DOI: 10.1214/EJP.v13-478

Electronic Journal of Probability

Vol. 13 (2008), Paper no. 4, pages 79–106.

Journal URL
http://www.math.washington.edu/~ejpecp/

Radius and profile of random planar maps with faces of arbitrary degrees

Grégory Miermont
CNRS & LM
Université de Paris-Sud
Bat. 425, 91405 Orsay Cedex, France.
gregory.miermont@math.u-psud.fr

Mathilde Weill
DMA, École normale supérieure
45 rue d'Ulm, 75005 Paris, France.
weill@dma.ens.fr

Abstract

We prove some asymptotic results for the radius and the profile of large random planar maps with faces of arbitrary degrees. Using a bijection due to Bouttier, Di Francesco & Guitter between rooted planar maps and certain four-type trees with positive labels, we derive our results from a conditional limit theorem for four-type spatial Galton-Watson trees.

Key words: Random planar map, invariance principle, multitype spatial Galton-Watson tree, Brownian snake.

AMS 2000 Subject Classification: Primary 60F17, 60J80, 05J30.

Submitted to EJP on June 25, 2007, final version accepted January 8, 2008.

1 Introduction

1.1 Overview

A *planar map* is a proper embedding, without edge crossings, of a connected graph in the 2-dimensional sphere $\mathbb{S}^2$. Loops and multiple edges are allowed. A map comes with more structure than the original graph, which is given by its *faces*, i.e. the connected components of the complement of the embedding in $\mathbb{S}^2$. If $\mathbf{m}$ is a planar map, we write $\mathcal{F}_{\mathbf{m}}$ for the set of its faces, and $\mathcal{V}_{\mathbf{m}}$ for the set of its vertices. The degree $\deg(f)$ of a face $f \in \mathcal{F}_{\mathbf{m}}$ equals the number of edges incident to it, where an edge whose removal disconnects the graph must be counted twice (as it appears twice in a cyclic exploration of its incident face). A *rooted* planar map is a pair $(\mathbf{m}, \vec{e})$ where $\mathbf{m}$ is a planar map and $\vec{e}$ is a distinguished oriented edge. The origin o of $\vec{e}$ is called the root vertex. For technical reasons, we also consider the *vertex map* $\dagger$ made of one vertex bounding a face of degree 0, as a rooted planar map.

Two rooted maps are identified if there exists an orientation-preserving homeomorphism of $\mathbb{S}^2$ that sends the first map onto the second, in a way that respects the root edges. With this identification, the set $\mathcal{M}_r$ of rooted maps is countable, so we can enumerate certain distinguished subfamilies and sample them at random. Random maps are used in physics, in the field of 2-dimensional quantum gravity, as discretized versions of an ill-defined random surface [2]. On a mathematical level, this requires a detailed understanding of geometric properties of maps. One possible approach is to consider maps as metric spaces by endowing the set of their vertices with the usual graph distance: if a and a' are two vertices of a map $\mathbf{m}$, $d(a, a')$ is the minimal number of edges on a path from a to a'.

The laws on maps that we want to consider are Boltzmann laws parameterized by a sequence $\mathbf{q} = (q_i, i \geq 1)$ of nonnegative weights such that $q_i > 0$ for at least one $i \geq 3$. For any planar map $\mathbf{m}$, we define $W_{\mathbf{q}}(\mathbf{m})$ by $W_{\mathbf{q}}(\dagger) = 1$ and

$$W_{\mathbf{q}}(\mathbf{m}) = \prod_{f \in \mathcal{F}_{\mathbf{m}}} q_{\deg(f)} \,.$$

Our basic assumption is that $\mathbf{q}$ is be *admissible*, that is

$$Z_{\mathbf{q}} = \sum_{\mathbf{m} \in \mathcal{M}_r} \# \mathcal{V}_{\mathbf{m}} W_{\mathbf{q}}(\mathbf{m}) < \infty.$$

In this case, we let

$$Z_{\mathbf{q}}^{(r)} = \sum_{\mathbf{m} \in \mathcal{M}_r} W_{\mathbf{q}}(\mathbf{m}) < \infty \,,$$

and define the Boltzmann probability distribution $\mathbb{B}_{\mathbf{q}}^r$ on the set $\mathcal{M}_r$ by

$$\mathbb{B}_{\mathbf{q}}^r(\{\mathbf{m}\}) = \frac{W_{\mathbf{q}}(\mathbf{m})}{Z_{\mathbf{q}}^{(r)}}.$$

Our main goal is to obtain asymptotic results for certain geometric functionals of $\mathbb{B}_{\mathbf{q}}^r$-distributed maps conditioned to have a large number of vertices. The typical quantities of interest will be the radius $\mathcal{R}_{\mathbf{m}}$ of the map $\mathbf{m}$, defined as the maximal distance between o and another vertex of $\mathbf{m}$, that is

$$\mathcal{R}_{\mathbf{m}} = \max\{d(o, a) : a \in \mathcal{V}_{\mathbf{m}}\} \,,$$

and the profile of $\mathbf{m}$, which is the measure $\lambda_{\mathbf{m}}$ on $\{0,1,2,\ldots\}$ defined by

$$\lambda_{\mathbf{m}}(\{k\}) = \#\{a \in \mathcal{V}_{\mathbf{m}} : d(o,a) = k\},\ k \geq 0.$$

Note that $\mathcal{R}_{\mathbf{m}}$ is the supremum of the support of $\lambda_{\mathbf{m}}$. It is also convenient to introduce the rescaled profile. If $\mathbf{m}$ has n vertices, this is the probability measure on $\mathbb{R}_+$ defined by

$$\lambda_{\mathbf{m}}^{(n)}(A) = \frac{\lambda_{\mathbf{m}}(n^{1/4}A)}{n}$$

for any Borel subset A of $\mathbb{R}_+$. Theorem 1.2 below provides the limits in distribution for $n^{-1/4}\mathcal{R}_{\mathbf{m}}$ and $\lambda_{\mathbf{m}}^{(n)}$ under the measure $\mathbb{B}_{\mathbf{q}}^{r}$ conditioned on $\{\mathcal{V}_{\mathbf{m}} = n\}$ as $n \to \infty$, for a wide class of weights $\mathbf{q}$. The limiting distributions are given in terms of the so-called one-dimensional Brownian snake driven by a normalized excursion. For instance, the limiting distribution of the renormalized radius is a multiple of the range of the Brownian snake. The latter is a continuous limit of models of spatial trees which was introduced by Le Gall, and is also related to the so-called ISE of Aldous.

Such results were obtained earlier by Chassaing & Schaeffer in the pioneering work [4] in the special case of quadrangulations, corresponding to the case $q_4 = 1$ and $q_i = 0$ for $i \neq 4$, and by Weill [19] in the case of bipartite maps where $q_i = 0$ for odd i. Similar results are proved in Marckert & Miermont [14] for bipartite maps and in Miermont [17] for the general case, but in quite different settings. Indeed [14] and [17] deal with maps that are both rooted and pointed (see definitions below), and consider distances from the distinguished point rather that from the root vertex.

Similarly as in [4; 14; 19; 17], bijections between labeled trees and maps serve as a major tool in our approach, and explain the role played by the Brownian snake in the limit. In the case of quadrangulations, these bijections were studied by Cori & Vauquelin [5] and then by Schaeffer [18]. They have been recently extended to general planar maps by Bouttier, Di Francesco & Guitter [3]. More precisely, Bouttier, Di Francesco & Guitter show that planar maps are in one-to-one correspondence with well-labeled mobiles, where a well-labeled mobile is a four-type spatial tree whose vertices are assigned positive labels satisfying certain compatibility conditions (see section 2.3 for a precise definition). This bijection has the nice feature that labels in the mobile correspond to distances from the root in the map. Then the above mentioned asymptotics for random maps reduce to a limit theorem for well-labeled mobiles, which is stated as Theorem 3.3 below. This statement can be viewed as an invariance principle for multitype spatial Galton-Watson trees obtained by Miermont [16, Theorem 4], but in a conditioned version where spatial labels are all positive (working with both rooted *and* pointed maps was the combinatorial trick allowing [17] to lift the positivity condition). The basic methods we rely on are derived from Le Gall's work [9] and are quite close to that of [19]. However, there are some notable differences which make the study more intricate. One of the key differences lies in a change in a re-rooting lemma for discrete trees, which is considerably more delicate in the present setting where multiple types are allowed (see Section 3.1). The present paper will focus essentially on these differences, while the parts which can be derived *mutatis mutandis* from [9; 19] will be omitted.

1.2 Setting

1.2.1 Assumptions on $\mathbf{q}$

Since Boltzmann distributions on bipartite maps have been the object of [17], we will assume from now on that $q_{2\kappa+1} > 0$ for some $\kappa \geq 1$.

We first need to define some auxiliary material. A *rooted pointed* planar map is a triple $(\mathbf{m}, \tau, \vec{e})$ where $(\mathbf{m}, \vec{e})$ is a rooted planar map and τ is a distinguished vertex. We let $\mathcal{M}_{r,p}$ be the set of rooted, pointed planar maps, and allow $\dagger$ among its elements. In what follows, we will focus on the subset $\mathcal{M}_{r,p}^+$ of $\mathcal{M}_{r,p}$ defined by :

$$\mathcal{M}_{r,p}^+ = \{(\mathbf{m}, \tau, \vec{e}) \in \mathcal{M}_{r,p} : d(\tau, \vec{e}_+) = d(\tau, \vec{e}_-) + 1\} \cup \{\dagger\},$$

where $\vec{e}_-, \vec{e}_+$ are the origin and target of the oriented edge $\vec{e}$. Note that the quantity $Z_{\mathbf{q}}$ defined above also equals

$$Z_{\mathbf{q}} = \sum_{(\mathbf{m},\tau,\vec{e}) \in \mathcal{M}_{r,p}} W_{\mathbf{q}}(\mathbf{m}),$$

because the choice of any vertex in a rooted map yields a distinct element of $\mathcal{M}_{r,p}$. Set also

$$Z_{\mathbf{q}}^+ = \sum_{(\mathbf{m},\tau,\vec{e}) \in \mathcal{M}_{r,p}^+} W_{\mathbf{q}}(\mathbf{m}).$$

If $\mathbf{q}$ is admissible, then this quantity is finite as well, we define the Boltzmann distribution $\mathbb{B}_{\mathbf{q}}^+$ on the set $\mathcal{M}_{r,p}^+$ by

$$\mathbb{B}_{\mathbf{q}}^+(\{\mathbf{m}\}) = \frac{W_{\mathbf{q}}(\mathbf{m})}{Z_{\mathbf{q}}^+}.$$

The family of weights $\mathbf{q}$ that we consider is the same as in [17], and we recall it briefly here. For $k, k' \geq 0$ we set $N_\bullet(k,k') = \binom{2k+k'+1}{k+1}$ and $N_\diamondsuit(k,k') = \binom{2k+k'}{k}$. For every weight sequence we define

$$\begin{aligned} f_{\mathbf{q}}^\bullet(x,y) &= \sum_{k,k' \geq 0} x^k y^{k'} N_\bullet(k,k') \binom{k+k'}{k} q_{2+2k+k'}, \quad x, y \geq 0 \\ f_{\mathbf{q}}^\diamondsuit(x,y) &= \sum_{k,k' \geq 0} x^k y^{k'} N_\diamondsuit(k,k') \binom{k+k'}{k} q_{1+2k+k'}, \quad x, y \geq 0. \end{aligned}$$

From Proposition 1 in [17], a sequence $\mathbf{q}$ is admissible if and only if the system

$$\begin{aligned} \frac{z^+ - 1}{z^+} &= f_{\mathbf{q}}^\bullet(z^+, z^\diamondsuit) \\ z^\diamondsuit &= f_{\mathbf{q}}^\diamondsuit(z^+, z^\diamondsuit), \end{aligned}$$

has a solution $(z^+, z^\diamondsuit) \in (0, +\infty)^2$ for which the matrix $\mathsf{M}_{\mathbf{q}}(z^+, z^\diamondsuit)$ defined by

$$\mathsf{M}_{\mathbf{q}}(z^+, z^\diamondsuit) = \begin{pmatrix} 0 & 0 & z^+ - 1 \\ \frac{z^+}{z^\diamondsuit} \partial_x f_{\mathbf{q}}^\diamondsuit(z^+, z^\diamondsuit) & \partial_y f_{\mathbf{q}}^\diamondsuit(z^+, z^\diamondsuit) & 0 \\ \frac{(z^+)^2}{z^+-1} \partial_x f_{\mathbf{q}}^\bullet(z^+, z^\diamondsuit) & \frac{z^+ z^\diamondsuit}{z^+-1} \partial_y f_{\mathbf{q}}^\bullet(z^+, z^\diamondsuit) & 0 \end{pmatrix}$$

has a spectral radius $\varrho \le 1$. Furthermore this solution is unique and

$$\begin{aligned} z^+ &= Z^+_{\mathbf{q}}, \\ z^\diamondsuit &= Z^\diamondsuit_{\mathbf{q}}, \end{aligned}$$

where $(Z^\diamondsuit_{\mathbf{q}})^2 = Z_{\mathbf{q}} - 2Z^+_{\mathbf{q}} + 1$. An admissible weight sequence $\mathbf{q}$ is said to be *critical* if the matrix $\mathsf{M}_{\mathbf{q}}(Z^+_{\mathbf{q}}, Z^\diamondsuit_{\mathbf{q}})$ has a spectral radius $\varrho = 1$. An admissible weight sequence $\mathbf{q}$ is said to be *regular critical* if $\mathbf{q}$ is critical and if $f^\bullet_{\mathbf{q}}(Z^+_{\mathbf{q}} + \varepsilon, Z^\diamondsuit_{\mathbf{q}} + \varepsilon) < \infty$ for some $\varepsilon > 0$.

1.2.2 The Brownian snake and the conditioned Brownian snake

Let $x \in \mathbb{R}$. The Brownian snake with initial point x is a pair $(\mathbf{b}, \mathbf{r}^x)$, where $\mathbf{b} = (\mathbf{b}(s), 0 \le s \le 1)$ is a normalized Brownian excursion and $\mathbf{r}^x = (\mathbf{r}^x(s), 0 \le s \le 1)$ is a real-valued process such that, conditionally given $\mathbf{b}$, $\mathbf{r}^x$ is Gaussian with mean and covariance given by

- $\mathbf{E}[\mathbf{r}^x(s)] = x$ for every $s \in [0,1]$,
- $\mathbf{Cov}(\mathbf{r}^x(s), \mathbf{r}^x(s')) = \inf_{s \le t \le s'} \mathbf{b}(t)$ for every $0 \le s \le s' \le 1$.

We know from [7] that $\mathbf{r}^x$ admits a continuous modification. From now on we consider only this modification. In the terminology of [7] $\mathbf{r}^x$ is the terminal point process of the one-dimensional Brownian snake driven by the normalized Brownian excursion $\mathbf{b}$ and with initial point x.

Write $\mathbf{P}$ for the probability measure under which the collection $(\mathbf{b}, \mathbf{r}^x)_{x \in \mathbb{R}}$ is defined. As mentioned in [19], for every $x > 0$, we have

$$\mathbf{P}\left(\inf_{s \in [0,1]} \mathbf{r}^x(s) \ge 0\right) > 0\,.$$

We may then define for every $x > 0$ a pair $(\overline{\mathbf{b}}^x, \overline{\mathbf{r}}^x)$ which is distributed as the pair $(\mathbf{b}, \mathbf{r}^x)$ under the conditioning that $\inf_{s \in [0,1]} \mathbf{r}^x(s) \ge 0$.

We equip $C([0,1], \mathbb{R})^2$ with the norm $\|(f,g)\| = \|f\|_u \vee \|g\|_u$ where $\|f\|_u$ stands for the supremum norm of f. The following theorem is a consequence of Theorem 1.1 in Le Gall & Weill [13].

