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A Generalized Itô's Formula in Two-Dimensions and Stochastic Lebesgue-Stieltjes Integrals *

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Abstract

In this paper, a generalized Itô's formula for continuous functions of two-dimensional continuous semimartingales is proved. The formula uses the local time of each coordinate process of the semimartingale, the left space first derivatives and the second derivative $\nabla_1^-\nabla_2^- f$, and the stochastic Lebesgue-Stieltjes integrals of two parameters. The second derivative $\nabla_1^-\nabla_2^- f$ is only assumed to be of locally bounded variation in certain variables. Integration by parts formulae are asserted for the integrals of local times. The two-parameter integral is defined as a natural generalization of both the Itô integral and the Lebesgue-Stieltjes integral through

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a type of Itô isometry formula.

Key words: local time, continuous semimartingale, generalized Itô's formula, stochastic Lebesgue-Stieltjes integral.

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1 Introduction

The classical Itô's formula for twice differentiable functions has been extended to less smooth functions by many mathematicians. Progress has been made mainly in one-dimension beginning with Tanaka's pioneering work [30] for $|X_t|$ to which the local time was beautifully linked. Further extensions were made to a time independent convex function $f(x)$ in [21] and [32] as the following Tanaka-Meyer formula:

$$f(X(t)) = f(X(0)) + \int_0^t f'_-(X(s))dX(s) + \int_{-\infty}^{\infty} L_t(x)d(f'_-(x)), \tag{1}$$

where the left derivative f'_- exists and is increasing due to the convexity assumption. This can be generalized easily to include the case when f'_- is of bounded variation where the integral $\int_{-\infty}^{\infty} L_t(x)d(f'_-(x))$ is a Lebesgue-Stieltjes integral. The extension to the time dependent case was given in [7]. Recently we proved that $L_t(x)$ is of finite p-variation (in the classical sense of Young and Lyons) for any $p > 2$ in [9]. This new result leads to the construction of $\int_{-\infty}^{\infty} L_t(x)d(f'_-(x))$ as a Young integral, so the Tanaka-Meyer formula still holds when f'_- is of finite q-variation for a constant $1 \leq q < 2$. Moreover in [10], we extended the above to the case when $2 \leq q < 3$ using Lyons' rough path integration theory.

The purpose of this paper is to extend formula (1) to two dimensions. This is a nontrivial extension as the local time in two-dimensions does not exist. But formally by using the occupation times formula (see (4)), the property that $\int_0^\infty 1_{R\setminus\{a\}}(X_1(s,\omega))d_sL_1(s,\omega) = 0$ a.s. and the "formal integration by parts formula", we observe that for a smooth function f,

$$\begin{aligned}
& \frac{1}{2}\int_0^t \Delta_1 f(X_1(s), X_2(s))d \langle X_1\rangle_s \\
= & \int_{-\infty}^{+\infty}\int_0^t \Delta_1 f(X_1(s), X_2(s))\mathrm{d}_s L_1(s,a)\mathrm{d}a \\
= & \int_{-\infty}^{+\infty}\int_0^t \Delta_1 f(a, X_2(s))\mathrm{d}_s L_1(s,a)\mathrm{d}a \\
= & \int_{-\infty}^{+\infty} L_1(t,a)\mathrm{d}_a\nabla_1 f(a, X_2(t)) - \int_{-\infty}^{+\infty}\int_0^t L_1(s,a)\mathrm{d}_{s,a}\nabla_1 f(a, X_2(s)).
\end{aligned} \tag{2}$$

Here the last step needs to be justified, and the final integral needs to be properly defined. It is worth noting that the right hand side does not include any second order derivative of f explicitly. Here $\nabla_1 f(a, X_2(s))$ is a semimartingale for any fixed a, following the Tanaka-Meyer formula. We study the kind of integral $\int_{-\infty}^{+\infty}\int_0^t g(s,a)\mathrm{d}_{s,a}h(s,a)$ in Section 2. Here $h(s,x)$ is a continuous martingale with cross variation $< h(\cdot,a), h(\cdot,b) >_s$ of locally bounded variation in (s,a,b), and $E\left[\int_0^t \int_{R^2} |g(s,a)g(s,b)||\mathrm{d}_{a,b,s} < h(\cdot,a), h(\cdot,b) >_s |\right] < \infty$. The integral is different from both the Lebesgue-Stieltjes integral and Itô's stochastic integral. But it is a natural extension to the two-parameter stochastic case and is therefore called a stochastic Lebesgue-Stieltjes integral. To our knowledge, this integral is new. It differs from integration with Brownian sheet defined by Walsh ([31]) and from integration with respect to a Poisson random measure (see [15]). A generalized Itô's formula in two dimensions is proved in Section 3. Moreover, we also prove the integration by parts formula for the stochastic Lebesgue-Stieltjes integrals involving local times (Theorems 3.2 and 3.3). It is noted that Peskir recently gave a generalized Itô's formula in multi-dimensions using local times on surfaces where the first order derivative might be discontinuous under the condition that their second derivative has a limit from both sides of the surfaces in [24]. Our formula does not need the condition on the existence of limits of second order derivatives when x goes to the surface. There are numerous examples for which the classical Itô's formula and Peskir's formula may not work immediately, but our formula can be used (see Examples 3.1 and 3.2).

Applications e.g. in the study of the asymptotics of the solutions of heat equations with caustics in two dimensions, are not included in this paper. These results will be published in some future work.

Other kinds of relevant results include work for absolutely continuous functions with the first derivative being locally bounded in [26], and for $W^{1,2}_{loc}$ functions of a Brownian motion for one dimension in [12] and [13] for multi-dimensions. It was proved in [12] that $f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}[f(B), B]_t$, where $[f(B), B]_t$ is the covariation of the processes $f(B)$ and B, and is equal to $\int_0^t f(B_s)d^*B_s - \int_0^t f(B_s)dB_s$ as a difference of backward and forward integrals. See [29] for the case of a continuous semimartingale. The multi-dimensional case was considered in [13], [29] and [22]. An integral $\int_{-\infty}^{\infty} f'(x)\mathrm{d}_x L_t(x)$ was introduced in [3] through the existence of the expression $f(X(t)) - f(X(0)) - \int_0^t \frac{\partial^-}{\partial x} f(X(s))dX(s)$, where $L_t(x)$ is the local time of the semimartingale X_t. This work was extended further to define the local time space integral $\int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial x} f(s, X(s))d_{s,x}L_s(x)$ for a time dependent function $f(s,x)$ using forward and backward integrals for Brownian motion in [5] and to semimartingales other than Brownian motion in [6]. This integral was also defined in [27] as a stochastic integral with excursion fields, and in [14] through Itô's formula without assuming the reversibility of the semimartingale which was required in [5]. Other relevant references include [11] where it was also proved that, if X is an one-dimensional Brownian motion, then $f(X(t))$ is a semimartingale if and only if $f \in W^{1,2}_{loc}$ and its weak derivative is of bounded variation using backward and forward integrals ([19]). But our results are new.

2 The definition of stochastic Lebesgue-Stieltjes integrals and the integration by parts formula

For a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, denote by $\mathcal{M}_2$ the Hilbert space of all processes $X = (X_t)_{0\leq t\leq T}$ such that $(X_t)_{0\leq t\leq T}$ is a $(\mathcal{F}_t)_{0\leq t\leq T}$ right continuous square integrable martingale with inner product $(X, Y) = E(X_T Y_T)$. A three-variable function $f(s, x, y)$ is called left continuous iff it is left continuous in all three variables together i.e. for any sequence $(s_1, x_1, y_1) \leq (s_2, x_2, y_2) \leq \cdots \leq (s_k, x_k, y_k) \leq (s, x, y)$ and $(s_k, x_k, y_k) \to (s, x, y)$, as $k \to \infty$, we have $f(s_k, x_k, y_k) \to f(s, x, y)$ as $k \to \infty$. Here $(s_1, x_1, y_1) \leq (s_2, x_2, y_2)$ means $s_1 \leq s_2$, $x_1 \leq x_2$ and $y_1 \leq y_2$. Define

$$\begin{aligned}\mathcal{V}_1 := \Big\{h: \quad & [0,t] \times (-\infty, \infty) \times \Omega \to R \ s.t. \ (s, x, \omega) \mapsto h(s, x, \omega) \\ & is\ \mathcal{B}([0,s] \times R) \times \mathcal{F}_s{-}measurable,\ and\ h(s,x)\ is \\ & \mathcal{F}_s{-}adapted\ for\ any\ x \in R\Big\},\end{aligned}$$

$$\mathcal{V}_2 := \Big\{ h: \quad h \in \mathcal{V}_1 \ is\ a\ continuous\ (in\ s)\ \mathcal{M}_2 - martingale\ for\ each\ x,\\ and\ the\ crossvariation\ < h(\cdot,x), h(\cdot,y) >_s is\ left\ continuous\\ and\ of\ locally\ bounded\ variation\ in\ (s,x,y) \Big\}.$$

In the following, we will always denote $< h(\cdot,x), h(\cdot,y) >_s$ by $< h(x), h(y) >_s$.

We now recall some classical results (see [1] and [20]). A three-variable function $f(s,x,y)$ is called monotonically increasing if whenever $(s_2,x_2,y_2) \geq (s_1,x_1,y_1)$, then

$$\begin{aligned} & f(s_2,x_2,y_2) - f(s_2,x_1,y_2) - f(s_2,x_2,y_1) + f(s_2,x_1,y_1) \\ & -f(s_1,x_2,y_2) + f(s_1,x_1,y_2) + f(s_1,x_2,y_1) - f(s_1,x_1,y_1) \geq 0. \end{aligned}$$

For a left-continuous and monotonically increasing function $f(s,x,y)$, one can define a Lebesgue-Stieltjes measure by setting

$$\begin{aligned} & \nu([s_1,s_2) \times [x_1,x_2) \times [y_1,y_2)) \\ = \ & f(s_2,x_2,y_2) - f(s_2,x_1,y_2) - f(s_2,x_2,y_1) + f(s_2,x_1,y_1) \\ & -f(s_1,x_2,y_2) + f(s_1,x_1,y_2) + f(s_1,x_2,y_1) - f(s_1,x_1,y_1). \end{aligned}$$

For $h \in \mathcal{V}_2$, define

$$< h(x), h(y) >_{t_1}^{t_2} := < h(x), h(y) >_{t_2} - < h(x), h(y) >_{t_1}, \ t_2 \geq t_1.$$

Note that, since $< h(x), h(y) >_s$ is left continuous and of locally bounded variation in (s,x,y), it can be decomposed to the difference of two increasing and left continuous functions $f_1(s,x,y)$ and $f_2(s,x,y)$ (see McShane [20] or Proposition 2.2 in Elworthy, Truman and Zhao [7] which also holds for multi-parameter functions). Note that each of f_1 and f_2 generates a measure so, for any measurable function $g(s,x,y)$, we can define

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{a_1}^{a_2} \int_{b_1}^{b_2} g(s,x,y) d_{x,y,s} < h(x), h(y) >_s \\ = \ & \int_{t_1}^{t_2} \int_{a_1}^{a_2} \int_{b_1}^{b_2} g(s,x,y) d_{x,y,s} f_1(s,x,y) - \int_{t_1}^{t_2} \int_{a_1}^{a_2} \int_{b_1}^{b_2} g(s,x,y) d_{x,y,s} f_2(s,x,y). \end{aligned}$$

In particular, a signed product measure in the space $[0,T] \times R^2$ can be defined as follows: for any $[t_1,t_2) \times [x_1,x_2) \times [y_1,y_2) \subset [0,T] \times R^2$

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \mathrm{d}_{x,y,s} < h(x), h(y) >_s \\ = \ & \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \mathrm{d}_{x,y,s} f_1(s,x,y) - \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \mathrm{d}_{x,y,s} f_2(s,x,y) \\ = \ & < h(x_2), h(y_2) >_{t_1}^{t_2} - < h(x_2), h(y_1) >_{t_1}^{t_2} \\ & - < h(x_1), h(y_2) >_{t_1}^{t_2} + < h(x_1), h(y_1) >_{t_1}^{t_2} \\ = \ & < h(x_2) - h(x_1), h(y_2) - h(y_1) >_{t_1}^{t_2}. \end{aligned} \tag{1}$$

Define

$$|\mathrm{d}_{x,y,s} < h(x), h(y) >_s| = \mathrm{d}_{x,y,s} f_1(s,x,y) + \mathrm{d}_{x,y,s} f_2(s,x,y). \tag{2}$$

