of nonamenable graphs.

**Key words:** Stochastic Ising model, nonamenable graphs, spectral gap, mixing time.

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# 1 Introduction

The goal of this paper is to analyze the effect of boundary conditions on the Glauber dynamics for the Ising model on nonamenable graphs. We will focus on a particular class of graphs which includes, among others, hyperbolic graphs with sufficiently high degree. Before discussing the motivation and the formulation of the results we shall give some necessary definitions.

Given a finite graph $G = (V, E)$, we consider spin configurations $\sigma = \{\sigma_x\}_{x \in V}$ which consist of an assignment of $\pm 1$-values to each vertex of $V$. In the Ising model the probability of finding the system in a configuration $\sigma \in \{\pm 1\}^V \equiv \Omega_G$ is given by the Gibbs measure

$$\mu_G(\sigma) = (Z_G)^{-1} \exp\left(\beta \sum_{(xy) \in E} \sigma_x \sigma_y + \beta h \sum_{x \in V} \sigma_x\right), \tag{1.1}$$

where $Z_G$ is a normalizing constant, and $\beta$ and $h$ are parameters of the model corresponding, respectively, to the inverse temperature and to the external field. Boundary conditions can also be taken into account by fixing the spin values at some specified boundary vertices of $G$. The term free boundary is used to indicate that no boundary is specified.

The Glauber dynamics for the Ising model on $G$ is a (discrete or continuous time) Markov chain on the set of spin configurations $\Omega_G$, reversible with respect to the Gibbs measure $\mu_G$. The corresponding generator is given by

$$(\mathscr{L} f)(\sigma) = \sum_{x \in V} c_x(\sigma)[f(\sigma^x) - f(\sigma)], \tag{1.2}$$

where $\sigma^x$ is the configuration obtained from $\sigma$ by a spin flip at the vertex $x$, and $c_x(\sigma)$ is the jump rate from $\sigma$ to $\sigma^x$.

Beyond of being the basis of Markov chain Monte Carlo algorithms, the Glauber dynamics provides a plausible model for the evolution of the underlying physical system toward the equilibrium. In both contexts, a central question is to determine the *mixing time*, i.e. the number of steps until the dynamics is close to its stationary measure.

In the past decades a lot of efforts have been devoted to the study of the dynamics for the classical Ising model, namely when $G = G_n$ is a cube of size $n$ in the finite-dimensional lattice $\mathbb{Z}^d$, and a remarkable connection between the equilibrium and the dynamical phenomena has been pointed out. As an example, on finite $n$-vertex cubes with free boundary in $\mathbb{Z}^d$, when $h = 0$ and $\beta$ is smaller than the critical value $\beta_c$ (one-phase region), the mixing time is of order $\log n$, while for $\beta > \beta_c$ (phase coexistence region) it is $\exp(n^{(d-1)/d})$ ([31; 24; 25; 23]).

More recently, an increasing attention has been devoted to the study of spin systems on graphs other than regular lattices. Among the various motivations which are beyond this new surge of interest, we stress that many new phenomena only appear when one considers graphs different from the Euclidean lattices, thus revealing the presence of an interplay between the geometry of the graph and the behavior of statistical systems.

Here we are interested in the problem of the *influence of boundary conditions* on the mixing time. It has been conjectured that in the presence of (+)-boundary condition on regular boxes of the lattice $\mathbb{Z}^d$, the mixing time should remain at most polynomial in $n$ for all temperatures rather than $\exp(n^{(d-1)/d})$ [9]. But even if some results supporting this conjecture have been achieved [5], a formal proof for the dynamics on the lattice is still missing.

However a different scenario can appear if one replaces the classical lattice structure with different graphs. The first rigorous result along this direction, has been obtained recently by Martinelli, Sinclair and Weitz [26] when studying the Glauber dynamics for the Ising model on regular trees. With this graph setting and in presence of $(+)$-boundary condition, they proved in fact that the mixing time remains of order $\log n$ also at low temperatures (phase coexistence region), in contrast to the free boundary case where it grows polynomially in $n$ [17; 4].

In this paper we extend the above result to a class of nonamenable graphs which includes trees, but also hyperbolic graphs with sufficiently high degree, and some suitable constructed expanders. Specifically, we consider the dynamics on an $n$-vertex ball of the graph with $(+)$-boundary condition, and prove that the spectral gap is $\Omega(1)$ (i.e. bounded away from zero uniformly in $n$) for all low enough temperatures and zero external field. This implies, by classical argument (see, e.g., [28]), an upper bound of order $n$ on the mixing time. Notice that this result is in contrast with the behavior of the free boundary dynamics on hyperbolic graphs, for which the spectral gap is decreasing in $n$ for all low temperatures, and bounded below by $n^{-\alpha(\beta)}$, with exponent $\alpha(\beta)$ arbitrarily increasing with $\beta$ [17; 4]. Moreover, we give an example of an expander, in the above class of graphs, for which we prove that the mixing time of the free boundary dynamics is at least exponentially large in $n$. This provides a first rigorous example of graph where the mixing time shrinks from exponential to linear in $n$ while passing from free to $(+)$-boundary condition.

We remark that what we believe to be determinant for the result obtained in [26] for the dynamics on trees, is in fact the nonamenability of the graph. On the other hand, the possible presence of cycles, which are absent on trees, makes the structure of some nonamenable graphs more similar to classical lattices. Our results show that cycles are not an obstacle for proving the influence of boundary conditions on the mixing time.

The work is organized as follows. In section 2 we give some basic definitions and state the main results. In section 3 we analyze the system at equilibrium and prove a mixing property of the plus phase. Then, in section 4, we deduce from this property a lower bound for the spectral gap of the dynamics and conclude the proof of our main result. Finally, in section 5, we give an example of a graph satisfying the hypothesis of the main theorem, and prove for it an exponential lower bound on the spectral gap for the free boundary dynamics.

## 2 The model: definitions and main result

### 2.1 Graph setting

Before describing the class of graphs in which we are interested, let us fix some notation and recall a few definitions concerning the graph structure.

Let $G = (V, E)$ be a general (finite or infinite) graph, where $V$ denotes the vertex set and $E$ the edge set. We will always implicitly assume that $G$ is connected. The *graph distance* between two vertices $x, y \in V$ is defined as the length of the shortest path from $x$ to $y$ and it is denoted by $d(x, y)$. If $x$ and $y$ are at distance one, i.e. if they are neighbors, we write $x \sim y$. The set of neighbors of $x$ is denoted by $N_x$, and $|N_x|$ is called the *degree* of $x$.
For a given subset $S \subset V$, let $E(S)$ be the set of all edges in $E$ which have both their end vertices in $S$ and define the *induced subgraph* on $S$ by $G(S) := (S, E(S))$. When it creates no confusion, we will identify $G(S)$ with its vertex set $S$.

For $S \subset V$ let us introduce the *vertex boundary* of $S$

$$\partial_V S = \{x \in V \setminus S : \exists y \in S \text{ s.t. } x \sim y\}$$

and the *edge boundary* of $S$

$$\partial_E S = \{e = (x, y) \in E \text{ s.t. } x \in S, \, y \in V \setminus S\}.$$

If $G = (V, E)$ is an infinite, locally finite graph, we can define the *edge isoperimetric constant* of $G$ (also called *Cheeger constant*) by

$$i_e(G) := \inf \left\{ \frac{|\partial_E(S)|}{|S|} \, ; \, S \subset V \text{ finite} \right\}. \tag{2.1}$$

**Definition 2.1.** *An infinite graph $G = (V, E)$ is amenable if its edge isoperimetric constant is zero, i.e. if for every $\epsilon > 0$ there is a finite set of vertices $S$ such that $|\partial_E S| < \epsilon |S|$. Otherwise $G$ is nonamenable.*

Roughly speaking, a nonamenable graph is such that the boundary of every subgraph is of comparable size to its volume. A typical example of amenable graph is the lattice $\mathbb{Z}^d$, while one can easily show that regular trees, with branching number bigger than two, are nonamenable. We emphasize that nonamenability seems to be strongly related to the qualitative behavior of models in statistical mechanics. See, e.g., [15; 19; 20; 29] for results concerning the Ising and the Potts models, and [6; 7; 11; 14] for percolation and random cluster models.

In this paper we focus on a class of nonamenable graphs, that we call *growing graphs*, defined as follows. Given an infinite graph $G = (V, E)$ and a vertex $o \in V$, let $B_r(o)$ denote the ball centered in $o$ and with radius $r \in \mathbb{N}$ with respect to the graph distance, namely the finite subgraph induced on $\{x \in V : d(o, x) \leq r\}$, and let $L_r(o) := \{x \in V : d(x, o) = r\} = \partial_V B_{r-1}(o)$.

**Definition 2.2.** *An infinite graph $G = (V, E)$ is growing with parameter $g > 0$ and root $o \in V$, if*

$$\min_{x \in L_r(o), \, r \in \mathbb{N}} \left\{ \left| N_x \cap L_{r+1}(o) \right| - \left| N_x \cap B_r(o) \right| \right\} = g. \tag{2.2}$$

*We call $G$ a $(g, o)$-growing graph.*

It is easy to prove that a growing graph in the sense of Definition 2.2 is also nonamenable. The simplest example of growing graph with parameter $g$, is an infinite tree with minimal vertex degree equal to $g + 2$, where the growing property is satisfied for every choice of the root on the vertex set. On the other hand, there are many examples of growing graphs which are not cycle-free. Between them we mention hyperbolic graphs, that we will prove to be growing provided that the vertex-degree is sufficiently high.

Hyperbolic graphs are a family of infinite planar graphs characterized by a cycle periodic structure. They can be briefly described as follows (for their detailed construction see, e.g., [22], or Section 2 of ref. [27]). Consider a planar graph in which each vertex has the same degree, denoted by $v$, and each face (or *tile*) is equilateral with constant number of sides denoted by $s$. If the parameters $v$ and $s$ satisfy the relation $(v - 2)(s - 2) > 4$, then the graph can be embedded in the hyperbolic plane $\mathbb{H}^2$ and it is called *hyperbolic graph* with parameters $v$ and $s$. It will be denoted by $\mathbb{H}(v, s)$.

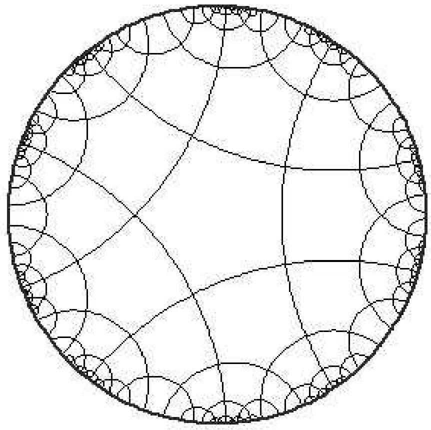

Figure 2.1: The hyperbolic graph $\mathbb{H}(4,5)$ in the Poincaré disc representation.

The typical representation of hyperbolic graphs make use of the Poincaré disc that is in bi-univocal correspondence with $\mathbb{H}^2$ (see Fig. 2.1).

Hyperbolic graphs are nonamenable, with edge isoperimetric constant explicitly computed in [11] as a function of $v$ and $s$. Moreover, the following holds:

**Lemma 2.3.** *For all couples $(v,s)$ such that $s \geq 4$ and $v \geq 5$, or $s = 3$ and $v \geq 9$, $\mathbb{H}(v,s)$ is a $(g,o)$-growing graph for every vertex $o \in V$ and with parameter $g = g(v,s)$.*

The proof of this Lemma is postponed to Section 5, where we will also construct a growing graph that will serve us as further example of influence of boundary conditions on the mixing time.

Let us stress, that due to the possible presence of cycles in a growing graph, a careful analysis of the correlations between spins will be required. This is actually the main distinction between our proof and the similar work on trees [26].

## 2.2 Ising model on nonamenable graphs

The Ising model on nonamenable graphs has been investigated in many papers (see, e.g, [20] for a survey). A general result, concerning the uniqueness/non-uniqueness phase transition of the model, is the following [15]:

**Theorem 2.4** (Jonasson and Steif)**.** *If $G$ is a connected nonamenable graph with bounded degree, then there exists an inverse temperature $\beta_0 > 0$, depending on the graph, such that for all $\beta \geq \beta_0$ there exists an interval of $h$ where $G$ exhibits a phase transition.*

Thus, contrary to what happens on the Euclidian lattice, the Ising model on nonamenable graphs undergoes a phase transition also at non zero value of the external field.

Though some properties of the Ising model are common to all nonamenable graphs, the particular behavior of the system may differ from one family to another one, also depending on other geometric parameters. Since we will be especially interested in hyperbolic graphs, we recall briefly the main results concerning the Ising model on these graphs, and stress which are the main differences from the model on classical lattices.

It has been proved (see [30; 33; 34]) that the Ising model on $\mathbb{H}(v,s)$ exhibits two different phase transitions appearing at inverse temperatures $\beta_c \leq \beta_c'$. The first one, $\beta_c$, corresponds to the occurrence of a uniqueness/non-uniqueness phase transition, while the second critical temperature refers to a change in the properties of the free boundary condition measure $\mu^f$. Specifically, it is defined as

$$\beta_c' := \inf\{\beta \geq \beta_c \,:\, \mu^f = (\mu^+ + \mu^-)/2\}\,, \tag{2.3}$$

where $\mu^+$ and $\mu^-$ denote the extremal measures obtained by imposing, respectively, $(+)$- and $(-)$-boundary condition. As is well explained in [34] (see also [8] for more details), using the Fortuin-Kasteleyn representation it is possible to show that $\beta_c' < \infty$ for all hyperbolic graphs, in contrast to the behavior of the model on regular trees where $\mu^f \neq (\mu^+ + \mu^-)/2$ for all finite $\beta \geq \beta_c$. From Definition 2.3 it turns out that for $\beta_c \leq \beta < \beta_c'$, when this interval is not empty (see [34]), the measure $\mu^f$ is not a convex combination of $\mu^+$ and $\mu^-$. This implies the existence of a translation invariant Gibbs state different from $\mu^+$ and $\mu^-$, in contrast to what happens on $\mathbb{Z}^d$ [2].

Another interesting result concerning the Ising model on hyperbolic graphs, is due to Sinai and Series [30]. For low enough temperatures and $h = 0$, they proved the existence of uncountably many mutually singular Gibbs states which they conjectured to be extremal. This points out, once more, the difference between the system on hyperbolic graphs and on classical lattices, where it is known that the extremal measures are at most a countable number.

In this paper we are interested in the region of the phase diagram where the dynamics is highly sensitive to the boundary condition, namely when the temperature is low and the magnetic field is zero (phase coexistence region). Let us explain the model in detail and give the necessary definitions and notation.