Theorem 1.1. *There exists a pair $(\overline{\mathbf{b}}^0, \overline{\mathbf{r}}^0)$ such that $(\overline{\mathbf{b}}^x, \overline{\mathbf{r}}^x)$ converges in distribution as $x \downarrow 0$ towards $(\overline{\mathbf{b}}^0, \overline{\mathbf{r}}^0)$.*

The pair $(\overline{\mathbf{b}}^0, \overline{\mathbf{r}}^0)$ is the so-called conditioned Brownian snake with initial point 0. Theorem 1.2 in [13] provides a useful construction of the conditioned object $(\overline{\mathbf{b}}^0, \overline{\mathbf{r}}^0)$ from the unconditioned one $(\mathbf{b}, \mathbf{r}^0)$. This construction also appears in Marckert & Mokkadem [15], though its outcome was not interpreted as the object appearing in Theorem 1.1. First recall that there is a.s. a unique s_* in $(0,1)$ such that

$$\mathbf{r}^0(s_*) = \inf_{s \in [0,1]} \mathbf{r}^0(s)$$

(see Lemma 16 in [15] or Proposition 2.5 in [13]). For every $s \in [0, \infty)$, write $\{s\}$ for the fractional part of s. According to Theorem 1.2 in [13], the conditioned snake $(\overline{\mathbf{b}}^0, \overline{\mathbf{r}}^0)$ may be constructed explicitly as follows : for every $s \in [0,1]$,

$$\begin{aligned}\overline{\mathbf{b}}^0(s) &= \mathbf{b}(s_*) + \mathbf{b}(\{s_* + s\}) - 2 \inf_{s \wedge \{s_*+s\} \le t \le s \vee \{s_*+s\}} \mathbf{b}(t), \\ \overline{\mathbf{r}}^0(s) &= \mathbf{r}^0(\{s_* + s\}) - \mathbf{r}^0(s_*).\end{aligned}$$

1.3 Statement of the main result

Recall from section 1.2.2 that $(\mathbf{b}, \mathbf{r}^0)$ denotes the Brownian snake with initial point 0.

Theorem 1.2. *Let $\mathbf{q}$ be a regular critical weight sequence. There exists a scaling constant $\mathrm{C}_{\mathbf{q}}$ such that the following results hold.*

(i) The law of $n^{-1/4}\,\mathcal{R}_{\mathbf{m}}$ under $\mathbb{B}^r_{\mathbf{q}}(\cdot \mid \#\mathcal{V}_{\mathbf{m}} = n)$ converges as $n \to \infty$ to the law of the random variable

$$\mathrm{C}_{\mathbf{q}} \left(\sup_{0 \le s \le 1} \mathbf{r}^0(s) - \inf_{0 \le s \le 1} \mathbf{r}^0(s) \right).$$

(ii) The law of the random probability measure $\lambda^{(n)}_{\mathbf{m}}$ under $\mathbb{B}^r_{\mathbf{q}}(\cdot \mid \#\mathcal{V}_{\mathbf{m}} = n)$ converges as $n \to \infty$ to the law of the random probability measure $\mathcal{I}$ defined by

$$\langle \mathcal{I}, g \rangle = \int_0^1 g\left(\mathrm{C}_{\mathbf{q}} \left(\mathbf{r}^0(t) - \inf_{0 \le s \le 1} \mathbf{r}^0(s) \right) \right) \mathrm{d}t.$$

(iii) The law of the rescaled distance $n^{-1/4}\,d(o,a)$ where a is a vertex chosen uniformly at random among all vertices of $\mathbf{m}$, under $\mathbb{B}^r_{\mathbf{q}}(\cdot \mid \#\mathcal{V}_{\mathbf{m}} = n)$ converges as $n \to \infty$ to the law of the random variable

$$\mathrm{C}_{\mathbf{q}} \sup_{0 \le s \le 1} \mathbf{r}^0(s).$$

Theorem 1.2 is analogous to Theorem 2.5 in [19] in the setting of non-bipartite maps. Beware that in Theorem 1.2 maps are conditioned on their number of vertices whereas in [19] they are conditioned on their number of faces. However the results stated in Theorem 2.5 in [19] remain valid by conditioning on the number of vertices (with different scaling constants). On the other hand, our arguments to prove Theorem 1.2 do not lead to the statement of these results by conditioning maps on their number of faces. A notable exception is the case of k-angulations ($\mathbf{q} = q\delta_k$ for some $k \ge 3$ and appropriate $q > 0$), where an application of Euler's formula shows that $\#\mathcal{F}_{\mathbf{m}} = (k/2 - 1)\#\mathcal{V}_{\mathbf{m}} + 2$, so that the two conditionings are essentially equivalent and result in a change in the scale factor $\mathrm{C}_{\mathbf{q}}$.

Recall that the results of Theorem 2.5 in [19] for the special case of quadrangulations were obtained by Chassaing & Schaeffer [4] (see also Theorem 8.2 in [9]). Last, observe that Theorem 1.2 is obviously related to Theorem 1 in [17]. Note however that [17] deals with rooted pointed maps instead of rooted maps as we do and studies distances from the distinguished point of the map rather than from the root vertex.

The rest of the paper is organized as follows. We recall the necessary formalism for multitype spatial trees in the next section, together with the main properties of the bijection of Bouttier, Di Francesco & Guitter. Section 3 is devoted to the statement and proof of a key result, which is an invariance principle for random conditioned multitype spatial trees. Theorem 1.2 is finally derived in Section 4.

2 Preliminaries

2.1 Multitype spatial trees

We start with some formalism for discrete trees. Set

$$\mathcal{U} = \bigcup_{n \geq 0} \mathbb{N}^n,$$

where by convention $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}^0 = \{\varnothing\}$. An element of $\mathcal{U}$ is a sequence $u = u^1 \ldots u^n$, and we set $|u| = n$ so that $|u|$ represents the generation of u. In particular $|\varnothing| = 0$. If $u = u^1 \ldots u^n$ and $v = v^1 \ldots v^m$ belong to $\mathcal{U}$, we write $uv = u^1 \ldots u^n v^1 \ldots v^m$ for the concatenation of u and v. In particular $\varnothing u = u \varnothing = u$. If v is of the form $v = uj$ for $u \in \mathcal{U}$ and $j \in \mathbb{N}$, we say that v is a child of u, or that u is the father of v, and we write $u = \check{v}$. More generally if v is of the form $v = uw$ for $u, w \in \mathcal{U}$, we say that v is a descendant of u, or that u is an ancestor of v. The set $\mathcal{U}$ comes with the natural lexicographical order such that $u \preccurlyeq v$ if either u is an ancestor of v, or if $u = wa$ and $v = wb$ with $a \in \mathcal{U}^*$ and $b \in \mathcal{U}^*$ satisfying $a^1 < b^1$, where we have set $\mathcal{U}^* = \mathcal{U} \setminus \{\varnothing\}$. We write $u \prec v$ if $u \preccurlyeq v$ and $u \neq v$.

A plane tree $\mathbf{t}$ is a finite subset of $\mathcal{U}$ such that

- $\varnothing \in \mathbf{t}$,
- $u \in \mathbf{t} \setminus \{\varnothing\} \Rightarrow \check{u} \in \mathbf{t}$,
- for every $u \in \mathbf{t}$ there exists a number $c_u(\mathbf{t}) \geq 0$ such that $uj \in \mathbf{t} \Leftrightarrow 1 \leq j \leq c_u(\mathbf{t})$.

Let $\mathbf{t}$ be a plane tree and let $\xi = \#\mathbf{t} - 1$. The *search-depth sequence* of $\mathbf{t}$ is the sequence $u_0, u_1, \ldots, u_{2\xi}$ of vertices of $\mathbf{t}$ which is obtained by induction as follows. First $u_0 = \varnothing$, and then for every $i \in \{0, 1, \ldots, 2\xi - 1\}$, u_{i+1} is either the first child of u_i that has not yet appeared in the sequence $u_0, u_1, \ldots, u_i$, or the father of u_i if all children of u_i already appear in the sequence $u_0, u_1, \ldots, u_i$. It is easy to verify that $u_{2\xi} = \varnothing$ and that all vertices of $\mathbf{t}$ appear in the sequence $u_0, u_1, \ldots, u_{2\xi}$ (of course some of them appear more than once). We can now define the *contour function* of $\mathbf{t}$. For every $k \in \{0, 1, \ldots, 2\xi\}$, we let $C(k) = |u_k|$ denote the generation of the vertex u_k. We extend the definition of C to the line interval $[0, 2\xi]$ by interpolating linearly between successive integers. Clearly $\mathbf{t}$ is uniquely determined by its contour function C.

Let $K \in \mathbb{N}$ and $[K] = \{1, 2, \ldots, K\}$. A K-type tree is a pair $(\mathbf{t}, \mathbf{e})$ where $\mathbf{t}$ is a plane tree and $\mathbf{e} : \mathbf{t} \to [K]$ assigns a type to each vertex. If $(\mathbf{t}, \mathbf{e})$ is a K-type tree and if $i \in [K]$ we set

$$\mathbf{t}^i = \{u \in \mathbf{t} : \mathbf{e}(u) = i\}.$$

We denote by $T^{(K)}$ the set of all K-type trees and we set

$$T_i^{(K)} = \left\{ (\mathbf{t}, \mathbf{e}) \in T^{(K)} : \mathbf{e}(\varnothing) = i \right\}.$$

Set

$$\mathcal{W}_K = \bigcup_{n \geq 0} [K]^n,$$

with the convention $[K]^0 = \{\varnothing\}$. An element of $\mathcal{W}_K$ is a sequence $\mathbf{w} = (w_1, \ldots, w_n)$ and we set $|\mathbf{w}| = n$. Consider the natural projection $p : \mathcal{W}_K \to \mathbb{Z}_+^K$ where $p(\mathbf{w}) = (p_1(\mathbf{w}), \ldots, p_K(\mathbf{w}))$ and

$$p_i(\mathbf{w}) = \#\{j \in \{1, \ldots, |\mathbf{w}|\} : w_j = i\}.$$

Note that $p(\varnothing) = (0, \ldots, 0)$ with this definition. Let $u \in \mathcal{U}$ and let $(\mathbf{t}, \mathbf{e}) \in T^{(K)}$ such that $u \in \mathbf{t}$. We then define $\mathbf{w}_u(\mathbf{t}) \in \mathcal{W}_K$ by

$$\mathbf{w}_u(\mathbf{t}) = (\mathbf{e}(u1), \ldots, \mathbf{e}(uc_u(\mathbf{t}))),$$

and we set $\mathbf{z}_u(\mathbf{t}) = p(\mathbf{w}_u(\mathbf{t}))$.

A K-type spatial tree is a triple $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell})$ where $(\mathbf{t}, \mathbf{e}) \in T^{(K)}$ and $\boldsymbol{\ell} : \mathbf{t} \to \mathbb{R}$. If v is a vertex of $\mathbf{t}$, we say that $\boldsymbol{\ell}_v$ is the *label* of v. We denote by $\mathbb{T}^{(K)}$ the set of all K-type spatial trees and we set

$$\mathbb{T}_i^{(K)} = \left\{(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \mathbb{T}^{(K)} : \mathbf{e}(\varnothing) = i\right\}.$$

If $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \mathbb{T}^{(K)}$ we define the *spatial contour function* of $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell})$ as follows. Recall that $u_0, u_1, \ldots, u_{2\xi}$ denotes the search-depth sequence of $\mathbf{t}$. First if $k \in \{0, \ldots, 2\xi\}$, we put $V(k) = \boldsymbol{\ell}_{u_k}$. We then complete the definition of V by interpolating linearly between successive integers.

2.2 Multitype spatial Galton-Watson trees

Let $\boldsymbol{\zeta} = (\zeta^{(i)}, i \in [K])$ be a family of probability measures on the set $\mathcal{W}_K$. We associate with $\boldsymbol{\zeta}$ the family $\boldsymbol{\mu} = (\mu^{(i)}, i \in [K])$ of probability measures on the set $\mathbb{Z}_+^K$ in such a way that each $\mu^{(i)}$ is the image measure of $\zeta^{(i)}$ under the mapping p. We make the basic assumption that

$$\max_{i \in [K]} \mu^{(i)}\left(\left\{\mathbf{z} \in \mathbb{Z}_+^K : \sum_{j=1}^{K} z_j \neq 1\right\}\right) > 0,$$

and we say that $\boldsymbol{\zeta}$ (or $\boldsymbol{\mu}$) is non-degenerate. If for every $i \in [K]$, $\mathbf{w} \in \mathcal{W}_K$ and $\mathbf{z} = p(\mathbf{w})$ we have

$$\zeta^{(i)}(\{\mathbf{w}\}) = \frac{\mu^{(i)}(\{\mathbf{z}\})}{\#\left(p^{-1}(\mathbf{z})\right)},$$

then we say that $\boldsymbol{\zeta}$ is the *uniform ordering* of $\boldsymbol{\mu}$.

For every $i, j \in [K]$, let

$$m_{ij} = \sum_{\mathbf{z} \in \mathbb{Z}_+^K} z_j \mu^{(i)}(\{\mathbf{z}\}),$$

be the mean number of type-j children of a type-i individual, and let $\mathsf{M}_{\boldsymbol{\mu}} = (m_{ij})_{1 \le i,j \le K}$. The matrix $\mathsf{M}_{\boldsymbol{\mu}}$ is said to be irreducible if for every $i, j \in [K]$ there exists $n \in \mathbb{N}$ such that $m_{ij}^{(n)} > 0$ where we have written $m_{ij}^{(n)}$ for the ij-entry of $\mathsf{M}_{\boldsymbol{\mu}}^n$. We say that $\boldsymbol{\zeta}$ (or $\boldsymbol{\mu}$) is irreducible if $\mathsf{M}_{\boldsymbol{\mu}}$ is. Under this assumption the Perron-Frobenius theorem ensures that $\mathsf{M}_{\boldsymbol{\mu}}$ has a real, positive eigenvalue ϱ with maximal modulus. The distribution $\boldsymbol{\zeta}$ (or $\boldsymbol{\mu}$) is called sub-critical if $\varrho < 1$ and critical if $\varrho = 1$.

Assume that $\boldsymbol{\zeta}$ is non-degenerate, irreducible and (sub-)critical. We denote by $P_{\boldsymbol{\zeta}}^{(i)}$ the law of a K-type Galton-Watson tree with offspring distribution $\boldsymbol{\zeta}$ and with ancestor of type i, meaning that for every $(\mathbf{t}, \mathbf{e}) \in T_i^{(K)}$,

$$P_{\boldsymbol{\zeta}}^{(i)}(\{(\mathbf{t}, \mathbf{e})\}) = \prod_{u \in \mathbf{t}} \zeta^{(\mathbf{e}(u))}\left(\mathbf{w}_u(\mathbf{t})\right),$$

The fact that this formula defines a probability measure on $T_i^{(K)}$ is justified in [16].

