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Random Graph-Homomorphisms and Logarithmic Degree

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Abstract

A graph homomorphism between two graphs is a map from the vertex set of one graph to the vertex set of the other graph, that maps edges to edges. In this note we study the range of a uniformly chosen homomorphism from a graph G to the infinite line $\mathbb{Z}$. It is shown that if the maximal degree of G is 'sub-logarithmic', then the range of such a homomorphism is super-constant.

Furthermore, some examples are provided, suggesting that perhaps for graphs with super-logarithmic degree, the range of a typical homomorphism is bounded. In particular, a sharp transition is shown for a specific family of graphs $C_{n,k}$ (which is the tensor product of the n-cycle and a complete graph, with self-loops, of size k). That is, given any function $\psi(n)$ tending to infinity, the range of a typical homomorphism of $C_{n,k}$ is super-constant for $k = 2\log(n) - \psi(n)$, and is 3 for $k = 2\log(n) + \psi(n)$.

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1 Introduction

A graph homomorphism from a graph G to a graph H is a map from the vertex set of G to the vertex set of H, that maps edges to edges. By a *homomorphism of* G we mean a graph homomorphism from G to the infinite line $\mathbb{Z}$. Thus, a homomorphism of G maps adjacent vertices to adjacent integers. We note that the uniform measure on the set of all homomorphisms of G, that send some fixed vertex to 0, generalizes the concept of random walks on $\mathbb{Z}$. Indeed, a random homomorphism of the k-line is a random walk of length k on $\mathbb{Z}$. So, random homomorphisms of a graph G, are also referred to as G-indexed random walks. Tree-indexed random walks were studied by Benjamini and Peres in (2). For results concerning random homomorphisms of general graphs see (1; 8). (3) deals with connections between random homomorphisms and the Gaussian random field. For other related 2-dimensional height models in physics see (7; 9).

A key quantity for our first result, Theorem 2.1, is $V(r)$, the maximal size of a ball of radius r in G. Theorem 2.1 states that for every r such that $V(r)$ is at most $\frac{1}{2}\log(|G|)$, the range of a random homomorphism is greater than r, with high probability. If d is the maximal degree in G, then $V(r)$ is at most $(d+1)^r$. Thus, Theorem 2.1 implies that for graphs of 'small enough' degree, the range of a homomorphism is typically 'large' (see Corollary 2.2). We stress that this is only a sufficient condition for large range, and not a necessary one. For example, consider the $\log(n)$-regular tree of size n. Already for $r = 1$, a ball of radius r has at least $\log(n)$ vertices, so the assertion of Theorem 2.1 is trivial. However, the range of a typical homomorphism of this tree is of size at least $\Omega(\log(n)/\log\log(n))$.

The next natural question is: How tight is this lower bound? That is, are there examples of graphs of logarithmic degree that have bounded range (as the size of the graph grows to infinity)? This can be decided via a result of Kahn (6). Kahn's results states that there exists a constant $b \in \mathbb{N}$, such that the range of a random homomorphism of Q_d, the discrete cube of dimension d, is at most b, with probability tending to 1 as d tends to infinity (note that the size of Q_d is 2^d and the degree of Q_d is d). Galvin (4) later calculated $b = 5$.

Kahn's result raises a new question: What happens to the range of a random homomorphism, if the degree is logarithmic, but the diameter is large? (The discrete cube has logarithmic degree, but also has logarithmic diameter.) To answer this question, we study the graph $C_{n,k}$ in Section 3 (the graph $C_{n,k}$ is the tensor product of the n-cycle and the complete graph of size k with self-loops). We show a sharp transition in k, of the range of a random homomorphism of $C_{n,k}$. Namely, for any monotone function $\psi(n)$ tending to infinity, if $k = 2\log(n) - \psi(n)$ the range is $2^{\Omega(\psi(n))}$, with high probability, and if $k = 2\log(n) + \psi(n)$ the range is 3, with high probability. In particular, $C_{n,3\log n}$ is a graph of almost linear diameter and logarithmic degree such that the range of a random homomorphism of $C_{n,3\log n}$ is 3, with high probability.

The rest of this paper is organized as follows: We first introduce some notation. Section 2 contains our lower bound. Section 3 proves the upper and lower bounds on the range of random homomorphisms of the graph $C_{n,k}$. Section 4 lists some further possible research directions concerning random homomorphisms of graphs.