Let $G = (V,E)$ be an infinite $(g,o)$-growing graph with maximal degree $\Delta$. For any $r \in \mathbb{N}$, we denote by $B_r = (V_r, E_r) \subset G$ the ball with radius $r$ centered in $o$. When it does not create confusion, we identify the subgraphs of $G$ with their vertex sets. Given a finite ball $B \equiv B_m$ and an Ising spin configuration $\tau \in \Omega_G$, let $\Omega_B^\tau \subset \{\pm 1\}^{B \cup \partial_V B}$ be the set of configurations that agree with $\tau$ on $\partial_V B$. Analogously, for any subset $A \subseteq V_m$ and any $\eta \in \Omega_B^\tau$, we denote by $\Omega_A^\eta \subset \{\pm 1\}^{A \cup \partial_V A}$ the set of configurations that agree with $\eta$ on $\partial_V A$. The Ising model on $A$ with $\eta$-boundary condition (b.c.) and zero external field is thus specified by the Gibbs probability measure $\mu_A^\eta$, with support on $\Omega_A^\eta$, defined as

$$\mu_A^\eta(\sigma) = \frac{1}{Z(\beta)} \exp\Big(\beta \sum_{(x,y) \in E(\overline{A})} \sigma_x \sigma_y\Big)\,, \tag{2.4}$$

where $Z(\beta)$ is a normalizing constant and the sum runs over all pairs of nearest neighbors in the induced subgraph on $\overline{A} = A \cup \partial_V A$.

Similarly, the Ising model on $A$ with free boundary condition is specified by the Gibbs measure $\mu_A$ supported on the set of configurations $\Omega_A := \{\pm 1\}^A$. This is defined as in (2.4) by replacing the sum over $E(\overline{A})$ in a sum over $E(A)$, namely cutting away the influence of the boundary $\partial_V A$. Notice that when $A = V_m$, $\mu_{V_m}^\eta$ is simply the Gibbs measure on $B$ with boundary condition $\tau$ ($\eta$ agrees with $\tau$ on $\partial_V V_m \equiv \partial_V B$) and $\mu_{V_m}$ is the Gibbs measure on $B$ with free boundary condition.

We denote by $\mathscr{F}_A$ the $\sigma$-algebra generated by the set of projections $\{\pi_x\}_{x \in A}$ from $\{\pm 1\}^A$ to $\{\pm 1\}$, where $\pi_x : \sigma \mapsto \sigma_x$, and write $f \in \mathscr{F}_A$ to indicate that $f$ is $\mathscr{F}_A$-measurable. Finally, we recall that if $f : \Omega_B^\tau \to \mathbb{R}$ is a measurable function, the *expectation of* $f$ w.r.t. $\mu_A^\eta$ is given by $\mu_A^\eta(f) = \sum_{\sigma\in\Omega} \mu_A^\eta(\sigma) f(\sigma)$ and the *variance of* $f$ w.r.t. $\mu_A^\eta$ is given by $\mathrm{Var}_A^\eta = \mu_A^\eta(f^2) - \mu_A^\eta(f)^2$. We usually think of them as functions of $\eta$, that is $\mu_A(f)(\eta) = \mu_A^\eta(f)$ and $\mathrm{Var}_A(f)(\eta) = \mathrm{Var}_A^\eta(f)$. In particular $\mu_A(f)\,, \mathrm{Var}_A(f) \in \mathscr{F}_{A^c}$.

In the following discussion we will be concerned with the Ising model on $B$ with $(+)$-b.c. and we will use the abbreviations $\Omega^+$, $\mathscr{F}$ and $\mu$ instead of $\Omega_B^+$, $\mathscr{F}_{\overline{B}}$ and $\mu_B^+$, and thus $\mu(f)$ and $\mathrm{Var}(f)$ instead of $\mu_B^+(f)$ and $\mathrm{Var}_B^+(f)$.

### 2.3 Glauber dynamics and mixing time

The Glauber dynamics on $B$ with $(+)$-boundary condition is a continuous time Markov chain $(\sigma(t))_{t\geq 0}$ on $\Omega^+$ with Markov generator $\mathscr{L}$ given by

$$(\mathscr{L}f)(\sigma) = \sum_{x\in B} c_x(\sigma)\left[f(\sigma^x) - f(\sigma)\right]\,, \tag{2.5}$$

where $\sigma^x$ denotes the configuration obtained from $\sigma$ by flipping the spin at the site $x$ and $c_x(\sigma)$ is the jump rate from $\sigma$ to $\sigma^x$. We sometimes prefer the short notation $\nabla_x f(\sigma) = [f(\sigma^x) - f(\sigma)]$. The jump rates are required to be of finite-range, uniformly positive, bounded, and they should satisfy the detailed balance condition w.r.t. the Gibbs measure $\mu$. Although all our results apply to any choice of jump rates satisfying these hypothesis, for simplicity we will work with a specific choice called *heat-bath dynamics*:

$$c_x(\sigma) := \mu_x^\sigma(\sigma^x) = \frac{1}{1+\omega_x(\sigma)} \quad \text{where} \quad \omega_x(\sigma) := \exp(2\beta\sigma_x \sum_{y\sim x}\sigma_y\,). \tag{2.6}$$

It is easy to check that the Glauber dynamics is ergodic and reversible w.r.t. the Gibbs measure $\mu$, and so converges to $\mu$ by the Perron-Frobenius Theorem. The key point is now to determine the rate of convergence of the dynamics.

A useful tool to approach this problem is the *spectral gap* of the generator $\mathscr{L}$, that can be defined as the inverse of the first nonzero eigenvalue of $\mathscr{L}$.

**Remark 2.5.** *Notice that the generator $\mathscr{L}$ is a non-positive self-adjoint operator on $\ell^2(\Omega^+,\mu)$. Its spectrum thus consists of discrete eigenvalues of finite multiplicity that can be arranged as $0 = \lambda_0 \geq -\lambda_1 \geq -\lambda_2 \geq \ldots, \geq -\lambda_{N-1}$, if $|\Omega^+| = N$, with $\lambda_i \geq 0$.*

An equivalent definition of spectral gap is given through the so called *Poincaré inequality* for the measure $\mu$. For a function $f : \Omega^+ \mapsto \mathbb{R}$, define the *Dirichlet form* of $f$ associated to $\mathscr{L}$ by

$$\mathscr{D}(f) := \frac{1}{2}\sum_{x\in B}\mu\left(c_x[\nabla_x f]^2\right) = \sum_{x\in B}\mu(\mathrm{Var}_x(f))\,, \tag{2.7}$$

where the second equality holds under our specific choice of jump rates. The *spectral gap* of the generator, $c_{gap}(\mu)$, is then defined as the inverse of the best constant $c$ in the *Poincaré inequality*

$$\mathrm{Var}(f) \leq c\mathscr{D}(f)\,, \quad \forall f \in \ell^2(\Omega^+,\mu), \tag{2.8}$$

or equivalently

$$c_{gap}(\mu) := \inf\left\{\frac{\mathscr{D}(f)}{\mathrm{Var}(f)}; \mathrm{Var}(f) \neq 0\right\}. \tag{2.9}$$

Denoting by $P_t$ the Markov semigroup associated to $\mathscr{L}$, with transition kernel $P_t(\sigma,\eta) = e^{t\mathscr{L}}(\sigma,\eta)$, it easy to show that

$$\mathrm{Var}(P_t f) \leq e^{-2c_{gap}(\mu)t}\,\mathrm{Var}(f). \tag{2.10}$$

The last inequality shows that the spectral gap gives a measure of the exponential decay of the variance, and justifies the name *relaxation time* for the inverse of the spectral gap.

Moreover, let $h_t^\sigma$ denote the density of the distribution at time $t$ of the process starting at $\sigma$ w.r.t. $\mu$, i.e. $h_t^\sigma(\eta) = \frac{P_t(\sigma,\eta)}{\mu(\eta)}$. For $1 \leq p \leq \infty$ and a function $f \in \ell^p(\Omega^+, \mu)$, let $\|f\|_p$ denote the $\ell^p$ norm of $f$ and define the time of convergence

$$\tau_p = \min\left\{t > 0 : \sup_\sigma \|h_t^\sigma - 1\|_p \leq e^{-1}\right\}, \tag{2.11}$$

that for $p = 1$ is called *mixing time*. A well known and useful result relating $\tau_p$ to the spectral gap (see, e.g., [28]), when specializing to the Glauber dynamics yields the following:

**Theorem 2.6.** *On an n-vertex ball $B \subset G$ with $(\tau)$-boundary condition,*

$$c_{gap}(\mu)^{-1} \leq \tau_1 \leq c_{gap}(\mu)^{-1} \times cn, \tag{2.12}$$

*where $\mu = \mu_B^\tau$ and $c$ is a positive constant independent of $n$.* □

We stress that a different choice of jump rates (here we considered the heat-bath dynamics) only affects the spectral gap by at most a constant factor. The bound stated in Theorem 2.6 is thus equivalent, apart for a multiplicative constant, for any choice of the Glauber dynamics.

Before presenting our main result, we recall that the Glauber dynamics for the Ising model on regular trees and hyperbolic graphs has been recently investigated by Peres et al. [17; 4]. In particular, they consider the free boundary dynamics on a finite ball $B \subset G$, $G$ hyperbolic graph or regular tree, and prove that at all temperatures, the inverse spectral gap (relaxation time) scales at most polynomially in the size of $B$, with exponent $\alpha(\beta) \uparrow \infty$ as $\beta \to \infty$. Let us stress again that under the same conditions, the dynamics on a cube of size $n$ in the $d$-dimensional lattice, relaxes in a time exponentially large in the surface area $n^{(d-1)/d}$.

## 2.4 Main results

We are finally in position to state our main results.

**Theorem 2.7.** *Let $G$ be an infinite $(o,g)$-growing graph with maximal degree $\Delta$. Then, for all $\beta \gg 1$, the Glauber dynamics on the n-vertex ball $B$ with (+)-boundary condition and zero external field has spectral gap $\Omega(1)$.*

As a corollary we obtain that, under the same hypothesis of the theorem above, the mixing time of the dynamics is bounded linearly in $n$ (see Theorem 2.6).

This result, applied to hyperbolic graphs with sufficiently high degree, provides a convincing example of the influence of the boundary condition on the mixing time. Indeed, for all $\beta \gg 1$, due to the fact that the free boundary measure on $\mathbb{H}(v,s)$ is a convex combination of $\mu^+$ and $\mu^-$ (see section 2.2), it is not hard to prove that the spectral gap for an $n$-vertex ball in the hyperbolic graph with free boundary condition, is decreasing with $n$. Most likely it will be of order $n^{-\alpha(\beta)}$, with $\alpha(\beta) \uparrow \infty$ as $\beta \to \infty$, as in the lower bound given in [17; 4]. The presence of $(+)$-boundary condition thus gives rise to a jump of the spectral gap, and consequently it speeds up the dynamics.

**Remarks.**

(i) *We recall that on $\mathbb{Z}^d$ not much is known about the mixing time when $\beta > \beta_c$, $h = 0$ and the boundary condition is $(+)$, though it has been conjectured that the it should be polynomial in $n$ (see [9] and [5]).*

(ii) *A result similar to Theorem 2.7 has been obtained for the spectral gap, and thus for the mixing time, of the dynamics on a regular b-ary tree (see [26]). In particular it has been proved that while under free-boundary condition the mixing time on a tree of size $n$ jumps from $\log n$ to $n^{\Theta(\beta)}$ when passing a certain critical temperature, it remains of order $\log n$ at all temperatures and at all values of the magnetic field under $(+)$-boundary condition. However we stress that while trees do not have any cycle, growing graphs, and in general nonamenable graphs, can have many cycles, as well as the Euclidean lattices. The theorem above can thus be looked upon as an extension of this result to a class of graphs which in some respects are similar to Euclidean lattices.*

(iii) *At high enough temperatures (one phase region) the spectral gap of the dynamics on a ball $B \subset G$, where $G$ is an infinite graph with bounded degree, is $\Omega(1)$ for all boundary conditions, as can be proved by path coupling techniques [32]. This suggests that the result of Theorem 2.7 should hold for all temperatures, as for the dynamics on regular trees, and not only for $\beta \gg 1$. At the moment, what happens in the intermediate region of temperature, still remains an open question.*

The following result provides a further example of influence of boundary conditions on the dynamics.

**Theorem 2.8.** *For all finite $g \in \mathbb{N}$, there exists an infinite $(g,o)$-growing graph $G$ with bounded degree, such that, for all $\beta \gg 1$, the Glauber dynamics on the $n$-vertex ball $B$ with free boundary condition and zero external field has spectral gap $O\left(e^{-\theta n}\right)$, with $\theta = \theta(g,\beta) > 0$.*

Combining this with Theorem 2.6 and Theorem 2.7, we get a first rigorous example where the mixing time jumps abruptly from exponential to linear in $n$ while passing from one boundary condition to another.

We now proceed to sketch briefly the ideas and techniques used along the paper.

The proof of our main result, Theorem 2.7, is based on the variational definition of the spectral gap and it is aimed to show that the Gibbs measure relative to the system satisfies a Poincaré inequality with constant $c$ independent of the size of $B$. We will first analyze the equilibrium properties of the system conditioned on having $(+)$-boundary, and under this condition we will deduce a special kind of correlation decay between spins. The proof of this spatial mixing property rests on a disagreement argument and on a Peierls type argument.

The second main step to prove Theorem 2.7, is deriving a Poincaré inequality for the Gibbs measure from the obtained notion of spatial mixing. This will be achieved by first deducing, via coupling techniques, a like-Poincaré inequality for the marginal Gibbs measure with support on suitable subsets, and then iterating the argument to recover the required estimate on the variance.

The proof of Theorem 2.8 is given by the explicit construction of a growing graph with the property of remaining "expander" even when the boundary of a finite ball is erased, as in the free measure. Using a suitable test function, we will prove the stated exponentially small upper bound on the spectral gap.

## 3 Mixing properties of the plus phase

In this section we analyze the effect of the $(+)$-boundary condition on the equilibrium properties of the system. In particular, we prove that the Gibbs measure $\mu \equiv \mu_B^+$ satisfies a kind of *spatial mixing property*, i.e. a form of weak dependence between spins placed at distant sites.

Before presenting the main result of this section, we need some more notation and definitions. Recall that for every integer $i$, we denoted by $B_i = (V_i, E_i)$ the ball of radius $i$ centered in $o$, and by $B = B_m$ the ball of radius $m$ such that $|V_m| = n$. Let us define the following objects:

(i) the $i$-th *level* $L_i = \{x \in V \,:\, d(x,o) = i\} \equiv \partial_V B_{i-1}$;

(ii) the vertex-set $F_i \subseteq B$ given by $F_i := \{x \in B_{i-1}^c \cap B\}$;

(iii) the $\sigma$-algebra $\mathscr{F}_i$ generated by the functions $\pi_x$ for $x \in F_i^c = B_{i-1}$.