Let us now recall from [16] how one can couple K-type trees with a spatial displacement in order to turn them into random elements of $\mathbb{T}^{(K)}$. To this end, consider a family $\boldsymbol{\nu} = (\nu_{i,\mathbf{w}}, i \in [K], \mathbf{w} \in \mathcal{W}_K)$ where $\nu_{i,\mathbf{w}}$ is a probability measure on $\mathbb{R}^{|\mathbf{w}|}$. If $(\mathbf{t}, \mathbf{e}) \in T^{(K)}$ and $x \in \mathbb{R}$, we denote by $R_{\nu,x}((\mathbf{t}, \mathbf{e}), \mathrm{d}\boldsymbol{\ell})$ the probability measure on $\mathbb{R}^{\mathbf{t}}$ which is characterized as follows. For every $i \in [K]$ and $u \in \mathbf{t}$ such that $\mathbf{e}(u) = i$, consider $\mathbf{Y}_u = (Y_{u1}, \ldots, Y_{u|\mathbf{w}|})$ (where we have written $\mathbf{w}_u(\mathbf{t}) = \mathbf{w}$) a random variable distributed according to $\nu_{i,\mathbf{w}}$, in such a way that $(\mathbf{Y}_u, u \in \mathbf{t})$ is a collection of independent random variables. We set $L_\varnothing = x$ and for every $v \in \mathbf{t} \setminus \{\varnothing\}$,

$$L_v = \sum_{u \in \rrbracket \varnothing, v \rrbracket} Y_u,$$

where $\rrbracket \varnothing, v \rrbracket$ is the set of all ancestors of v distinct from the root $\varnothing$. The probability measure $R_{\boldsymbol{\nu},x}((\mathbf{t}, \mathbf{e}), \mathrm{d}\boldsymbol{\ell})$ is then defined as the law of $(L_v, v \in \mathbf{t})$. We finally define for every $x \in \mathbb{R}$ a probability measure $\mathbb{P}_{\boldsymbol{\zeta},\boldsymbol{\nu},x}^{(i)}$ on the set $\mathbb{T}_i^{(K)}$ by setting,

$$\mathbb{P}_{\boldsymbol{\zeta},\boldsymbol{\nu},x}^{(i)}(\mathrm{d}\mathbf{t}\, \mathrm{d}\mathbf{e}\, \mathrm{d}\boldsymbol{\ell}) = P_{\boldsymbol{\zeta}}^{(i)}(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{e}) R_{\boldsymbol{\nu},x}((\mathbf{t}, \mathbf{e}), \mathrm{d}\boldsymbol{\ell}).$$

2.3 The Bouttier-Di Francesco-Guitter bijection

We start with a definition. We consider the set $T_M \subset T_1^{(4)}$ of 4-type trees in which, for every $(\mathbf{t}, \mathbf{e}) \in T_M$ and $u \in \mathbf{t}$,

1. if $\mathbf{e}(u) = 1$ then $\mathbf{z}_u(\mathbf{t}) = (0, 0, k, 0)$ for some $k \geq 0$,
2. if $\mathbf{e}(u) = 2$ then $\mathbf{z}_u(\mathbf{t}) = (0, 0, 0, 1)$,
3. if $\mathbf{e}(u) \in \{3, 4\}$ then $\mathbf{z}_u(\mathbf{t}) = (k, k', 0, 0)$ for some $k, k' \geq 0$.

Let now $\mathbb{T}_M \subset \mathbb{T}_1^{(4)}$ be the set of 4-type spatial trees $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell})$ such that $(\mathbf{t}, \mathbf{e}) \in T_M$ and in which, for every $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \mathbb{T}_M$ and $u \in \mathbf{t}$,

4. $\boldsymbol{\ell}_u \in \mathbb{Z}$,
5. if $\mathbf{e}(u) \in \{1, 2\}$ then $\boldsymbol{\ell}_u = \boldsymbol{\ell}_{ui}$ for every $i \in \{1, \ldots, c_u(\mathbf{t})\}$,
6. if $\mathbf{e}(u) \in \{3, 4\}$ and $c_u(\mathbf{t}) = k$ then by setting $u0 = u(k+1) = \check{u}$ and $x_i = \boldsymbol{\ell}_{ui} - \boldsymbol{\ell}_{u(i-1)}$ for $1 \leq i \leq k+1$, we have
   (a) if $\mathbf{e}(u(i-1)) = 1$ then $x_i \in \{-1, 0, 1, 2, \ldots\}$,

(b) if $\mathbf{e}(u(i-1)) = 2$ then $x_i \in \{0, 1, 2, \ldots\}$.

We will be interested in the set $\overline{\mathbb{T}}_M = \{(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \mathbb{T}_M : \ell_\varnothing = 1 \text{ and } \boldsymbol{\ell}_v \geq 1 \text{ for all } v \in \mathbf{t}^1\}$. Notice that condition **6.** implies that if $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \overline{\mathbb{T}}_M$ then $\boldsymbol{\ell}_v \geq 0$ for all $v \in \mathbf{t}$.

We will now describe the Bouttier-Di Francesco-Guitter bijection from the set $\overline{\mathbb{T}}_M$ onto $\mathcal{M}_r$. This bijection can be found in [3] in the more general setting of Eulerian maps.

Let $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \overline{\mathbb{T}}_M$. Recall that $\xi = \#\mathbf{t} - 1$. Let $u_0, u_1, \ldots, u_{2\xi}$ be the search-depth sequence of $\mathbf{t}$. It is immediate to see that $\mathbf{e}(u_k) \in \{1, 2\}$ if k is even and that $\mathbf{e}(u_k) \in \{3, 4\}$ if k is odd. We define the sequence $v_0, v_1, \ldots, v_\xi$ by setting $v_k = u_{2k}$ for every $k \in \{0, 1, \ldots, \xi\}$. Notice that $v_0 = v_\xi = \varnothing$.

Suppose that the tree $\mathbf{t}$ is drawn in the plane and add an extra vertex ∂, not on $\mathbf{t}$. We associate with $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell})$ a planar map whose set of vertices is

$$\mathbf{t}^1 \cup \{\partial\},$$

and whose edges are obtained by the following device : for every $k \in \{0, 1, \ldots, \xi - 1\}$,

- if $\mathbf{e}(v_k) = 1$ and $\boldsymbol{\ell}_{v_k} = 1$, or if $\mathbf{e}(v_k) = 2$ and $\boldsymbol{\ell}_{v_k} = 0$, draw an edge between v_k and ∂ ;
- if $\mathbf{e}(v_k) = 1$ and $\boldsymbol{\ell}_{v_k} \geq 2$, or if $\mathbf{e}(v_k) = 2$ and $\boldsymbol{\ell}_{v_k} \geq 1$, draw an edge between v_k and the first vertex in the sequence $v_{k+1}, \ldots, v_\xi$ with type 1 and label $\boldsymbol{\ell}_{v_k} - \mathbb{1}_{\{\mathbf{e}(v_k)=1\}}$.

Notice that condition **6.** in the definition of the set $\overline{\mathbb{T}}_M$ entails that $\boldsymbol{\ell}_{v_{k+1}} \geq \boldsymbol{\ell}_{v_k} - \mathbb{1}_{\{\mathbf{e}(v_k)=1\}}$ for every $k \in \{0, 1, \ldots, \xi - 1\}$, and recall that $\min\{\boldsymbol{\ell}_{v_j} : j \in \{k+1, \ldots, \xi\} \text{ and } \mathbf{e}(v_j) = 1\} = 1$. The preceding properties ensure that whenever $\mathbf{e}(v_k) = 1$ and $\boldsymbol{\ell}(v_k) \geq 2$ or $\mathbf{e}(v_k) = 2$ and $\boldsymbol{\ell}(v_k) \geq 1$ there is at least one type-1 vertex among $\{v_{k+1}, \ldots, v_\xi\}$ with label $\boldsymbol{\ell}_{v_k} - \mathbb{1}_{\{\mathbf{e}(v_k)=1\}}$. The construction can be made in such a way that edges do not intersect. Notice that condition **2.** in the definition of the set T^M entails that a type-2 vertex is connected by the preceding construction to exactly two type-1 vertices with the same label, so that we can erase all type-2 vertices. The resulting planar graph is a planar map. We view this map as a rooted planar map by declaring that the distinguished edge is the one corresponding to $k = 0$, pointing from ∂, in the preceding construction.

It follows from [3] that the preceding construction yields a bijection Ψ_r between $\overline{\mathbb{T}}_M$ and $\mathcal{M}_r$. Furthermore it is not difficult to see that Ψ_r satisfies the following two properties : let $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \overline{\mathbb{T}}_M$ and let $\mathbf{m} = \Psi_r((\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}))$,

(i) the set $\mathcal{F}_\mathbf{m}$ is in one-to-one correspondence with the set $\mathbf{t}^3 \cup \mathbf{t}^4$, more precisely, with every $v \in \mathbf{t}^3$ (resp. $v \in \mathbf{t}^4$) such that $\mathbf{z}_u(\mathbf{t}) = (k, k', 0, 0)$ is associated a unique face of $\mathbf{m}$ whose degree is equal to $2k + k' + 2$ (resp. $2k + k' + 1$),

(ii) for every $l \geq 1$, the set $\{a \in \mathcal{V}_\mathbf{m} : d(\partial, a) = l\}$ is in one-to-one correspondence with the set $\{v \in \mathbf{t}^1 : \boldsymbol{\ell}_v = l\}$.

2.4 Boltzmann laws on multitype spatial trees

Let $\mathbf{q}$ be a regular critical weight sequence. We associate with $\mathbf{q}$ four probability measures on $\mathbb{Z}_+^4$ defined by :

$$\begin{aligned}
\mu_{\mathbf{q}}^{(1)}(\{(0,0,k,0)\}) &= \frac{1}{Z_{\mathbf{q}}^+}\left(1-\frac{1}{Z_{\mathbf{q}}^+}\right)^k, \ k\geq 0,\\
\mu_{\mathbf{q}}^{(2)}(\{(0,0,0,1)\}) &= 1,\\
\mu_{\mathbf{q}}^{(3)}(\{(k,k',0,0)\}) &= \frac{(Z_{\mathbf{q}}^+)^k(Z_{\mathbf{q}}^\diamondsuit)^{k'}N_\bullet(k,k')\binom{k+k'}{k}q_{2+2k+k'}}{f_{\mathbf{q}}^\bullet(Z_{\mathbf{q}}^+,Z_{\mathbf{q}}^\diamondsuit)}, \ k,k'\geq 0,\\
\mu_{\mathbf{q}}^{(4)}(\{(k,k',0,0)\}) &= \frac{(Z_{\mathbf{q}}^+)^k(Z_{\mathbf{q}}^\diamondsuit)^{k'}N_\diamondsuit(k,k')\binom{k+k'}{k}q_{1+2k+k'}}{f_{\mathbf{q}}^\diamondsuit(Z_{\mathbf{q}}^+,Z_{\mathbf{q}}^\diamondsuit)}, \ k,k'\geq 0.
\end{aligned}$$

We set $\boldsymbol{\mu}_{\mathbf{q}}=\left(\mu_{\mathbf{q}}^{(1)},\mu_{\mathbf{q}}^{(2)},\mu_{\mathbf{q}}^{(3)},\mu_{\mathbf{q}}^{(4)}\right)$ and $\mathsf{M}_{\boldsymbol{\mu}_{\mathbf{q}}}=(m_{ij})_{1\leq i,j\leq 4}$. The matrix $\mathsf{M}_{\boldsymbol{\mu}_{\mathbf{q}}}$ is given by

$$\mathsf{M}_{\boldsymbol{\mu}_{\mathbf{q}}}=\begin{pmatrix}
0 & 0 & Z_{\mathbf{q}}^+-1 & 0\\
0 & 0 & 0 & 1\\
\frac{(Z_{\mathbf{q}}^+)^2}{Z_{\mathbf{q}}^+-1}\partial_x f_{\mathbf{q}}^\bullet(Z_{\mathbf{q}}^+,Z_{\mathbf{q}}^\diamondsuit) & \frac{Z_{\mathbf{q}}^+Z_{\mathbf{q}}^\diamondsuit}{Z_{\mathbf{q}}^+-1}\partial_y f_{\mathbf{q}}^\bullet(Z_{\mathbf{q}}^+,Z_{\mathbf{q}}^\diamondsuit) & 0 & 0\\
\frac{Z_{\mathbf{q}}^+}{Z_{\mathbf{q}}^\diamondsuit}\partial_x f_{\mathbf{q}}^\diamondsuit(Z_{\mathbf{q}}^+,Z_{\mathbf{q}}^\diamondsuit) & \partial_y f_{\mathbf{q}}^\diamondsuit(Z_{\mathbf{q}}^+,Z_{\mathbf{q}}^\diamondsuit) & 0 & 0
\end{pmatrix}.$$

We see that $\mathsf{M}_{\boldsymbol{\mu}_{\mathbf{q}}}$ is irreducible and has a spectral radius $\varrho=1$. Thus $\boldsymbol{\mu}_{\mathbf{q}}$ is critical. Let us denote by $\mathbf{a}=(a_1,a_2,a_3,a_4)$ the right eigenvector of $\mathsf{M}_{\boldsymbol{\mu}_{\mathbf{q}}}$ with eigenvalue 1 chosen so that $a_1+a_2+a_3+a_4=1$.

Let $\boldsymbol{\zeta}_{\mathbf{q}}$ be the uniform ordering of $\boldsymbol{\mu}_{\mathbf{q}}$. Note that if $\mathbf{w}\in\mathcal{W}_4$ satisfies $w_j\in\{1,2\}$ for every $j\in\{1,\ldots,|\mathbf{w}|\}$, then, by setting $k=p_1(\mathbf{w})$ and $k'=p_2(\mathbf{w})$, we have

$$\begin{aligned}
\zeta_{\mathbf{q}}^{(3)}(\{\mathbf{w}\}) &= \frac{(Z_{\mathbf{q}}^+)^k(Z_{\mathbf{q}}^\diamondsuit)^{k'}N_\bullet(k,k')q_{2+2k+k'}}{f_{\mathbf{q}}^\bullet(Z_{\mathbf{q}}^+,Z_{\mathbf{q}}^\diamondsuit)},\\
\zeta_{\mathbf{q}}^{(4)}(\{\mathbf{w}\}) &= \frac{(Z_{\mathbf{q}}^+)^k(Z_{\mathbf{q}}^\diamondsuit)^{k'}N_\diamondsuit(k,k')q_{1+2k+k'}}{f_{\mathbf{q}}^\diamondsuit(Z_{\mathbf{q}}^+,Z_{\mathbf{q}}^\diamondsuit)}.
\end{aligned}$$

Let us now define a collection $\boldsymbol{\nu}=(\nu_{i,\mathbf{w}},i\in\{1,2,3,4\},\mathbf{w}\in\mathcal{W}_4)$ as follows.

- For $i\in\{1,2\}$ the measure $\nu_{i,\mathbf{w}}$ is the Dirac mass at $\mathbf{0}\in\mathbb{R}^{|\mathbf{w}|}$.
- Let $\mathbf{w}\in\mathcal{W}_4$ be such that $p(\mathbf{w})=(k,k',0,0)$. Then $\nu_{3,\mathbf{w}}$ is the distribution of the random vector $(X_1,X_1+X_2,\ldots,X_1+X_2+\ldots+X_{k+k'})$, where $(X_j+\mathbb{1}_{\{w_{j-1}=1\}},1\leq j\leq k+k'+1)$ (with $w_0=1$) is uniformly distributed on the set
$$A_{k,k'}=\left\{(n_1,\ldots,n_{k+k'})\in\mathbb{Z}_+^{k+k'+1}:n_1+\ldots+n_{k+k'+1}=k+1\right\}.$$
- Let $\mathbf{w}\in\mathcal{W}_4$ be such that $p(\mathbf{w})=(k,k',0,0)$. Then $\nu_{4,\mathbf{w}}$ is the distribution of the random vector $(X_1,X_1+X_2,\ldots,X_1+X_2+\ldots+X_{k+k'})$, where $(X_j+\mathbb{1}_{\{w_{j-1}=1\}},1\leq j\leq k+k'+1)$ (with $w_0=2$) is uniformly distributed on the set
$$B_{k,k'}=\left\{(n_1,\ldots,n_{k+k'})\in\mathbb{Z}_+^{k+k'+1}:n_1+\ldots+n_{k+k'+1}=k\right\}.$$

- If $i \in \{3,4\}$ and if $\mathbf{w} \in \mathcal{W}_4$ does not satisfy $p_3(\mathbf{w}) = p_4(\mathbf{w}) = 0$ then $\nu_{i,\mathbf{w}}$ is arbitrarily defined.

Note that $\#A_{k,k'} = N_{\bullet}(k,k')$ and $\#B_{k,k'} = N_{\diamondsuit}(k,k')$.