Moreover, for $h \in \mathcal{V}_2$, define:

$$\begin{aligned}
\mathcal{V}_3(h) := \Big\{ g \quad &: \quad g \in \mathcal{V}_1, \textit{ and there exists } N \textit{ such that } (-N, N) \textit{ covers} \\
& \quad \textit{the compact support of } g(s, \cdot, \omega) \textit{ for a.a. } \omega, \textit{ and } s \in [0, T] \textit{ and} \\
E \quad & \left[\int_0^t \int_{R^2} |g(s,x)g(s,y)||\mathrm{d}_{x,y,s} < h(x), h(y) >_s | \right] < \infty \Big\}. \\
\mathcal{V}_4(h) := \Big\{ g \quad &: \quad g \in \mathcal{V}_1 \textit{ has a compact support in } x \textit{ for a.a. } \omega, \textit{ and} \\
E \quad & \left[\int_0^t \int_{R^2} |g(s,x)g(s,y)||\mathrm{d}_{x,y,s} < h(x), h(y) >_s | \right] < \infty \Big\}.
\end{aligned}$$

Consider now a simple function in $\mathcal{V}_3$, and always assume that, for any $s > 0$, $g(s, -N) = g(s, N) = 0$,

$$g(s, x, \omega) = \sum_{i=0}^{n-1} e_{0,i} 1_{\{0\}}(s) 1_{(x_i, x_{i+1}]}(x) + \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} e_{j,i} 1_{(t_j, t_{j+1}]}(s) 1_{(x_i, x_{i+1}]}(x) \tag{3}$$

where $\{t_n\}_{m=0}^{\infty}$ with $t_0 = 0$ and $\lim_{m \to \infty} t_m = \infty$, $-N = x_0 < x_1 < x_2 < \cdots < x_n = N$, $e_{j,i}$ are $\mathcal{F}_{t_j}$-measurable. For $h \in \mathcal{V}_2$, define an integral as:

$$\begin{aligned}
I_t(g) &:= \int_0^t \int_{-\infty}^{\infty} g(s,x) \mathrm{d}_{s,x} h(s,x) \qquad (4) \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{n-1} e_{j,i} \Big[h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) - h(t_{j+1} \wedge t, x_i) + h(t_j \wedge t, x_i) \Big].
\end{aligned}$$

This integral is called the stochastic Lebesgue-Stieltjes integral of the simple function g. It is easy to see for simple functions $g_1, g_2 \in \mathcal{V}_3(h)$, that

$$I_t(\alpha g_1 + \beta g_2) = \alpha I_t(g_1) + \beta I_t(g_2), \tag{5}$$

for any $\alpha, \beta \in R$. The following lemma plays a key role in extending the integral of simple functions to functions in $\mathcal{V}_3(h)$. It is equivalent to the Itô's isometry formula in the case of the stochastic integral.

Lemma 2.1. *If $h \in \mathcal{V}_2$, $g \in \mathcal{V}_3(h)$ is simple, then $I_t(g)$ is a continuous martingale with respect to $(\mathcal{F}_t)_{0 \le t \le T}$ and*

$$E\Big(\int_0^t \int_{-\infty}^{\infty} g(s,x) \mathrm{d}_{s,x} h(s,x) \Big)^2 = E \int_0^t \int_{R^2} g(s,x) g(s,y) \mathrm{d}_{x,y,s} < h(x), h(y) >_s . \tag{6}$$

Proof: From the definition of $\int_0^t \int_{-\infty}^{\infty} g(s,x) \mathrm{d}_{s,x} h(s,x)$, it is easy to see that I_t is a continuous martingale with respect to $(\mathcal{F}_t)_{0 \le t \le T}$. As $h(s,x,\omega)$ is a continuous martingale in $\mathcal{M}_2$, using a standard conditional expectation argument to remove the cross product parts, we get:

$$
\begin{aligned}
& E\left[\left(\int_0^t \int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}h(s,x)\right)^2\right] \\
= \;& E\sum_{j=0}^{\infty}\left(\sum_{i=0}^{n-1} e_{j,i}\Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j\wedge t, x_{i+1}) - h(t_{j+1}\wedge t, x_i) + h(t_j \wedge t, x_i)\Big]\right)^2 \\
= \;& E\sum_{j=0}^{\infty}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} e_{j,i}e_{j,k}\cdot\right. \\
& \quad \Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j\wedge t, x_{i+1}) - h(t_{j+1}\wedge t, x_i) + h(t_j \wedge t, x_i)\Big]\cdot \\
& \quad \left.\Big[h(t_{j+1}\wedge t, x_{k+1}) - h(t_j\wedge t, x_{k+1}) - h(t_{j+1}\wedge t, x_k) + h(t_j \wedge t, x_k)\Big]\right) \\
= \;& E\sum_{j=0}^{\infty}\left\{\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} e_{j,i}e_{j,k}\cdot\right. \\
& \quad \Big[\big(h(t_{j+1}\wedge t, x_{i+1}) - h(t_j\wedge t, x_{i+1})\big)\big(h(t_{j+1}\wedge t, x_{k+1}) - h(t_j\wedge t, x_{k+1})\big) \\
& \quad -\big(h(t_{j+1}\wedge t, x_{i+1}) - h(t_j\wedge t, x_{i+1})\big)\big(h(t_{j+1}\wedge t, x_k) - h(t_j\wedge t, x_k)\big) \\
& \quad -\big(h(t_{j+1}\wedge t, x_i) - h(t_j\wedge t, x_i)\big)\big(h(t_{j+1}\wedge t, x_{k+1}) - h(t_j\wedge t, x_{k+1})\big) \\
& \quad \left.+\big(h(t_{j+1}\wedge t, x_i) - h(t_j\wedge t, x_i)\big)\big(h(t_{j+1}\wedge t, x_k) - h(t_j\wedge t, x_k)\big)\Big]\right\} \\
= \;& E\int_0^t \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} g(s,x_{i+1})g(s,x_{k+1})\Big[\mathrm{d}_s < h(x_{i+1}), h(x_{k+1}) >_s - \mathrm{d}_s < h(x_{i+1}), h(x_k) >_s \\
& \qquad\qquad -\mathrm{d}_s < h(x_i), h(x_{k+1}) >_s + \mathrm{d}_s < h(x_i), h(x_k) >_s \Big] \\
= \;& E\sum_{j=0}^{\infty}\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} e_{j,i}e_{j,k}\Big[< h(x_{i+1}), h(x_{k+1}) >_{t_j\wedge t}^{t_{j+1}\wedge t} - < h(x_{i+1}), h(x_k) >_{t_j\wedge t}^{t_{j+1}\wedge t} \\
& \qquad\qquad - < h(x_i), h(x_{k+1}) >_{t_j\wedge t}^{t_{j+1}\wedge t} + < h(x_i), h(x_k) >_{t_j\wedge t}^{t_{j+1}\wedge t} \Big] \\
= \;& E\left[\int_0^t \int_{R^2} g(s,x)g(s,y)\mathrm{d}_{x,y,s} < h(x), h(y) >_s\right].
\end{aligned}
$$

So the desired result is proved. $\diamond$

The idea now is to use (6) to extend the definition of the integrals of simple functions to integrals of functions in $\mathcal{V}_3(h)$ and finally in $\mathcal{V}_4(h)$, for any $h \in \mathcal{V}_2$. We achieve this goal in several steps:

Lemma 2.2. *Let $h \in \mathcal{V}_2$, $f \in \mathcal{V}_3(h)$ be bounded uniformly in ω, $f(\cdot,\cdot,\omega)$ be continuous for each ω on its compact support. Then there exist a sequence of bounded simple functions $\varphi_{m,n} \in \mathcal{V}_3(h)$ such that*

$$
E\int_0^t \int_{R^2} |\, (f-\varphi_{m,n})(s,x)(f-\varphi_{m',n'})(s,y)|\, |\, \mathrm{d}_{x,y,s} < h(x), h(y) >_s |\to 0,
$$

as $m, n, m', n' \to \infty$.

Proof: Let $0 = t_0 < t_1 < \cdots < t_m = t$, and $-N = x_0 < x_1 < \cdots < x_n = N$ be a partition of $[0,t] \times [-N, N]$. Assume when $n, m \to \infty$, $\max_{0 \le j \le m-1}(t_{j+1} - t_j) \to 0$, $\max_{0 \le i \le n-1}(x_{i+1} - x_i) \to 0$. Define

$$\varphi_{m,n}(s,x) := \sum_{i=0}^{n-1} f(0, x_i) 1_{\{0\}}(s) 1_{(x_i, x_{i+1}]}(x) + \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} f(t_j, x_i) 1_{(t_j, t_{j+1}]}(s) 1_{(x_i, x_{i+1}]}(x). \tag{7}$$

Then $\varphi_{m,n}(s,x)$ are simple and $\varphi_{m,n}(s,x) \to f(s,x)$ *a.s. as* $m, n \to \infty$. The result follows from applying Lebesgue's dominated convergence theorem. $\diamond$

Lemma 2.3. *Let* $h \in \mathcal{V}_2$ *and* $k \in \mathcal{V}_3(h)$ *be bounded uniformly in* ω*. Then there exist functions* $f_n \in \mathcal{V}_3(h)$ *such that* $f_n(\cdot, \cdot, \omega)$ *are continuous for all* ω *and* n*, and*

$$E \int_0^t \int_{R^2} |\,(k - f_n)(s,x)(k - f_{n'})(s,y)|\,|\,\mathrm{d}_{x,y,s} < h(x), h(y) >_s |\to 0,$$

as $n, n' \to \infty$.

Proof: Define

$$f_n(s,x) = n^2 \int_{x-\frac{1}{n}}^{x} \int_{s-\frac{1}{n}}^{s} k(\tau, y) d\tau dy.$$

Then $f_n(s,x)$ is continuous in s, x, and when $n \to \infty$, $f_n(s,x) \to k(s,x)$ a.s.. So for sufficiently large n, $f_n(s,x)$ also has compact support in $(-N, N)$ for all $s \in [0,T]$. The desired convergence follows from applying Lebesgue's dominated convergence theorem. $\diamond$

Lemma 2.4. *Let* $h \in \mathcal{V}_2$ *and* $g \in \mathcal{V}_3(h)$*. Then there exist functions* $k_n \in \mathcal{V}_3(h)$*, bounded uniformly in* ω *for each* n*, and*

$$E \int_0^t \int_{R^2} |\,(g - k_n)(s,x)(g - k_{n'})(s,y)|\,|\,\mathrm{d}_{x,y,s} < h(x), h(y) >_s |\to 0,$$

as $n, n' \to \infty$.

Proof: Define

$$k_n(t,x,\omega) := \begin{cases} -n & \text{if } g(t,x,\omega) < -n \\ g(t,x,\omega) & \text{if } -n \le g(t,x,\omega) \le n \\ n & \text{if } g(t,x,\omega) > n. \end{cases} \tag{8}$$

Then as $n \to \infty$, $k_n(t,x,\omega) \to g(t,x,\omega)$ for each (t,x,ω). Note $|k_n(t,x,\omega)| \le |g(t,x,\omega)|$ and $k_n \in \mathcal{V}_3(h)$. So applying Lebesgue's dominated convergence theorem, we obtain the desired result. $\diamond$

Lemma 2.5. *Let* $h \in \mathcal{V}_2$ *and* $g \in \mathcal{V}_4(h)$*. Then there exist functions* $g_N \in \mathcal{V}_3(h)$ *such that*

$$E \int_0^t \int_{R^2} |\,(g - g_N)(s,x)(g - g_{N'})(s,y)|\,|\,\mathrm{d}_{x,y,s} < h(x), h(y) >_s |\to 0, \tag{9}$$

as $N, N' \to \infty$.