We will be mainly concerned with the Gibbs distribution on $F_i$ with boundary condition $\eta \in \Omega^+$, which we will shortly denote by $\mu_i^\eta = \mu_{F_i}^\eta = \mu(\cdot|\eta \in \mathscr{F}_i)$; analogously we will denote by $\mathrm{Var}_i^\eta$ the variance w.r.t. $\mu_i^\eta$.

Notice that $\{F_i\}_{i=0}^{m+1}$ is a decreasing sequence of subsets such that $V_m = F_0 \supset F_1 \supset \ldots \supset F_{m+1} = \emptyset$, and in particular $\mu_i(\mu_{i+1}(f)) = \mu_i(f)$, for all finite $i$, and $\mu_{m+1}(f) = f$. The set of variables $\{\mu_i(f)\}_{i\geq 0}$ is a Martingale with respect to the filtration $\{\mathscr{F}_i\}_{i\geq 0}$.

For a given $i \in \{0,\ldots,m\}$ and a given subset $S \subset L_i$, we set $U = F_{i+1} \cup S$ and consider the Gibbs measure conditioned on the configuration outside $U$ being $\tau \in \Omega^+$, which as usually will be denoted by $\mu_U^\tau$.

We are now able to state the following:

**Proposition 3.1.** *Let $G$ a $(g,o)$-growing graph with maximal degree $\Delta$. Then there exists a constant $\delta = \delta(\Delta) > 0$ such that, for every $\beta > \frac{\delta}{2g}$, every $\tau \in \Omega^+$, and every pair of vertices $x \in S \subset L_i$ and $y \in L_i \setminus S$, $i \in \{0,\ldots,m\}$,*

$$|\mu_U^\tau(\sigma_x = +) - \mu_U^{\tau^y}(\sigma_x = +)| \leq c e^{-\beta' d(x,y)}, \tag{3.1}$$

*with $\beta' := 2g\beta - \delta > 0$ and for some constant $c > 0$.*

Let us briefly justify the above result. Since the boundary of $B$ is proportional to its volume, the $(+)$-b.c. on $B$ is strong enough to influence spins at arbitrary distance. In particular, as we will prove, the effect of the (+)-boundary on a given spin $\sigma_x$, weakens the influence on $\sigma_x$ coming from other spins (placed in vertices arbitrary near to $x$) and gives rise to the decay correlation stated in Proposition 3.1. Notice that the correlation decay increases with $\beta$.

The proof of Proposition 3.1 is divided in two parts. First, we define a suitable event and show that the correlation between two spins is controlled by the probability of this event. Then, in the second part, we estimate this probability using a Peierls type argument. Throughout the discussion $c$ will denote a constant which is independent of $|B| = n$, but may depend on the parameters $\Delta$ and $g$ of the graph, and on $\beta$. The particular value of $c$ may change from line to line as the discussion progresses.

## 3.1 Proof of Proposition 3.1

Let us consider two vertices $x \in S \subset L_i$ and $y \in L_i \setminus S$, such that $d(x,y) = \ell$, and a configuration $\tau \in \Omega^+$. Let $\tau^{y,+}$ be the configuration that agrees with $\tau$ in all sites but $y$ and has a (+)-spin on $y$; define analogously $\tau^{y,-}$ and denote by $\mu_U^{y,+}$ and $\mu_U^{y,-}$ the measures conditioned on having respectively $\tau^{y,+}$- and $\tau^{y,-}$-b.c.. With this notation and from the obvious fact that the event $\{\sigma : \sigma_x = +\}$ is increasing, we get that

$$|\mu_U^{\tau}(\sigma_x = +) - \mu_U^{\tau^y}(\sigma_x = +)| = \mu_U^{y,+}(\sigma_x = +) - \mu_U^{y,-}(\sigma_x = +). \tag{3.2}$$

In the rest of the proof we will focus on the correlation in the r.h.s. of (3.2).

In order to introduce and have a better understanding of the ideas and techniques that we will use along the proof, we first consider the case $\ell = 1$, which is simpler but with a similar structure to the general case $\ell > 1$.

### 3.1.1 Correlation decay: the case $\ell = 1$

Assume that $\ell = 1$, namely that $x$ and $y$ are neighbors. Denoting by $\mu_U^-$ the measure with (−)-b.c. on $U^c = B_i \setminus \{S\}$ and (+)-b.c. on $\partial_V B$, we get

$$\begin{aligned} \mu_U^{y,+}(\sigma_x = +) - \mu_U^{y,-}(\sigma_x = +) &= \mu_U^{y,-}(\sigma_x = -) - \mu_U^{y,+}(\sigma_x = -) \\ &\leq \mu_U^{y,-}(\sigma_x = -) \\ &\leq \mu_U^{-}(\sigma_x = -), \end{aligned} \tag{3.3}$$

where the last inequality follows by monotonicity. The problem is thus reduced to estimate the probability of the event $\{\sigma : \sigma_x = -\}$ w.r.t. $\mu_U^-$.

Let $\mathscr{K}$ be the set of connected subsets of $U$ containing $x$ and write

$$\mathscr{K} = \bigsqcup_{p \geq 1} \mathscr{K}_p \qquad \text{with} \quad \mathscr{K}_p = \{C \in \mathscr{K} \ \text{s.t.}\ |C| = p\}.$$

For any configuration $\sigma \in \Omega^+$, we denote by $K^{(\sigma)}$ the maximal negative component in $\mathscr{K}$ admitted by $\sigma$, i.e.

$$K^{(\sigma)} \in \mathscr{K} \quad \text{s.t.} \begin{cases} \sigma_z = - & \forall\, z \in K^{(\sigma)} \\ \sigma_z = + & \forall\, z \in \partial_V K^{(\sigma)} \cap U \end{cases} \tag{3.4}$$

With this notation the event $\{\sigma : \sigma_x = -\}$ can be expressed by means of disjoint events as

$$\{\sigma : \sigma_x = -\} = \bigsqcup_{p\geq 1} \bigsqcup_{C\in\mathscr{K}_p} \{\sigma : K^{(\sigma)} = C\}, \tag{3.5}$$

and then

$$\mu_U^-(\sigma_x = -) = \sum_{p\geq 1} \sum_{C\in\mathscr{K}_p} \mu_U^-(K^{(\sigma)} = C). \tag{3.6}$$

Let us introduce the symbol $\sigma \sim C$ for a configuration $\sigma$ such that $\sigma_C = -$ and $\sigma_{\partial_V C\cap U} = +$. The main step in the proof is to show the following claim:

**Claim 3.2.** *If $G$ is a $(g,o)$-growing graph with maximal degree $\Delta$, then, for any subset $C \subset U$,*

$$\mu_U^-(\sigma \sim C) \leq e^{-2g\,\beta|C|}. \tag{3.7}$$

The proof of Claim 3.2 is postponed to subsection 3.2. Let us assume for the moment its validity and complete the proof of the case $\ell = 1$. By Claim 3.2 and from the definition of $K^{(\sigma)}$, we get

$$\mu_U^-(K^{(\sigma)} = C) \leq e^{-2g\,\beta|C|}. \tag{3.8}$$

We now recall the following Lemma due to Kesten (see [16]).

**Lemma 3.3.** *Let $G$ an infinite graph with maximum degree $\Delta$ and let $\mathscr{C}_p$ be the set of connected sets with $p$ vertices containing a fixed vertex $v$. Then $|\mathscr{C}_p| \leq (e(\Delta+1))^p$.*

Applying Lemma 3.3 to the set $\mathscr{K}_p$, we obtain the bound $|\mathscr{K}_p| \leq e^{\delta p}$, with $\delta = 1 + \log(\Delta+1)$. Continuing from (3.6), we finally get that for all $\beta' = 2g\beta - \delta > 0$, i.e. for all $\beta > \frac{\delta}{2g}$,

$$\begin{aligned} \mu_U^-(\sigma_x = -) &\leq \sum_{p\geq 1} \sum_{C\in\mathscr{K}_p} e^{-2g\,\beta p} \\ &\leq \sum_{p\geq 1} e^{-2g\beta p} e^{\delta p} \\ &\leq c e^{-\beta'} \end{aligned} \tag{3.9}$$

which concludes the proof of (3.1) in the case $\ell = 1$.

Notice that the argument above only involves the spin at $x$, and thus applies for all pairs of $x, y \in L_i$, independently of their distance. Anyway, when $d(x,y) > 1$ this method does not provide the decay with the distance stated in Proposition 3.1, and a different approach is required.

### 3.1.2 Correlation decay: the case $\ell > 1$

Let us now consider two vertices $x \in S \subset L_i$ and $y \in L_i \setminus S$, such that $d(x,y) = \ell > 1$. Before defining new objects, we want to clarify the main idea beyond the proof. Since the measure $\mu_U^\tau$ fixes the configuration on all sites in $U^c \equiv B_i \setminus S$, the vertex $y$ can communicate with $x$ only through paths going from $x$ to $y$ and crossing vertices in $U$. However, the effect of this communication can be very small compared to the information arriving to $x$ from the $(+)$-boundary. In particular, if every path starting from $y$ crosses a $(+)$-spin before arriving to $x$, then the communication between them is interrupted. Let us formalize this assertion.

We denote by $\mathscr{C}$ the set of connected subsets $C \subseteq U \cup \{y\}$ such that $y \in C$, and call an element $C \in \mathscr{C}$ a *component* of $y$. For every configuration $\sigma \in \Omega^+$, we define $C^{(\sigma)}$ as the maximal component of $y$ which is negative on $C^{(\sigma)} \cap U$, i.e

$$C^{(\sigma)} \in \mathscr{C} \quad \text{s.t.} \begin{cases} \sigma_z = - & \forall z \in C^{(\sigma)} \cap U \\ \sigma_z = + & \forall z \in \partial_V C^{(\sigma)} \cap U \end{cases} . \tag{3.10}$$

Observe that the spin on $y$ is not fixed under the event $\{\sigma : C^{(\sigma)} = C\}$. Finally, let $\mathscr{C}^{\emptyset} := \{C \in \mathscr{C} \text{ s.t. } x \notin C\}$ and define the event

$$A := \{\sigma : C^{(\sigma)} \in \mathscr{C}^{\emptyset}\} = \bigsqcup_{C \in \mathscr{C}^{\emptyset}} \{\sigma : C^{(\sigma)} = C\}. \tag{3.11}$$

Then we have

$$\begin{aligned} \mu_U^{y,-}(\sigma_x = + | A) &= \sum_{C \in \mathscr{C}^{\emptyset}} \mu_U^{y,-}(\sigma_x = +, C^{(\sigma)} = C \,|A) \\ &= \frac{\sum_{C \in \mathscr{C}^{\emptyset}} \mu_U^{y,-}(\sigma_x = +, C^{(\sigma)} = C\,)}{\sum_{C \in \mathscr{C}^{\emptyset}} \mu_U^{y,-}(\,C^{(\sigma)} = C\,)} \\ &= \frac{\sum_{C \in \mathscr{C}^{\emptyset}} \mu_U^{y,-}(\sigma_x = + \,|\, C^{(\sigma)} = C\,)\mu_U^{y,-}(\,C^{(\sigma)} = C\,)}{\sum_{C \in \mathscr{C}^{\emptyset}} \mu_U^{y,-}(\,C^{(\sigma)} = C\,)} \\ &\geq \min_{C \in \mathscr{C}^{\emptyset}} \mu_U^{y,-}(\sigma_x = + \,|\, C^{(\sigma)} = C). \end{aligned} \tag{3.12}$$

Notice that when the measure $\mu_U^{y,-}$ is conditioned on the event $\{\sigma : C^{(\sigma)} = C\}$, the spin configuration on $\partial_V C$ is completely determined by the boundary condition: on $\partial_V C \cap U$ it is given by all $(+)$-spins and on $\partial_V C \cap U^c$ it corresponds to $\tau^{y,-}$. Hence, spins on $U \setminus (C \cup \partial_V C)$ become independent of spins on $C$, and we get

$$\begin{aligned} \mu_U^{y,-}(\,\cdot\,|\, C^{(\sigma)} = C) &= \mu_{K_x}^{y,-}(\,\cdot\,|\sigma_z = +\,,\, z \in \partial_V C \cap U) \\ &= \mu_U^{y,+}(\,\cdot\,|\sigma_z = +\,,\, z \in (C \cup \partial_V C) \cap U) \\ &\geq \mu_U^{y,+}(\,\cdot\,), \end{aligned} \tag{3.13}$$

where the last inequality follows by stochastic domination. Being $\{\sigma : \sigma_x = +\}$ an increasing event, and from (3.12) and (3.13), we get

$$\mu_U^{y,-}(\sigma_x = + | A) \geq \mu_U^{y,+}(\sigma_x = +),$$

which with the obvious fact that $\mu_U^{y,-}(\sigma_x = +) \geq \mu_U^{y,-}(\sigma_x = + | A)\mu_U^{y,-}(A)$, implies

$$\mu_U^{y,+}(\sigma_x = +) - \mu_U^{y,-}(\sigma_x = +) \leq \mu_U^{y,-}(A^c). \tag{3.14}$$

By monotonicity and being $A^c$ a decreasing event, we get the inequality $\mu_U^{y,-}(A^c) \leq \mu_U^{-}(A^c)$, where, we recall, $\mu_U^{-}$ denotes the measure on $U$ conditioned on having all $(-)$-spins on $U^c$. We now focus on $\mu_U^{-}(A^c)$.