Let us now introduce some notation. We have $P^{(i)}_{\boldsymbol{\mu}_\mathbf{q}}(\#\mathbf{t}^1 = n) > 0$ for every $n \geq 1$ and $i \in \{1,2\}$. Then we may define, for every $n \geq 1$, $i \in \{1,2\}$ and $x \in \mathbb{R}$,

$$\begin{aligned} P^{(i),n}_{\boldsymbol{\mu}_\mathbf{q}}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}) &= P^{(i)}_{\boldsymbol{\mu}_\mathbf{q}}\left(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e} \mid \#\mathbf{t}^1 = n\right), \\ \mathbb{P}^{(i),n}_{\boldsymbol{\mu}_\mathbf{q},\boldsymbol{\nu},x}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell}) &= \mathbb{P}^{(i)}_{\boldsymbol{\mu}_\mathbf{q},\boldsymbol{\nu},x}\left(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell} \mid \#\mathbf{t}^1 = n\right). \end{aligned}$$

Furthermore, we set for every $(\mathbf{t},\boldsymbol{\ell},\mathbf{e}) \in \mathbb{T}^{(4)}$,

$$\underline{\boldsymbol{\ell}} = \min\left\{\boldsymbol{\ell}_v : v \in \mathbf{t}^1 \setminus \{\varnothing\}\right\},$$

with the convention $\min \varnothing = \infty$. Finally we define for every $n \geq 1$, $i \in \{1,2\}$ and $x \geq 0$,

$$\begin{aligned} \overline{\mathbb{P}}^{(i)}_{\boldsymbol{\mu}_\mathbf{q},\boldsymbol{\nu},x}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell}) &= \mathbb{P}^{(i)}_{\boldsymbol{\mu}_\mathbf{q},\boldsymbol{\nu},x}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell} \mid \underline{\boldsymbol{\ell}} > 0), \\ \overline{\mathbb{P}}^{(i),n}_{\boldsymbol{\mu}_\mathbf{q},\boldsymbol{\nu},x}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell}) &= \overline{\mathbb{P}}^{(i)}_{\boldsymbol{\mu}_\mathbf{q},\boldsymbol{\nu},x}\left(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell} \mid \#\mathbf{t}^1 = n\right). \end{aligned}$$

The following proposition can be proved from Proposition 3 of [17] in the same way as Corollary 2.3 of [19].

Proposition 2.1. *The probability measure $\mathbb{B}^r_\mathbf{q}(\cdot \mid \#\mathcal{V}_\mathbf{m} = n)$ is the image of $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu}_\mathbf{q},\boldsymbol{\nu},1}$ under the mapping Ψ_r.*

3 A conditional limit theorem for multitype spatial trees

Let $\mathbf{q}$ be a regular critical weight sequence. Recall from section 2.4 the definition of the offspring distribution $\boldsymbol{\mu}_\mathbf{q}$ associated with $\mathbf{q}$ and the definition of the spatial displacement distributions $\boldsymbol{\nu}$. To simplify notation we set $\boldsymbol{\mu} = \boldsymbol{\mu}_\mathbf{q}$.

In view of applying a result of [16], we have to take into account the fact that the spatial displacements $\boldsymbol{\nu}$ are not centered distributions, and to this end we will need a shuffled version of the spatial displacement distributions $\boldsymbol{\nu}$. Let $i \in [K]$ and $\mathbf{w} \in \mathcal{W}$. Set $n = |\mathbf{w}|$. We set $\overleftarrow{\mathbf{w}} = (w_n,\ldots,w_1)$ and we denote by $\overleftarrow{\nu}_{i,\mathbf{w}}$ the image of the measure $\nu_{i,\mathbf{w}}$ under the mapping $S_n : (x_1,\ldots,x_n) \mapsto (x_n,\ldots,x_1)$. Last we set

$$\overleftrightarrow{\nu}_{i,\mathbf{w}}(\mathrm{d}\mathbf{y}) = \frac{\nu_{i,\mathbf{w}}(\mathrm{d}\mathbf{y}) + \overleftarrow{\nu}_{i,\overleftarrow{\mathbf{w}}}(\mathrm{d}\mathbf{y})}{2}.$$

We write $\overleftarrow{\boldsymbol{\nu}} = (\overleftarrow{\nu}_{i,\mathbf{w}}, i \in [K], \mathbf{w} \in \mathcal{W})$ and $\overleftrightarrow{\boldsymbol{\nu}} = (\overleftrightarrow{\nu}_{i,\mathbf{w}}, i \in [K], \mathbf{w} \in \mathcal{W})$.

If $(\mathbf{t},\mathbf{e},\boldsymbol{\ell})$ is a multitype spatial tree, we denote by C its contour function and by V its spatial contour function. Recall that $C([0,1],\mathbb{R})^2$ is equipped with the norm $\|(f,g)\| = \|f\|_u \vee \|g\|_u$. The following result is a special case of Theorem 4 in [16].

Theorem 3.1. *Let* $\mathbf{q}$ *be a regular critical weight sequence. There exists two scaling constants* $\mathrm{A}_{\mathbf{q}} > 0$ *and* $\mathrm{B}_{\mathbf{q}} > 0$ *such that for* $i \in \{1,2\}$*, the law under* $\mathbb{P}^{(i),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},0}$ *of*

$$\left(\left(\mathrm{A}_{\mathbf{q}}\,\frac{C(2(\#\mathbf{t}-1)s)}{n^{1/2}}\right)_{0\le s\le 1},\left(\mathrm{B}_{\mathbf{q}}\,\frac{V(2(\#\mathbf{t}-1)s)}{n^{1/4}}\right)_{0\le s\le 1}\right)$$

converges as $n \to \infty$ *to the law of* $(\mathbf{b}, \mathbf{r}^0)$*. The convergence holds in the sense of weak convergence of probability measures on* $C([0,1],\mathbb{R})^2$*.*

Note that Theorem 4 in [16] deals with the so-called height process instead of the contour process. However, we can deduce Theorem 3.1 from [16] by classical arguments (see e.g. [8]). Moreover, the careful reader will notice that the spatial displacements $\overleftrightarrow{\boldsymbol{\nu}}$ depicted above are not all centered, and thus may compromise the application of [16, Theorem 4]. However, it is explained in [17, Sect. 3.3] how a simple modification of these laws can turn them into centered distributions, by appropriate translations. More precisely, one can couple the spatial trees associated with $\overleftrightarrow{\boldsymbol{\nu}}$ and its centered version so that the labels of vertices differ by at most $1/2$ in absolute value, which of course does not change the limiting behavior of the label function rescaled by $n^{-1/4}$.

In this section, we will prove a conditional version of Theorem 3.1. Before stating this result, we establish a corollary of Theorem 3.1. To this end we set

$$\begin{aligned} Q_{\boldsymbol{\mu}}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}) &= P^{(1)}_{\boldsymbol{\mu}}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e} \mid c_{\varnothing}(\mathbf{t}) = 1),\\ \mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell}) &= \mathbb{P}^{(1)}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},0}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell} \mid c_{\varnothing}(\mathbf{t}) = 1). \end{aligned}$$

Notice that this conditioning makes sense since $\mu^{(1)}(\{(0,0,1,0)\}) > 0$. We may also define for every $n \ge 1$,

$$\begin{aligned} Q^n_{\boldsymbol{\mu}}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}) &= Q_{\boldsymbol{\mu}}\left(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e} \mid \#\mathbf{t}^1 = n\right),\\ \mathbb{Q}^n_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell}) &= \mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}\left(\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{e}\,\mathrm{d}\boldsymbol{\ell} \mid \#\mathbf{t}^1 = n\right). \end{aligned}$$

The following corollary can be proved from Theorem 3.1 in the same way as Corollary 2.2 in [19].

Corollary 3.2. *Let* $\mathbf{q}$ *be a regular critical weight sequence. The law under* $\mathbb{Q}^n_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}$ *of*

$$\left(\left(\mathrm{A}_{\mathbf{q}}\,\frac{C(2(\#\mathbf{t}-1)s)}{n^{1/2}}\right)_{0\le s\le 1},\left(\mathrm{B}_{\mathbf{q}}\,\frac{V(2(\#\mathbf{t}-1)s)}{n^{1/4}}\right)_{0\le s\le 1}\right)$$

converges as $n \to \infty$ *to the law of* $(\mathbf{b}, \mathbf{r}^0)$*. The convergence holds in the sense of weak convergence of probability measures on* $C([0,1],\mathbb{R})^2$*.*

Recall from section 1.2.2 that $(\overline{\mathbf{b}}^0, \overline{\mathbf{r}}^0)$ denotes the conditioned Brownian snake with initial point 0.

Theorem 3.3. *Let* $\mathbf{q}$ *be a regular critical weight sequence. For every* $x \geq 0$*, the law under* $\overline{\mathbb{P}}^{(1),n}_{\mu^{\mathbf{q}},\overleftrightarrow{\nu},x}$ *of*

$$\left(\left(\mathrm{A}_{\mathbf{q}} \frac{C(2(\#\mathbf{t}-1)s)}{n^{1/2}} \right)_{0 \leq s \leq 1}, \left(\mathrm{B}_{\mathbf{q}} \frac{V(2(\#\mathbf{t}-1)s)}{n^{1/4}} \right)_{0 \leq s \leq 1} \right)$$

converges as $n \to \infty$ *to the law of* $(\overline{\mathbf{b}}^0, \overline{\mathbf{r}}^0)$*. The convergence holds in the sense of weak convergence of probability measures on* $C([0,1],\mathbb{R})^2$*.*

In the same way as in the proof of Theorem 3.3 in [19], we will follow the lines of the proof of Theorem 2.2 in [9] to prove Theorem 3.3.

3.1 Rerooting spatial trees

If $(\mathbf{t},\mathbf{e}) \in T_M$, we write $\partial\mathbf{t} = \{u \in \mathbf{t} : c_u(\mathbf{t}) = 0\}$ for the set of all leaves of $\mathbf{t}$, and we write $\partial_1\mathbf{t} = \partial\mathbf{t} \cap \mathbf{t}^1$ for the set of leaves of $\mathbf{t}$ which are of type 1. Let $w_0 \in \mathbf{t}$. Recall that $\mathcal{U}^* = \mathcal{U} \setminus \{\varnothing\}$. We set

$$\mathbf{t}^{(w_0)} = \mathbf{t} \setminus \{w_0 u \in \mathbf{t} : u \in \mathcal{U}^*\},$$

and we write $\mathbf{e}^{(w_0)}$ for the restriction of the function $\mathbf{e}$ to the truncated tree $\mathbf{t}^{(w_0)}$.

Let $v_0 = u^1 \dots u^{2p} \in \mathcal{U}^*$ and $(\mathbf{t},\mathbf{e}) \in T_M$ such that $v_0 \in \mathbf{t}^1$. We define $k = k(v_0,\mathbf{t})$ and $l = l(v_0,\mathbf{t})$ in the following way. Write $\xi = \#\mathbf{t} - 1$ and $u_0, u_1, \dots, u_{2\xi}$ for the search-depth sequence of $\mathbf{t}$. Then we set

$$\begin{aligned} k &= \min\{i \in \{0,1,\dots,2\xi\} : u_i = v_0\}, \\ l &= \max\{i \in \{0,1,\dots,2\xi\} : u_i = v_0\}, \end{aligned}$$

which means that k is the time of the first visit of v_0 in the evolution of the contour of $\mathbf{t}$ and that l is the time of the last visit of v_0. Note that $l \geq k$ and that $l = k$ if and only if $v_0 \in \partial\mathbf{t}$. For every $s \in [0, 2\xi - (l-k)]$, we set

$$\widehat{C}^{(v_0)}(s) = C(k) + C([\![k-s]\!]) - 2 \inf_{u \in [k \wedge [\![k-s]\!], k \vee [\![k-s]\!]]} C(u),$$

where C is the contour function of $\mathbf{t}$ and $[\![k-s]\!]$ stands for the unique element of $[0,2\xi)$ such that $[\![k-s]\!] - (k-s) = 0$ or 2ξ. Then there exists a unique plane tree $\widehat{\mathbf{t}}^{(v_0)}$ whose contour function is $\widehat{C}^{(v_0)}$. Informally, $\widehat{\mathbf{t}}^{(v_0)}$ is obtained from $\mathbf{t}$ by removing all vertices that are descendants of v_0, by re-rooting the resulting tree at v_0, and finally by reversing the planar orientation. Furthermore we see that $\widehat{v}_0 = 1u^{2p} \dots u^2$ belongs to $\widehat{\mathbf{t}}^{(v_0)}$. In fact, $\widehat{v}_0$ is the vertex of $\widehat{\mathbf{t}}^{(v_0)}$ corresponding to the root of the initial tree. At last notice that $c_\varnothing(\widehat{\mathbf{t}}^{(v_0)}) = 1$.

We now define the function $\widehat{\mathbf{e}}^{(v_0)}$. To this end, for $u \in [\![\varnothing, v_0]\!] \setminus \{v_0\}$, let $j(u,v_0) \in \{1,\dots,c_u(\mathbf{t})\}$ be such that $uj(u,v_0) \in [\![\varnothing, v_0]\!]$. Then set

$$\begin{aligned} [\![\varnothing, v_0]\!]^3_2 &= \left\{u \in [\![\varnothing, v_0]\!] \cap \mathbf{t}^3 : \mathbf{e}(uj(u,v_0)) = 2\right\} \\ [\![\varnothing, v_0]\!]^4_1 &= \left\{u \in [\![\varnothing, v_0]\!] \cap \mathbf{t}^4 : \mathbf{e}(uj(u,v_0)) = 1\right\}. \end{aligned}$$

For every $u \in \widehat{\mathbf{t}}^{(v_0)}$, we denote by $\overline{u}$ the vertex which corresponds to u in the tree $\mathbf{t}$. We then set $\widehat{\mathbf{e}}^{(v_0)}(u) = \mathbf{e}(\overline{u})$, except in the following cases :

$$\begin{cases} \text{if } \overline{u} \in [\![\varnothing, v_0]\!]^2_3 \text{ then } \widehat{\mathbf{e}}^{(v_0)}(u) = 4, \\ \text{if } \overline{u} \in [\![\varnothing, v_0]\!]^4_1 \text{ then } \widehat{\mathbf{e}}^{(v_0)}(u) = 3. \end{cases} \quad (1)$$

Since $v_0 \in \mathbf{t}^1$ we have $\#[\![\varnothing, v_0]\!]_2^3 = \#[\![\varnothing, v_0]\!]_1^4$. Indeed, if $1 = e_0, e_1, \ldots, e_{2p} = 1$ is the sequence of types of elements of $[\![\varnothing, v_0]\!]$ listed according to their generations, then this list is a concatenation of patterns of the form $13\overline{24}1$, where by $\overline{24}$ we mean an arbitrary (possibly empty) repetition of the pattern 24. If at least one 24 occurs then the second and antepenultimate element of the pattern $13\overline{24}1$ correspond respectively to exactly one element of $[\![\varnothing, v_0]\!]_2^3$ and $[\![\varnothing, v_0]\!]_1^4$, while no term of a pattern 131 corresponds to such elements.