Proof: Define

$$g_N(s,x,\omega) := g(s,x,\omega)1_{[-N+1,N-1]}(x). \tag{10}$$

Then $|g_N| \le |g|$ and $g_N \to g$ a.s., as $N \to \infty$. So applying Lebesgue's dominated convergence theorem, we obtain the desired result. $\diamond$

From Lemmas 2.4, 2.3, 2.2, for each $h \in \mathcal{V}_2$, $g \in \mathcal{V}_3(h)$, we can construct a sequence of simple functions $\{\varphi_{m,n}\}$ in $\mathcal{V}_3(h)$ such that,

$$E\int_0^t \int_{R^2} |\,(g-\varphi_{m,n})(s,x)(g-\varphi_{m',n'})(s,y)|\,|\,\mathrm{d}_{x,y,s} < h(x), h(y) >_s |\to 0,$$

as $m,n,m',n' \to \infty$. For $\varphi_{m,n}$ and $\varphi_{m',n'}$, we can define stochastic Lebesgue-Stieltjes integrals $I_t(\varphi_{m,n})$ and $I_t(\varphi_{m',n'})$. From Lemma 2.1 and (5), it is easy to see that

$$\begin{aligned}
& E\left[I_T(\varphi_{m,n}) - I_T(\varphi_{m',n'})\right]^2 \\
= \;& E\left[I_T(\varphi_{m,n} - \varphi_{m',n'})\right]^2 \\
= \;& E\int_0^T \int_{R^2} (\varphi_{m,n} - \varphi_{m',n'})(s,x)(\varphi_{m,n} - \varphi_{m',n'})(s,y) d_{x,y,s} < h(x), h(y) >_s \\
= \;& E\int_0^T \int_{R^2} [(\varphi_{m,n} - g) - (\varphi_{m',n'} - g)](s,x) \cdot \\
& \qquad\qquad [(\varphi_{m,n} - g) - (\varphi_{m',n'} - g)](s,y) d_{x,y,s} < h(x), h(y) >_s \\
= \;& E\int_0^T \int_{R^2} (\varphi_{m,n} - g)(s,x)(\varphi_{m,n} - g)(s,y) d_{x,y,s} < h(x), h(y) >_s \\
& -E\int_0^T \int_{R^2} (\varphi_{m,n} - g)(s,x)(\varphi_{m',n'} - g)(s,y) d_{x,y,s} < h(x), h(y) >_s \\
& -E\int_0^T \int_{R^2} (\varphi_{m',n'} - g)(s,x)(\varphi_{m,n} - g)(s,y) d_{x,y,s} < h(x), h(y) >_s \\
& +E\int_0^T \int_{R^2} (\varphi_{m',n'} - g)(s,x)(\varphi_{m',n'} - g)(s,y) d_{x,y,s} < h(x), h(y) >_s \\
\le \;& E\int_0^T \int_{R^2} |\,(\varphi_{m,n} - g)(s,x)(\varphi_{m,n} - g)(s,y)\,||\, d_{x,y,s} < h(x), h(y) >_s| \\
& +E\int_0^T \int_{R^2} |\,(\varphi_{m,n} - g)(s,x)(\varphi_{m',n'} - g)(s,y)\,||\, d_{x,y,s} < h(x), h(y) >_s| \\
& +E\int_0^T \int_{R^2} |\,(\varphi_{m',n'} - g)(s,x)(\varphi_{m,n} - g)(s,y)\,||\, d_{x,y,s} < h(x), h(y) >_s| \\
& +E\int_0^T \int_{R^2} |\,(\varphi_{m',n'} - g)(s,x)(\varphi_{m',n'} - g)(s,y)\,||\, d_{x,y,s} < h(x), h(y) >_s| \\
\to \;& 0,
\end{aligned}$$

as $m,n,m',n' \to \infty$. Therefore $\{I_\cdot(\varphi_{m,n})\}_{m,n=1}^\infty$ is a Cauchy sequence in $\mathcal{M}_2$ whose norm is denoted by $\|\cdot\|$. So there exists a process $I(g) = \{I_t(g), 0 \le t \le T\}$ in $\mathcal{M}_2$, defined modulo indistinguishability, such that

$$\|\, I(\varphi_{m,n}) - I(g)\,\| \to 0, \;\; as\ m,n \to \infty.$$

By the same argument as for the stochastic integral, one can easily prove that $I(g)$ is well-defined (independent of the choice of the simple functions), and (6) is true for $I(g)$. We now can have the following definition.

Definition 2.1. *Let $h \in \mathcal{V}_2$, $g \in \mathcal{V}_3(h)$. Then the integral of g with respect to h can be defined in $\mathcal{M}_2$ as:*

$$\int_0^t \int_{-\infty}^{\infty} g(s,x) \mathrm{d}_{s,x} h(s,x) = \lim_{m,n\to\infty} \int_0^t \int_{-\infty}^{\infty} \varphi_{m,n}(s,x) \mathrm{d}_{s,x} h(s,x).$$

Here $\{\varphi_{m,n}\}$ is a sequence of simple functions in $\mathcal{V}_3(h)$, s.t.

$$E\int_0^t \int_{R^2} |\ (g-\varphi_{m,n})(s,x)(g-\varphi_{m',n'})(s,y)|\ |\ \mathrm{d}_{x,y,s} < h(x), h(y) >_s | \to 0,$$

as $m, n, m', n' \to \infty$. Note $\varphi_{m,n}$ may be constructed by combining the three approximation procedures in Lemmas 2.4, 2.3, 2.2. For $g \in \mathcal{V}_4(h)$, we can then define the integral in $\mathcal{M}_2$ as:

$$\int_0^t \int_{-\infty}^{\infty} g(s,x) \mathrm{d}_{s,x} h(s,x) \quad = \quad \lim_{N\to\infty} \int_0^t \int_{-\infty}^{\infty} g(s,x) 1_{[-N+1,N-1]}(x) \mathrm{d}_{s,x} h(s,x).$$

It is a continuous martingale with respect to $(\mathcal{F}_t)_{0\le t\le T}$ and for each $0 \le t \le T$,

$$E\Big(\int_0^t \int_{-\infty}^{\infty} g(s,x) \mathrm{d}_{s,x} h(s,x)\Big)^2 = E\int_0^t \int_{R^2} g(s,x) g(s,y) \mathrm{d}_{x,y,s} < h(x), h(y) >_s . \tag{11}$$

The following results will be useful in the proof of our main theorem in the next section.

Proposition 2.1. *If $h \in \mathcal{V}_2$, $g \in \mathcal{V}_4(h)$, and $g(t,x)$ is C^2 in x, $\Delta g(t,x)$ is bounded uniformly in t, then a.s.*

$$-\int_{-\infty}^{+\infty} \int_0^t \nabla g(s,x) \mathrm{d}_s h(s,x) dx = \int_0^t \int_{-\infty}^{+\infty} g(s,x) \mathbf{d}_{s,x} h(s,x). \tag{12}$$

Moreover, for any $g \in \mathcal{V}_4(h)$, $h \in \mathcal{V}_2$ and C^1 in x, $\nabla h \in \mathcal{M}_2$,

$$\int_{-\infty}^{+\infty} \int_0^t g(s,x) \mathrm{d}_s \nabla h(s,x) dx = \int_0^t \int_{-\infty}^{+\infty} g(s,x) \mathbf{d}_{s,x} h(s,x). \tag{13}$$

Proof: If g is a simple function in $\mathcal{V}_3(h)$ as given in (3), and note that $e_{j,0} = e_{j,n} = 0$, we have

$$\begin{aligned}
& \int_0^t \int_{-\infty}^{\infty} g(s,x) \mathrm{d}_{s,x} h(s,x) \\
= \;& \sum_{i=0}^{n-1} \sum_{j=0}^{\infty} e_{j,i} \Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) - h(t_{j+1}\wedge t, x_i) + h(t_j\wedge t, x_i) \Big] \\
= \;& -\sum_{i=0}^{n-1} \sum_{j=0}^{\infty} e_{j,i+1} \Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \Big] \\
& + \sum_{i=0}^{n-1} \sum_{j=0}^{\infty} e_{j,i} \Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \Big] \\
= \;& -\sum_{i=0}^{n-1} \sum_{j=0}^{\infty} \Big[e_{j,i+1} - e_{j,i} \Big] \Big[h(t_{j+1}\wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \Big].
\end{aligned}$$

If $g(t,x)$ is C^2 in x, let

$$\varphi_{m,n}(s,x) := \sum_{i=0}^{n-1} g(0,x_i)1_{\{0\}}(s)1_{(x_i,x_{i+1}]}(x) + \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} g(t_j,x_i)1_{(t_j,t_{j+1}]}(s)1_{(x_i,x_{i+1}]}(x),$$

then

$$\varphi_{m,n}(s,x) \to g(s,x) \ a.s. \ as \ m,n\to\infty.$$

Moreover, by the intermediate value theorem,

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_0^t g(s,x)\mathrm{d}_{s,x}h(s,x)\\
= &-\lim_{\delta_t,\delta_x\to 0}\sum_{i=0}^{n-1}\sum_{j=0}^{\infty}\Big[g(t_j\wedge t,x_{i+1})-g(t_j\wedge t,x_i)\Big]\\
&\qquad\Big[h(t_{j+1}\wedge t,x_{i+1})-h(t_j\wedge t,x_{i+1})\Big] \qquad (limit\ in\ \mathcal{M}_2)\\
= &-\lim_{\delta_t,\delta_x\to 0}\sum_{i=0}^{n-1}\sum_{j=0}^{\infty}\Big[\int_0^1\nabla g(t_j\wedge t,x_i+\alpha(x_{i+1}-x_i))d\alpha\Big]\Big[h(t_{j+1}\wedge t,x_{i+1})-h(t_j\wedge t,x_{i+1})\Big]\cdot\\
&\qquad(x_{i+1}-x_i)\\
= &-\lim_{\delta_x\to 0}\sum_{i=0}^{n-1}\int_0^t\Big[\int_0^1\nabla g(s,x_i+\alpha(x_{i+1}-x_i))d\alpha\Big]\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i) \quad (limit\ in\ \mathcal{M}_2)\\
= &-\lim_{\delta_x\to 0}\sum_{i=0}^{n-1}\int_0^t\nabla g(s,x_{i+1})\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i)\\
&-\lim_{\delta_x\to 0}\sum_{i=0}^{n-1}\int_0^t\Big[\int_0^1\big(\nabla g(s,x_i+\alpha(x_{i+1}-x_i))-\nabla g(s,x_{i+1})\big)d\alpha\Big]\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i).
\end{aligned}$$

Here $\delta_t = \max_{1\le j\le m}|t_{j+1}-t_j|$, $\delta_x = \max_{1\le i\le m}|x_{i+1}-x_i|$. To prove (12), first notice that

$$\lim_{\delta_x\to 0}\sum_{i=0}^{n-1}\int_0^t\nabla g(s,x_{i+1})\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i) = \int_{-\infty}^{+\infty}\int_0^t\nabla g(s,x)\mathrm{d}_s h(s,x)dx.$$

Second, by the intermediate value theorem again, and from the assumption that $\Delta g(s,x)$ is bounded uniformly in s, the second term can be estimated as:

$$\begin{aligned}
& E\left[\sum_{i=0}^{n-1}\int_0^t \Big[\int_0^1 \big(\nabla g(s,x_i+\alpha(x_{i+1}-x_i))-\nabla g(s,x_{i+1})\big)d\alpha\Big]\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i)\right]^2 \\
= \;& E\sum_{i=0}^{n-1}\sum_{k=0}^{n-1}\Big[\int_0^t \Big[\int_0^1 \big(\nabla g(s,x_i+\alpha(x_{i+1}-x_i))-\nabla g(s,x_{i+1})\big)d\alpha\Big]\mathrm{d}_s h(s,x_{i+1})(x_{i+1}-x_i)\cdot \\
& \quad \int_0^t \Big[\int_0^1 \big(\nabla g(s,x_k+\alpha(x_{k+1}-x_k))-\nabla g(s,x_{k+1})\big)d\alpha\Big]\mathrm{d}_s h(s,x_{k+1})(x_{k+1}-x_k)\Big] \\
= \;& \sum_{i=0}^{n-1}\sum_{k=0}^{n-1}E\int_0^t \Big[\int_0^1 \big(\nabla g(s,x_i+\alpha(x_{i+1}-x_i))-\nabla g(s,x_{i+1})\big)d\alpha\Big]\cdot \\
& \quad \Big[\int_0^1 \big(\nabla g(s,x_k+\alpha(x_{k+1}-x_k))-\nabla g(s,x_{k+1})\big)d\alpha\Big] \\
& \quad \mathrm{d}_s < h(x_{i+1}), h(x_{k+1}) >_s (x_{i+1}-x_i)(x_{k+1}-x_k) \\
\le \;& E\Big[(\sup_i \sup_{\substack{s \\ \eta\in(x_i,x_{i+1})}} |\Delta g(s,\eta)|)|x_{i+1}-x_i| \cdot (\sup_k \sup_{\substack{s \\ \eta\in(x_k,x_{k+1})}} |\Delta g(s,\eta)|)|x_{k+1}-x_k| \\
& \quad \cdot| < h(x_{i+1}) >_t < h(x_{k+1}) >_t |^{\frac{1}{2}}\Big]\cdot\left(\sum_{i=0}^{n-1}\sum_{k=0}^{n-1}(x_{i+1}-x_i)(x_{k+1}-x_k)\right) \\
\to \;& 0,\ as\ \delta_x \to 0.
\end{aligned}$$

So (12) is proved.