Let $\mathscr{C}^{\neq\emptyset}$ denote the set of components of $y$ containing $x$, and for every $p \in \mathbb{N}$, let $\mathscr{C}_p$ be the set of components in $\mathscr{C}^{\neq\emptyset}$ with $p$ vertices, i.e

$$\mathscr{C}_p := \{C \in \mathscr{C}^{\neq\emptyset} \text{ s.t. } |C| = p\} \qquad \mathscr{C}^{\neq\emptyset} := \bigsqcup_{p>0} \mathscr{C}_p.$$

Notice that if $C \in \mathscr{C}^{\neq\emptyset}$, then $|C| \geq \ell + 1$, since $d(x, y) = \ell$. Thus, $A^c$ can be expressed by means of disjoint events as

$$A^c = \bigsqcup_{p \geq \ell+1} \bigsqcup_{C \in \mathscr{C}_p} \{\sigma : C^{(\sigma)} = C\}, \tag{3.15}$$

and we get

$$\mu_U^-(A^c) = \sum_{p \geq \ell+1} \sum_{C \in \mathscr{C}_p} \mu_U^-(C^{(\sigma)} = C). \tag{3.16}$$

Since $\partial_V(C \setminus \{y\}) \cap U \subseteq \partial_V C \cap U$, we observe that the event $\{\sigma : C^{(\sigma)} = C\} \equiv \{\sigma : \sigma_{C\setminus\{y\}} = -, \sigma_{\partial_V C \cap U} = +\}$ is a subset of $\{\sigma : \sigma_{C\setminus\{y\}} = -, \sigma_{\partial_V(C\setminus\{y\})\cap U} = +\} \equiv \{\sigma : \sigma \sim C \setminus \{y\}\}$. Applying the result stated in Claim 3.2 to the set $C \setminus \{y\}$, we obtain the bound

$$\mu_U^-(C^{(\sigma)} = C) \leq e^{-2g\,\beta(|C|-1)}, \tag{3.17}$$

which holds under the same hypothesis of the claim. Continuing from (3.16), we then have that for all $\beta' = 2g\beta - \delta > 0$, i.e. for all $\beta > \frac{\delta}{2g}$,

$$\begin{aligned} \mu_U^-(A^c) &\leq \sum_{p \geq \ell+1} \sum_{C \in \mathscr{C}_p} e^{-2g\,\beta(p-1)} \\ &\leq e^{\delta} \sum_{p \geq \ell} e^{-(2g\beta-\delta)p} \\ &\leq c e^{-\beta'\ell}, \end{aligned} \tag{3.18}$$

where in the second line we used the bound $|\mathscr{C}_p| \leq e^{\delta p}$ due to Lemma 3.3. This concludes the proof of Proposition 3.1. In the next subsection we will go back and prove Claim 3.2.

## 3.2 Proof of Claim 3.2

To estimate the probability $\mu_U^-(\sigma \sim C)$, we now appeal to a kind of Peierls argument that runs as follows (see also [15]). Given a subset $C \subseteq U$, we consider the edge boundary $\partial_E C$ and define

$$\begin{aligned} \partial_+ C &:= \{e = (z, w) \in \partial_E C \;:\; z, w \in U\} \\ \partial_- C &:= \{e = (z, w) \in \partial_E C \;:\; z \text{ or } w \in U^c\} \end{aligned}. \tag{3.19}$$

The meaning of this notation can be better understood if we consider a configuration $\sigma \in \Omega_U^-$ such that $C^{(\sigma)} = C$ (see (3.10)). In this case $\sigma$ has $(-)$-spins on both the end-vertices of every edge in $\partial_- C$ and a $(+)$-spin in one end-vertex of every edge in $\partial_+ C$. Similarly if we consider $\sigma$ such that $K^{(\sigma)} = C$ (see (3.4)).

For every $\sigma \in \Omega_U^-$ such that $\sigma \sim C$, let $\sigma^* \in \Omega_U^-$ denote the configuration obtained by a global spin flip of $\sigma$ on the subset $C$, and observe that the map $\sigma \to \sigma^*$ is injective. This flipping changes the

Hamiltonian contribute of the interactions just along the edges in $\partial_E C$. In particular $\sigma^*$ loses the positive contribute of the edges in $\partial_+ C$ and gains the contribute of the edges in $\partial_- C$, and then we get

$$H_U^-(\sigma^*) = H_U^-(\sigma) - 2(|\partial_+ C| - |\partial_- C|). \tag{3.20}$$

From this, we have

$$\begin{aligned}
\mu_U^-(\sigma \sim C) &= \sum_{\{\sigma:\sigma\sim C\}} \frac{e^{-\beta H_U^-(\sigma)}}{Z_U^-} \\
&\leq \frac{\sum_{\{\sigma:\sigma\sim C\}} e^{-\beta H_U^-(\sigma)}}{\sum_{\{\sigma:\sigma\sim C\}} e^{-\beta H_U^-(\sigma^*)}} \\
&= e^{-2\beta(|\partial_+ C| - |\partial_- C|)},
\end{aligned} \tag{3.21}$$

where in the first inequality we reduced the partition function to a summation over $\{\sigma : \sigma \sim C\}$ and then we applied (3.20).

The following Lemma concludes the proof of Claim 3.2.

**Lemma 3.4.** *Let $G$ a $(g, o)$-growing graph with maximal degree $\Delta$. Then, for every subset $C \subseteq U$,*

$$|\partial_+ C| - |\partial_- C| \geq g|C|. \tag{3.22}$$

*Proof.* For a subset $C \subseteq U$, we define the downward boundary of $C$, $\partial_\downarrow C$, as the edges of $\partial_E C$ such that the endpoint in $C$ is in a higher level (strictly small index) than the endpoint not in $C$, i.e.

$$\partial_\downarrow C = \{(u, v) \in \partial_E C : \exists j \text{ s.t. } u \in L_j \cap C,\ v \in L_{j+1}\}.$$

We then define the not-downward boundary of $C$, $\partial_\Rsh C$, as the edges in $\partial_E C$ which are not-downward edges, i.e. $\partial_\Rsh C = \partial_E C \setminus \partial_\downarrow C$.
Notice that $\partial_\downarrow C \subseteq \partial_+ C$, while $\partial_\Rsh C \supseteq \partial_- C$. In particular, inequality (3.22) follows from the bound

$$|\partial_\downarrow C| - |\partial_\Rsh C| \geq g|C|. \tag{3.23}$$

For all $j \geq 0$, define $C_j = C \cap L_j$ and notice that, by the growing property of $G$,

$$|\partial_\downarrow C_j| - |\partial_\Rsh C_j| \geq g|C_j|. \tag{3.24}$$

Moreover, one can easily realize that

$$|\partial_\downarrow C| - |\partial_\Rsh C| = \sum_{j\geq 0} |\partial_\downarrow C_j| - |\partial_\Rsh C_j|. \tag{3.25}$$

In fact, all edges belonging to $\partial_E C_j$, for some $j$, but not belonging to $\partial_E C$, are summed once as downward edges and subtracted once as non-downward edges. In conclusion, the last two inequalities imply bound (3.23) and conclude the proof of the lemma. $\square$

## 4 Fast mixing inside the plus phase

In this section we will prove that the spectral gap of the Glauber dynamics, in the situation described by Theorem 2.7, is bounded from zero uniformly in the size of the system. From Definition 2.9 of spectral gap, this is equivalent to showing that for all inverse temperature $\beta \gg 1$, the Poincaré inequality

$$\mathrm{Var}(f) \leq c\,\mathscr{D}(f), \quad \forall f \in L^2(\Omega^+, \mathscr{F}, \mu)$$

holds with constant $c$ independent of the size of $B$.

First, we give a brief sketch of the proof. The rest of the section is divided into two parts. In the first part, from the mixing property deduced in section 3 and by means of coupling techniques, we derive a Poincaré inequality for some suitable marginal Gibbs measures. Then, in the second part, we will run a recursive argument that together with some estimates, also derived from Proposition 3.1, will yield the Poincaré inequality for the global Gibbs measure $\mu$.

### 4.1 Plan of the Proof

Let us first recall the following decomposition property of the variance which holds for all subsets $D \subseteq C \subseteq B$,

$$\mathrm{Var}_C^\eta(f) = \mu_C^\eta[\mathrm{Var}_D(f)] + \mathrm{Var}_C^\eta[\mu_D(f)]\,. \tag{4.1}$$

Applying recursively (4.1) to subsets $B \equiv F_0 \supset F_1 \supset \ldots \supset F_{m+1} = \emptyset$ and recalling the relations $\mu_i(\mu_{i+1}(f)) = \mu_i(f)$ and $\mu_{m+1}(f) = f$, we obtain

$$\begin{aligned}
\mathrm{Var}(f) &= \mu[\mathrm{Var}_m(f)] + \mathrm{Var}[\mu_m(f)] \\
&= \mu[\mathrm{Var}_m(\mu_{m+1}(f))] + \mu[\mathrm{Var}_{m-1}(\mu_m(f))] + \mathrm{Var}[\mu_{m-1}(\mu_m(f))] \\
&= \vdots \\
&= \sum_{i=0}^{m} \mu[\mathrm{Var}_i(\mu_{i+1}(f))]\,.
\end{aligned} \tag{4.2}$$

Notice that (4.2) can also be seen as a decomposition of the Martingale given by the set of variables $\{\mu_i(f)\}_{i\geq 0}$ respect to the filtration $\{\mathscr{F}_i\}_{i\geq 0}$.

To simplify the notation we define $g_i := \mu_i(f)$ for all $i = 0, \ldots, m+1$. Notice that $g_i \in \mathscr{F}_i$. Inserting $g_i$ in (4.2), we then have that

$$\mathrm{Var}(f) = \sum_{i=0}^{m} \mu[\mathrm{Var}_i(g_{i+1})]\,. \tag{4.3}$$

The proof of the Poincaré inequality for $\mu$, with constant independent of the size of the system, is given in the following two steps:

1. Proving that $\forall\, \tau \in \Omega^+$ and $i \in \{0, \ldots, m\}$, there exist suitable vertex-subsets $\{K_x\}_{x \in L_i}$, $K_x \ni x$, such that the like-Poincaré inequality

$$\mathrm{Var}_i^\tau(g_{i+1}) \leq c \sum_{x \in L_i} \mu_i^\tau(\mathrm{Var}_{K_x}(g_{i+1})) \tag{4.4}$$

   holds with constant $c$ uniformly bounded in the size of $L_i$;

2. Relating the variance of $g_i = \mu_i(f)$ to the variance of $f$ in order to get an inequality of the kind

$$\sum_{i=0}^{m} \sum_{x \in L_i} \mu(\mathrm{Var}_{K_x}(g_{i+1})) \le c\mathscr{D}(f) + \varepsilon \sum_{i=0}^{m} \sum_{x \in L_i} \mu(\mathrm{Var}_{K_x}(g_{i+1})) \tag{4.5}$$

with $\varepsilon$ a small quantity for $\beta \gg 1$.

Notice that from (4.5) the inequality

$$\sum_{i=0}^{m} \sum_{x \in L_i} \mu(\mathrm{Var}_{K_x}(g_{i+1})) \le c(1-\varepsilon)^{-1}\mathscr{D}(f)$$

follows with $c(1-\varepsilon)^{-1} = \Omega(1)$ for all $\beta \gg 1$. Together with Eqs. (4.3) and (4.4), this will establish the required Poincaré inequality for $\mu$ and therefore will conclude the proof of Theorem 2.7.

## 4.2 Step 1: From correlation decay to Poincaré inequality

In this section we prove that under the same hypothesis of Proposition 3.1, the marginal of the conditioned Gibbs measure on some suitable subsets, satisfies a Poincaré inequality with constant independent of the size of these subsets.

To state the result, let us fix a subset $S \subseteq L_i$ and a configuration $\tau \in \Omega^+$. We then define the measure

$$\nu_S^\tau(\sigma) := \sum_{\eta : \eta_S = \sigma_S} \mu(\eta \,|\, \tau \in \mathscr{F}_{B_i \setminus S}), \tag{4.6}$$

which is the marginal of the Gibbs measure $\mu^\tau_{F_{i+1} \cup S}$ on $S$, and denote by $\mathrm{Var}_{\nu_S^\tau}$ the variance w.r.t. $\nu_S^\tau$. We state the following:

**Theorem 4.1.** *For all $\beta \gg 1$ and for every subset $S \subseteq L_i$, $\tau \in \Omega^+$ and $f \in L^2(\Omega, \mathscr{F}_S, \nu_S^\tau)$, the measure $\nu_S^\tau$ satisfies the Poincaré inequality*

$$\mathrm{Var}_{\nu_S^\tau}(f) \le c_0 \sum_{x \in S} \nu_S^\tau(\mathrm{Var}_x(f)). \tag{4.7}$$

*with $c_0 = c_0(\beta, \Delta, g) = 1 + O(e^{-c\beta})$.*

**Remark 4.2.** *Before proceeding with the proof of Theorem 4.1, we point out that this result includes, as a particular case, inequality (4.4) for subsets $K_x = F_{i+1} \cup \{x\}$. To see that, choose $S = L_i$ so that $\mu(\cdot \,|\mathscr{F}_{B_i \setminus S}) \equiv \mu_i$. An easy computation shows that, for every function $f \in \mathscr{F}_{i+1}$, $\nu^\tau_{L_i}(f) \equiv \mu_i^\tau(f)$ and $\nu^\tau_{L_i}(\mathrm{Var}_x(f)) = \mu_i^\tau(\mathrm{Var}_{K_x}(f))$. Inequality (4.7) then corresponds to the like-Poincaré inequality*

$$\mathrm{Var}_i^\tau(g_{i+1}) \le c_0 \sum_{x \in L_i} \mu_i^\tau(\mathrm{Var}_{K_x}(g_{i+1})),$$

*which concludes the first step of the proof of Theorem 2.7.*

### 4.2.1 Proof of Theorem 4.1

The proof of Theorem 4.1 rests on the so called *coupling technique*. This is a useful method to bound from above the mixing time of Markov processes, introduced for the first time in this setting by Aldous [1] and subsequently refined to the *path coupling* [3; 21]. See also [18] for a wider discussion on the coupling method.

A *coupling* of two measure $\mu_1$ and $\mu_2$ on $\Omega$ is any joint distribution $\rho$ on $\Omega \times \Omega$ whose marginal are $\mu_1$ and $\mu_2$ respectively. Here, we want to construct a coupling of two Glauber dynamics on $\Omega_S$ with same reversible measure $\nu_S^\tau$ but different initial configurations. We denote by $\mathscr{L}_S$ the generator of this dynamics. We also recall that for all $\eta \in \Omega_S^\tau$, $x \in S$ and $a \in \{\pm 1\}$, the jump rates of the heat bath version of the dynamics (see Def. 2.6 ) are given by

$$\begin{aligned} c_x(\eta, a) &= \nu_S^\tau(\sigma_x = a \,|\, \eta \in \mathscr{F}_{S\setminus x}) \\ &= \mu(\sigma_x = a \,|\, \eta \in \mathscr{F}_{S\setminus x}, \tau \in \mathscr{F}_{B_i \setminus S}) \\ &= \mu_{K_x}^\eta(\sigma_x = a), \end{aligned} \tag{4.8}$$

where in the second line we applied the definition of $\nu_S^\tau$ and used that $\{\sigma_x = a\} \in \mathscr{F}_S$, and in the last line we adopt the notation $K_x := F_{i+1} \cup \{x\}$.