Notice that if $(\widehat{\mathbf{t}}^{(v_0)}, \widehat{\mathbf{e}}^{(v_0)}) = (\mathcal{T}, \mathrm{e})$, then $(\mathbf{t}^{(v_0)}, \mathbf{e}^{(v_0)}) = (\widehat{\mathcal{T}}^{(\widehat{v}_0)}, \widehat{\mathrm{e}}^{(\widehat{v}_0)})$. Moreover, if $u \in \mathcal{T} \setminus \{\varnothing, \widehat{v}_0\}$ then we have $c_u(\mathcal{T}) = c_{\overline{u}}(\widehat{\mathcal{T}}^{(\widehat{v}_0)})$. Recall that if $\mathbf{w} = (w_1, \ldots, w_n)$ we write $\overleftarrow{\mathbf{w}} = (w_n, \ldots, w_1)$. To be more accurate, it holds that $\mathbf{w}_u(\mathcal{T}) = \overleftarrow{\mathbf{w}}_{\overline{u}}(\widehat{\mathcal{T}}^{(\widehat{v}_0)})$ except in the following cases :

$$\begin{cases} \text{if } \overline{u} \in [\![\varnothing, v_0]\!] \setminus ([\![\varnothing, v_0]\!]_2^3 \cap [\![\varnothing, v_0]\!]_1^4) \text{ then } \mathbf{w}_u(\mathcal{T}) = \overleftarrow{\mathbf{w}}_{\overline{u}}^{j(\overline{u}, v_0), \mathbf{e}(\overline{u})}(\widehat{\mathcal{T}}^{(\widehat{v}_0)}), \\ \text{if } \overline{u} \in [\![\varnothing, v_0]\!]_2^3 \text{ then } \mathbf{w}_u(\mathcal{T}) = \overleftarrow{\mathbf{w}}_{\overline{u}}^{j(\overline{u}, v_0), 1}(\widehat{\mathcal{T}}^{(\widehat{v}_0)}), \\ \text{if } \overline{u} \in [\![\varnothing, v_0]\!]_1^4 \text{ then } \mathbf{w}_u(\mathcal{T}) = \overleftarrow{\mathbf{w}}_{\overline{u}}^{j(\overline{u}, v_0), 2}(\widehat{\mathcal{T}}^{(\widehat{v}_0)}), \end{cases} \tag{2}$$

where for $\mathbf{w} \in \mathcal{W}$, $n = |\mathbf{w}|$, and $1 \leq j \leq n$, we set

$$\begin{cases} \mathbf{w}^{j,1} = (w_{j+1}, \ldots, w_n, 1, w_1, \ldots, w_{j-1}), \\ \mathbf{w}^{j,2} = (w_{j+1}, \ldots, w_n, 2, w_1, \ldots, w_{j-1}). \end{cases}$$

In particular, if $\overline{u} \in [\![\varnothing, v_0]\!]_2^3$ (resp. $[\![\varnothing, v_0]\!]_1^4$) with $p(\mathbf{w}_{\overline{u}}(\widehat{\mathcal{T}}^{(\widehat{v}_0)})) = (k, k', 0, 0)$ then $p(\mathbf{w}_u(\mathcal{T})) = (k+1, k'-1, 0, 0)$ (resp. $(k-1, k'+1, 0, 0)$), while $p(\mathbf{w}_{\overline{u}}(\widehat{\mathcal{T}}^{(\widehat{v}_0)})) = p(\mathbf{w}_u(\mathcal{T}))$ otherwise.

Recall the definition of the probability measure $Q_{\boldsymbol{\mu}}$.

Lemma 3.4. *Let $v_0 \in \mathcal{U}^*$ be of the form $v_0 = 1u^2 \ldots u^{2p}$ for some $p \in \mathbb{N}$. Assume that*

$$Q_{\boldsymbol{\mu}}\left(v_0 \in \mathbf{t}^1\right) > 0.$$

Then the law of the re-rooted multitype tree $(\widehat{\mathbf{t}}^{(v_0)}, \widehat{\mathbf{e}}^{(v_0)})$ under $Q_{\boldsymbol{\mu}}(\cdot \mid v_0 \in \mathbf{t}^1)$ coincides with the law of the multitype tree $(\mathbf{t}^{(\widehat{v}_0)}, \mathbf{e}^{(\widehat{v}_0)})$ under $Q_{\boldsymbol{\mu}}(\cdot \mid \widehat{v}_0 \in \mathbf{t}^1)$.

Proof: Let $(\mathcal{T}, \mathrm{e}) \in T_M$ such that $\widehat{v}_0 \in \partial_1 \mathcal{T}$. We have

$$Q_{\boldsymbol{\mu}}\Big((\widehat{\mathbf{t}}^{(v_0)}, \widehat{\mathbf{e}}^{(v_0)}) = (\mathcal{T}, \mathrm{e})\Big) = Q_{\boldsymbol{\mu}}\Big((\mathbf{t}^{(v_0)}, \mathbf{e}^{(v_0)}) = (\widehat{\mathcal{T}}^{(\widehat{v}_0)}, \widehat{\mathrm{e}}^{(\widehat{v}_0)})\Big).$$

And

$$Q_{\boldsymbol{\mu}}\Big((\mathbf{t}^{(v_0)}, \mathbf{e}^{(v_0)}) = (\widehat{\mathcal{T}}^{(\widehat{v}_0)}, \widehat{\mathrm{e}}^{(\widehat{v}_0)})\Big) = \prod_{\overline{u} \in \widehat{\mathcal{T}}^{(\widehat{v}_0)} \setminus \{\varnothing, v_0\}} \zeta^{(\widehat{\mathrm{e}}^{(\widehat{v}_0)}(u))}(\mathbf{w}_u(\widehat{\mathcal{T}}^{(\widehat{v}_0)})),$$

$$Q_{\boldsymbol{\mu}}\Big((\mathbf{t}^{(\widehat{v}_0)}, \mathbf{e}^{(\widehat{v}_0)}) = (\mathcal{T}, \mathrm{e})\Big) = \prod_{u \in \mathcal{T} \setminus \{\varnothing, \widehat{v}_0\}} \zeta^{(\mathrm{e}(u))}(\mathbf{w}_u(\mathcal{T})).$$

By the above discussion around (2), the terms corresponding to $u, \overline{u}$ in these two products are all equal, except for those corresponding to vertices $\overline{u} \in [\![\varnothing, v_0]\!]_3^2 \cup [\![\varnothing, v_0]\!]_4^1$.

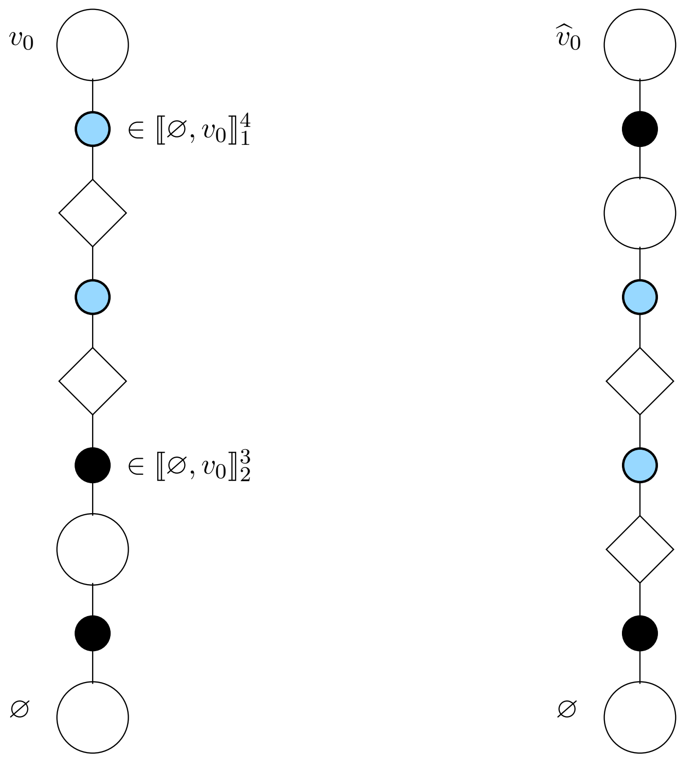

Figure 1: The branch leading from $\varnothing$ to v_0, and the corresponding branch in the tree $\widehat{\mathbf{t}}^{(v_0)}$: the branch is put upside-down and the vertices of $[\![\varnothing, v_0]\!]_2^3$ and $[\![\varnothing, v_0]\!]_1^4$ interchange their roles.

Let $k \geq 0$ and $k' \geq 1$. We have $N_{\diamondsuit}(k+1, k'-1) = N_{\bullet}(k, k')$ which implies that

$$\begin{aligned}
\frac{\mu^{(4)}(k+1, k'-1, 0, 0)}{\binom{k+k'}{k+1}} &= \frac{(Z_{\mathbf{q}}^{+})^{k+1}(Z_{\mathbf{q}}^{\diamondsuit})^{k'-1} N_{\diamondsuit}(k+1, k'-1) q_{1+2(k+1)+k'-1}}{f_{\mathbf{q}}^{\diamondsuit}(Z_{\mathbf{q}}^{+}, Z_{\mathbf{q}}^{\diamondsuit})} \\
&= \frac{Z_{\mathbf{q}}^{+} f_{\mathbf{q}}^{\bullet}(Z_{\mathbf{q}}^{+}, Z_{\mathbf{q}}^{\diamondsuit})}{Z_{\mathbf{q}}^{\diamondsuit} f_{\mathbf{q}}^{\diamondsuit}(Z_{\mathbf{q}}^{+}, Z_{\mathbf{q}}^{\diamondsuit})} \, \frac{\mu^{(3)}(k, k', 0, 0)}{\binom{k+k'}{k}} \\
&= \frac{Z_{\mathbf{q}}^{+} - 1}{(Z_{\mathbf{q}}^{\diamondsuit})^2} \, \frac{\mu^{(3)}(k, k', 0, 0)}{\binom{k+k'}{k}}.
\end{aligned}$$

Likewise let $k \geq 1$ and $k' \geq 0$. We have $N_{\bullet}(k-1, k'+1) = N_{\diamondsuit}(k, k')$ which implies that

$$\begin{aligned}
\frac{\mu^{(3)}(k-1, k'+1, 0, 0)}{\binom{k+k'}{k-1}} &= \frac{(Z_{\mathbf{q}}^{+})^{k-1}(Z_{\mathbf{q}}^{\diamondsuit})^{k'+1} N_{\bullet}(k-1, k'+1) q_{2+2(k-1)+k'+1}}{f_{\mathbf{q}}^{\bullet}(Z_{\mathbf{q}}^{+}, Z_{\mathbf{q}}^{\diamondsuit})} \\
&= \frac{Z_{\mathbf{q}}^{\diamondsuit} f_{\mathbf{q}}^{\diamondsuit}(Z_{\mathbf{q}}^{+}, Z_{\mathbf{q}}^{\diamondsuit})}{Z_{\mathbf{q}}^{+} f_{\mathbf{q}}^{\bullet}(Z_{\mathbf{q}}^{+}, Z_{\mathbf{q}}^{\diamondsuit})} \, \frac{\mu^{(4)}(k, k', 0, 0)}{\binom{k+k'}{k}} \\
&= \frac{(Z_{\mathbf{q}}^{\diamondsuit})^2}{Z_{\mathbf{q}}^{+} - 1} \, \frac{\mu^{(4)}(k, k', 0, 0)}{\binom{k+k'}{k}}.
\end{aligned}$$

Using the relation between $p(\mathbf{w}_{\overline{u}}(\widehat{\mathcal{T}}^{(\widehat{v}_0)}))$ and $p(\mathbf{w}_u(\mathcal{T}))$ discussed above for elements of $[\![\varnothing, v_0]\!]_2^3 \cup$

$[\![\varnothing, v_0]\!]_1^4$, we obtain

$$\begin{aligned}
Q_{\boldsymbol{\mu}}\left((\mathbf{t}^{(v_0)}, \mathbf{e}^{(v_0)}) = (\widehat{\mathcal{T}}^{(\widehat{v}_0)}, \widehat{\mathrm{e}}^{(\widehat{v}_0)})\right)
&= \left(\frac{Z_{\mathbf{q}}^{+}-1}{(Z_{\mathbf{q}}^{\diamondsuit})^2}\right)^{\#[\![\varnothing,v_0]\!]_2^3 - \#[\![\varnothing,v_0]\!]_1^4} Q_{\boldsymbol{\mu}}\left((\mathbf{t}^{(\widehat{v}_0)}, \mathbf{e}^{(\widehat{v}_0)}) = (\mathcal{T}, \mathrm{e})\right) \\
&= Q_{\boldsymbol{\mu}}\left((\mathbf{t}^{(\widehat{v}_0)}, \mathbf{e}^{(\widehat{v}_0)}) = (\mathcal{T}, \mathrm{e})\right),
\end{aligned}$$

implying that

$$Q_{\boldsymbol{\mu}}\left((\widehat{\mathbf{t}}^{(v_0)}, \widehat{\mathbf{e}}^{(v_0)}) = (\mathcal{T}, \mathrm{e})\right) = Q_{\boldsymbol{\mu}}\left((\mathbf{t}^{(\widehat{v}_0)}, \mathbf{e}^{(\widehat{v}_0)}) = (\mathcal{T}, \mathrm{e})\right). \tag{3}$$

To conclude the proof we use (3) to get that

$$\begin{aligned}
Q_{\boldsymbol{\mu}}(v_0 \in \mathbf{t}) &= \sum_{\{(\mathcal{T},\mathrm{e}) \in T_M : v_0 \in \partial_1 \mathcal{T}\}} \mathbb{Q}_{\boldsymbol{\mu}}\left((\mathbf{t}^{(v_0)}, \mathbf{e}^{(v_0)}) = (\mathcal{T}, \mathrm{e})\right) \\
&= \sum_{\{(\mathcal{T},\mathrm{e}) \in T_M : v_0 \in \partial_1 \mathcal{T}\}} \mathbb{Q}_{\boldsymbol{\mu}}\left((\widehat{\mathbf{t}}^{(v_0)}, \widehat{\mathbf{e}}^{(v_0)}) = (\widehat{\mathcal{T}}^{(v_0)}, \widehat{\mathrm{e}}^{(v_0)})\right) \\
&= \sum_{\{(\mathcal{T},\mathrm{e}) \in T_M : v_0 \in \partial_1 \mathcal{T}\}} \mathbb{Q}_{\boldsymbol{\mu}}\left((\mathbf{t}^{(\widehat{v}_0)}, \mathbf{e}^{(\widehat{v}_0)}) = (\widehat{\mathcal{T}}^{(v_0)}, \widehat{\mathrm{e}}^{(v_0)})\right) \\
&= \sum_{\{(\mathcal{T}',\mathrm{e}') \in T_M : \widehat{v}_0 \in \partial_1 \mathcal{T}'\}} \mathbb{Q}_{\boldsymbol{\mu}}\left((\mathbf{t}^{(\widehat{v}_0)}, \mathbf{e}^{(\widehat{v}_0)}) = (\mathcal{T}', \mathrm{e}')\right) \\
&= Q_{\boldsymbol{\mu}}(\widehat{v}_0 \in \mathbf{t}).
\end{aligned}$$

□

If $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \mathbb{T}_M$ and $v_0 \in \mathbf{t}^1$, the re-rooted multitype spatial tree $(\widehat{\mathbf{t}}^{(v_0)}, \widehat{\mathbf{e}}^{(v_0)}, \widehat{\boldsymbol{\ell}}^{(v_0)})$ is defined as follows. If $u \in \widehat{\mathbf{t}}^{(v_0)}$, recall that $\overline{u}$ denotes the vertex which corresponds to u in the tree $\mathbf{t}$ and that $\check{u}$ denotes its father (in the tree $\widehat{\mathbf{t}}^{(v_0)}$).

- If $\widehat{\mathbf{e}}^{(v_0)}(u) \in \{1, 2\}$ then $\widehat{\boldsymbol{\ell}}_u^{(v_0)} = \boldsymbol{\ell}_{\overline{u}} - \boldsymbol{\ell}_{v_0}$.
- If $\widehat{\mathbf{e}}^{(v_0)}(u) \in \{3, 4\}$ then $\widehat{\boldsymbol{\ell}}_u^{(v_0)} = \boldsymbol{\ell}_{\check{u}}^{(v_0)}$.