To prove (13), first consider $g \in \mathcal{V}_3(h)$ to be sufficiently smooth jointly in (s,x). Then (12) and the integration by parts formula give

$$\begin{aligned}
& \int_0^t \int_{-\infty}^{+\infty} g(s,x)\mathbf{d}_{s,x}h(s,x) \\
= \;& -\int_{-\infty}^{+\infty}\int_0^t \nabla g(s,x)\mathrm{d}_s h(s,x)dx \\
= \;& -\int_{-\infty}^{+\infty}[\nabla g(s,x)h(s,x)]_0^t dx + \int_{-\infty}^{+\infty}\int_0^t \left(\frac{\partial}{\partial s}\nabla g(s,x)\right)h(s,x)dsdx. \qquad (14)
\end{aligned}$$

But by the integration by parts formula and the Fubini theorem,

$$\begin{aligned}
& \int_{-\infty}^{+\infty}\int_0^t \left(\frac{\partial}{\partial s}\nabla g(s,x)\right)h(s,x)dsdx \\
= \;& -\int_0^t \int_{-\infty}^{+\infty}\frac{\partial}{\partial s}g(s,x)\nabla h(s,x)dxds \\
= \;& -\int_{-\infty}^{+\infty}\int_0^t \frac{\partial}{\partial s}g(s,x)\nabla h(s,x)dsdx \\
= \;& -\int_{-\infty}^{+\infty}[g(s,x)\nabla h(s,x)]_0^t dx + \int_{-\infty}^{+\infty}\int_0^t g(s,x)d_s\nabla h(s,x)dx. \qquad (15)
\end{aligned}$$

By (14), (15) and the integration by parts formula, it follows that for g being sufficiently smooth

$$\int_0^t \int_{-\infty}^{+\infty} g(s,x)\mathbf{d}_{s,x}h(s,x) = \int_{-\infty}^{+\infty} \int_0^t g(s,x)d_s\nabla h(s,x)dx.$$

But any bounded function $g \in \mathcal{V}_3(h)$ can be approximated by a sequence of smooth functions $g_n \in \mathcal{V}_3(h)$. The desired result for $g \in \mathcal{V}_3(h)$ follows from (11) and

$$\begin{aligned}
& E|\int_{-\infty}^{+\infty} \int_0^t (g_n(s,x) - g(s,x))d_s\nabla h(s,x)dx|^2 \\
\leq\ & 2N \int_{-\infty}^{+\infty} E|\int_0^t (g_n(s,x) - g(s,x))d_s\nabla h(s,x)|^2 dx \\
=\ & 2N \int_{-\infty}^{+\infty} E \int_0^t |g_n(s,x) - g(s,x)|^2 d_s < \nabla h(x) >_s dx \\
\rightarrow\ & 0,
\end{aligned}$$

when $n \rightarrow \infty$. From Lemmas 2.4, 2.5, we can obtain that (12) and (13) also hold for $g \in \mathcal{V}_4(h)$. $\diamond$

3 The generalized Itô's formula in two-dimensional space

Let $X(s) = (X_1(s), X_2(s))$ be a two-dimensional continuous semimartingale with $X_i(s) = X_i(0) + M_i(s) + V_i(s) (i = 1,2)$ on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Here $M_i(s)$ is a continuous local martingale and $V_i(s)$ is an adapted continuous process of locally bounded variation (in s). Let $L_i(t,a)$ be the local time of $X_i(t)$ (i=1,2)

$$L_i(t,a) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{[a,a+\epsilon)}(X_i(s))d <M_i>_s \quad a.s. \quad i = 1,2 \tag{1}$$

for each t and $a \in R$. Then it is well known that, for each fixed $a \in R$, $L_i(t,a,\omega)$ is continuous, increasing in t, and right continuous with left limit (càdlàg) with respect to a ([16], [26]). Therefore we can define a Lebesgue-Stieltjes integral $\int_0^\infty \phi(s)d_sL_i(s,a,\omega)$ for each a for any Borel-measurable function ϕ. In particular

$$\int_0^\infty 1_{R\setminus\{a\}}(X_i(s))d_sL_i(s,a,\omega) = 0 \quad a.s. \quad i = 1,2. \tag{2}$$

Furthermore if ϕ is differentiable, then we have the following integration by parts formula

$$\int_0^t \phi(s)d_sL_i(s,a,\omega) = \phi(t)L_i(t,a,\omega) - \int_0^t \phi'(s)L_i(s,a,\omega)\mathrm{d}s \quad a.s.. \tag{3}$$

Moreover, if $g(s,x_i,\omega)$ is measurable and bounded, by the occupation times formula (e.g. see [16], [26])),

$$\int_0^t g(s,X_i(s))d <M_i>_s = 2\int_{-\infty}^{\infty} \int_0^t g(s,a)d_sL_i(s,a,\omega)\mathrm{d}a \quad a.s. \quad i = 1,2. \tag{4}$$

If $g(\cdot,x)$ is absolutely continuous for each x, $\frac{\partial}{\partial s}g(s,x)$ is locally bounded and measurable in $[0,t]\times R$, then using the integration by parts formula, we have

$$\begin{aligned}
&\int_0^t g(s,X_i(s))d <M_i>_s \\
= \;& 2\int_{-\infty}^{\infty}\int_0^t g(s,a)d_sL_i(s,a,\omega)\mathrm{d}a \\
= \;& 2\int_{-\infty}^{\infty} g(t,a)L_i(t,a,\omega)\mathrm{d}a \; - 2\int_{-\infty}^{\infty}\int_0^t \frac{\partial}{\partial s}g(s,a)L_i(s,a,\omega)\mathrm{d}s\mathrm{d}a \; a.s.,
\end{aligned}$$

for $i=1,2$. On the other hand, by the Tanaka formula

$$L_1(t,a)=(X_1(t)-a)^+-(X_1(0)-a)^+-\hat{M}_1(t,a)-\hat{V}_1(t,a),$$

where $\hat{Z}_1(t,a)=\int_0^t 1_{\{X_1(s)>a\}}dZ_1(s),\ Z_1=M_1,V_1,X_1$. By a standard localizing argument, we may assume without loss of generality that there is a constant N for which

$$\sup_{0\le s\le t}|X_1(s)|\le N,\ <M_1>_t\le N,\ Var_tV_1\le N,$$

where Var_tV_1 is the total variation of V_1 on $[0,t]$. From the property of local time (see Chapter 3 in [16]), for any $\gamma\ge 1$,

$$E|\hat{M}_1(t,a)-\hat{M}_1(t,b)|^{2\gamma}=E|\int_0^t 1_{\{a<X_s\le b\}}d <M_1>_s |^{\gamma}\le C(b-a)^{\gamma},\ a<b$$

where the constant C depends on γ and on the bound N. From Kolmogorov's tightness criterion (see [17]), we know that the sequence $Y_n(a):=\frac{1}{n}\hat{M}_1(t,a)$, $n=1,2,\cdots$, is tight. Moreover for any $a_1,a_2,\cdots,a_k$,

$$\begin{aligned}
&P(\sup_{a_i}|\frac{1}{n}\hat{M}_1(t,a_i)|\le 1) \\
= \;& P(|\frac{1}{n}\hat{M}_1(t,a_1)|\le 1,|\frac{1}{n}\hat{M}_1(t,a_2)|\le 1,\cdots,|\frac{1}{n}\hat{M}_1(t,a_k)|\le 1|) \\
\ge \;& 1-\sum_{i=1}^k P(|\frac{1}{n}\hat{M}_1(t,a_i)|>1) \\
\ge \;& 1-\frac{1}{n^2}\sum_{i=1}^k E[\hat{M}_1^2(t,a_i)] \\
\ge \;& 1-\frac{k}{n^2}C(N-a),
\end{aligned}$$

so by the weak convergence theorem of random fields (see Theorem 1.4.5 in [17]), we have

$$\lim_{n\to\infty}P(\sup_a|\hat{M}_1(t,a)|\le n)=1.$$

Furthermore it is easy to see that

$$\frac{1}{n}\hat{V}_1(t,a)\le\frac{1}{n}Var_tV_1(t,a)\to 0,\ when\ n\to\infty,$$

so it follows that,

$$\lim_{n\to\infty} P(\sup_a |L_1(t,a)| \le n) = 1.$$

Therefore in our localization argument, we can also assume that $L_1(t,a)$ and $L_2(t,a)$ are bounded uniformly in a.

We now assume the following conditions on $f : R \times R \to R$:

Condition (i) the function $f(\cdot,\cdot) : R \times R \to R$ is jointly continuous and absolutely continuous in x_1, x_2 respectively;

Condition (ii) the left derivative $\nabla_i^- f(x_1, x_2)$ is locally bounded, jointly left continuous, and of locally bounded variation in x_i $(i = 1, 2)$;

Condition (iii) the left derivaties $\nabla_1^- f(x_1, x_2)$ is absolutely continuous in x_2, and $\nabla_2^- f(x_1, x_2)$ is absolutely continuous in x_1;

Condition (iv) the derivatives $\nabla_1^-\nabla_2^- f(x_1, x_2)$ is jointly left continuous, and of locally bounded variation in x_1, x_2 respectively and also in (x_1, x_2).

From the assumption of $\nabla_1^- f$, we can use the Tanaka-Meyer formula to have,

$$\begin{aligned}\nabla_1^- f(a, X_2(t)) \quad - \quad \nabla_1^- f(a, X_2(0)) &= \int_0^t \nabla_1^-\nabla_2^- f(a, X_2(s))dX_2(s) \\ &+ \int_{-\infty}^{\infty} L_2(t, x_2)\mathrm{d}_{x_2}\nabla_1^-\nabla_2^- f(a, x_2) \quad a.s..\end{aligned}$$

Therefore $\nabla_1^- f(a, X_2(t))$ is a continuous semimartingale, which can be decomposed as

$$\nabla_1^- f(a, X_2(t)) = \nabla_1^- f(a, X_2(0)) + h(t, a) + v(t, a), \tag{5}$$

where h is a continuous local martingale and v is a continuous process of locally bounded variation (in t). In fact $h(t,a) = \int_0^t \nabla_1^-\nabla_2^- f(a, X_2(s))dM_2(s)$. Define

$$\begin{aligned} F_s(a,b) &:= \; < h(a), h(b) >_s \, = \, < \nabla_1^- f(a, X_2(\cdot)), \nabla_1^- f(b, X_2(\cdot)) >_s \\ &= \int_0^s \nabla_1^-\nabla_2^- f(a, X_2(r))\nabla_1^-\nabla_2^- f(b, X_2(r))\mathrm{d} <M_2>_r, \qquad (6) \\ F(a,b)_{s_k}^{s_{k+1}} &:= \; < h(a), h(b) >_{s_k}^{s_{k+1}} \, = \, < \nabla_1^- f(a, X_2(\cdot)), \nabla_1^- f(b, X_2(\cdot)) >_{s_k}^{s_{k+1}} \\ &= \int_{s_k}^{s_{k+1}} \nabla_1^-\nabla_2^- f(a, X_2(r))\nabla_1^-\nabla_2^- f(b, X_2(r))\mathrm{d} <M_2>_r . \qquad (7) \end{aligned}$$

We need to prove $h \in \mathcal{V}_2$. To see this, as $\nabla_1^-\nabla_2^- f(x_1, x_2)$ is of locally bounded variation in x_1, so for any compact set $[-N, N]$, $\nabla_1^-\nabla_2^- f(x_1, x_2)$ is of bounded variation in x_1 for $x_1 \in [-N, N]$. Let $\mathcal{P}$ be the partition on $[-N, N]^2 \times [0, t]$, $\mathcal{P}_i$ be a partition on $[-N, N]$ $(i = 1, 2)$, $\mathcal{P}_3$ be a partition on $[0,t]$ such that $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$. Then we have:

$$
\begin{aligned}
& \mathrm{Var}_{s,a,b}(F_s(a,b)) \\
= & \sup_{\mathcal{P}} \sum_k \sum_i \sum_j \Big| F(a_{i+1},b_{j+1})_{s_k}^{s_{k+1}} - F(a_{i+1},b_j)_{s_k}^{s_{k+1}} - F(a_i,b_{j+1})_{s_k}^{s_{k+1}} \\
& \qquad\qquad + F(a_i,b_j)_{s_k}^{s_{k+1}} \Big| \\
= & \sup_{\mathcal{P}} \sum_k \sum_i \sum_j \Big| \int_{s_k}^{s_{k+1}} \nabla_1^- \nabla_2^- f(a_{i+1}, X_2(r)) \nabla_1^- \nabla_2^- f(b_{j+1}, X_2(r)) \mathrm{d} <M_2>_r \\
& \quad - \int_{s_k}^{s_{k+1}} \nabla_1^- \nabla_2^- f(a_{i+1}, X_2(r)) \nabla_1^- \nabla_2^- f(b_j, X_2(r)) \mathrm{d} <M_2>_r \\
& \quad - \int_{s_k}^{s_{k+1}} \nabla_1^- \nabla_2^- f(a_i, X_2(r)) \nabla_1^- \nabla_2^- f(b_{j+1}, X_2(r)) \mathrm{d} <M_2>_r \\
& \quad + \int_{s_k}^{s_{k+1}} \nabla_1^- \nabla_2^- f(a_i, X_2(r)) \nabla_1^- \nabla_2^- f(b_j, X_2(r)) \mathrm{d} <M_2>_r \Big| \\
= & \sup_{\mathcal{P}} \sum_k \sum_i \sum_j \Big| \int_{s_k}^{s_{k+1}} \Big(\nabla_1^- \nabla_2^- f(a_{i+1}, X_2(r)) - \nabla_1^- \nabla_2^- f(a_i, X_2(r)) \Big) \\
& \quad \Big(\nabla_1^- \nabla_2^- f(b_{j+1}, X_2(r)) - \nabla_1^- \nabla_2^- f(b_j, X_2(r)) \Big) \mathrm{d} <M_2>_r \Big| \\
\leq & \int_0^s \sup_{\mathcal{P}_1} \sum_i \Big| \nabla_1^- \nabla_2^- f(a_{i+1}, X_2(r)) - \nabla_1^- \nabla_2^- f(a_i, X_2(r)) \Big| \\
& \quad \sup_{\mathcal{P}_2} \sum_j \Big| \nabla_1^- \nabla_2^- f(b_{j+1}, X_2(r)) - \nabla_1^- \nabla_2^- f(b_j, X_2(r)) \Big| \mathrm{d} <M_2>_r \\
= & \int_0^s \Big(\mathrm{Var}_a(\nabla_1^- \nabla_2^- f(a, X_2(r))) \Big)^2 \mathrm{d} <M_2>_r < \infty.
\end{aligned}
$$