We now consider the coupled process $(\eta(t), \xi(t))_{t \geq 0}$ on $\Omega_S \times \Omega_S$ defined as follows. Given the initial configurations $(\eta, \xi)$, we let the two dynamics evolve at the same time and update the configurations at the same vertex. We then chose the coupling jump rates $\tilde{c}_x((\eta, a), (\xi, b))$ to go from $(\eta, \xi)$ to $(\eta^{x,a}, \xi^{x,b})$, with $a, b \in \{\pm 1\}$, as the optimal coupling (see [18]) between the jump rates $\mu_{K_x}^\eta(\sigma_x = a)$ and $\mu_{K_x}^\xi(\sigma_x = b)$. More explicitly, for $a \in \{\pm 1\}$, they are given by

$$\begin{cases} \tilde{c}_x((\eta, a), (\xi, a)) = \min\{\mu_{K_x}^\eta(\sigma_x = a) \,;\, \mu_{K_x}^\xi(\sigma_x = a)\} \\ \tilde{c}_x((\eta, a), (\xi, -a)) = \max\{0 \,;\, \mu_{K_x}^\eta(\sigma_x = a) - \mu_{K_x}^\xi(\sigma_x = a)\} \end{cases} \tag{4.9}$$

We denote by $\widetilde{\mathscr{L}}$ the generator of the coupled process, and by $\widetilde{P}_t$ the correspondent Markov semigroup. Notice that from our choice of coupling jump rates, we get that the probability of disagreement in $x$, after one update in $x$ of $(\eta, \xi)$, is given by

$$P_{dis}^x(\eta, \xi) := |\mu_{K_x}^\eta(\sigma_x = +) - \mu_{K_x}^\xi(\sigma_x = +)| \,. \tag{4.10}$$

Let us now consider the subset $H \subset \Omega_S \times \Omega_S$ given by all couples of configurations which differ by a single spin flip in some vertex of $S$. One can easily verify that the graph $(\Omega_S, H)$ is connected and that the induced graph distance between configurations $(\eta, \xi) \in \Omega_S \times \Omega_S$, $D(\eta, \xi)$, just corresponds to their Hamming distance. Let us also denote by $\mathbb{E}_{\eta,\xi}[D(\eta(t), \xi(t))]$ the average distance at time $t$ between two coupled configurations of the process starting at $(\eta, \xi)$. We claim the following:

**Claim 4.3.** *For all $\beta \gg 1$, there exists a positive constant $\alpha \equiv \alpha(\beta, g, \Delta)$ such that, for every initial configurations $(\eta, \xi) \in H$, the process $(\eta(t), \xi(t))_{t \geq 0}$ satisfies the inequality*

$$\frac{d}{dt} \mathbb{E}_{\eta,\xi}[D(\eta(t), \xi(t))] \Big|_{t=0} \leq -\alpha \,. \tag{4.11}$$

*Proof of Claim 4.3.* The derivative in $t$ of the average distance, computed for $t=0$, can be written as

$$\begin{aligned}\frac{d}{dt}\mathbb{E}_{\eta,\xi}[D(\eta(t),\xi(t))]\Big|_{t=0} &= \frac{d}{dt}\left(\widetilde{P}_t D\right)(\eta,\xi)\Big|_{t=0} = (\widetilde{\mathscr{L}}D)(\eta,\xi)\\ &= \sum_{x\in S}\sum_{a,b\in\{\pm1\}}\tilde{c}_x((\eta,a)(\xi,b))[D(\eta^{x,a},\xi^{x,b})-D(\eta,\xi)]\,. \end{aligned}\tag{4.12}$$

Since $(\eta,\xi)\in H$, there exists a vertex $y\in S$ such that $\xi=\eta^y$. If $x=y$, then $P^x_{dis}(\eta,\eta^x)=0$ and the distance between the updated configurations decreases of one. While if $x\neq y$, with probability $P^x_{dis}(\eta,\eta^y)$ the updated configurations have different spin at $x$ and their distance increases by one. Continuing from (4.12) we get, for all $\beta\gg1$,

$$\begin{aligned}\frac{d}{dt}\mathbb{E}_{\eta,\xi}[D(\eta(t),\xi(t))]\Big|_{t=0} &= -1+\sum_{\substack{x\in S\\ x\neq y}}P^x_{dis}(\eta,\eta^y)\\ &\leq -1+c\sum_{\ell\geq1}e^{-\beta'\ell}\Delta^{\ell}\\ &\leq -(1-ce^{-\beta'}),\end{aligned}\tag{4.13}$$

where in the second line we used the bound

$$P^x_{dis}(\eta,\eta^y)=|\mu^{\eta}_{K_x}(\sigma_x=+)-\mu^{\eta^y}_{K_x}(\sigma_x=+)|\leq ce^{-\beta' d(x,y)}\,,\tag{4.14}$$

which holds for all $\beta'=\delta\beta-2g>0$ as stated in Proposition 3.1. Claim 4.3 follows taking $\alpha=(1-ce^{-\beta'})$ and $\beta$ sufficiently large. □

Using the *path coupling* technique (see [3]) we can extend the result of Claim 4.3 to arbitrary initial configurations $(\eta,\xi)\in\Omega_S\times\Omega_S$, and obtain

$$\frac{d}{dt}\mathbb{E}_{\eta,\xi}[D(\eta(t),\xi(t))]\Big|_{t=0}\leq-\alpha D(\eta,\xi)\,.\tag{4.15}$$

From (4.15) it now follows straightforwardly that $\mathbb{E}_{\eta,\xi}[D(\eta(t),\xi(t))]\leq e^{-\alpha t}D(\eta,\xi)$, and then we get

$$\mathbb{P}(\eta(t)\neq\xi(t))\leq\mathbb{E}_{\eta,\xi}(D(\eta(t),\xi(t)))\leq e^{-\alpha t}D(\eta,\xi)\,.\tag{4.16}$$

To bound the spectral gap $c_{gap}(\nu^\tau_S)$ of the dynamics on $S$, we consider an eigenfunction $f$ of $\mathscr{L}_S$ with eigenvalue $-c_{gap}(\nu^\tau_S)$, so that

$$\mathbb{E}_\sigma f(\eta(t))=e^{t\mathscr{L}_S}f(\eta)=e^{-c_{gap}(\nu^\tau_S)t}f(\eta)\,.$$

Since the identity function has eigenvalue zero, and therefore it is orthogonal to $f$, it follows that $\nu^\tau_S(f)=0$ and $\nu^\tau_S(\mathbb{E}_\xi f(\xi(t)))=0$, where $\nu^\tau_S$ is the invariant measure for $\mathscr{L}_S$. From these considerations and inequality (4.16), we have

$$\begin{aligned}e^{t\mathscr{L}_S}f(\sigma) &= \mathbb{E}_\eta f(\eta(t))-\nu^\tau_S(\mathbb{E}_\xi f(\xi(t)))\\ &= \sum_\xi \nu^\tau_S(\xi)[E_\eta f(\eta(t))-E_\xi f(\xi(t))]\\ &\leq 2\|f\|_\infty\sup_{\eta,\xi}\mathbb{P}(\eta(t)\neq\xi(t))\\ &\leq 2\|f\|_\infty|S|e^{-\alpha t}\,.\end{aligned}\tag{4.17}$$

From the last computation, which holds for all $\eta \in \Omega_S^\tau$ and for all $t$, we finally obtain that $c_{gap}(\nu_S^\tau) \geq \alpha$ independently of the size of $S$, which implies the Poincaré inequality (4.7) with constant $c_0 = \alpha^{-1} = 1 + O(e^{-c\beta})$. This concludes the proof of Theorem 4.1. □

## 4.3 Step 2: Poincaré inequality for the global Gibbs measure

With the previous analysis we obtained a like-Poincaré inequality for the marginal of the measure $\mu_i$ on the level $L_i$ (see Remark 4.2), which inserted in formula (4.3) provides the bound

$$\mathrm{Var}(f) \leq c_0 \sum_{i=0}^{m} \sum_{x \in L_i} \mu \left[ \mu_i(\mathrm{Var}_{K_x}(g_{i+1})) \right] . \tag{4.18}$$

Using the same notation as in [26], let us denote the sum in the r.h.s. of (4.18) by $\mathrm{P_{var}}(f)$. The aim of the following analysis is to study $\mathrm{P_{var}}(f)$ in order to find an inequality of the kind $\mathrm{P_{var}}(f) \leq c\mathscr{D}(f) + \varepsilon \mathrm{P_{var}}(f)$, with $\varepsilon = \varepsilon(\beta, g, \Delta) < 1$ independent of the size of the system. This would imply that

$$\mathrm{Var}(f) \leq c_0 \cdot \mathrm{P_{var}}(f) \leq c_0 \frac{c}{1-\varepsilon} \mathscr{D}(f) \,,$$

and then would conclude the proof of Theorem 2.7.

In this last part of the section, we will first relate the local variance of $g_i = \mu_i(f)$ with the local variance of $f$. This will produce a covariance term that will be analyzed using a recursive argument.

### 4.3.1 Reduction to covariance

In order to reconstruct the Dirichlet form of $f$ from (4.18), we want to extract the local variance of $f$ from the local variance of $g_{i+1}$. Notice that w.r.t. the measure $\mu_{K_x}$, the function $g_{i+1}$ just depends on $x$. Fixing $x \in L_i$ and $\tau \in \Omega^+$, and defining $p(\tau) := \mu_{K_x}^\tau(\sigma_x = +)$ and $q(\tau) := \mu_{K_x}^\tau(\sigma_x = -)$, we can write

$$\mu_i \left( \mathrm{Var}_{K_x}(g_{i+1}) \right) = \sum_\tau \mu_i(\tau) p(\tau) q(\tau) (\nabla_x \, g_{i+1}(\tau))^2 \,. \tag{4.19}$$

Using the martingale property $g_{i+1} = \mu_{i+1}(g_{i+2})$, the variance $\mathrm{Var}_{K_x}(g_{i+1})$ can be split in two terms, stressing the dependence on $x$ of $g_{i+2}$ and of the conditioned measure $\mu_{i+1}$. Let us formalize this idea.

For a given configuration $\tau \in \Omega^+$ we introduce the symbols

$$\tau^+ := \begin{cases} \tau_y^+ = \tau_y & \text{if } y \neq x \\ \tau_y^+ = + & \text{if } y = x \end{cases} \qquad \tau^- := \begin{cases} \tau_y^- = \tau_y & \text{if } y \neq x \\ \tau_y^- = - & \text{if } y = x \end{cases}$$

and define the density

$$h_x(\sigma) := \frac{\mu_{i+1}^{\tau^+}(\sigma)}{\mu_{i+1}^{\tau^-}(\sigma)} \,, \quad \text{with} \quad \mu_{i+1}^{\tau^-}(h_x) = 1 \,. \tag{4.20}$$

Whit this notation and continuing from (4.19), we get

$$\begin{aligned}
\mu_i(\mathrm{Var}_{K_x}(g_{i+1})) &= \sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left[\nabla_x\,\mu_{i+1}(g_{i+2})(\tau)\right]^2 \\
&= \sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left[\mu_{i+1}^{\tau^-}(g_{i+2})-\mu_{i+1}^{\tau^+}(g_{i+2})\right]^2 \\
&= \sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left[\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2})-\mu_{i+1}^{\tau^-}(h_x,g_{i+2})\right]^2 \\
&\le 2\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left[\left(\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2})\right)^2+\left(\mu_{i+1}^{\tau^-}(h_x,g_{i+2})\right)^2\right] \qquad (4.21)
\end{aligned}$$

Consider the first term of (4.21) and notice that $\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2})=\mu_{i+1}^{\tau^+}(\nabla_x f)$. To understand this fact, it is enough to observe that the dependence on $x$ of $g_{i+2}=\mu_{i+2}(f)$ comes only from $f$, since the b.c. on $B_{i+1}$ is fixed equal to $\tau^+$. Replacing $\nabla_x g_{i+2}$ by $\nabla_x f$ and applying the Jensen inequality, we get

$$\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left(\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2})\right)^2 \le \sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left(\mu_{i+1}^{\tau^+}\left(\nabla_x f\right)^2\right). \qquad (4.22)$$

Now, notice that $p(\tau)\mu_{i+1}^{\tau^+}(\sigma^+)=\mu_{K_x}^{\tau}(\sigma^+)=\mu_x^{\sigma}(\sigma_x=+)\left(\mu_{K_x}^{\tau}(\sigma^+)+\mu_{K_x}^{\tau}(\sigma^-)\right)$. This, together with the fact that $\nabla_x f$ does not depend on $x$, means that the expression on the r.h.s. of (4.22) equals to

$$\sum_\tau \mu_i(\tau)q(\tau)\sum_\sigma \mu_{K_x}^{\tau}(\sigma)\mu_x^{\sigma}(\sigma_x=+)\left(\nabla_x f(\sigma)\right)^2 \le \sum_\tau \mu_i(\tau)\inf_{\sigma\in\Omega_{K_x}^{\tau}}\left\{\frac{q(\tau)}{\mu_x^{\sigma}(\sigma_x=-)}\right\}. \qquad (4.23)$$

Since $\sigma$ agrees with $\tau$ on $K_x^c$, then

$$\frac{q(\tau)}{\mu_x^{\sigma}(\sigma_x=-)}=\frac{\mu_{K_x}^{\tau}(\sigma_x=-)}{\mu_{K_x}^{\tau}(\sigma_x=-|\sigma)}=\frac{\mu_{K_x}^{\tau}(\sigma)}{\mu_{K_x}^{\tau}(\sigma|\sigma_x=-)}\le h_x(\sigma), \qquad (4.24)$$

and hence, $\inf_\sigma\left\{\frac{q(\tau)}{\mu_x^{\sigma}(\sigma_x=-)}\right\}\le \|h_x\|_\infty$.