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1.1 Notation and Definitions

Logarithms are always of base 2. For an integer $k \in \mathbb{N}$, denote $[k] = \{1, \ldots, k\}$. For two integers $x, y \in \mathbb{Z}$, denote by $[x, y]$ the set of integers at least x and at most y. For $n \in \mathbb{N}$, denote by $\mathbb{Z}_n$ the additive group whose elements are $[0, n-1]$, and addition is modulo n.

1.1.1 Graphs

All graphs considered are simple and connected. Let G be a graph. For simplicity of notation, we use G to denote the vertex set of the graph G. In particular, we write $v \in G$, if v is a vertex of the graph G. For two vertices $v, u \in G$, we write $v \sim u \in G$, if $\{u, v\}$ is an edge in the graph G. When the graph is clear, we use $v \sim u$. The *size* of the graph G, denoted $|G|$, is the number of vertices in G. The *diameter* of G is the maximal distance between any two vertices in G. For a vertex $v \in G$ and an integer $r \in \mathbb{N}$, a *ball of radius r centered at v* is the subgraph of G induced by the set of all vertices at distance at most r from v.

1.1.2 Homomorphisms

For two graphs G and H, a *graph homomorphism from G to H* is a mapping $f : G \to H$ that preserves edges; that is, every two vertices $v \sim u \in G$ admit $f(v) \sim f(u) \in H$. For two vertices $v_0 \in G$ and $x_0 \in H$, we denote by $\mathrm{Hom}_{v_0}^{x_0}(G, H)$ the set of all homomorphisms f from G to H such that $f(v_0) = x_0$. A homomorphism from G to H is also called a *H-coloring of G*.

We denote by $\mathbb{Z}$ both the set of integers, and the graph whose vertex set is the integers and edge set is $\big\{\{z, z+1\} \;\big|\; z \in \mathbb{Z}\big\}$. We mostly consider $\mathrm{Hom}_{v_0}(G, \mathbb{Z}) \stackrel{\text{def}}{=} \mathrm{Hom}_{v_0}^{0}(G, \mathbb{Z})$. Note that

$$\mathrm{Hom}_{v_0}(G, \mathbb{Z}) = \big\{ f : G \to \mathbb{Z} \;\big|\; \forall\, u \sim v \in G \;\; |f(u) - f(v)| = 1 \text{ and } f(v_0) = 0 \big\} .$$

For a mapping $f : G \to \mathbb{Z}$, define

$$f(G) = \big\{ f(v) \;\big|\; v \in G \big\} \qquad \text{and} \qquad R(f) = |f(G)| .$$

We call both $f(G)$ and $R(f)$ the *range* of f. We use the notation $\in_R$ to denote an element chosen uniformly at random. E.g., $f \in_R \mathrm{Hom}_{v_0}(G, \mathbb{Z})$ is a random homomorphism from G to $\mathbb{Z}$ such that $f(v_0) = 0$, chosen uniformly at random. (When G is finite and connected, the set $\mathrm{Hom}_{v_0}(G, \mathbb{Z})$ is finite, and $f \in_R \mathrm{Hom}_{v_0}(G, \mathbb{Z})$ is well defined.) For example, consider the case where G is the interval of length n; that is

$$V(G) = [0, n] \qquad \text{and} \qquad E(G) = \big\{ \{i, i+1\} \;\big|\; 0 \le i \le n-1 \big\} .$$

Then, $\mathrm{Hom}_0(G, \mathbb{Z})$ is the set of all paths in $\mathbb{Z}$ starting from 0, of length n. Therefore, $f \in_R \mathrm{Hom}_0(G, \mathbb{Z})$ is a n-step random walk on $\mathbb{Z}$, starting at 0. Thus, for a general (connected and finite) graph G, a random homomorphism $f \in_R \mathrm{Hom}_{v_0}(G, \mathbb{Z})$, is also called a *$G$-indexed random walk*.

For a graph G, we say that a homomorphism f from G to itself is an *automorphism*, if f is invertible, and f^{-1} is a homomorphism as well. We say that a graph G is *vertex transitive*, if all the vertices of G "look" the same; that is, for any two vertices $v, u \in G$, there exists an automorphism f of G such that $f(v) = u$. We say that a graph G is *edge transitive*, if all the edges of G "look" the same; that is, for any two edges $\{v_1, v_2\}$ and $\{u_1, u_2\}$ in G, there exists an automorphism f of G such that $\{f(v_1), f(v_2)\} = \{u_1, u_2\}$.