Therefore under the localization assumption, $\int_{-\infty}^{\infty} \int_0^t L_1(s,a) d_{s,a} h(s,a)$ can be defined by Definition 2.1, i.e. it is a stochastic Lebesgue-Stieltjes integral. On the other hand, under the localization assumption and condition (iii) and (iv), let's prove that

$$
v(s,a) = \int_0^s \nabla_1^- \nabla_2^- f(a, X_2(r)) dV_2(r) + \int_{-\infty}^{\infty} L_2(s, x_2) \mathrm{d}_{x_2} \nabla_1^- \nabla_2^- f(a, x_2) := v_1(s,a) + v_2(s,a)
$$

is of bounded variation in (s,a) for $s \in [0,t]$, $a \in [-N,N]$. In fact,

$$
\begin{aligned}
Var_{s,a} v_1(s,a) &= \sup_{\mathcal{P}_1 \times \mathcal{P}_3} \sum_k \sum_i |v_1(s_{k+1}, a_{i+1}) - v_1(s_k, a_{i+1}) - v_1(s_{k+1}, a_i) + v_1(s_k, a_i)| \\
&= \sup_{\mathcal{P}_1 \times \mathcal{P}_3} \sum_k \sum_i \Big| \int_{s_{k+1}}^{s_k} \Big[\nabla_1^- \nabla_2^- f(a_{i+1}, X_2(r)) - \nabla_1^- \nabla_2^- f(a_i, X_2(r)) \Big] dV_2(r) \Big| \\
&\leq \int_0^t \sup_{\mathcal{P}_1} \sum_i |\nabla_1^- \nabla_2^- f(a_{i+1}, X_2(r)) - \nabla_1^- \nabla_2^- f(a_i, X_2(r))| |dV_2(r)| \\
&< \infty,
\end{aligned}
$$

as $\nabla_1^- \nabla_2^- f(x_1, x_2)$ is locally bounded and of bounded variation in x_1. Moreover, in the case when $\nabla_1^- \nabla_2^- f(x_1, x_2)$ is increasing in (x_1, x_2),

$$
\begin{aligned}
Var_{s,a} v_2(s,a) &= \sup_{\mathcal{P}_1 \times \mathcal{P}_3} \sum_k \sum_i |v_2(s_{k+1}, a_{i+1}) - v_2(s_k, a_{i+1}) - v_2(s_{k+1}, a_i) + v_2(s_k, a_i)| \\
&= \sup_{\mathcal{P}_1 \times \mathcal{P}_3} \sum_k \sum_i \int_{-\infty}^{\infty} \Big(L_2(s_{k+1}, x_2) - L_2(s_k, x_2) \Big) \\
&\qquad\qquad \mathrm{d}_{x_2} \Big(\nabla_1^- \nabla_2^- f(a_{i+1}, x_2) - \nabla_1^- \nabla_2^- f(a_i, x_2) \Big) \\
&\leq \sum_i \int_{-\infty}^{\infty} L_2(t, x_2) \mathrm{d}_{x_2} \Big(\nabla_1^- \nabla_2^- f(a_{i+1}, x_2) - \nabla_1^- \nabla_2^- f(a_i, x_2) \Big) \\
&\leq \max_{x_2} L_2(t, x_2) (\nabla_1^- \nabla_2^- f(N, N) - \nabla_1^- \nabla_2^- f(N, -N) \\
&\qquad\qquad - \nabla_1^- \nabla_2^- f(-N, N) + \nabla_1^- \nabla_2^- f(-N, -N)) \\
&< \infty.
\end{aligned}
$$

In the general case when $\nabla_1^- \nabla_2^- f(x_1, x_2)$ is of bounded variation in (x_1, x_2), we can assert that $v_2(s,a)$ is also of bounded variation in (s,a) by applying the above result to the difference of two increasing functions. So $\int_0^t \int_{-\infty}^{\infty} L_1(s,a) d_{s,a} v(s,a)$ is a Lebesgue-Stieltjes integral. Hence, $\int_0^t \int_{-\infty}^{\infty} L_1(s,a) d_{s,a} \nabla_1^- f(a, X_2(s))$ can be well defined. A localization argument implies that it is a semimartingale. Now we recall that the local time $L_1(s,a)$ can be decomposed as in [9],

$$
L_1(s,a) = \tilde{L}_1(s,a) + \sum_{x_k^* \leq a} \widehat{L}_1(s, x_k^*) := \tilde{L}_1(s,a) + \bar{L}_1(s,a),
$$

where $\tilde{L}_1(s,a)$ is jointly continuous in s, a, and $\{x_k^*\}$ are the discontinuous points of $L_1(s,a)$. From [26],

$$
\widehat{L}_1(t,x) = L_1(t,x) - L_1(t,x-) = \int_0^t 1_{\{x\}}(X_s) dV_s. \tag{8}
$$

Again we use the localization argument and assume that the support of the local time is included in $(-N, N)$. Let $g_1(s,a) := \nabla_1^- f(a, X_2(s))$, by a computation in (4.5) in [9], for any partition $\{0 = t_0 < t_1 < \cdots < t_m = t, -N = a_0 < a_1 < a_2 < \cdots < a_l = N\}$,

$$
\begin{aligned}
& \sum_{i=0}^{l-1} \sum_{j=0}^{m-1} g_1(t_{j+1}, a_{i+1}) \Big[\tilde{L}_1(t_{j+1}, a_{i+1}) - \tilde{L}_1(t_j, a_{i+1}) - \tilde{L}_1(t_{j+1}, a_i) + \tilde{L}_1(t_j, a_i) \Big] \\
= & \sum_{i=0}^{l-1} \sum_{j=0}^{m-1} \tilde{L}_1(t_j, a_i) \Big[g_1(t_{j+1}, a_{i+1}) - g_1(t_j, a_{i+1}) - g_1(t_{j+1}, a_i) + g_1(s_j, a_i) \Big] \\
& - \sum_{i=0}^{l-1} \tilde{L}_1(t, a_i) \Big[g_1(t, a_{i+1}) - g_1(t, a_i) \Big].
\end{aligned} \tag{9}
$$

Note that the first Riemann sum of the right hand side tends to $\int_0^t \int_{-N}^{N} \tilde{L}_1(s,a) d_{s,a} g_1(s,a)$, and the second Riemann sum of the right hand side has the limit $\int_{-N}^{N} \tilde{L}_1(s,a) d_a g_1(s,a)$, when

$\delta_t = \max_j(t_{j+1} - t_j) \to 0$ and $\delta_x = \max_i(x_{i+1} - x_i) \to 0$. Therefore the left hand side converges as well when $\delta_t \to 0$, $\delta_x \to 0$. Denote the limit by $\int_0^t \int_{-N}^N g_1(s,a) d_{s,a}\tilde{L}_1(s,a)$ on $\{\omega$: $L_1(t,a)$ has support which is included in $(-N,N)\}$. Taking the limit as $N \to \infty$ we can define $\int_0^t \int_{-\infty}^{\infty} g_1(s,a) d_{s,a}\tilde{L}_1(s,a)$ for almost all $\omega \in \Omega$ and it is easy to see that

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a}\tilde{L}_1(s,a) &= \int_0^t \int_{-\infty}^{\infty} \tilde{L}_1(s,a) d_{s,a}\nabla_1^- f(a, X_2(s)) \\ &\quad - \int_{-\infty}^{\infty} \tilde{L}_1(t,a) d_a \nabla_1^- f(a, X_2(t)).\end{aligned} \tag{10}$$

From Lemma 2.2 in [9], we know that $\bar{L}_1(t,a)$ is of bounded variation in (t,a) for almost every $\omega \in \Omega$. So $\int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a}\bar{L}_1(s,a)$ is a Lebesgue-Stieltjes integral. Therefore the integral

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a}L_1(s,a) &= \int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a}\tilde{L}_1(s,a) \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) d_{s,a}\bar{L}_1(s,a)\end{aligned}$$

can be well defined.

We will prove the following generalized Itô's formula in two-dimensional space.

Theorem 3.1. *Under conditions (i)-(iv), for any continuous two-dimensional semimartingale $X(t) = (X_1(t), X_2(t))$, we have almost surely*

$$\begin{aligned}& f(X(t)) - f(X(0)) \\ = \;& \sum_{i=1}^2 \int_0^t \nabla_i^- f(X(s)) dX_i(s) - \int_{-\infty}^{+\infty} \int_0^t \nabla_1^- f(a, X_2(s)) \mathrm{d}_{s,a} L_1(s,a) \\ & - \int_{-\infty}^{+\infty} \int_0^t \nabla_2^- f(X_1(s), a) \mathrm{d}_{s,a} L_2(s,a) + \int_0^t \nabla_1^- \nabla_2^- f(X(s)) d <M_1, M_2>_s .\end{aligned} \tag{11}$$

Proof: By a standard localization argument, we can assume $X_1(t)$, $X_2(t)$, their quadratic variations $<X_1>_t$, $<X_2>_t$, $<X_1, X_2>_t$ and the local times L_1, L_2 are bounded processes and f, $\nabla_i^- f$, $Var_{x_i}\nabla_i^- f$, $\nabla_1^-\nabla_2^- f$, $Var_{x_i}\nabla_1^-\nabla_2^- f$, $Var_{(x_1,x_2)}\nabla_1^-\nabla_2^- f$ $(i = 1, 2)$ are bounded.

We divide the proof into several steps:

(A) Define

$$\rho(x) = \begin{cases} c e^{\frac{1}{(x-1)^2-1}} & \text{if } x \in (0,2), \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Here c is chosen such that $\int_0^2 \rho(x)dx = 1$. Take $\rho_n(x) = n\rho(nx)$ as mollifiers. Define

$$f_n(x_1, x_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_n(x_1 - y)\rho_n(x_2 - z) f(y,z) dydz. \; n \geq 1,$$

Then $f_n(x_1, x_2)$ are smooth and

$$f_n(x_1, x_2) = \int_0^2 \int_0^2 \rho(y)\rho(z)f(x_1 - \frac{y}{n}, x_2 - \frac{z}{n})dydz,\ n \geq 1. \tag{13}$$

Because of the absolute continuity assumption, we can differentiate under the integral (13) to see f, $\nabla_i f_n$, $Var_{x_i}\nabla_i f_n$, $\nabla_1\nabla_2 f_n$, $Var_{x_i}\nabla_1\nabla_2 f_n$, $Var_{(x_1,x_2)}\nabla_1\nabla_2 f_n$ $(i = 1, 2)$ are bounded. Furthermore using Lebesgue's dominated convergence theorem, one can prove that as $n \to \infty$,

$$\begin{aligned}
f_n(x_1, x_2) &\to f(x_1, x_2), && (14)\\
\nabla_1 f_n(x_1, x_2) &\to \nabla_1^- f(x_1, x_2), && (15)\\
\nabla_2 f_n(x_1, x_2) &\to \nabla_2^- f(x_1, x_2), && (16)\\
\nabla_1\nabla_2 f_n(x_1, x_2) &\to \nabla_1^-\nabla_2^- f(x_1, x_2), && (17)
\end{aligned}$$

and each $(x_1, x_2) \in R^2$.