To bound $\|h_x\|_\infty$, we first write

$$h_x(\sigma)=\frac{\mu_{k_x}^{\tau}(\sigma|\sigma_x=+)}{\mu_{k_x}^{\tau}(\sigma|\sigma_x=-)}=\frac{\mu_{k_x}^{\tau}(\sigma_x=+|\sigma)}{\mu_{k_x}^{\tau}(\sigma_x=-|\sigma)}\cdot\frac{\mu_{k_x}^{\tau}(\sigma_x=-)}{\mu_{k_x}^{\tau}(\sigma_x=+)}. \qquad (4.25)$$

Taking the supremum over $\sigma$ of $h_x$, we then have

$$\|h_x\|_\infty\le\frac{1}{\mu_{i+1}^{\tau^-}(\sigma_y=+;\forall y\in N_x\cap L_{i+1})}\le\frac{1}{1-\sum_{y\in N_x\cap L_{i+1}}\mu_{i+1}^{-}(\sigma_y=-)}, \qquad (4.26)$$

where we denoted by $\mu_{i+1}^{-}$ the measure conditioned on having all minus spins in $B_i$ and plus spins in $\partial_V B$. Inequality (3.9), applied to $U=F_{i+1}$, implies that

$$\mu_{i+1}^{-}(\sigma_y=-)\le ce^{-\beta'}, \qquad (4.27)$$

with $\beta' = 2g\beta - \delta$ as in Proposition 3.1. Combining (4.26) and (4.27), we get that for all $\beta \geq \frac{\delta}{2g}$,

$$\|h_x\|_\infty \leq 1 + ce^{-\beta'} . \tag{4.28}$$

Altogether, inequalities (4.22)-(4.28) imply that

$$\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left(\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2})\right)^2 \leq c_1\mu_i\left(\mathrm{Var}_x(f)\right) , \tag{4.29}$$

with $c_1 = c_1(\beta, \Delta, g) = 1 + O(e^{-c\beta})$. Thus, summing both sides of (4.21) over $x \in L_i$ and $i \in \{0, \ldots, m\}$, and applying inequality (4.29), we obtain

$$\mathrm{P_{var}}(f) \leq 2c_1\,\mathscr{D}(f) + 2\sum_{i=0}^{m}\sum_{x\in L_i}\mu\left[\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left(\mu_{i+1}^{\tau^-}(h_x, g_{i+2})\right)^2\right] . \tag{4.30}$$

Notice that since $g_{m+2} \equiv f$ is constant w.r.t. $\mu_{m+1}$, then $\mu_{m+1}^{\tau^-}(h_x, g_{m+2}) \equiv 0$ and the value $m$ can be removed from the summation over $i$ in the r.h.s. of (4.30).
It now remains to analyze the covariance $\mu_{i+1}^{\tau^-}(h_x, g_{i+2})$.

### 4.3.2 Recursive argument

Before going on with the proof, we need some more definitions and notation. For every $x \in L_i$, let $D_x$ denote the set of nearest neighbors of $x$ in the level $L_{i+1}$ (descendants of $x$). Given $x \in L_i$ and $\ell \in \mathbb{N}$, let us define the following objects:

(i) $D_{x,\ell} := \{y \in L_{i+1} : d(y, D_x) \leq \ell\}$ is the $\ell$-neighborhood of $D_x$ in $L_{i+1}$;

(ii) $\mathscr{F}_{x,\ell} := \sigma\left(\sigma_y : y \in B_{i+1} \setminus D_{x,\ell}\right)$ is the $\sigma$-algebra generated by the spins on $B_{i+1} \setminus D_{x,\ell}$;

(iii) $\mu_{x,\ell}(\cdot) := \mu(\cdot|\mathscr{F}_{x,\ell})$ is the Gibbs measure conditioned on the $\sigma$-algebra $\mathscr{F}_{x,\ell}$.

We remark that $D_{x,0} = D_x$, and that there exists some $\ell_0 \leq 2(i+1)$ such that, for all integers $\ell \geq \ell_0$, $D_{x,\ell} = L_{i+1}$ and $\mu_{x,\ell} = \mu_{i+1}$.
We also remark that for any function $f \in L^1(\Omega, \mathscr{F}_{i+1}, \mu)$, the set of variables $\{\mu_{x,\ell}(f)\}_{\ell\in\mathbb{N}}$ is a Martingale with respect to the filtration $\{\mathscr{F}_{x,\ell}\}_{\ell=0,1,\ldots,\ell_0}$.

Let us now come back to our proof and recall the following property of the covariance. For all subsets $D \subseteq C \subseteq B$,

$$\mu_C^\eta(f, g) = \mu_C^\eta(\mu_D(f, g)) + \mu_C^\eta(\mu_D(f), \mu_D(g)) . \tag{4.31}$$

Since the support of $\mu_{i+1}$ strictly contains the support of $\mu_{x,0}$, we can apply the property (4.31) to the square covariance $(\mu_{i+1}^{\tau^-}(h_x, g_{i+2}))^2$ appearing in (4.30), in order to get

$$(\mu_{i+1}^{\tau^-}(h_x, g_{i+2}))^2 \leq 2(\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x, g_{i+2})))^2 + 2(\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x), \mu_{x,0}(g_{i+2})))^2 . \tag{4.32}$$

The first term in the r.h.s. of (4.32) can be bounded, by the Schwartz inequality, as

$$(\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x, g_{i+2})))^2 \leq \mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,0}(h_x)) \cdot \mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,0}(g_{i+2})) . \tag{4.33}$$

The second term can be rearranged and bounded as follows:

$$\begin{aligned}
[\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x),\mu_{x,0}(g_{i+2}))]^2 &= \left[\mu_{i+1}^{\tau^-}\left(\mu_{x,0}(h_x)-\mu_{i+1}(h_x),g_{i+2}\right)\right]^2 \\
&= \left[\mu_{i+1}^{\tau^-}\left(\sum_{\ell=1}^{\ell_0}\left(\mu_{x,\ell-1}(h_x)-\mu_{x,\ell}(h_x)\right),g_{i+2}\right)\right]^2 \\
&\le \sum_{\ell=1}^{\ell_0}\ell^2\left[\mu_{i+1}^{\tau^-}(\mu_{x,\ell-1}(h_x)-\mu_{x,\ell}(h_x),g_{i+2})\right]^2 \\
&= \sum_{\ell=1}^{\ell_0}\ell^2\left[\mu_{i+1}^{\tau^-}\left(\mu_{x,\ell}(\mu_{x,\ell-1}(h_x),g_{i+2})\right)\right]^2, \qquad (4.34)
\end{aligned}$$

where in the second line, due to the fact that $\mu_{x,\ell_0}=\mu_{i+1}$ for some $\ell_0$, we substituted $\mu_{x,0}(h_x)-\mu_{i+1}(h_x)$ by the telescopic sum $\sum_{\ell=1}^{\ell_0}(\mu_{x,\ell-1}(h_x)-\mu_{x,\ell}(h_x))$. Applying again the Cauchy-Schwartz inequality to the last term in (4.34), we get

$$\begin{aligned}
&[\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x),\mu_{x,0}(g_{i+2}))]^2\le \\
&\qquad \le\sum_{\ell=1}^{\ell_0}\ell^2\,\mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(\mu_{x,\ell-1}(h_x))\right)\cdot\mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(g_{i+2})\right) \qquad (4.35)
\end{aligned}$$

To conclude the estimate on the covariance, it remains to analyze the three quantities appearing in (4.33) and (4.35):

(i) $\mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(g_{i+2})\right)$, for all $\ell=0,1,\dots,\ell_0$;

(ii) $\mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,0}(h_x))$;

(iii) $\mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(\mu_{x,\ell-1}(h_x))\right)$, for all $\ell=1,\dots,\ell_0$,

We proceed estimating separately these three terms.

**First term: Poincaré inequality for the marginal measure on $D_{x,\ell}$.**
Let us consider the variance $\mathrm{Var}_{x,\ell}(g_{i+2})$ appearing in (i). By definition, the function $g_{i+2}$ depends on the spin configuration on $B_{i+1}$. Since the measure $\mu_{x,\ell}^{\eta}$ fixes the configuration on $B_{i+1}\setminus D_{x,\ell}$, it follows that

$$\mu_{x,\ell}^{\eta}(g_{i+2})=\mu_{x,\ell|_{D_{x,\ell}}}^{\eta}(g_{i+2}).$$

Thus, for every configuration $\eta\in\Omega^+$, we can apply the Poincaré inequality stated in Theorem (4.1) to $\mathrm{Var}_{x,\ell}^{\eta}(g_{i+2})$, and obtain the inequality

$$\mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(g_{i+2})\right)\le c_0\sum_{y\in D_{x,\ell}}\mu_{i+1}^{\tau^-}(\mathrm{Var}_{K_y}(g_{i+2})), \qquad (4.36)$$

with $c_0=1+O(e^{-c\beta})$ independent of the size of system.

**Second term: computation of the variance of $h_x$.**
Notice that, from definition (4.20), it turns out that $h_x$ only depends on the spin configuration on $D_x$. In particular, for all $\eta$ which agrees with $\tau^-$ on $B_i$, $\mu^\eta_{x,0}(h_x) = 0$ and

$$\mathrm{Var}^\eta_{x,0}(h_x) \le \|h_x\|^2_\infty - 1\,. \tag{4.37}$$

Together with inequality (4.28), this yields the bound:

$$\mathrm{Var}^\eta_{x,0}(h_x) \le c e^{-\beta'} =: k_\beta\,. \tag{4.38}$$

**Third term: the variance of $\mu_{x,\ell-1}(h_x)$.**
We now consider the variance $\mathrm{Var}^\eta_{x,\ell}(\mu_{x,\ell-1}(h_x))$, with $\eta \in \Omega^+$ and $\ell \ge 1$. Applying the result of Theorem 4.1, we obtain

$$\begin{aligned}\mathrm{Var}^\eta_{x,\ell}(\mu_{x,\ell-1}(h_x)) &\le c_0 \sum_{z\in D_{x,\ell}} \mu^\eta_{x,\ell}(\mathrm{Var}_{K_z}(\mu_{x,\ell-1}(h_x)))\\ &= c_0 \sum_{z\in D_{x,\ell}\setminus D_{x,\ell-1}} \mu^\eta_{x,\ell}(\mathrm{Var}_{K_z}(\mu_{x,\ell-1}(h_x)))\,,\end{aligned} \tag{4.39}$$

where in the last line we used that the function $\mu_{x,\ell-1}(h_x)$ does not depend on the spin configuration on $D_{x,\ell-1}$.
Let $z \in D_{x,\ell} \setminus D_{x,\ell-1}$, and for any configuration $\zeta \in \Omega^\eta_{D_{x,\ell}}$, let us denote by $\zeta^+$ and $\zeta^-$ the configurations that agree with $\zeta$ in all sites but $z$, and have respectively a $(+)$-spin and a $(-)$-spin on $z$. The summand in (4.39) can be trivially bounded as

$$\mu^\eta_{x,\ell}(\mathrm{Var}_{K_z}(\mu_{x,\ell-1}(h_x))) \le \frac{1}{2} \sup_{\zeta\in\Omega^\eta_{x,\ell}} (\mu^{\zeta^+}_{x,\ell-1}(h_x) - \mu^{\zeta^-}_{x,\ell-1}(h_x))^2\,. \tag{4.40}$$

Moreover, by stochastic domination, the inequality $\mu^{\zeta^+}_{x,\ell-1}(h_x) \ge \mu^{\zeta^-}_{x,\ell-1}(h_x)$ holds. Let $\nu(\sigma,\sigma')$ denote a monotone coupling with marginal measures $\mu^{\zeta^+}_{x,\ell-1}$ and $\mu^{\zeta^-}_{x,\ell-1}$. We then have

$$\begin{aligned}\mu^{\zeta^+}_{x,\ell-1}(h_x) - \mu^{\zeta^-}_{x,\ell-1}(h_x) &= \sum_{\sigma,\sigma'} \nu(\sigma,\sigma')\,(h_x(\sigma) - h_x(\sigma'))\\ &\le \|h_x\|_\infty\, \nu(\sigma_y \ne \sigma'_y\,, y \in D_x)\\ &\le \Delta\|h_x\|_\infty \max_{y\in D_x} \Big(\nu(\sigma_y = +) - \nu(\sigma'_y = +)\Big)\\ &= \Delta\|h_x\|_\infty \max_{y\in D_x} (\mu^{\zeta^+}_{x,\ell-1}(\sigma_y = +) - \mu^{\zeta^-}_{x,\ell-1}(\sigma_y = +)),\end{aligned} \tag{4.41}$$

where we used that the function $h_x$ only depends on the spins on $D_x$.
From Proposition 3.1, with $U = F_{i+2} \cup D_{x,\ell}$, and since $d(z,y) \ge d(z,D_x) \ge \ell$, the probability of disagreement appearing in (4.41) can be bounded as

$$\mu^{\zeta^+}_{x,\ell-1}(\sigma_y = +) - \mu^{\zeta^-}_{x,\ell-1}(\sigma_y = +) \le c\, e^{-\beta'\ell}\,. \tag{4.42}$$

Putting together formulas (4.39)-(4.42), and applying inequality (4.28), we obtain that for all $\eta \in \Omega_+$,

$$\mathrm{Var}^{\eta}_{x,\ell}(\mu_{x,\ell-1}(h_x)) \le k'_\beta e^{-2\beta'\ell} \tag{4.43}$$

with $k'_\beta = c(1+O(e^{-c\beta}))$.

**Conclusion**.
Let us go back to inequalities (4.33) and (4.35). Applying bounds (4.36),(4.38) and (4.43), we get respectively

- $(\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x, g_{i+2})))^2 \le k_\beta \sum_{y\in D_x} \mu^{\tau^-}_{i+1}(\mathrm{Var}_{K_y}(g_{i+2}))$,
- $[\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x), \mu_{x,0}(g_{i+2}))]^2 \le k'_\beta \sum_{\ell=1}^{\ell_0} \ell^2 e^{-2\beta'\ell} \sum_{y\in D_{x,\ell}} \mu^{\tau^-}_{i+1}(\mathrm{Var}_{K_y}(g_{i+2}))$,

where we included in $k_\beta$ and $k'_\beta$ all constants non depending on $\beta$.