Let $n = c_u(\widehat{\mathbf{t}}^{(v_0)})$. Observe that when $\overline{u} \notin [\![\varnothing, v_0]\!]$, then the spatial displacements between u and its offspring is left unchanged by the re-rooting, meaning that

$$\left(\widehat{\boldsymbol{\ell}}_{ui}^{(v_0)} - \widehat{\boldsymbol{\ell}}_u^{(v_0)}, 1 \le i \le n\right) = \left(\boldsymbol{\ell}_{\overline{ui}}^{(\widehat{v}_0)} - \boldsymbol{\ell}_{\overline{u}}^{(\widehat{v}_0)}, 1 \le i \le n\right).$$

Otherwise, if $\overline{u} \in [\![\varnothing, v_0]\!]$, set $j = j(u, v_0)$, and define the mapping

$$\phi_{n,j} : (x_1, \ldots, x_n) \mapsto (x_{j-1} - x_j, \ldots, x_1 - x_j, -x_j, x_n - x_j, \ldots, x_{j+1} - x_j).$$

Then observe that the spatial displacements are affected in the following way:

$$\left(\widehat{\boldsymbol{\ell}}_{ui}^{(v_0)} - \widehat{\boldsymbol{\ell}}_u^{(v_0)}, 1 \le i \le n\right) = \phi_{n,j}\left(\left(\boldsymbol{\ell}_{\overline{ui}}^{(\widehat{v}_0)} - \boldsymbol{\ell}_{\overline{u}}^{(\widehat{v}_0)}, 1 \le i \le n\right)\right). \tag{4}$$

If $(\mathbf{t},\mathbf{e},\boldsymbol{\ell})\in\mathbb{T}_M$ and $w_0\in\mathbf{t}$, we also consider the multitype spatial tree $(\mathbf{t}^{(w_0)},\mathbf{e}^{(w_0)},\boldsymbol{\ell}^{(w_0)})$ where $\boldsymbol{\ell}^{(w_0)}$ is the restriction of $\boldsymbol{\ell}$ to the tree $\mathbf{t}^{(w_0)}$.

Recall the definition of the probability measure $\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}$.

Lemma 3.5. *Let $v_0\in\mathcal{U}^*$ be of the form $v_0=1u^2\ldots u^{2p}$ for some $p\in\mathbb{N}$. Assume that*

$$Q_{\boldsymbol{\mu}}\left(v_0\in\mathbf{t}^1\right)>0.$$

Then the law of the re-rooted multitype spatial tree $(\widehat{\mathbf{t}}^{(v_0)},\widehat{\mathbf{e}}^{(v_0)},\widehat{\boldsymbol{\ell}}^{(v_0)})$ under the measure $\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}(\cdot\mid v_0\in\mathbf{t}^1)$ coincides with the law of the multitype spatial tree $(\mathbf{t}^{(\widehat{v}_0)},\mathbf{e}^{(\widehat{v}_0)},\boldsymbol{\ell}^{(\widehat{v}_0)})$ under the measure $\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}(\cdot\mid\widehat{v}_0\in\mathbf{t}^1)$.

This lemma is a simple consequence of Lemma 3.4 and our observations around (4) on the spatial displacements $\overleftrightarrow{\boldsymbol{\nu}}$, combined with the discussion of how the set of children of various vertices are affected by re-rooting, see (1) and (2).

Lemma 3.6. *Let $\mathbf{w}\in\mathcal{W}$ such that $p_3(\mathbf{w})=p_4(\mathbf{w})=0$. Set $n=|\mathbf{w}|$ and let $j\in\{1,\ldots,n\}$.*

- *(i) The image of the measure $\overleftrightarrow{\nu}_{3,\mathbf{w}}$ under the mapping $\phi_{n,j}$ is*
  - *(a) the measure $\overleftrightarrow{\nu}_{3,\mathbf{w}^{j,1}}$ if $w_j=1$,*
  - *(b) the measure $\overleftrightarrow{\nu}_{4,\mathbf{w}^{j,2}}$ if $w_j=2$.*
- *(ii) The image of the measure $\overleftrightarrow{\nu}_{4,\mathbf{w}}$ under the mapping $\phi_{n,j}$ is*
  - *(a) the measure $\overleftrightarrow{\nu}_{3,\mathbf{w}^{j,1}}$ if $w_j=1$,*
  - *(b) the measure $\overleftrightarrow{\nu}_{4,\mathbf{w}^{j,2}}$ if $w_j=2$.*

Proof: We first suppose that $w_j=1$. Set $k=p_1(\mathbf{w})$, $k'=p_2(\mathbf{w})$ and $w_0=1$. Define $\widetilde{\phi}_{n,j}=S_n\circ\phi_{n,j}$, where as before S_n stands for the mapping $(x_1,\ldots,x_n)\mapsto(x_n,\ldots,x_1)$. We consider a uniform vector $(X_l+\mathbb{1}_{\{w_{l-1}=1\}},1\le l\le n+1)$ on the set $A_{k,k'}$ and we set $X^{(j)}=(X_{j+1},\ldots,X_{n+1},X_1,\ldots,X_j)$. Since $w_0=w_j=1$, the vector

$$\left(X_l^{(j)}+\mathbb{1}_{\{w_{l-1}^{j,1}=1\}},1\le l\le n+1\right)$$

is uniformly distributed on the set $A_{k,k'}$ and the measure $\nu_{3,\mathbf{w}^{j,1}}$ is the law of the vector

$$\left(X_1^{(j)},X_1^{(j)}+X_2^{(j)},\ldots,X_1^{(j)}+\ldots+X_n^{(j)}\right).$$

Furthermore we notice that $X_1+X_2+\ldots+X_{n+1}=0$. This implies that

$$\left(X_1^{(j)},X_1^{(j)}+X_2^{(j)},\ldots,X_1^{(j)}+\ldots+X_n^{(j)}\right)=\widetilde{\phi}_{n,j}(X_1,X_1+X_2,\ldots,X_1+\ldots+X_n),$$

which means that the measure $\nu_{3,\mathbf{w}^{j,1}}$ is the image of $\nu_{3,\mathbf{w}}$ under the mapping $\widetilde{\phi}_{n,j}$. Since $\widetilde{\phi}_{n,j}\circ S_n=S_n\circ\phi_{n,n-j+1}$, we obtain together with what precedes that the measure $\overleftarrow{\nu}_{3,\overleftarrow{\mathbf{w}}^{n-j+1,1}}$ is the image of $\overleftarrow{\nu}_{3,\overleftarrow{\mathbf{w}}}$ under the mapping $\widetilde{\phi}_{n,j}$. Thus $\overleftrightarrow{\nu}_{3,\mathbf{w}^{j,1}}$ is the image of $\overleftrightarrow{\nu}_{3,\mathbf{w}}$ under the mapping $\widetilde{\phi}_{n,j}$. Hence, it is the image of the same measure under $\phi_{n,j}$, being invariant under the action of S_n. Thus we get the first part of the lemma. The other assertions can be proved in the same way. □

If $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \mathbb{T}_M$, we set

$$\underline{\boldsymbol{\ell}} = \min\left\{\boldsymbol{\ell}_v : v \in \mathbf{t}^1 \setminus \{\varnothing\}\right\},$$

$$\Delta_1 = \left\{v \in \mathbf{t}^1 : \boldsymbol{\ell}_v = \min\left\{\boldsymbol{\ell}_w : w \in \mathbf{t}^1\right\}\right\}.$$

We also denote by v_m the first element of Δ_1 in the lexicographical order.

The following two Lemmas can be proved from Lemma 3.5 in the same way as Lemma 3.3 and Lemma 3.4 in [9].

Lemma 3.7. *For any nonnegative measurable functional F on $\mathbb{T}_M$,*

$$\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}\left(F\left(\widehat{\mathbf{t}}^{(v_m)}, \widehat{\mathbf{e}}^{(v_m)}, \widehat{\boldsymbol{\ell}}^{(v_m)}\right)\mathbb{1}_{\{\#\Delta_1=1, v_m\in\partial_1\mathbf{t}\}}\right) = \mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}\left(F(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell})(\#\partial_1\mathbf{t})\mathbb{1}_{\{\underline{\boldsymbol{\ell}}>0\}}\right).$$

Lemma 3.8. *For any nonnegative measurable functional F on $\mathbb{T}_M$,*

$$\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}\left(\sum_{v_0\in\Delta_1\cap\partial_1\mathbf{t}} F\left(\widehat{\mathbf{t}}^{(v_0)}, \widehat{\mathbf{e}}^{(v_0)}, \widehat{\boldsymbol{\ell}}^{(v_0)}\right)\right) = \mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}\left(F(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell})(\#\partial_1\mathbf{t})\mathbb{1}_{\{\underline{\boldsymbol{\ell}}\geq 0\}}\right).$$

3.2 Estimates for the probability of staying on the positive side

In this section we will derive upper and lower bounds for the probability $\mathbb{P}^n_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},x}(\underline{\boldsymbol{\ell}} > 0)$ as $n \to \infty$. We first state a lemma which is a direct consequence of Lemma 6 in [16].

Lemma 3.9. *There exist two constants $c_0 > 0$ and $c_1 > 0$ such that*

$$n^{3/2} P_{\boldsymbol{\mu}}\left(\#\mathbf{t}^1 = n\right) \underset{n\to\infty}{\longrightarrow} c_0,$$

$$n^{3/2} Q_{\boldsymbol{\mu}}\left(\#\mathbf{t}^1 = n\right) \underset{n\to\infty}{\longrightarrow} c_1.$$

We now establish a preliminary estimate concerning the number of leaves of type 1 in a tree with n vertices of type 1. Write $\mathbf{0}$ for the origin of $\mathbb{R}^4$.

Lemma 3.10. *There exists a constant $\beta > 0$ such that for all n sufficiently large,*

$$P_{\boldsymbol{\mu}}\left(|\#\partial_1\mathbf{t} - \mu^{(1)}(\{\mathbf{0}\})n| > n^{3/4}, \#\mathbf{t}^1 = n\right) \leq e^{-n^\beta}.$$

Proof: Let $(\mathbf{t}, \mathbf{e}) \in T_M$. Recall that $\xi = \#\mathbf{t} - 1$. Let $v(0) = \varnothing \prec v(1) \prec \ldots \prec v(\xi)$ be the vertices of $\mathbf{t}$ listed in lexicographical order. For every $n \in \{0, 1, \ldots, \xi\}$, we define $R_n = (R_n(k))_{k\geq 1}$ as follows. For every $k \in \{1, \ldots, |v(n)|\}$, we write $v(n, k)$ for the ancestor of $v(n)$ at generation k and we let

$$v_1(n, k) \prec \ldots \prec v_m(n, k)$$

be the younger brothers of $v(n, k)$ listed in lexicographical order. Here younger brothers are those brothers which have not yet been visited at time n in search-depth sequence. Then we set

$$R_n(k) = (\mathbf{e}(v_1(n, k)), \ldots, \mathbf{e}(v_m(n, k)))$$

if $m \geq 1$ and $R_n(k) = \varnothing$ if $m = 0$. For every $k > |v(n)|$, we set $R_n(k) = \varnothing$. By abuse of notations we assimilate R_n with $(R_n(1), \dots, R_n(|v(n)|))$ and let $R_n = \varnothing$ if $|v(n)| = 0$. Standard arguments (see e.g. [11] for similar results) show that $(R_n, \mathbf{e}(v(n)), |v(n)|)_{0 \leq n \leq \xi}$ has the same distribution as $(R'_n, e'_n, h'_n)_{0 \leq n \leq T'-1}$, where $(R'_n, e'_n, h'_n)_{n \geq 0}$ is a Markov chain starting at $(\varnothing, 1, 0)$, whose transition probabilities are specified as follows :

- $((\mathbf{r}_1, \dots, \mathbf{r}_h), i, h) \to ((\mathbf{r}_1, \dots, \mathbf{r}_h, \mathbf{r}^+_{h+1}), \mathbf{r}_{h+1}(1), h+1)$ with probability $\boldsymbol{\zeta}^{(i)}(\{\mathbf{r}_{h+1}\})$ where $\mathbf{r}^+_{h+1} = (\mathbf{r}_{h+1}(2), \dots, \mathbf{r}_{h+1}(|\mathbf{r}_{h+1}|))$, for $\mathbf{r}_{h+1} \in \mathcal{W}_4 \setminus \{\varnothing\}$, $i \in \{1,2,3,4\}$, $\mathbf{r}_1, \dots, \mathbf{r}_h \in \mathcal{W}_4$ and $h \geq 0$,
- $((\mathbf{r}_1, \dots, \mathbf{r}_h), i, h) \to ((\mathbf{r}_1, \dots, \mathbf{r}_{k-1}, \mathbf{r}^+_k), \mathbf{r}_k(1), k)$ with probability $\boldsymbol{\zeta}^{(i)}(\{\varnothing\})$, whenever $h \geq 1$ and $\{m \geq 1 : \mathbf{r}_m \neq \varnothing\} \neq \varnothing$, and where $k = \sup\{m \geq 1 : \mathbf{r}_m \neq \varnothing\}$, for $i \in \{1,2,3,4\}$, $\mathbf{r}_1, \dots, \mathbf{r}_h \in \mathcal{W}_4$,
- $((\varnothing, \dots, \varnothing), i, h) \to (\varnothing, 1, 0)$ with probability $\boldsymbol{\zeta}^{(i)}(\{\varnothing\})$ for $i \in \{1,2,3,4\}$ and $h \geq 0$,

and finally

$$T' = \inf\left\{n \geq 1 : h'_n = 0\right\}.$$

Write $\mathbf{P}'$ for the probability measure under which $(R'_n, e'_n, h'_n)_{n \geq 0}$ is defined. We define a sequence of stopping times $(\tau'_j)_{j \geq 0}$ by $\tau'_0 = 0$ and $\tau'_{j+1} = \inf\{n > \tau'_j : e'_n = 1\}$ for every $j \geq 0$. At last we set for every $j \geq 0$,

$$X'_j = \mathbb{1}\left\{h_{\tau'_j+1} \leq h_{\tau'_j}\right\}.$$

Thus we have,

$$\begin{aligned} & P_{\boldsymbol{\mu}}\left(|\#\partial_1 \mathbf{t} - \boldsymbol{\mu}^{(1)}(\{\mathbf{0}\})n| > n^{3/4}, \#\mathbf{t}^1 = n\right) \\ = \ & \mathbf{P}'\left(\Big|\sum_{j=0}^{n-1} X'_j - \boldsymbol{\mu}^{(1)}(\{\mathbf{0}\})n\Big| > n^{3/4}, \tau'_{n-1} < T' \leq \tau'_n\right) \\ \leq \ & \mathbf{P}'\left(\Big|\sum_{j=0}^{n-1} X'_j - \boldsymbol{\mu}^{(1)}(\{\mathbf{0}\})n\Big| > n^{3/4}\right). \end{aligned}$$

Thanks to the strong Markov property, under the probability measure $\mathbf{P}'(\cdot \mid e'_0 = 1)$, the random variables X'_j are independent and distributed according to the Bernoulli distribution with parameter $\boldsymbol{\zeta}^{(1)}(\{\varnothing\}) = \boldsymbol{\mu}^{(1)}(\{\mathbf{0}\})$. So we get the result using a standard moderate deviations inequality and Lemma 3.9. □

We will now state a lemma which plays a crucial role in the proof of the main result of this section. To this end, recall the definition of v_m and the definition of the probability measure $\mathbb{Q}^n_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}$.