2 Lower Bounds for Graphs with Small Degree

In this section we show that for graphs of 'small enough' degree, the range of a homomorphism is typically 'large'. In fact, we prove something slightly stronger:

Theorem 2.1. *Let $\{G_n\}$ be a family of graphs such that $\lim_{n\to\infty} |G_n| = \infty$. For $r \in \mathbb{N}$, define $V_n(r)$ to be the maximal size of a ball of radius r in G_n. Let $v_n \in G_n$ and let $f_n \in_R \mathrm{Hom}_{v_n}(G_n, \mathbb{Z})$ be a random homomorphism. Let $r = r(n) \in \mathbb{N}$. Assume that there exists a constant $c < 1$ such that every large enough $n \in \mathbb{N}$ admits $V_n(r) \le c \log |G_n|$. Then*

$$\Pr[R(f_n) \le r] = o(1).$$

We defer the proof of Theorem 2.1 to Section 2.4. First we discuss the tightness of Theorem 2.1. In Section 3 we consider the family of graphs $\{C_{n,k}\}$, where $n \in \mathbb{N}$ is even, and $k = k(n) \in \mathbb{N}$. For $n \in \mathbb{N}$, the size of $C_{n,k}$ is kn, and the size of a ball of radius 3 in $C_{n,k}$ is at most $7k$; that is, $V_n(3) \le 7k$. In Theorem 3.1 we show an upper bound on the range of a random homomorphism of $C_{n,k}$, for logarithmic k. More specifically, we show that for $k = 2\log n + \log\log\log n$,

$$\Pr[R(f_n) > 3] = o(1), \tag{1}$$

where $f_n \in_R \mathrm{Hom}_{(0,1)}(C_{n,k}, \mathbb{Z})$ is a random homomorphism. Thus, Theorem 2.1 is wrong if instead of $c < 1$ we use $c \le 14$. Indeed, assume towards a contradiction that Theorem 2.1 holds for every $c \le 14$. Since $V_n(3) \le 7k < 14\log(kn)$, by the assumption we have

$$\Pr[R(f_n) \le 3] = o(1),$$

where $f_n \in_R \mathrm{Hom}_{(0,1)}(C_{n,k}, \mathbb{Z})$ is a random homomorphism. This is a contradiction to (1). We note that since $C_{n,k}$ is vertex transitive (and edge transitive), Theorem 2.1 is tight in the above sense for vertex transitive graphs (and for edge transitive graphs).

2.1 Lower Bounds for Graphs with Small Degree

The following corollary of Theorem 2.1 shows that the range of a random homomorphism from a graph of "small" degree to $\mathbb{Z}$ is "large". For example, consider any family of graphs $\{G_n\}$, such that the degree of G_n is $\log\log |G_n|$. Then, the corollary states that the range of a random homomorphism from G_n to $\mathbb{Z}$ is super-constant (as n tends to infinity), with high probability.

Corollary 2.2. *Let $\{G_n\}$ be a family of graphs such that $\lim_{n\to\infty} |G_n| = \infty$. Let $n \in \mathbb{N}$, and let $d = d(n)$ be the maximal degree of G_n. Let $v_n \in G_n$ and let $f_n \in_R \mathrm{Hom}_{v_n}(G_n, \mathbb{Z})$ be a random homomorphism. Then*

$$\Pr\left[R(f_n) \le \frac{\log\log |G_n| - 1}{\log(d+1)}\right] = o(1).$$

Proof. For $r \in \mathbb{N}$, denote by $V_n(r)$ the maximal size of a ball of radius r in G_n. Since the maximal degree of G_n is $d = d(n)$, every $r \in \mathbb{N}$ admits $V_n(r) \le (d+1)^r$. Denote $r = r(n) = \frac{\log\log|G_n|-1}{\log(d+1)}$. Since $(d+1)^r = \frac{1}{2}\log|G_n|$, the corollary follows, by Theorem 2.1 (with $c = 1/2$). □

2.2 An Example - the Torus

A specific example for the use Theorem 2.1 is in the case of the torus. For an integer $n \in \mathbb{N}$, define the $n \times n$ torus, denoted T_n, as follows: The vertex set is $\mathbb{Z}_n \times \mathbb{Z}_n$, and the edge set is defined by the relations

$$\forall\, i,j \in \mathbb{Z}_n \quad (i,j) \sim (i+1,j) \quad (i,j) \sim (i,j+1)$$

(where addition is modulo n). Note that T_n is both vertex transitive and edge transitive. The following corollary shows that the range of a random homomorphism of the $n \times n$ torus is at least $\Omega(\log^{1/2} n)$, with high probability.