(B) It turns out for any $g(t, x_1)$ being continuous in t and C^1 in x_1 and having a compact support, using the integration by parts formula and Lebesgue's dominated convergence theorem, we see that

$$\begin{aligned}
\lim_{n\to+\infty}\int_{-\infty}^{+\infty} g(t, x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1, X_2(t)) &= -\lim_{n\to+\infty}\int_{-\infty}^{\infty} \nabla g(t, x_1)\nabla_1 f_n(x_1, X_2(t))dx_1\\
&= -\int_{-\infty}^{\infty} \nabla g(t, x_1)\nabla_1^- f(x_1, X_2(t))dx_1\ a.s.. \quad (18)
\end{aligned}$$

Note $\nabla_1^- f(x_1, x_2)$ is of locally bounded variation in x_1 and $g(t, x_1)$ has a compact support in x_1 and Riemann-Stieltjes integrable with respect to $\nabla^- f$, so

$$-\int_{-\infty}^{+\infty} \nabla g(t, x_1)\nabla_1^- f(x_1, X_2(t))dx_1 = \int_{-\infty}^{+\infty} g(t, x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1, X_2(t)).$$

Thus

$$\lim_{n\to+\infty}\int_{-\infty}^{+\infty} g(t, x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1, X_2(t)) = \int_{-\infty}^{\infty} g(t, x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1, X_2(t)). \tag{19}$$

(C) If $g(s, x_1)$ is C^2 in x_1, $\Delta g(s, x_1)$ is bounded uniformly in s, $\frac{\partial}{\partial s}\nabla g(s, x_1)$ is continuous in s and has a compact support in x_1, and $E\left[\int_0^t \int_{R^2} |g(s, x)g(s, y)||\mathrm{d}_{x,y,s} < h(x), h(y) >_s |\right] < \infty$, where $h \in \mathcal{V}_2$, then applying Lebesgue's dominated convergence theorem and Proposition 2.1

and the integration by parts formula,

$$
\begin{aligned}
&\lim_{n\to+\infty}\int_{-\infty}^{+\infty}\int_0^t g(s,x_1)\mathbf{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s))\\
=&\ -\lim_{n\to+\infty}\int_{-\infty}^{\infty}\int_0^t \nabla g(s,x_1)\mathrm{d}_s\nabla_1 f_n(x_1,X_2(s))dx_1\\
=&\ -\lim_{n\to+\infty}\Big(\int_{-\infty}^{\infty}\nabla g(s,x_1)\nabla_1 f_n(x_1,X_2(s))\Big|_0^t dx_1\\
&\qquad -\int_0^t\int_{-\infty}^{+\infty}\frac{\partial}{\partial s}\nabla g(s,x_1)\nabla_1 f_n(x_1,X_2(s))dx_1\mathrm{d}s\Big)\\
=&\ -\int_{-\infty}^{\infty}\nabla g(s,x_1)\nabla_1^- f(x_1,X_2(s))\Big|_0^t dx_1\\
&+\int_0^t\int_{-\infty}^{+\infty}\frac{\partial}{\partial s}\nabla g(s,x_1)\nabla_1^- f(x_1,X_2(s))dx_1\mathrm{d}s\\
=&\ -\int_{-\infty}^{+\infty}\int_0^t\nabla g(s,x_1)\mathrm{d}_s\nabla_1^- f(x_1,X_2(s))dx_1\\
=&\ \int_0^t\int_{-\infty}^{+\infty}g(s,x_1)\mathbf{d}_{s,x_1}\nabla_1^- f(x_1,X_2(s))\quad a.s.,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\lim_{n\to+\infty}\int_{-\infty}^{+\infty}\int_0^t g(s,x_1)\mathbf{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s))\\
=&\ \int_0^t\int_{-\infty}^{+\infty}g(s,x_1)\mathbf{d}_{s,x_1}\nabla_1^- f(x_1,X_2(s))\quad a.s.. \qquad (20)
\end{aligned}
$$

(D) In the following we will prove that (19) also holds for any continuous function $g(t,x_1)$ with a compact support in x_1. Moreover, if $g\in\mathcal{V}_3$ and continuous, (20) also holds.

To see (19), first note any continuous function with a compact support can be approximated by smooth functions with a compact support uniformly by the following standard smoothing procedure

$$g_m(t,x_1)=\int_{-\infty}^{\infty}\rho_m(y-x_1)g(t,y)dy=\int_0^2\rho(z)g(t,x_1+\frac{z}{m})dz.$$

Note that there is a compact set $G\subset R^1$ such that

$$
\begin{aligned}
\max_{x_1\in G}|g_m(t,x_1)-g(t,x_1)|\to 0 &\quad \text{as}\ \ m\to+\infty,\\
g_m(t,x_1)=g(t,x_1)=0 &\quad \text{for}\ \ x_1\notin G.
\end{aligned}
$$

Note

$$
\begin{aligned}
\int_{-\infty}^{+\infty}g(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) =&\ \int_{-\infty}^{+\infty}g_m(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) \qquad (21)\\
&+\int_{-\infty}^{+\infty}(g(t,x_1)-g_m(t,x_1))\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)).
\end{aligned}
$$

It is easy to see from (19) and Lebesgue's dominated convergence theorem, that

$$\begin{aligned}
& \lim_{m\to\infty}\lim_{n\to\infty}\int_{-\infty}^{+\infty} g_m(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) \\
= & \lim_{m\to\infty}\int_{-\infty}^{\infty} g_m(t,x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1,X_2(t)) \\
= & \int_{-\infty}^{\infty} g(t,x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1,X_2(t)) \ a.s.. \qquad (22)
\end{aligned}$$

Moreover,

$$\begin{aligned}
& |\int_{-\infty}^{+\infty}\Big(g(t,x_1)-g_m(t,x_1)\Big)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))| \\
\leq & \Big(\max_{x_1\in G}|g(t,x_1)-g_m(t,x_1)|\Big)Var_{x_1\in G}\nabla_1 f_n(x_1,X_2(t)). \qquad (23)
\end{aligned}$$

But,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\Big(\max_{x_1\in G}|g(t,x_1)-g_m(t,x_1)|\Big)Var_{x_1\in G}\nabla_1 f_n(x_1,X_2(t))=0 \ a.s..$$

So inequality (23) leads to

$$\lim_{m\to\infty}\limsup_{n\to\infty}|\int_{-\infty}^{+\infty}\Big(g(t,x_1)-g_m(t,x_1)\Big)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))|=0 \ a.s.. \qquad (24)$$

Now we use (21), (22) and (24)

$$\begin{aligned}
& \limsup_{n\to\infty}\int_{-\infty}^{+\infty} g(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) \\
= & \lim_{m\to\infty}\limsup_{n\to\infty}\int_{-\infty}^{+\infty} g_m(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) \\
& +\lim_{m\to\infty}\limsup_{n\to\infty}\int_{-\infty}^{+\infty}\Big(g(t,x_1)-g_m(t,x_1)\Big)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t)) \\
= & \int_{-\infty}^{\infty} g(t,x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1,X_2(t)) \ a.s..
\end{aligned}$$

Similarly we also have

$$\liminf_{n\to\infty}\int_{-\infty}^{+\infty} g(t,x_1)\mathrm{d}_{x_1}\nabla_1 f_n(x_1,X_2(t))=\int_{-\infty}^{\infty} g(t,x_1)\mathrm{d}_{x_1}\nabla_1^- f(x_1,X_2(t)) \ a.s.. \qquad (25)$$

So (19) holds for a continuous function g with a compact support in x_1.

Now we prove that (20) also holds for a continuous function $g\in\mathcal{V}_3$. Define

$$g_m(s,x_1)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_m(y-x_1)\rho_m(\tau-s)g(\tau,y)d\tau dy.$$

Then there is a compact $G \subset R^1$ such that

$$\max_{0\le s\le t, x_1\in G} |g_m(s,x_1) - g(s,x_1)| \to 0 \quad \text{as} \quad m \to +\infty,$$
$$g_m(s,x_1) = g(s,x_1) = 0 \qquad \text{for} \quad x_1 \notin G.$$

Then it is trivial to see

$$\begin{aligned} &\int_0^t \int_{-\infty}^{+\infty} g(s,x_1)\mathbf{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)) \\ = &\int_0^t \int_{-\infty}^{+\infty} g_m(s,x_1)\mathbf{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)) \\ &+ \int_0^t \int_{-\infty}^{+\infty} (g(s,x_1) - g_m(s,x_1))\mathbf{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)). \end{aligned}$$

But from (20), we can see that

$$\begin{aligned} &\lim_{m\to\infty}\lim_{n\to\infty}\int_0^t \int_{-\infty}^{+\infty} g_m(s,x_1)\mathbf{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)) \\ = &\lim_{m\to\infty}\int_0^t \int_{-\infty}^{+\infty} g_m(s,x_1)\mathbf{d}_{s,x_1}\nabla_1^- f(x_1,X_2(s)) \quad a.s. \\ = &\int_0^t \int_{-\infty}^{+\infty} g(s,x_1)\mathbf{d}_{s,x_1}\nabla_1 f(x_1,X_2(s)). \qquad (\mathit{limit\ in}\ \mathcal{M}_2) \end{aligned} \tag{26}$$

The last limit holds because of the following:

$$\begin{aligned} &E\Big[\int_0^t \int_{-\infty}^{+\infty} (g_m(s,x_1) - g(s,x_1))\mathbf{d}_{s,x_1}\nabla_1^- f(x_1,X_2(s))\Big]^2 \\ = \ &E\Big[\int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g_m-g)(s,a)(g_m-g)(s,b)\mathbf{d}_{a,b,s} < \nabla_1^- f(a,X_2(\cdot)), \nabla_1^- f(b,X_2(\cdot)) >_s \Big] \\ = \ &E\Big[\int_0^t \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} (g_m-g)(s,a)(g_m-g)(s,b) \\ &\qquad \mathbf{d}_{a,b}\nabla_1^-\nabla_2^- f(a,X_2(s))\nabla_1^-\nabla_2^- f(b,X_2(s))\mathrm{d} <M_2>_s \Big] \\ = \ &E\Big[\int_0^t \Big(\int_{-\infty}^{+\infty} (g_m-g)(s,a)\mathrm{d}_a\nabla_1^-\nabla_2^- f(a,X_2(s))\Big)^2 \mathrm{d} <M_2>_s \Big] \\ \to \ &0, \end{aligned}$$

as $m \to \infty$. Here we used (11) and (6) to obtain the first equality. On the other hand, in $\mathcal{M}_2$

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_0^t \int_{-\infty}^{+\infty} (g(s,x_1) - g_m(s,x_1))\mathbf{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s)) = 0. \tag{27}$$

In fact,

$$\begin{aligned} &E\Big[\int_0^t \int_{-\infty}^{+\infty} (g(s,x_1) - g_m(s,x_1))\mathbf{d}_{s,x_1}\nabla_1 f_n(x_1,X_2(s))\Big]^2 \\ = \ &E\int_0^t \Big[\int_{-\infty}^{+\infty} (g-g_m)(s,a)\mathrm{d}_a\nabla_1\nabla_2 f_n(a,X_2(s))\Big]^2 d <M_2>_s . \end{aligned}$$

Noting that $\nabla_1\nabla_2 f_n(a, X_2(s))$ is of bounded variation in a, we can use an argument similar to the one in the proof of (24) and (25) to prove (27).