Lemma 3.11. *There exists a constant $c > 0$ such that for all n sufficiently large,*

$$\mathbb{Q}^n_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}(v_m \in \partial_1 \mathbf{t}) \geq c.$$

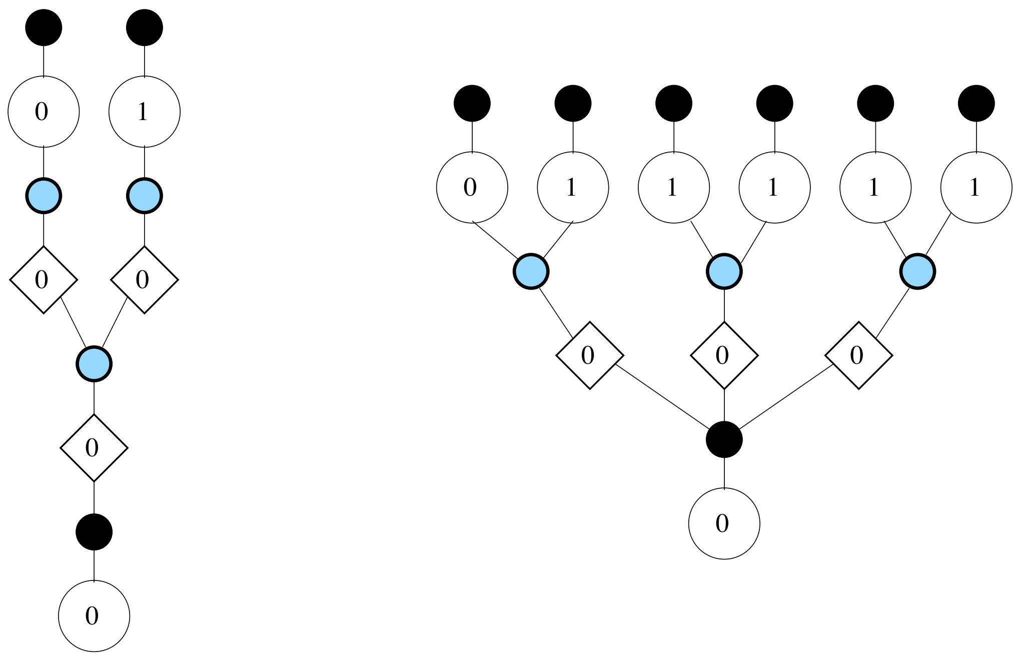

Figure 2: The events F (left) and Γ for $k=2$ (right)

Proof: We first treat the case where $q_{2k+1}=0$ for every $k\geq 2$ which implies that $q_3>0$. Consider the event

$$E=\Big\{\mathbf{z}_\varnothing(\mathbf{t})=(0,0,1,0),\mathbf{z}_1(\mathbf{t})=(0,1,0,0),\mathbf{z}_{11}(\mathbf{t})=(0,0,0,1),\mathbf{z}_{111}(\mathbf{t})=(0,2,0,0),\\ \mathbf{z}_{1111}(\mathbf{t})=\mathbf{z}_{1112}(\mathbf{t})=(0,0,0,1),\mathbf{z}_{11111}(\mathbf{t})=\mathbf{z}_{11121}(\mathbf{t})=(1,0,0,0),\\ \mathbf{z}_{111111}(\mathbf{t})=\mathbf{z}_{111211}(\mathbf{t})=(0,0,1,0)\Big\}.$$

Let $u\in\mathcal{U}$ and let $(\mathbf{t},\mathbf{e},\boldsymbol{\ell})\in\overline{\mathbb{T}}_M$ such that $u\in\mathbf{t}$. We set $\mathbf{t}^{[u]}=\{v\in\mathcal{U}:uv\in\mathbf{t}\}$ and for every $v\in\mathbf{t}^{[u]}$ we set $\mathbf{e}^{[u]}(v)=\mathbf{e}(uv)$ and $\boldsymbol{\ell}^{[u]}(v)=\boldsymbol{\ell}(uv)-\boldsymbol{\ell}(u)$. On the event E we can define $(\mathbf{t}_1,\mathbf{e}_1,\boldsymbol{\ell}_1)=(\mathbf{t}^{[u_1]},\mathbf{e}^{[u_1]},\boldsymbol{\ell}^{[u_1]})$ and $(\mathbf{t}_2,\mathbf{e}_2,\boldsymbol{\ell}_2)=(\mathbf{t}^{[u_2]},\mathbf{e}^{[u_2]},\boldsymbol{\ell}^{[u_2]})$, where we have written $u_1=111111$ and $u_2=111211$. Let F be the event defined by

$$F=E\cap\Big\{\boldsymbol{\ell}_1=\boldsymbol{\ell}_{11}=\boldsymbol{\ell}_{111}=\boldsymbol{\ell}_{1111}=\boldsymbol{\ell}_{1112}=\boldsymbol{\ell}_{11111}=\boldsymbol{\ell}_{11121}=\boldsymbol{\ell}_{111111}=0,\boldsymbol{\ell}_{111211}=1\Big\}.$$

We observe that $\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}(F)>0$ and that under $\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}(\cdot\mid F)$, the spatial trees $(\mathbf{t}_1,\mathbf{e}_1,\boldsymbol{\ell}_1)$ and $(\mathbf{t}_2,\mathbf{e}_2,\boldsymbol{\ell}_2)$ are independent and distributed according to $\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}$. Furthermore

$$\left\{\#\mathbf{t}^1=n,v_m\in\partial_1\mathbf{t}\right\}\supset F\cap\{v_{m,1}\in\partial_1\mathbf{t}_1\}\cap\{\underline{\boldsymbol{\ell}}_2\geq 0\}\cap\left\{\#\mathbf{t}_1^1+\#\mathbf{t}_2^1=n-1\right\},$$

where $v_{m,1}$ is the first vertex of $\mathbf{t}_1^1\setminus\{\varnothing\}$ that achieves the minimum of $\underline{\boldsymbol{\ell}}_1$. So we obtain that

$$\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}\left(\#\mathbf{t}^1=n,v_m\in\partial_1\mathbf{t}\right)\\ \geq\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}(F)\sum_{j=1}^{n-2}\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}\left(\#\mathbf{t}^1=j,v_m\in\partial_1\mathbf{t}\right)\mathbb{Q}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}}}\left(\#\mathbf{t}^1=n-1-j,\underline{\boldsymbol{\ell}}\geq 0\right). \tag{5}$$

Let us now turn to the second case for which there exists $k \geq 2$ such that $q_{2k+1} > 0$. Let $K = 2k - 1$. On the event

$$\Lambda = \Big\{ \mathbf{z}_\varnothing(\mathbf{t}) = (0,0,1,0), \mathbf{z}_1(\mathbf{t}) = (0,K,0,0), \mathbf{z}_{11}(\mathbf{t}) = \ldots = \mathbf{z}_{1K}(\mathbf{t}) = (0,0,0,1), \\ \mathbf{z}_{111}(\mathbf{t}) = \ldots = \mathbf{z}_{1K1}(\mathbf{t}) = (k,0,0,0), \mathbf{z}_{1111}(\mathbf{t}) = \mathbf{z}_{1112}(\mathbf{t}) = (0,0,1,0) \Big\}$$

we can define $((\mathbf{t}^{[u_{ij}]}, \mathbf{e}^{[u_{ij}]}, \boldsymbol{\ell}^{[u_{ij}]}))_{1 \leq i \leq K, 1 \leq j \leq k}$ where we have written $u_{ij} = 1i1j$. Let Γ be the event

$$\Lambda \cap \{\boldsymbol{\ell}_1 = 0\} \cap \bigcap_{1 \leq i \leq K} \{\boldsymbol{\ell}_{1i} = \boldsymbol{\ell}_{1i1} = 0\} \cap \{\boldsymbol{\ell}_{1111} = 0\} \cap \bigcap_{2 \leq i \leq k} \{\boldsymbol{\ell}_{111i} = 1\} \cap \bigcap_{2 \leq i \leq K, 1 \leq j \leq k} \{\boldsymbol{\ell}_{u_{ij}} = 1\}.$$

We observe that $\mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}(\Gamma) > 0$. Furthermore, under the probability measure $\mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}(\cdot \mid \Gamma)$, the spatial trees $((\mathbf{t}^{[u_{ij}]}, \mathbf{e}^{[u_{ij}]}, \boldsymbol{\ell}^{[u_{ij}]}))_{1 \leq i \leq K, 1 \leq j \leq k}$ are independent, $(\mathbf{t}^{[u_{11}]}, \mathbf{e}^{[u_{11}]}, \boldsymbol{\ell}^{[u_{11}]})$ and $(\mathbf{t}^{[u_{12}]}, \mathbf{e}^{[u_{12}]}, \boldsymbol{\ell}^{[u_{12}]})$ are distributed according to $\mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}$, and $((\mathbf{t}^{[u_{ij}]}, \mathbf{e}^{[u_{ij}]}, \boldsymbol{\ell}^{[u_{ij}]}))_{1 \leq i \leq K, 1 \leq j \leq k}$ are distributed according to $\mathbb{P}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}, 0}$. Last

$$\big\{\#\mathbf{t}^1 = n, v_m \in \partial_1 \mathbf{t}\big\} \supset \Gamma \cap \Big\{ v_m^{u_{11}} \in \partial_1 \mathbf{t}^{[u_{11}]} \Big\} \cap \Big\{ \underline{\boldsymbol{\ell}}^{[u_{12}]} \geq 0 \Big\} \\ \cap \Big\{ \#\mathbf{t}^{[u_{11}],1} + \#\mathbf{t}^{[u_{12}],1} = n + 1 - kK \Big\} \cap \bigcap_{2 \leq i \leq K, 1 \leq j \leq k} \Big\{ \mathbf{t}^{[u_{ij}]} = \{\varnothing\} \Big\}.$$

So we obtain that

$$\begin{aligned} & \mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}} \big(\#\mathbf{t}^1 = n, v_m \in \partial_1 \mathbf{t}\big) \\ \geq \;& C \sum_{j=2}^{n-kK} \mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}} \big(\#\mathbf{t}^1 = j, v_m \in \partial_1 \mathbf{t}\big) \, \mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}} \big(\#\mathbf{t}^1 = n + 1 - kK - j, \underline{\boldsymbol{\ell}} \geq 0\big) \end{aligned} \tag{6}$$

where we have written $C = \boldsymbol{\mu}^{(1)}(\{\mathbf{0}\})^{k(K-1)} \mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}(\Gamma)$.

We can now conclude the proof of Lemma 3.11 in both cases from respectively (5) and (6) by following the lines of the proof of Lemma 4.3 in [9]. □

We can now state the main result of this section.

Proposition 3.12. *Let $M > 0$. There exist four constants $\gamma_1 > 0$, $\gamma_2 > 0$, $\widetilde{\gamma}_1 > 0$ and $\widetilde{\gamma}_2 > 0$ such that for all n sufficiently large and for every $x \in [0, M]$,*

$$\begin{aligned} \frac{\widetilde{\gamma}_1}{n} &\leq \mathbb{Q}^n_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}(\underline{\boldsymbol{\ell}} > 0) \leq \frac{\widetilde{\gamma}_2}{n}, \\ \frac{\gamma_1}{n} &\leq \mathbb{P}^n_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}, x}(\underline{\boldsymbol{\ell}} > 0) \leq \frac{\gamma_2}{n}. \end{aligned}$$

Proof: We prove exactly in the same way as in [9] the existence of $\widetilde{\gamma}_2$ and the existence of a constant $\gamma_3 > 0$ such that for all n sufficiently large, we have

$$\mathbb{Q}^n_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}(\underline{\boldsymbol{\ell}} \geq 0) \geq \frac{\gamma_3}{n}.$$

Let us now fix $M > 0$. Let $k \geq 1$ be such that $q_{2k+1} > 0$. We choose an integer p such that $pk \geq M$. First note that $\overleftrightarrow{\boldsymbol{\nu}}_{3,\mathbf{w}}(\{\mathbf{0}\}) = 1/(2k-1)$ if $\mathbf{w} = (0, 2k-1, 0, 0)$ and that $\overleftrightarrow{\boldsymbol{\nu}}_{4,\mathbf{w}}(\{k, k-1, \dots, 1\}) = 1/(2\#A_{k,0})$ if $\mathbf{w} = (k, 0, 0, 0)$. For every $l \in \mathbb{N}$, we define $1^l \in \mathcal{U}$ by $1^l = 11\dots1$, $|1^l| = l$. By arguing on the event

$$\begin{aligned}E' = \Big\{ & \mathbf{z}_\varnothing(\mathbf{t}) = \ldots = \mathbf{z}_{1^{4p}}(\mathbf{t}) = (0,0,1,0), \mathbf{z}_1(\mathbf{t}) = \ldots = \mathbf{z}_{1^{4p-3}}(\mathbf{t}) = (0, 2k-1, 0, 0),\\ & \mathbf{z}_{11}(\mathbf{t}) = \ldots = \mathbf{z}_{1(2k-1)}(\mathbf{t}) = \ldots = \mathbf{z}_{1^{4p-3}1}(\mathbf{t}) = \ldots = \mathbf{z}_{1^{4p-3}(2k-1)} = (0,0,0,1),\\ & \mathbf{z}_{111}(\mathbf{t}) = \ldots = \mathbf{z}_{1(2k-1)1}(\mathbf{t}) = \ldots = \mathbf{z}_{1^{4p-3}11}(\mathbf{t}) = \ldots = \mathbf{z}_{1^{4p-3}(2k-1)1} = (k,0,0,0)\Big\}\\ & \cap \bigcap_{i=0}^{p-1} \{\mathbf{z}_{1^{4i+3}2}(\mathbf{t}) = \ldots = \mathbf{z}_{1^{4i+3}k} = \mathbf{0}\} \cap \bigcap_{i=0}^{p-1} \bigcap_{j=2}^{2k-1} \bigcap_{l=1}^{k} \{\mathbf{z}_{1^{4i+1}j1l} = \mathbf{0}\},\end{aligned}$$

we show that

$$\mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}\left(\underline{\boldsymbol{\ell}} > 0, \#\mathbf{t}^1 = n\right) \geq \frac{C(\boldsymbol{\mu}, \boldsymbol{\nu}, k)^p}{\mu^{(1)}(\{(0,0,1,0)\})} \mathbb{P}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}, pk}\left(\underline{\boldsymbol{\ell}} > 0, \#\mathbf{t}^1 = n - pk(2k-1)\right),$$

where $C(\boldsymbol{\mu}, \boldsymbol{\nu}, k)$ is equal to

$$\frac{\boldsymbol{\mu}^{(1)}(\{(0,0,1,0)\})\boldsymbol{\mu}^{(3)}(\{(0,2k-1,0,0)\})(\boldsymbol{\mu}^{(4)}(\{(k,0,0,0)\}))^{2k-1}\boldsymbol{\mu}^{(1)}(\{\mathbf{0}\}))^{k(2k-1)-1}}{(2k-1)(2\#A_{k,0})^{2k-1}}.$$

This implies thanks to Lemma 3.9 that for all n sufficiently large,

$$\mathbb{P}^n_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}, pk}\left(\underline{\boldsymbol{\ell}} > 0\right) \leq \frac{2\boldsymbol{\mu}^{(1)}(\{(0,0,1,0)\})\widetilde{\gamma}_2}{C(\boldsymbol{\mu}, \boldsymbol{\nu}, k)n},$$

which ensures the existence of γ_2.

Last by arguing on the event

$$\begin{aligned}F = \Big\{ & \mathbf{z}_\varnothing(\mathbf{t}) = \mathbf{z}_{1^4} = (0,0,1,0), \mathbf{z}_1(\mathbf{t}) = (0, 2k-1, 0, 0),\\ & \mathbf{z}_{11}(\mathbf{t}) = \ldots = \mathbf{z}_{1(2k-1)}(\mathbf{t}) = (0,0,0,1), \mathbf{z}_{111}(\mathbf{t}) = \ldots = \mathbf{z}_{1(2k-1)1}(\mathbf{t}) = (k,0,0,0)\Big\}\\ & \cap \bigcap_{j=2}^{k} \{\mathbf{z}_{1^3 j}(\mathbf{t}) = \mathbf{0}\} \cap \bigcap_{i=2}^{2k-1} \bigcap_{j=1}^{k} \{\mathbf{z}_{1i1j}(\mathbf{t}) = \mathbf{0}\},\end{aligned}$$

we show that

$$\begin{aligned}& \mathbb{P}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}, 0}\left(\underline{\boldsymbol{\ell}} > 0, \#\mathbf{t}^1 = n\right)\\ & \geq \; C(\boldsymbol{\mu}, \boldsymbol{\nu}, k)\boldsymbol{\mu}^{(1)}(\{(0,0,1,0)\})\mathbb{Q}_{\boldsymbol{\mu}, \overleftrightarrow{\boldsymbol{\nu}}}\left(\underline{\boldsymbol{\ell}} \geq 0, \#\mathbf{t}^1 = n - k(2k-1)\right),\end{aligned}$$

which ensures the existence of γ_1. We get the existence of $\widetilde{\gamma}_1$ by the same arguments. □

3.3 Proof of Theorem 3.3

To prove Theorem 3.3 from what precedes, we can adapt section 7 of [9] in exactly the same way as in the proof of Theorem 3.3 in [19]. A key result in the proof of Theorems 2.2 in [9] and 3.3 in [19] is a spatial Markov property for spatial Galton-Watson trees. Let $a > 0$ and $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell}) \in \mathbb{T}_M$. As in section 5 of [9] let $v_1, \ldots, v_M$ denote the exit vertices from $(-\infty, a)$ listed in lexicographical order, and let $(\mathbf{t}^a, \mathbf{e}^a, \boldsymbol{\ell}^a)$ correspond to the multitype spatial tree $(\mathbf{t}, \mathbf{e}, \boldsymbol{\ell})$ which has been truncated at the first exit from $(-\infty, a)$. Let $v \in \mathbf{t}$. Recall from section 3.2 the definition of the multitype spatial tree $(\mathbf{t}^{[v]}, \mathbf{e}^{[v]}, \boldsymbol{\ell}^{[v]})$. We set $\overline{\boldsymbol{\ell}}^{[v]}_u = \boldsymbol{\ell}^{[v]}_u + \boldsymbol{\ell}_v$ for every $u \in \mathbf{t}^{[v]}$.