For all $\beta \gg 1$, there exists a constant $\varepsilon \equiv \varepsilon(\beta, g, \Delta) = O(e^{-c\beta})$ such that $k_\beta \le \varepsilon$ and $k'_\beta \ell^2 e^{-\beta'\ell} \le k'_\beta e^{-\beta'} \le \varepsilon$. Substituting $\varepsilon$ in the inequalities above and summing the two terms as in (4.32), we obtain

$$\left(\mu^{\tau^-}_{i+1}(h_x, g_{i+2})\right)^2 \le \varepsilon \sum_{\ell=0}^{\ell_0} e^{-\beta'\ell} \sum_{y\in D_{x,\ell}} \mu^{\tau^-}_{i+1}(\mathrm{Var}_{K_y}(g_{i+2})).$$

Inserting this result in the second term of formula (4.30) and rearranging the summation, we get

$$\begin{aligned}
&\sum_{i=0}^{m-1} \sum_{x\in L_i} \mu\left[\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left(\mu^{\tau^-}_{i+1}(h_x, g_{i+2})\right)^2\right] \\
&\qquad \le \quad \varepsilon \sum_{i=0}^{m-1} \sum_{x\in L_i} \sum_{\ell=0}^{\ell_0} \sum_{y\in D_{x,\ell}} e^{-\beta'\ell} \mu(\mathrm{Var}_{K_y}(g_{i+2})) \\
&\qquad \le \quad \varepsilon \sum_{i=0}^{m-1} \sum_{y\in L_{i+1}} \mu(\mathrm{Var}_{K_y}(g_{i+2})) \sum_{\ell=0}^{\ell_0} e^{-\beta'\ell} n(\ell), \qquad (4.44)
\end{aligned}$$

where in the last line we denoted by $n(\ell)$ the factor which bounds the number of vertices $x$ such that a fixed vertex $y$ belongs to $D_{x,\ell}$. Since $n(\ell)$ grows at most like $\Delta^\ell$, the product $e^{-\beta'\ell}n(\ell)$ decays exponentially with $\ell$ for all $\beta \gg 1$. Thus the sum over $\ell \in \{0, \dots, \ell_0\}$ can be bounded by a finite constant $c$ which will be included in the factor $\varepsilon$ in front of the summations. Continuing from (4.44), we get

$$\begin{aligned}
\sum_{i=0}^{m-1} \sum_{x\in L_i} \mu\left[\sum_\tau \mu_i(\tau)p(\tau)q(\tau)\left(\mu^{\tau^-}_{i+1}(h_x, g_{i+2})\right)^2\right] \quad &\le \quad \varepsilon \sum_{i=1}^{m} \sum_{y\in L_i} \mu(\mathrm{Var}_{K_y}(g_{i+1})) \\
&\le \quad \varepsilon\, \mathrm{P}_{\mathrm{var}}(f). \qquad (4.45)
\end{aligned}$$

Inserting this result in (4.30) and noticing that $\varepsilon = O(e^{-c\beta}) < 1$ for $\beta$ large enough, we obtain

$$\mathrm{P_{var}}(f) \le 2c_1 \mathscr{D}(f) + \varepsilon \mathrm{P_{var}}(f) \Longrightarrow \mathrm{P_{var}}(f) \le \frac{2c_1}{1-\varepsilon}\mathscr{D}(f).$$

Together with inequality (4.18), this implies that

$$\mathrm{Var}(f) \le c_0 \mathrm{P_{var}}(f) \le c_2 \mathscr{D}(f),$$

that is the desired Poincaré inequality with $c_2 = c_2(\beta, g, \Delta) = 2(1 + O(e^{-c\beta}))$ independent of the size of the system . Notice also that the lower bound on the spectral gap, $1/c_2$, increases with $\beta$ and converges to $1/2$ when $\beta \uparrow \infty$. This concludes the proof of Theorem 2.7. □

# 5 Influence of boundary conditions on the mixing time

In this section we discuss two examples of influence of the boundary condition on the mixing time derived from Theorem 2.7. In particular, we first prove Lemma 2.3, which implies the applicability of Theorem 2.7 to hyperbolic graphs with sufficiently high degree. Then we prove Theorem 2.8 providing an explicit example of growing graph which exhibits the behavior stated in the theorem.

## 5.1 Hyperbolic graphs

In this subsection we want to prove Lemma 2.3. Before starting the proof, let us recall the following definition:

**Definition 5.1.** *The number of ends, $\mathscr{E}(G)$, of a graph $G = (V, E)$, is defined as*

$$\mathscr{E}(G) := \sup_{\substack{K \subset V \\ K \text{finite}}} \{\, \textit{number of infinite connected components of } G \setminus K \,\},$$

*where $G \setminus K$ denotes the graph obtained from $G$ by removing the vertices which belong to $K$ and the edges incident to these vertices.*

It is well known that hyperbolic graphs have *one-end*, as well as all the lattices $\mathbb{Z}^d$ with $d \ge 2$. This property, together with the planarity and the cycle-periodic structure of the hyperbolic graphs, will be the main ingredient of the next proof.

*Proof of Lemma 2.3.* Consider a planar embedding of $\mathbb{H}(v, s)$ and assume, without loss of generality, that the vertices of any ball $B_r$ around a fixed vertex $o \in V$, are in the infinite face of the graph.

With the same notation introduced in the previous sections, let us consider $x \in L_r = \partial_V B_{r-1}$. Let $D_x = N_x \cap L_{r+1}$ be the set of "descendants of $x$", and define

$$S_x := N_x \cap L_r \quad \text{and} \quad P_x := N_x \cap L_{r-1},$$

so that $N_x = D_x \cup S_x \cup P_x$. Thus, when specialized to hyperbolic graphs $\mathbb{H}(v, s)$, the definition 2.2 of growing graph corresponds to the condition:

$$\min_{x \in L_r, r \in \mathbb{N}} \{D_x - S_x - P_x\} = \min_{x \in L_r, r \in \mathbb{N}} \{v - 2(S_x + P_x)\} = g. \tag{5.1}$$

Let us start verifying the following general properties:

(i) Every vertex $x \in L_r$, has at least one neighbor in $L_{r+1}$, i.e. $|D_x| \geq 1$;

(ii) Every vertex $x \in L_r$, has at most two neighbor in $L_r$, i.e. $|S_x| \leq 2$;

(iii) Every vertex $x \in L_r$, has at most two neighbors in $L_{r-1}$, i.e. $|P_x| \leq 2$.

Property (i) is proved by contradiction. Assume the existence of a vertex $x \in L_r$ such that $D_x = \emptyset$. By the periodic structure of the graph it follows that there are at least $v$ vertices in $L_r$ with no descendants, spaced out by vertices in $L_r$ connected to $\mathbb{H}(v,s) \setminus B_r$ by at least one edge. Then, it exists a finite path, from $x$ to another vertex in $L_r$ with no descendants, dividing two infinite components. This is in contradiction we the property of having one end, thus we conclude that $|D_x| \geq 1$.

Property (ii) is also proved contradiction. Assume the existence of a vertex $x \in L_r$ linked to three vertices in $L_r$. Then there exist two faces, included in $B_r$, both passing through $x$ and one of its neighbors on $L_r$, say $y$ (see Fig. 5.2, left frame). Consequently, or $D_y \neq \emptyset$, contradicting the planarity of the graph or the property of having one-end, or $D_y = \emptyset$, contradicting property (i). We conclude that $|S_x| \leq 2$.

Property (iii) is clearly satisfied if $v = 3$, as a consequence of property (i). Let $v \geq 4$ and proceed by induction. For all vertices $x$ in the first level, we have obviously that $|P_x| = 1$. Assume that property (iii) holds for all vertices of $L_{r-1}$, and consider a vertex $x \in L_r$. If we assume by contradiction that $P_x = 3$, then there exist two faces, included in $B_r$, both passing through $x$ and one of its parents, say $y$ (see Fig. 5.2, right frame). Consequently, the vertex $y$ can not have neighbors in $L_r$ other than $x$, because this would contradict the planarity of the graph or the property of having one-end. On the other hand, if $s > 3$, $y$ can not be linked to the other parents of $x$ because this would create a triangular face. Globally, and by the inductive hypothesis that $|P_y| \leq 2$, it follows that $|N_y| \leq 3$, which is a contradiction. Then $P_x \leq 2$, which completes the induction step for $s > 3$. If $s = 3$, $y$ is connected with the two other parents of $x$ to create two triangular faces. Thus, by the inductive hypothesis that $|P_y| \leq 2$, it follows that $|N_y| = 5$, which is a contradiction since $v \geq 7$ for all triangular tilings (recall the condition $(v-2)(s-2) > 4$). Then $|P_x| \leq 2$, and by induction this concludes the proof of property (iii) also for $s = 3$.

Properties (ii) and (iii) imply that $S_x + P_x \leq 4$. From (5.1), we then get that for all $v \geq 9$, $\mathbb{H}(v,s)$ is a growing graph with parameter $g \geq v - 8$. Moreover, it can be easily verified that if $s = 3$ (triangular tilings), then $g = v - 8$, since in this case $S_x + P_x = 4$ for some $x \in V$.

If $s > 4$, the above condition can be improved by proving the following properties:

(iv) If $s = 2k$, with $k \geq 2$, then for every $x \in L_r$, $S_x = 0$;

(v) If $s = 2k + 1$, with $k \geq 2$, then for every $x \in L_r$, $S_x + P_x \leq 2$.

Let us prove property (iv) by induction. For all vertices $x$ in the first level, we have obviously that $|S_x| = 0$, otherwise we would have a triangular face. Assume that property (iv) holds for all vertices of $L_{r-1}$, and consider a vertex $x \in L_r$. If we assume that $x$ has a neighbor in $L_r$, say $y$, then there exists a face $F$, included in $B_r$, passing through $x$, $y$, and the respective parents (see Fig. 5.3, left frame). The remaining sides of $F$ are all included in $B_{r-1}$, and by the inductive hypothesis they can only connect vertices on subsequent levels. In particular, this face can only have an odd number of sides, which is in contradiction with the hypothesis that $s = 2k$. We conclude that $S_x = 0$ for all $x \in V$.

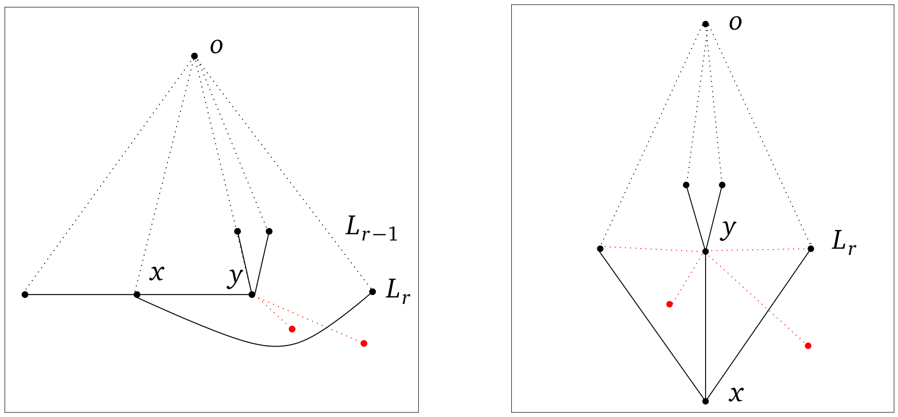


Figure 5.2: The presence of more than two neighbors of $x$ in the same level of $x$ (left frame) or in the level above $x$ (right frame) is not consistent with the structure of hyperbolic graphs. The red lines correspond to edges which are forbidden by the geometric properties of the graph.

Property (v) is clearly satisfied if $v = 3$, as a consequence of property (i). Thus let $v \geq 4$ and assume by contradiction that $|S_x| + |P_x| = 3$, which means, coherently to properties (ii) and (iii), that $S_x = 2$ and $P_x = 1$ or viceversa. In both cases, there exist two faces, included in $B_r$, passing through $x$ and one of its parents, say $y$ (see Fig. 5.3, central and right frames). Then the vertex $y$ can not have neighbors in $L_r$ other than $x$, because this would create a triangular face, or would contradict the planarity of the graph or the property of having one-end. Analogously, $y$ can not have neighbors in $L_{r-1}$, because this would create a triangular or a square face, while $s = 2k + 1$ by hypothesis. Then we get that $|N_y| = 1 + |P_y| \leq 3$, which is in contradiction with the hypothesis that $v \geq 4$. We conclude that $|S_x| + |P_x| \leq 2$.

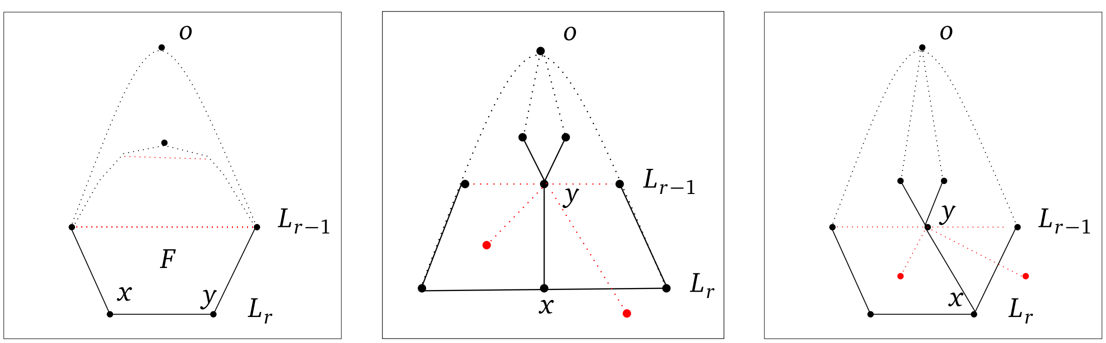


Figure 5.3: Forbidden patterns. In the left frame it is shown that $S_x \neq \emptyset$ only if the number of side in each face is odd. The right and central frames show that the condition $|S_x| + |P_x| \geq 3$ is possible only if $s = 3$.

All together, properties (iii)-(v) imply that if $s \geq 4$, then $S_x + P_x \leq 2$. Thus, for all couples $(v, s)$ such that $s \geq 4$ and $v \geq 5$, $\mathbb{H}(v, s)$ is a growing graph with parameter $g \leq v - 4$. It can be easily verified that if $s \geq 4$ then $g = v - 4$, since $S_x + P_x = 2$ for some $x \in V$. $\square$

Together with Theorem 2.7, the above lemma implies that for all $(v, s)$ such that $\mathbb{H}(v, s)$ is a growing graph and for all $\beta \gg 1$, the dynamics on an $n$-vertex ball of $\mathbb{H}(v,s)$ with $(+)$-boundary condition, has spectral gap uniformly positive in $n$ and mixing time at most linear in $n$. Comparing this result with the behavior of the dynamics with free boundary condition [4], we get a convincible example of the influence of boundary conditions on the mixing time.

## 5.2 Expanders

In this subsection we prove Theorem 2.8 by providing an explicit example of growing graph which exhibits the behavior stated in the theorem.

### 5.2.1 Construction

Let us start with some definitions.

Let $G = (V, E)$ be a finite graph. The edge isoperimetric constant of $G$, defined in (2.1) for infinite graphs, is given by

$$i_e(G) := \min_{\substack{\emptyset \neq S \subset V \\ |S| \leq \frac{|V|}{2}}} \left\{ \frac{|\partial_E(S)|}{|S|} \right\} . \tag{5.2}$$

For $c > 0$ and $k, n \in \mathbb{N}$, we then have the following definition:

**Definition 5.2.** *A finite graph $G = (V, E)$ is called an $(n, k, c)$-expander if it is regular with degree $k$, $|V| = n$, and $i_e(G) = c$.*

Notice that, since $G$ is finite, $i_e(G) > 0$ if and only if $G$ is connected. In particular, every connected $k$-regular graph on $n$ vertices is an expander for some $c > 0$. However, one is usually interested in a family of expanders, that is a sequence of $(n, k, c)$-expander graphs such that $k$ is fixed, $c \geq \epsilon > 0$, and $n \to \infty$. That $c \geq \epsilon > 0$ ensures that these graphs are highly connected, or, in other terms, that the sequence of expanders converges to an infinite regular nonamenable graph. It is easy to see that this happens only if $k \geq 3$. On the other hand, for all $k \geq 3$, a sequence of $(n, k, c)$-expanders exists, as it has been proved by showing that $k$-regular random graphs on $n$ vertices are expanders with probability tending to 1 as $n \to \infty$ (probabilistic method). See [12] for a nice survey on expanders and their applications.