(E) Now we use the multi-dimensional Itô's formula to the function $f_n(X(s))$, then a.s.

$$\begin{aligned} & f_n(X(t)) - f_n(X(0)) \\ = \; & \sum_{i=1}^{2} \int_0^t \nabla_i f_n(X(s)) dX_i(s) + \frac{1}{2}\int_0^t \Delta_1 f_n(X(s)) d <M_1>_s \\ & + \frac{1}{2}\int_0^t \Delta_2 f_n(X(s)) d <M_2>_s + \int_0^t \nabla_1\nabla_2 f_n(X(s)) d <M_1, M_2>_s . \end{aligned} \tag{28}$$

As $n \to \infty$, it is easy to see from Lebesgue's dominated convergence theorem and (14), (15), (16), (17) that, $(i = 1, 2)$

$$\begin{aligned} f_n(X(t)) - f_n(X(0)) & \to f(X(t)) - f(X(0)) \quad a.s., \\ \int_0^t \nabla_i f_n(X(s)) dV_i(s) & \to \int_0^t \nabla_i^- f(X(s)) dV_i(s) \quad a.s., \\ \int_0^t \nabla_1\nabla_2 f_n(X(s)) d <M_1, M_2>_s & \to \int_0^t \nabla_1^-\nabla_2^- f(X(s)) d <M_1, M_2>_s \quad a.s. \end{aligned}$$

and

$$E\int_0^t (\nabla_i f_n(X(s)))^2 d <M_i>_s \to E\int_0^t (\nabla_i^- f(X(s))^2 d <M_i>_s .$$

Therefore in $\mathcal{M}_2$,

$$\int_0^t \nabla_i f_n(X(s)) dM_i(s) \to \int_0^t \nabla_i^- f(X(s)) dM_i(s), (i = 1, 2).$$

To see the convergence of $\frac{1}{2}\int_0^t \Delta_1 f_n(X(s)) d <M_1>_s$, first from integration by parts formula and (13), we have

$$\begin{aligned} \frac{1}{2}\int_0^t \Delta_1 f_n(X(s)) d <M_1>_s & = \int_{-\infty}^{+\infty}\int_0^t \Delta_1 f_n(a, X_2(s)) \mathrm{d}_s L_1(s, a) \mathrm{d}a \\ & = \int_{-\infty}^{+\infty} L_1(t, a) \mathrm{d}_a \nabla_1 f_n(a, X_2(t)) \\ & \quad - \int_{-\infty}^{+\infty}\int_0^t L_1(s, a) \mathbf{d}_{s,a} \nabla_1 f_n(a, X_2(s)). \end{aligned}$$

But local time $L_1(s, a)$ can be decomposed as

$$L_1(s, a) = \tilde{L}_1(s, a) + \sum_{x_k^* \le a} \widehat{L}_1(s, x_k^*) := \tilde{L}_1(s, a) + \bar{L}_1(s, a), \tag{29}$$

where $\tilde{L}_1(s,a)$ is jointly continuous in s, a, and $\{x_k^*\}$ are the discontinuous points of $L_1(s,a)$. From (D) and (10), we have as $n \to \infty$,

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} \tilde{L}_1(t,a) \mathrm{d}_a \nabla_1 f_n(a, X_2(t)) - \int_{-\infty}^{+\infty} \int_0^t \tilde{L}_1(s,a) \mathbf{d}_{s,a} \nabla_1 f_n(a, X_2(s)) \\
\to \quad & \int_{-\infty}^{\infty} \tilde{L}_1(t,a) \mathrm{d}_a \nabla_1^- f(a, X_2(t)) - \int_{-\infty}^{+\infty} \int_0^t \tilde{L}_1(s,a) \mathbf{d}_{s,a} \nabla_1^- f(a, X_2(s)) \quad (\textit{limit in } \mathcal{M}_2) \\
= \quad & -\int_{-\infty}^{\infty} \int_0^t \nabla_1^- f(a, X_2(s)) \mathbf{d}_{s,a} \tilde{L}_1(s,a). \qquad (30)
\end{aligned}
$$

On the other hand, from Lemma 2.2 in [9], we know that $\bar{L}_1(s,a)$ is of bounded variation in a for each s and of bounded variation in (s,a) for almost every $\omega \in \Omega$. And also because $\nabla_1 f_n(a, X_2(s))$ is continuous in (s,a), $\int_0^t \int_{-\infty}^{\infty} \nabla_1 f_n(a, X_2(s)) \mathbf{d}_{s,a} \bar{L}_1(s,a)$ is Riemann-Stieltjes integral. Hence in (9), replacing $\tilde{L}_1(s,a)$ by $\bar{L}_1(s,a)$, $g_1(s,a)$ by $\nabla_1 f_n(a, X_2(s))$, we still can obtain an integration by parts formula as follows

$$
\begin{aligned}
& \int_0^t \int_{-\infty}^{\infty} \bar{L}_1(s,a) \mathbf{d}_{s,a} \nabla_1 f_n(a, X_2(s)) \\
= & \int_0^t \int_{-\infty}^{\infty} \nabla_1 f_n(a, X_2(s)) \mathbf{d}_{s,a} \bar{L}_1(s,a) + \int_{-\infty}^{\infty} \bar{L}_1(t,a) d_a \nabla_1 f_n(a, X_2(t))
\end{aligned}
$$

Note here the integral $\int_0^t \int_{-\infty}^{\infty} \bar{L}_1(s,a) \mathbf{d}_{s,a} \nabla_1 f_n(a, X_2(s))$ is also a Riemann-Stieltjes integral though it is stochastic. Therefore

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \bar{L}_1(t,a) d_a \nabla_1 f_n(a, X_2(t)) - \int_0^t \int_{-\infty}^{\infty} \bar{L}_1(s,a) \mathbf{d}_{s,a} \nabla_1 f_n(a, X_2(s)) \\
= \quad & -\int_0^t \int_{-\infty}^{\infty} \nabla_1 f_n(a, X_2(s)) \mathbf{d}_{s,a} \bar{L}_1(s,a) \\
\to \quad & -\int_0^t \int_{-\infty}^{\infty} \nabla_1^- f(a, X_2(s)) \mathbf{d}_{s,a} \bar{L}_1(s,a) \qquad (31)
\end{aligned}
$$

as $n \to \infty$ by Lebesgue's dominated convergence theorem. So by (30) and (31),

$$
\frac{1}{2} \int_0^t \Delta_1 f_n(X(s)) d <M_1>_s \to -\int_{-\infty}^{\infty} \int_0^t \nabla_1^- f(x_1, X_2(t)) \mathbf{d}_{s,x_1} L_1(s, x_1),
$$

as $n \to \infty$. The term $\frac{1}{2} \int_0^t \Delta_2 f_n(s, X(s)) d <M_2>_s$ can be treated similarly. So we proved the desired formula. $\diamond$

The following theorem gives the new representation of $f(X_t)$, which leads to integration by parts formula for integrations of local times.

Theorem 3.2. *Under conditions (i)-(iv), for any continuous two-dimensional semimartingale*

$X(t) = (X_1(t), X_2(t))$, we have almost surely

$$\begin{aligned}
f(X(t)) &= f(X(0)) + \sum_{i=1}^{2} \int_0^t \nabla_i^- f(X(s)) dX_i(s) \\
&\quad + \int_{-\infty}^{\infty} L_1(t,a) \mathrm{d}_a \nabla_1^- f(a, X_2(t)) - \int_{-\infty}^{+\infty} \int_0^t L_1(s,a) \mathbf{d}_{s,a} \nabla_1^- f(a, X_2(s)) \\
&\quad + \int_{-\infty}^{\infty} L_2(t,a) \mathrm{d}_a \nabla_2^- f(X_1(t), a) - \int_{-\infty}^{+\infty} \int_0^t L_2(s,a) \mathbf{d}_{s,a} \nabla_2^- f(X_1(s), a) \\
&\quad + \int_0^t \nabla_1^- \nabla_2^- f(X(s)) d <M_1, M_2>_s \,. \qquad (32)
\end{aligned}$$

In particular, from (10), (11), we have the integration by parts formulae

$$\begin{aligned}
&\int_{-\infty}^{\infty} g(t,a) \mathrm{d}_a \nabla_1^- f(a, X_2(t)) - \int_{-\infty}^{+\infty} \int_0^t g(s,a) \mathbf{d}_{s,a} \nabla_1^- f(a, X_2(s)) \\
=& \; - \int_{-\infty}^{+\infty} \int_0^t \nabla_1^- f(a, X_2(s)) \mathbf{d}_{s,a} g(s,a),
\end{aligned}$$

for $g(s,a) = L_1(s,a), \tilde{L}_1(s,a), \bar{L}_1(s,a)$ respectively.

Proof: For (32), we only need to prove the convergence in (30) holds for $\bar{L}_1(s,x)$. First let's prove, when $n \to \infty$, in $\mathcal{M}_2$,

$$\int_{-\infty}^{+\infty} \int_0^t \bar{L}_1(s,a) \mathbf{d}_{s,a} \nabla_1 f_n(a, X_2(s)) \to \int_{-\infty}^{+\infty} \int_0^t \bar{L}_1(s,a) \mathbf{d}_{s,a} \nabla_1^- f(a, X_2(s)).$$

From the assumption of $\nabla_1^- f$ and the definition of f_n, recall (5) and from Itô's formula we have $\nabla_1 f_n(a, X_2(t)) = \nabla_1 f_n(a, X_2(0)) + h_n(t,a) + v_n(t,a)$, where h_n, h are continuous local martingales and v_n, v are continuous processes with locally bounded variation (in t). From previous computations, we know that $h_n, h \in \mathcal{V}_2$, i.e. $< (h_n - h)(a), (h_n - h)(b) >_s$ is of bounded variation in (s,a,b) and $v_n(s,a)$, $v(s,a)$ are of bounded variation in (s,a). So

$$\begin{aligned}
&E \Big| \int_{-\infty}^{+\infty} \int_0^t \bar{L}_1(s,a) \mathbf{d}_{s,a} h_n(s,a) - \int_{-\infty}^{+\infty} \int_0^t \bar{L}_1(s,a) \mathbf{d}_{s,a} h(s,a) \Big|^2 \\
=& \; E \int_0^t \int_{R^2} \bar{L}_1(s,a) \bar{L}_1(s,b) \mathbf{d}_{a,b,s} < h_n(a) - h(a), h_n(b) - h(b) >_s \,.
\end{aligned}$$

Let $(-N, N)$ cover the compact support of local time $L_1(t,\cdot)$, N is fixed for each ω, and

$$\begin{aligned}
&G(s,a,b) := \bar{L}_1(s,a) \bar{L}_1(s,b) \\
&G(a,b)_{s_k}^{s_{k+1}} := \bar{L}_1(s_{k+1},a) \bar{L}_1(s_{k+1},b) - \bar{L}_1(s_k,a) \bar{L}_1(s_k,b) \\
&H_n(s,a,b) := < h_n(a) - h(a), h_n(b) - h(b) >_s \,.
\end{aligned}$$

We can show that $G(s,a,b)$ is of bounded variation in (s,a,b). In fact, let $\mathcal{P}$ be a partition on $[-N,N]^2 \times [0,t]$, where $\mathcal{P}_i$ is a partition on $[-N,N]$ $(i = 1,2)$, $\mathcal{P}_3$ is a partition on $[0,t]$ such that $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$. Then from (8) and standard computations we can show

$$\mathrm{Var}_{s,a,b} G(s,a,b) \;\leq\; 2 \Big(\sum_{-N < x_m^* \leq N} \int_0^t 1_{\{x_m^*\}}(X_s) |dV_s| \Big)^2 \leq 2 \Big(\int_0^t 1_{(-N,N]}(X_s) |dV_s| \Big)^2 < \infty.$$

Therefore, G can be decomposed as differences of increasing (in all three variables) functions. But we can prove more results that will be used. Define

$$\tilde{G}_1(s,a,b) := V_G([0,s]\times[-N,a]\times[-N,b]) + G(s,a,b),$$
$$\tilde{G}_2(s,a,b) := V_G([0,s]\times[-N,a]\times[-N,b]) - G(s,a,b),$$

where $V_G([0,s]\times[-N,a]\times[-N,b])$ denotes the total variation of G on $[0,s]\times[-N,a]\times[-N,b]$. Then it is easy to see that $G(s,a,b) = \frac{1}{2}[\tilde{G}_1(s,a,b) - \tilde{G}_2(s,a,b)]$, and $\tilde{G}_1$, $\tilde{G}_2$ are increasing in (s,a,b). Moreover, by additivity of variation, one can see that for $s_2 \geq s_1$,

$$\begin{aligned}
& \tilde{G}_1(s_2,a,b) - \tilde{G}_1(s_1,a,b) \\
= \; & V_G([s_1,s_2]\times[-N,x]\times[-N,y]) + G(s_2,a,b) - G(s_1,a,b) - G(s_2,a,-N) \\
& + G(s_1,a,-N) - G(s_2,-N,b) + G(s_1,a,-N) - G(s_2,-N,-N) + G(s_1,-N,-N) \\
\geq \; & 0.
\end{aligned}$$

That is to say, $\tilde{G}_1(s_2,a,b)$ is increasing in s for each a and b. In the same way, we can show $\tilde{G}_1(s,a,b)$ is increasing in a for each s and b, and in b for each s and a. Therefore $\tilde{G}_1(s,a,b)$ is increasing in s, a, b respectively. Similarly, $\tilde{G}_2(s,a,b)$ is also increasing in s, a, b respectively. Define

$$G_1(s,a,b) = \lim_{s'\downarrow s, a'\downarrow a, b'\downarrow b} \tilde{G}_1(s',a',b'),$$
$$G_2(s,a,b) = \lim_{s'\downarrow s, a'\downarrow a, b'\downarrow b} \tilde{G}_2(s',a',b').$$

Then G_1 and G_2 are right continuous in (s,a,b), and increasing in s, a, b separately, and $G(s,a,b) = \frac{1}{2}[G_1(s,a,b) - G_2(s,a,b)]$. Now we claim for any $c > 0$, $A = \{(s,a,b) : G_1(s,a,b) < c\}$ is an open set. To see this, for any $(s,a,b) \in A$, take $\varepsilon = \frac{1}{2}(c - G_1(s,a,b)) > 0$. First as $G(s,a,b)$ is right continuous in (s,a,b), so there exists $\delta > 0$ such that

$$|G_1(s',a',b') - G_1(s,a,b)| < \varepsilon,$$

when $s \leq s' < s+\delta$, $a \leq a' < a+\delta$, $b \leq b' < b+\delta$. That is to say, $[s,s+\delta)\times[a,a+\delta)\times[b,b+\delta) \subset A$. But for any $s' \leq s$, $a' \leq a$, $b' \leq b$, by the monotonicity of G_1 in each variable separately,

$$G_1(s',a',b') \leq G_1(s,a',b') \leq G_1(s,a,b') \leq G_1(s,a,b) < c.$$

Therefore, $(-\infty, s+\delta)\times(-\infty,a+\delta)\times(-\infty,b+\delta) \in A$. This implies that A is an open set. Thus for any $c \geq 0$, $\{(s,a,b) : G_1(s,a,b) \geq c\}$ is a closed set (when $c = 0$, $\{(s,a,b) : G_1(s,a,b) \geq c\} = [0,t]\times[-N,N]^2$).