Lemma 3.13. *Let $x \in [0, a)$ and $p \in \{1, \ldots, n\}$. Let $n_1, \ldots, n_p$ be positive integers such that $n_1 + \ldots + n_p \leq n$. Assume that*

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},x}\left(M = p,\ \#\mathbf{t}^{[v_1],1} = n_1, \ldots,\ \#\mathbf{t}^{[v_p],1} = n_p\right) > 0.$$

Then, under the probability measure $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\boldsymbol{\nu},x}(\cdot \mid M = p, \#\mathbf{t}^{[v_1],1} = n_1, \ldots, \#\mathbf{t}^{[v_p],1} = n_p)$, and conditionally on $(\mathbf{t}^a, \mathbf{e}^a, \boldsymbol{\ell}^a)$, the spatial trees

$$\left(\mathbf{t}^{[v_1]}, \mathbf{e}^{[v_1]}, \overline{\boldsymbol{\ell}}^{[v_1]}\right), \ldots, \left(\mathbf{t}^{[v_p]}, \mathbf{e}^{[v_p]}, \overline{\boldsymbol{\ell}}^{[v_p]}\right)$$

are independent and distributed respectively according to $\overline{\mathbb{P}}^{(\mathbf{e}(v_1)),n_1}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},\boldsymbol{\ell}_{v_1}}, \ldots, \overline{\mathbb{P}}^{(\mathbf{e}(v_p)),n_p}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},\boldsymbol{\ell}_{v_p}}$.

Observe that in our context, if v is an exit vertex then $\mathbf{e}(v) \in \{1, 2\}$. This is the reason why Theorem 3.3 is stated under both probability measures $\mathbb{P}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},x}$ and $\mathbb{P}^{(2),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},x}$. Thus the statement of Lemma 7.1 of [9] (and of Lemma 3.18 of [19]) is modified in the following way. Set for every $n \geq 1$ and every $s \in [0, 1]$,

$$\begin{aligned} C^{(n)}(s) &= \mathrm{A}_{\mathbf{q}} \frac{C((\#\mathbf{t} - 1)s)}{n^{1/2}}, \\ V^{(n)}(s) &= \mathrm{B}_{\mathbf{q}} \frac{V((\#\mathbf{t} - 1)s)}{n^{1/4}}. \end{aligned}$$

Last define from section 1.2.2, on a suitable probability space $(\boldsymbol{\Omega}, \mathbf{P})$, a collection of processes $(\overline{\mathbf{b}}^x, \overline{\mathbf{r}}^x)_{x>0}$.

Lemma 3.14. *Let $F : \mathcal{C}([0, 1], \mathbb{R})^2 \to \mathbb{R}$ be a Lipschitz function. Let $0 < c' < c''$. Then for $i \in \{1, 2\}$,*

$$\sup_{c'n^{1/4} \leq y \leq c''n^{1/4}} \left| \overline{\mathbb{E}}^{(i),n}_{\boldsymbol{\mu},\boldsymbol{\nu},y}\left(F\left(C^{(n)}, V^{(n)}\right)\right) - \mathbf{E}\left(F\left(\overline{\mathbf{b}}^{\mathrm{B}_{\mathbf{q}}y/n^{1/4}}, \overline{\mathbf{r}}^{\mathrm{B}_{\mathbf{q}}y/n^{1/4}}\right)\right) \right| \underset{n\to\infty}{\longrightarrow} 0.$$

4 Proof of Theorem 1.2

We finally derive Theorem 1.2 from Theorem 3.3 in the same way as Theorem 2.5 in [19] is derived from Theorem 3.3. We first state a lemma, which is analogous to Lemma 3.20 in [19] in our more general setting. To this end we introduce some notation. Recall that if $\mathbf{t} \in T_M$, we set $\xi = \#\mathbf{t} - 1$ and we denote by $v(0) = \varnothing \prec v(1) \prec \ldots \prec v(\xi)$ the list of the vertices of $\mathbf{t}$ in lexicographical order. For $n \in \{0, 1, \ldots, \xi\}$, we set as in [16],

$$\Lambda_1^{\mathbf{t}}(n) = \#\left(\mathbf{t}^1 \cap \{v(0), v(1), \ldots, v(n)\}\right).$$

We extend $\Lambda_1^{\mathbf{t}}$ to the real interval $[0, \xi]$ by setting $\Lambda_1^{\mathbf{t}}(s) = \Lambda_1^{\mathbf{t}}(\lfloor s \rfloor)$ for every $s \in [0, \xi]$, and we set for every $s \in [0, 1]$

$$\overline{\Lambda}_1^{\mathbf{t}}(s) = \frac{\Lambda_1^{\mathbf{t}}(\xi s)}{\#\mathbf{t}^1}.$$

Recall that $u_0, u_1, \ldots, u_{2\xi}$ denotes the search-depth sequence of $\mathbf{t}$. We also define for $k \in \{0, 1, \ldots, 2\xi\}$,

$$K_{\mathbf{t}}(k) = 1 + \#\left\{l \in \{1, \ldots, k\} : C(l) = C(l-1) + 1 \text{ and } \mathbf{e}(u_l) = 1\right\}.$$

Note that $K_{\mathbf{t}}(k)$ is the number of vertices of type 1 in the search-depth sequence up to time k. As previously, we extend $K_{\mathbf{t}}$ to the real interval $[0, 2\xi]$ by setting $K_{\mathbf{t}}(s) = K_{\mathbf{t}}(\lfloor s \rfloor)$ for every $s \in [0, 2\xi]$, and we set for every $s \in [0, 1]$

$$\overline{K}_{\mathbf{t}}(s) = \frac{K_{\mathbf{t}}(2\xi s)}{\#\mathbf{t}^1}.$$

Lemma 4.1. *The law under* $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}$ *of* $\left(\overline{\Lambda}_1^{\mathbf{t}}(s), 0 \le s \le 1\right)$ *converges as* $n \to \infty$ *to the Dirac mass at the identity mapping of* $[0, 1]$*. In other words, for every* $\eta > 0$*,*

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(\sup_{s \in [0,1]} \left|\overline{\Lambda}_1^{\mathbf{t}}(s) - s\right| > \eta\right) \underset{n \to \infty}{\longrightarrow} 0. \tag{7}$$

Consequently, the law under $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}$ *of* $\left(\overline{K}_{\mathbf{t}}(s), 0 \le s \le 1\right)$ *converges as* $n \to \infty$ *to the Dirac mass at the identity mapping of* $[0, 1]$*. In other words, for every* $\eta > 0$*,*

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(\sup_{s \in [0,1]} \left|\overline{K}_{\mathbf{t}}(s) - s\right| > \eta\right) \underset{n \to \infty}{\longrightarrow} 0. \tag{8}$$

Proof: For $\mathbf{t} \in T_M$, we let $v^1(0) = \varnothing \prec v^1(1) \prec \ldots \prec v^1(\#\mathbf{t}^1 - 1)$ be the list of vertices of $\mathbf{t}$ of type 1 in lexicographical order. We define as in [16]

$$G_1^{\mathbf{t}}(k) = \#\left\{u \in \mathbf{t} : u \prec v^1(k)\right\}, \quad 0 \le k \le \#\mathbf{t}^1 - 1,$$

and we set $G_1^{\mathbf{t}}(\#\mathbf{t}^1) = \#\mathbf{t}$. Note that $v^1(k)$ does not belong to the set $\{u \in \mathbf{t} : u \prec v^1(k)\}$. Recall from section 2.4 the definition of the vector $\mathbf{a} = (a_1, a_2, a_3, a_4)$. From the second assertion of Proposition 6 in [16], for every $s \in [0, 1]$, there exists a constant $\varepsilon > 0$ such that for all n sufficiently large,

$$P^{(1)}_{\boldsymbol{\mu}}\left(|G_1^{\mathbf{t}}(\lfloor ns \rfloor) - a_1^{-1} ns| \ge n^{3/4}\right) \le e^{-n^{\varepsilon}}.$$

Thus we obtain thanks to Lemma 3.9 and Proposition 3.12 that for every $s \in [0, 1]$, there exists a constant $\varepsilon' > 0$ such that for all n sufficiently large,

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(|G_1^{\mathbf{t}}(\lfloor ns \rfloor) - a_1^{-1} ns| \ge n^{3/4}\right) \le e^{-n^{\varepsilon}}.$$

Let us fix $\eta > 0$. We then have for every $s \in [0,1]$,

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(|n^{-1}G_1^{\mathbf{t}}(\lfloor ns \rfloor) - a_1^{-1}s| \geq \eta\right) \underset{n\to\infty}{\longrightarrow} 0.$$

In particular for $s=1$ we have

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(|n^{-1}\#\mathbf{t} - a_1^{-1}| \geq \eta\right) \underset{n\to\infty}{\longrightarrow} 0,$$

which implies that for every $s \in [0,1]$,

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(|(\#\mathbf{t})^{-1}G_1^{\mathbf{t}}(\lfloor ns \rfloor) - s| \geq \eta\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Let us now set $k_\eta = \lceil \frac{2}{a_1\eta} \rceil$ and $s_m = mk_\eta^{-1}$ for every $m \in \{0,1,\ldots,k_\eta\}$. Since the mapping $s \in [0,1] \mapsto n^{-1}G_1^{\mathbf{t}}(\lfloor ns \rfloor)$ is non-decreasing, we have

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(\sup_{s\in[0,1]}\left|\frac{G_1^{\mathbf{t}}(\lfloor ns \rfloor)}{\#\mathbf{t}} - s\right| \geq \eta\right) \leq \overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(\sup_{0\leq m\leq k_\eta}\left|\frac{G_1^{\mathbf{t}}(\lfloor ns_m \rfloor)}{\#\mathbf{t}} - s_m\right| \geq \frac{\eta}{2}\right),$$

implying that

$$\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}\left(\sup_{s\in[0,1]}\left|(\#\mathbf{t})^{-1}G_1^{\mathbf{t}}(\lfloor ns \rfloor) - s\right| \geq \eta\right) \underset{n\to\infty}{\longrightarrow} 0.$$

We thus get (8) in the same way as (32) is obtained in [19]. Then we derive (8) from (7) in the same way as (33) is derived from (32) in [19]. □

We can now complete the proof of Theorem 1.2. Recall that $\mathcal{R}_{\mathbf{m}}$ denotes the radius of the map $\mathbf{m}$. Thanks to Proposition 2.1 we know that the law of $\mathcal{R}_{\mathbf{m}}$ under $\mathbb{B}^r_{\mathbf{q}}(\cdot \mid \#\mathcal{V}_{\mathbf{m}} = n)$ coincides with the law of $\sup_{v\in\mathbf{t}^1}\boldsymbol{\ell}_v$ under $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\boldsymbol{\nu},1}$. Furthermore we easily see (compare [17, Lemma 1]) that the law of $\sup_{v\in\mathbf{t}^1}\boldsymbol{\ell}_v$ under $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\boldsymbol{\nu},1}$ is the law of $\sup_{v\in\mathbf{t}^1}\boldsymbol{\ell}_v$ under $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}$. We thus get the first assertion of Theorem 1.2.

Let us turn to (ii). By Proposition 2.1 and property (ii) of the Bouttier-Di Francesco-Guitter bijection stated at the end of Section 2.3, the law of $\lambda^{(n)}_{\mathbf{m}}$ under $\mathbb{B}^r_{\mathbf{q}}(\cdot \mid \#\mathcal{V}_{\mathbf{m}} = n)$ is the law under $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\boldsymbol{\nu},1}$ of the probability measure $\mathcal{I}_n$ defined by

$$\langle \mathcal{I}_n, g\rangle = \frac{1}{\#\mathbf{t}^1 + 1}\left(g(0) + \sum_{v\in\mathbf{t}^1} g\left(n^{-1/4}\boldsymbol{\ell}_v\right)\right),$$

which coincides with the law of $\mathcal{I}_n$ under $\overline{\mathbb{P}}^{(1),n}_{\boldsymbol{\mu},\overleftrightarrow{\boldsymbol{\nu}},1}$. We thus complete the proof of (ii) by following the lines of the proof of Theorem 2.5 in [19]. Finally, assertion (iii) easily follows from (ii).

References

[1] Aldous, D.J. (1991) The continuum random tree. I. *Ann. Probab.*, 19(1):1–28. MR1085326

[2] Ambjørn, J., Durhuus, B., Jonsson, T. (1997) *Quantum geometry. A statistical field theory approach.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge. MR1465433

[3] Bouttier J., Di Francesco P., Guitter E. (2004) Planar maps as labeled mobiles. *Electron. J. Combin.* **11**, R69 MR2097335

[4] Chassaing, P., Schaeffer, G. (2004) Random planar lattices and integrated superBrownian excursion. *Probab. Th. Rel. Fields* **128**, 161–212. MR2031225

[5] Cori, R., Vauquelin, B. (1981) Planar maps are well labeled trees. *Canad. J. Math.* **33**, 1023-1042. MR0638363

[6] Gromov, M. (1999) *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA. MR1699320

[7] Le Gall, J.-F. (1999) *Spatial Branching Processes, Random Snakes and Partial Differential Equations. Lectures in Mathematics ETH Zürich.* Birkhäuser, Boston. MR1714707

[8] Le Gall, J.-F. (2005) Random trees and applications. *Probab. Surveys* **2**, 245–311. MR2203728

[9] Le Gall, J.-F. (2006) A conditional limit theorem for tree-indexed random walk. *Stoch. Process. Appl.* **116**, 539–567. MR2205115

[10] Le Gall, J.-F. (2007) The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169:621–670. MR2336042

[11] Le Gall, J.-F., Le Jan, Y. (1998) Branching processes in Lévy processes : the exploration process. *Ann. Probab.* **26**, 213–252. MR1617047

[12] Le Gall, J.-F., Paulin, F. (2006) Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. To appear in *Geom. Funct. Anal.* `math/0612315`.

[13] Le Gall, J.-F., Weill, M. (2006) Conditioned Brownian trees. *Ann. Inst. H. Poincaré Probab. Statist.* **42**, 455–489. MR2242956

[14] Marckert, J.-F., Miermont, G. (2007) Invariance principles for random bipartite planar maps. *Ann. Probab.* **35**, 1642–1705. MR2349571

[15] Marckert, J.-F., Mokkadem, A. (2006) Limits of normalized quadrangulations. The Brownian map. *Ann. Probab.* **34**, 2144–2202. MR2294979

[16] Miermont, G. (2006) Invariance principles for spatial multitype Galton-Watson trees. To appear in *Ann. Inst. H. Poincaré Probab. Statist.* `math.PR/0610807`

[17] Miermont, G. (2006) An invariance principle for random planar maps, in *Fourth Colloquium on Mathematics and Computer Sciences CMCS06 (Nancy, France), Discrete Math. Theor. Comput. Sci. Proc., AG*, Nancy, pp. 39-58 (electronic). Errata on `www.math.u-psud.fr/~miermont`.

[18] Schaeffer, G. (1998) *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux I.

[19] Weill, M. (2007) Asymptotics for rooted bipartite planar maps and scaling limits of two-type spatial trees. *Electron. J. Probab.* **12**, 862–925 (electronic). MR2318414