Let $T^\Delta$ denote an infinite rooted tree with constant degree $\Delta$ and root $o \in V$, and assume that $\Delta \geq 6$. Fix an integer $d < \Delta - 2$, and connect the vertices on each level $L_r = \{x \in V : d(x, o) = r\}$ of the tree in such a way that induced subgraph on $L_r$ is an $(\Delta(\Delta-1)^{r-1}, k_r, c_r)$-expander, with $3 \leq k_r \leq d$ and isoperimetric constant $c_r$ uniformly positive in $r$ . We denote the infinite graph obtained with this procedure by $X^{\Delta,d}$ (see Fig. 5.4).

Notice that $X^{\Delta,d}$ has degree $D \geq \Delta + d$, and by construction it is $(g, o)$-growing graph with $g \geq \Delta - 2 - d$. Moreover, one can easily realize that the isoperimetric constant $i_e(B_r)$ of the ball $B_r \subset X^{\Delta,d}$ centered in $o$, is uniformly positive in $r \geq 1$, namely, there is a constant $\epsilon = \epsilon(\Delta, d) > 0$ such that $i_e(B_r) > \epsilon$ for all $r \in \mathbb{N}$.

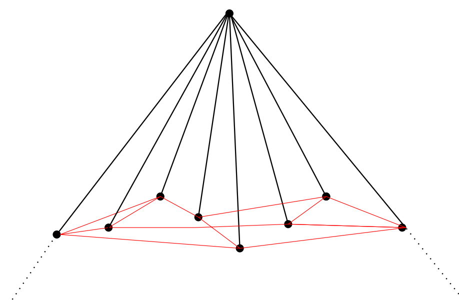

Figure 5.4: Construction of $X^{8,3}$ to first level. The red lines correspond to the edges of the expander in $L_1$.

### 5.2.2 Free boundary dynamics finite balls of $X^{\Delta,d}$

Let us consider the Glauber dynamics on the $n$-vertex ball $B \equiv B_m$ of the graph $X^{\Delta,d}$ with free boundary condition. We use the same notation of section 2.3, but here $\mu$ denotes the Gibbs measure over $\Omega \equiv \Omega_B = \{\pm 1\}^B$ with free boundary condition, that is, for all $\sigma \in \Omega$,

$$\mu(\sigma) = \frac{1}{Z(\beta)} \exp(\beta \sum_{(x,y)\in E(B)} \sigma_x \sigma_y). \tag{5.3}$$

With this notation, we can state the following result.

**Proposition 5.3.** *Let $B$ the $n$-vertex ball of the graph $X^{\Delta,d}$. Then, for all $\beta \gg 1$, the Glauber dynamics on $B$ with free boundary condition and zero external field, has spectral gap $O(e^{-\theta n})$, with $\theta = \theta(\Delta, d, \beta) > 0$.*

*Proof.* For every $\sigma \in \Omega$, let $m_B(\sigma) := \sum_{x\in B} \sigma_x$ be the magnetization of a configuration $\sigma$, and define the following characteristic function:

$$\mathbb{1}_{\{m_B>0\}}(\sigma) = \begin{cases} 1 & \text{if } m_B(\sigma) > 0 \\ 0 & \text{if } m_B(\sigma) \le 0 \end{cases}. \tag{5.4}$$

Without loss of generality we can restrict the analysis to odd values of $n$. Since in this case $m_B(\sigma) \neq 0$ for all $\sigma$, by symmetry it follows that $\mathrm{Var}(\mathbb{1}_{\{m_B>0\}}) = 1/4$ and then, from (2.9),

$$c_{gap}(\mu) \le 4\mathscr{D}(\mathbb{1}_{\{m_B>0\}}) = 2\sum_{x\in B} \mu\left(c_x\, [\nabla_x(\mathbb{1}_{\{m_B>0\}})]^2\right). \tag{5.5}$$

Notice that $\nabla_x(\mathbb{1}_{\{m_B>0\}})(\sigma) \neq 0$ only if $\sigma$ is such that $|m_B(\sigma)| = 1$, namely only if there are $(n+2)/2$ spins in $\sigma$ with the same value, and only if $x$ is one of the $(n+2)/2$ vertices with spin of the same sign of $m_B(\sigma)$. Then we get

$$\mathscr{D}(\mathbb{1}_{\{m_B>0\}}) \le \tfrac{n+2}{4}\, \mu(|m_B| = 1) = \tfrac{n+2}{2}\, \mu(m_B = 1). \tag{5.6}$$

For every configuration $\sigma \in \Omega$, we define the subsets

$$A^+(\sigma) := \{x \in B \text{ s.t. } \sigma_x = +\} \quad \text{and} \quad A^-(\sigma) = \{x \in B \text{ s.t. } \sigma_x = -\}.$$

Notice that the condition $m_B(\sigma)=1$ is satisfied if and only if $|A^+(\sigma)|=(n+1)/2$ and $|A^-(\sigma)|=(n-1)/2$. Thus, any configuration $\sigma$ with magnetization equal to 1, is univocally correspondent to a partition of the vertex set of $B$ into two subsets $S$ and $T$ of size $(n+1)/2$ and $(n-1)/2$, respectively, such that $A^+(\sigma)=S$ and $A^-(\sigma)=T$. Let $\mathscr{P}$ denote the set of these partitions, and define $E(S,T)$ as the set of edges between $S$ and $T$. Then we have

$$\begin{aligned}\mu(m_B=1) &= \sum_{\sigma:m_B(\sigma)=1} \frac{\exp(-\beta\sum_{(x,y)\in E(B)}\sigma_x\sigma_y)}{Z(\beta)} \\ &= \sum_{(S,T)\in\mathscr{P}} \frac{\exp(\beta(|E(S)|+|E(T)|-|E(S,T)|))}{Z(\beta)} \\ &\le \sum_{(S,T)\in\mathscr{P}} \frac{\exp(\beta(|E(S)|+|E(T)|-|E(S,T)|))}{\exp(\beta(|E(S)|+|E(T)|+|E(S,T)|))} \\ &= \sum_{(S,T)\in\mathscr{P}} \exp(-2\beta|E(S,T)|). \end{aligned} \tag{5.7}$$

Notice that $|E(S,T)|=|\partial_E T|$. In particular, since $B$ has uniformly positive isoperimetric constant, namely $i_e(B)\ge\epsilon>0$ for all $n\in\mathbb{N}$,

$$|E(S,T)|\ge\epsilon|T|=\tfrac{\epsilon}{2}(n-1).$$

Continuing from (5.7), we get

$$\begin{aligned}\mu(m_B=1) &\le \sum_{(S,T)\in\mathscr{P}} e^{-\beta\epsilon(n-1)} \\ &= \binom{n}{\frac{n-1}{2}} e^{-\beta\epsilon(n-1)} \\ &\le 2^n e^{-\beta\epsilon(n-1)} \\ &= c e^{-(\beta\epsilon-\log 2)n}, \end{aligned} \tag{5.8}$$

where in the third line we approximated the binomial factor using the Stirling formula. Inserting this bound in (5.6) we get that, for all $\beta>\frac{\log 2}{\epsilon}$, there exists a positive constant $\theta=\theta(\Delta,d,\beta)$ such that

$$\mathscr{D}(\mathbb{1}_{\{m_B>0\}}) \le \tfrac{n+2}{2} c e^{-(\beta\epsilon-\log 2)n} \le e^{-\theta n}, \tag{5.9}$$

which implies, by (5.5), that $c_{gap}(\mu)=O(e^{-\theta n})$. □

Theorem 2.8 follows from the above proposition, since by construction the infinite graph $X^{\Delta,d}$ is growing with parameter $g\ge\Delta-d-2$. Putting together Theorems 2.7 and Proposition 5.3, we get the first rigorous example in which the mixing time is at most linear in $n$ for the (+)-boundary condition, while at least exponential in $n$ for the free boundary condition.

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## References


[1] D. Aldous. *Random walks on finite groups and rapidly mixing Markov chains*. Seminar on probability, XVII, 243-297, Lecture Notes in Math., **986**, Springer, Berlin, 1983. MR0770418

[2] T. Bodineau. *Translation invariant Gibbs states for the Ising model*. Probab. Theory Related Fields **135** (2006), no. 2, 153-168. MR2218869

[3] R. Bubley, M. Dyer. *Path Coupling: a technique for proving rapid mixing in Markov Chains*. Proc. of the 38th Annual IEEE Symposium on Foundations of Computer Science, 1997, 223-231.

[4] N. Berger, C. Kenyon, E. Mossel, Y. Peres. *Glauber dynamics on trees and hyperbolic graphs*. Probab. Theory Related Fields **131** (2005), no. 3, 311-340. MR2123248

[5] T. Bodineau, F. Martinelli. *Some new results on the kinetic Ising model in a pure phase*. J. Statist. Phys. **109** (2002), no. 1-2, 207-235. MR1927919

[6] I. Benjamini, O. Schramm. *Percolation beyond* $\mathbb{Z}^d$*, many questions and a few answers*. Electron Comm. Probab. **1** (1996), no. 8, 71-82. MR1423907

[7] I. Benjamini, O. Schramm. *Percolation in the hyperbolic plane*. J. Amer. Math. Soc. **14** (2001), no. 2, 487-507. MR1815220

[8] J.T. Chayes, L. Chayes, J.P. Sethna, D.J. Thouless. *A mean field spin glass with short-range interactions*. Commun. Math. Phys. **106** (1986), no. 1, 41-89. MR0853978

[9] D. Fisher, D. Huse. *Dynamics of droplet fluctuations in pure and random Ising systems*. Phys. Rev. B **35** (1987), no. 13, 6841-6846.

[10] H.-O. Georgii. *Gibbs measures and Phase Transitions*. de Gruyter Studies in Mathematics, 9. Walter de Gruyter & Co., Berlin, 1988. MR0956646

[11] O. Häggström, J. Jonasson, R. Lyons. *Explicit isoperimetric constants and phase transitions in the random-cluster model*. Ann. Probab. **30** (2002), no. 1, 443-473. MR1894115

[12] S. Hoory, N. Linial, A. Wigderson. *Expander graphs and their applications*. Bull. Amer. Math. Soc. (N.S.) **43** (2006), no. 4, 439-561. MR2247919

[13] D. Ioffe. *Extremality of the disordered state for the Ising model on general trees*. Trees (Versailles, 1995), 3-14, Progr. Probab., **40**, Birkhäuser, Basel, 1996.

[14] J. Jonasson. *The random cluster model on a general graph and a phase transition characterization of nonamenability*. Stochastic Process. Appl., **79** (1999), no. 2, 335-354. MR1671859

[15] J. Jonasson, J.E. Steif. *Amenability and phase transition in the Ising model*. J. Theoret. Probab. **12** (1999), no. 2, 549-559. MR1684757

[16] H. Kesten. *Percolation theory for mathematicians*. Progr. Probab. Statist. **2.** Birkhäuser, Boston, Mass., 1982.

[17] C. Kenyon, E. Mossel, Y. Peres *Glauber dynamics on trees and hyperbolic graphs*. In 42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001), IEEE Computer Soc., Los Alamitos, CA, (2001), 568-578. MR1948746

[18] T. Lindvall. *Lectures on the coupling method*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1992. MR1180522

[19] R. Lyons. *The Ising model and percolation on trees and tree-like graphs*. Comm. Math. Phys. **125** (1989), no. 2, 337-353. MR1016874

[20] R. Lyons. *Phase transitions on Nonamenable Graphs*. J. Math. Phys. **41** (2000), no. 3, 1099-1126. MR1757952

[21] M. Luby, D. Randall, A. Sinclair. *Markov chain algorithms for planar lattice structures*. SIAM J. Comput. **31** (2001), no. 1, 167-192. MR1857394

[22] W. Magnus. *Noneuclidian tesselations and their groups*. Pure and Applied Mathematics, Vol. 61. Academic Press, New York-London, 1974. MR0352287

[23] F. Martinelli. *Lectures on Glauber dynamics for discrete spin models*. Lectures on probability and statistics (Saint-Flour, 1997), 93-191, Lecture Notes in Math., **1717**, Springer, Berlin, 1999. MR1746301

[24] F. Martinelli, E. Olivieri. *Approach to equilibrium of Glauber dynamics in the one phase region I: The attractive case*. Comm. Math. Phys. **161** (1994), no. 3, 447-486. MR1269387

[25] F. Martinelli, E. Olivieri. *Approach to equilibrium of Glauber dynamics in the one phase region II: The general case*. Comm. Math. Phys. **161** (1994), no. 3, 487-514. MR1269388

[26] F. Martinelli, A. Sinclair, D. Weitz. *Glauber dynamics on trees: boundary conditions and mixing time*. Comm. Math. Phys. **250** (2004), no. 2, 301-334. MR2094519

[27] R. Rietman, B. Nienhuis, J. Oitmaa. *Ising model on hyperlattices*. J. Phys. A **25** (1992), no. 24, 6577-6592. MR1210879

[28] L. Saloff-Coste. *Lectures on finite Markov chains*. Lectures on probability theory and statistics (Saint-Flour, 1996), 301-413, Lecture Notes in Math., **1665**, Springer, Berlin, 1997. MR1490046

[29] R. H. Schonmann. *Multiplicity of Phase Transitions and mean-field criticality on highly non-amenable graphs*. Comm. Math. Phys. **219** (2001), no. 2, 271-322. MR1833805

[30] C. M. Series, Ya. G. Sinai. *Ising models on the Lobachevsky plane*. Comm. Math. Phys. **128** (1990), no. 1, 63-76. MR1042443

[31] D.W. Stroock, B. Zegarlinski. *The logarithmic Sobolev inequality for discrete spin systems on a lattice*. Comm. Math. Phys. **149** (1992), no. 1, 175-194. MR1182416

[32] D. Weitz. *Combinatorial criteria for uniqueness of the Gibbs measure*. Random Structures Algorithms **27** (2005), no. 4, 445-475. MR2178257

[33] C. C. Wu. *Ising models on Hyperbolic Graphs*. J. Stat. Phys. **85** (1997), no. 1-2, 251-259. MR1413244

[34] C. C. Wu. *Ising models on Hyperbolic Graphs II*. J. Stat. Phys. **100** (2000), no. 5-6, 893-904. MR1798548