From the assumption, we know $H_n(s,a,b)$ is of bounded variation in (s,a,b) and when $n \to \infty$, $H_n \to 0$. We only consider the increasing part of H_n, still denote it by H_n. As $H_n(s,a,b)$ is left continuous and increasing, so it generates Lebesgue-Stieltjes measure, denote it by μ_n. It is easy to see that $\mu_n([s_1,s_2)\times[a_1,a_2)\times[b_1,b_2)) \to 0$, as $n \to \infty$, for any $[s_1,s_2)\times[a_1,a_2)\times[b_1,b_2) \subset [0,t]\times[-N,N]^2$. So $\mu_n \xrightarrow{W} 0$, as $n \to \infty$. Let P be a probability measure on $[0,t]\times[-N,N]^2$ and

$$P_n([s_1,s_2)\times[a_1,a_2)\times[b_1,b_2)) = \frac{(P+\mu_n)([s_1,s_2)\times[a_1,a_2)\times[b_1,b_2))}{(P+\mu_n)([0,t]\times[-N,N]\times[-N,N])}.$$

Then $P_n \xrightarrow{W} P$. Therefore, by the equivalent condition of weak convergence (cf. Proposition 1.2.4 in [15]), for any closed set E, $\limsup_{n\to\infty} P_n(E) \le P(E)$. Now without losing generality, we assume $0 \le G_1(s,a,b) \le 1$. Using the method in the proof of Proposition 1.2.4 in [15], we have for either $Q = P_n$ or P,

$$\begin{aligned}\sum_{i=1}^{k} \frac{i-1}{k} Q\{(s,a,b) : \frac{i-1}{k} \le G_1(s,a,b) < \frac{i}{k}\} &\le \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) Q(dsdadb) \\ &\le \sum_{i=1}^{k} \frac{i}{k} Q\{(s,a,b) : \frac{i-1}{k} \le G_1(s,a,b) < \frac{i}{k}\},\end{aligned}$$

and

$$\sum_{i=1}^{k} \frac{i}{k} Q\{(s,a,b) : \frac{i-1}{k} \le G_1(s,a,b) < \frac{i}{k}\} = \sum_{i=0}^{k-1} \frac{1}{k} Q\{(s,a,b) : G_1(s,a,b) \ge \frac{i}{k}\}.$$

But $E_i := \{(s,a,b) : G_1(s,a,b) \ge \frac{i}{k}\}$ is closed, so $\limsup_{n\to\infty} P_n(E_i) \le P(E_i), \quad i = 0, 1, \cdots, k-1$. Thus,

$$\begin{aligned}\limsup_{n\to\infty} \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) P_n(dsdadb) &\le \limsup_{n\to\infty} \sum_{i=0}^{k-1} \frac{1}{k} P_n\{(s,a,b) : G_1(s,a,b) \ge \frac{i}{k}\} \\ &\le \sum_{i=0}^{k-1} \frac{1}{k} P\{(s,a,b) : G_1(s,a,b) \ge \frac{i}{k}\} \\ &\le \frac{1}{k} + \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) P(dsdadb).\end{aligned}$$

As k is arbitrary, so

$$\limsup_{n\to\infty} \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) P_n(dsdadb) \le \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) P(dsdadb).$$

Applying above to $1 - G_1(s,a,b)$, we can prove

$$\liminf_{n\to\infty} \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) P_n(dsdadb) \ge \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) P(dsdadb).$$

Therefore,

$$\lim_{n\to\infty} \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) P_n(dsdadb) = \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) P(dsdadb).$$

It turns out that,

$$\lim_{n\to\infty} \int_0^t \int_{-N}^N \int_{-N}^N G_1(s,a,b) \mu_n(dsdadb) = 0.$$

The same result also holds for $G_2(s,a,b)$. Thus,

$$\lim_{n\to\infty}\int_0^t\int_{-N}^{N}\int_{-N}^{N}G(s,a,b)\mu_n(dsdadb)=0.$$

But when $H_n(s,a,b)$ is of bounded variation in (s,a,b), it can be decomposed to two increasing functions. Therefore, we have

$$\lim_{n\to\infty}\int_0^t\int_{-N}^{N}\int_{-N}^{N}G(s,a,b)\mathbf{d}_{a,b,s}H_n(s,a,b)=0.$$

Hence, when $n\to\infty$, in $\mathcal{M}_2$

$$\int_{-\infty}^{+\infty}\int_0^t\bar{L}_1(s,a)\mathbf{d}_{s,a}h_n(s,a)\to\int_{-\infty}^{+\infty}\int_0^t\bar{L}_1(s,a)\mathbf{d}_{s,a}h(s,a).$$

We can also easily prove that

$$\begin{aligned}&\int_{-\infty}^{+\infty}\int_0^t\bar{L}_1(s,a)\mathbf{d}_{s,a}v_n(s,a)\to\int_{-\infty}^{+\infty}\int_0^t\bar{L}_1(s,a)\mathbf{d}_{s,a}v(s,a),\\&\int_{-\infty}^{+\infty}\bar{L}_1(t,a)d_a\nabla_1 f_n(a,X_2(t))\to\int_{-\infty}^{+\infty}\bar{L}_1(t,a)d_a\nabla_1^- f(a,X_2(t)).\end{aligned}$$

Similarly we can deal with the terms with $\bar{L}_2(s,a)$. So (32) is proved and the integration by parts formulae follow easily. $\diamond$

The smoothing procedure in Theorem 3.1 can be used to prove that if $f:R\times R\to R$ is $C^{1,1}$, and the left derivatives $\frac{\partial^{2-}}{\partial x_i\partial x_j}f(x_1,x_2)$, $(i,j=1,2)$ exist and are locally bounded and left continuous, then

$$f(X(t))-f(X(0))=\sum_{i=1}^{2}\int_0^t\nabla_i f(X(s))dX_i(s)+\frac{1}{2}\sum_{i,j=1}^{2}\int_0^t\frac{\partial^{2-}}{\partial x_i\partial x_j}f(X(s))d<X_i,X_j>_s.$$

This can be seen from the convergence in the proof of Theorem 3.1 and the fact that $\frac{\partial^{2}}{\partial x_i\partial x_j}f_n(x_1,x_2)$
$\to\frac{\partial^{2-}}{\partial x_i\partial x_j}f(x_1,x_2)$ under the stronger condition on $\frac{\partial^{2-}}{\partial x_i\partial x_j}f$.
The next theorem is an easy consequence of the methods of the proofs of Theorem 3.1 and (33).

Theorem 3.3. *Let $f:R\times R\to R$ satisfy conditions (i) and $f(x_1,x_2)=f_h(x_1,x_2)+f_v(x_1,x_2)$. Assume f_h is $C^{1,1}$ and the left derivatives $\frac{\partial^{2-}}{\partial x_i\partial x_j}f_h(x_1,x_2)(i,j=1,2)$ exist and are left contin-*

uous and locally bounded; f_v satisfies conditions (ii)-(iv). Then

$$
\begin{aligned}
& f(X(t)) - f(X(0)) \\
= \ & \sum_{i=1}^{2} \int_0^t \nabla_i^- f(X(s)) dX_i(s) + \frac{1}{2} \sum_{i=1}^{2} \int_0^t \Delta_i^- f_h(X(s)) d <X_i>_s \\
& - \int_{-\infty}^{+\infty} \int_0^t \nabla_1^- f_v(a, X_2(s)) \mathrm{d}_{s,a} L_1(s,a) - \int_{-\infty}^{+\infty} \int_0^t \nabla_2^- f_v(X_1(s), a) \mathrm{d}_{s,a} L_2(s,a) \\
& + \int_0^t \nabla_1^- \nabla_2^- f(X(s)) d <M_1, M_2>_s \\
= \ & \sum_{i=1}^{2} \int_0^t \nabla_i^- f(X(s)) dX_i(s) + \frac{1}{2} \sum_{i=1}^{2} \int_0^t \Delta_i^- f_h(X(s)) d <X_i>_s \\
& + \int_{-\infty}^{\infty} L_1(t,a) \mathrm{d}_a \nabla_1^- f_v(a, X_2(t)) - \int_{-\infty}^{+\infty} \int_0^t L_1(s,a) \mathbf{d}_{s,a} \nabla_1^- f_v(a, X_2(s)) \\
& + \int_{-\infty}^{\infty} L_2(t,a) \mathrm{d}_a \nabla_2^- f_v(X_1(t), a) - \int_{-\infty}^{+\infty} \int_0^t L_2(s,a) \mathbf{d}_{s,a} \nabla_2^- f_v(X_1(s), a) \\
& + \int_0^t \nabla_1^- \nabla_2^- f(X(s)) d <M_1, M_2>_s \quad a.s..
\end{aligned}
\tag{33}
$$

Example 3.1. *Consider*

$$f(x_1, x_2) = (x_1 x_2)^+.$$

It is easy to see that

$$
\begin{aligned}
\nabla_1^- f(x_1, x_2) &= x_2 1_{\{x_1 x_2 > 0\}} 1_{\{x_2 > 0\}} + x_2 1_{\{x_1 x_2 \le 0\}} 1_{\{x_2 \le 0\}} \\
&= x_2 1_{\{x_1 > 0\}} 1_{\{x_2 > 0\}} + x_2 1_{\{x_1 \le 0\}} 1_{\{x_2 \le 0\}} \\
&= x_2^+ 1_{\{x_1 > 0\}} - x_2^- 1_{\{x_1 \le 0\}},
\end{aligned}
$$

so $\Delta_1^- f(0, x_2) = \infty$, which means that the classical Itô's formula doesn't work. But

$$\nabla_1^- \nabla_2^- f(x_1, x_2) = 1_{\{x_1 > 0\}} 1_{\{x_2 > 0\}} + 1_{\{x_1 \le 0\}} 1_{\{x_2 \le 0\}}.$$

This suggests that our generalized Itô's formula can be used.

Example 3.2. *Consider*

$$f(x_1, x_2) = x_2^{\frac{1}{3}} (x_1 x_2)^+.$$

It is easy to see that

$$\begin{aligned}
\nabla_1^- f(x_1,x_2) &= x_2^{\frac{1}{3}} x_2^+ 1_{\{x_1>0\}} - x_2^{\frac{1}{3}} x_2^- 1_{\{x_1\le 0\}},\\
\nabla_2^- f(x_1,x_2) &= \frac{1}{3} x_2^{-\frac{2}{3}} (x_1x_2)^+ + x_2^{\frac{1}{3}} x_1^+ 1_{\{x_2>0\}} - x_2^{\frac{1}{3}} x_1^- 1_{\{x_2\le 0\}}\\
&= \frac{4}{3} x_2^{-\frac{2}{3}} (x_1^+ x_2^+ + x_1^- x_2^-),\\
\Delta_2^- f(x_1,x_2) &= -\frac{8}{9} x_2^{-\frac{2}{3}} (x_1^+ 1_{\{x_2>0\}} - x_1^- 1_{\{x_2<0\}})\\
&\quad + \frac{4}{3} x_2^{-\frac{2}{3}} (x_1^+ 1_{\{x_2>0\}} - x_1^- 1_{\{x_2\le 0\}}),\\
\nabla_1^- \nabla_2^- f(x_1,x_2) &= \frac{4}{3} x_2^{\frac{1}{3}} 1_{\{x_1x_2>0\}} + \frac{4}{3} x_2^{\frac{1}{3}} 1_{\{x_1=0\}} 1_{\{x_2<0\}}.
\end{aligned}$$

So $\Delta_2^- f(x_1,0) = -\infty$ *when* $x_1 < 0$, *and* $\lim_{x_2\to 0-} \Delta_2^- f(x_1,x_2) = -\infty$ *when* $x_1 < 0$, $\lim_{x_2\to 0+} \Delta_2^- f(x_1,x_2) = \infty$ *when* $x_1 > 0$. *These calculations suggest that neither the classical Itô's formula, nor the formula in [24] can be applied immediately. But our generalized Itô's formula can be used here.*

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