Corollary 2.3. *Let $n \in \mathbb{N}$, and let T_n be the $n \times n$ torus. Let $f_n \in_R \mathrm{Hom}_{(0,0)}(T_n, \mathbb{Z})$ be a random homomorphism. Then*

$$\Pr[R(f_n) > 1/2 \log^{1/2} n] = 1 - o(1).$$

Proof. Note that the size of T_n is n^2. For $r \in \mathbb{N}$, denote by $V_n(r)$ the maximal size of a ball of radius r in T_n. The vertex set of the ball of radius r centered at $(0,0)$ is contained in

$$\left\{(i,j) \in T_n \;\middle|\; i,j \in \{-r,\ldots,0,\ldots,r\}\,(\text{modulo } n)\right\}.$$

Thus, since T_n is vertex transitive, $V_n(r) \le (2r+1)^2$. So, since every large enough n admits $(2(1/2\log^{1/2} n)+1)^2 \le 2/3\log(n^2)$, using Theorem 2.1 (with $r = 1/2\log^{1/2} n$ and $c = 2/3$), we have $\Pr[R(f_n) > r] = 1 - o(1)$. □

2.3 A Ball of Radius r Has a Homomorphism with Range $r+1$

Before we prove Theorem 2.1, we need the observation of Lemma 2.4.

For a graph G and a vertex $v \in G$, we denote by $B_r(v)$ the ball of radius r centered at v. We say that $B_r(v)$ *is of exact radius* r, if there exists $u \in B_r(v)$ such that the distance between v and u in G is at least r.

Lemma 2.4. *Let G be a graph, let v be a vertex in G, and let $r \in \mathbb{N}$. For an integer $s \in [0,r]$, define $B_s = B_s(v)$ to be the ball centered at v with radius s. Set $B = B_r(v)$ and $\Gamma = \left\{u \in B \;\middle|\; u \notin B_{r-1}\right\}$ (the boundary of B). Let f be a homomorphism from B to $\mathbb{Z}$. Assume that B is of exact radius r. Then there exists a homomorphism g from B to $\mathbb{Z}$ such that $g|_\Gamma = f|_\Gamma$, and $R(g) \ge r+1$.*

Proof. Since a translation of a homomorphism is a homomorphism with the same range size, assume without loss of generality that $\min_{u\in\Gamma} f(u) = 0$. We demonstrate an iterative process such that at the i^{th} step (for $i = 0,\ldots,r$) we have a homomorphism g_i that admits

1. $g_i|_\Gamma = f|_\Gamma$.

2. The minimal value of g_i on the ball B_{r-i} is i.

Thus, setting $g = g_r$, we have: By property 1, we have $g|_\Gamma = f|_\Gamma$. By property 2, we have $\max_{u\in B} g(u) \geq r$. Thus, since $\min_{u\in\Gamma} g(u) = \min_{u\in\Gamma} f(u) = 0$, we have $R(g) \geq |[0,r]| = r+1$ (which completes the proof of the lemma).

For the first step, we define $g_0 = |f|$; that is, for $u \in B$, define $g_0(u) = |f(u)|$. Note that g_0 is a homomorphism from B to $\mathbb{Z}$. Since $\min_{u\in\Gamma} f(u) = 0$, it follows that $g_0|_\Gamma = f|_\Gamma$. Furthermore, every $u \in B$ admits $g_0(u) \geq 0$. Thus, g_0 has the two properties described above.

At the i^{th} step ($i > 0$), define g_i as follows

$$\forall\ u \in B \quad g_i(u) = \begin{cases} g_{i-1}(u) & u \notin B_{r-i} \\ g_{i-1}(u) & u \in B_{r-i} \text{ and } g_{i-1}(u) \neq i-1 \\ i+1 & u \in B_{r-i} \text{ and } g_{i-1}(u) = i-1 \end{cases}$$

Since $\Gamma \cap B_{r-i} = \emptyset$, by induction $g_i|_\Gamma = g_{i-1}|_\Gamma = f|_\Gamma$. Let u be a vertex in B_{r-i} such that $g_{i-1}(u) = i-1$. Let w be a vertex in B such that $w \sim u$. Then, $w \in B_{r-(i-1)}$. Hence, by induction, $g_{i-1}(w) \geq i-1$. So, since g_{i-1} is a homomorphism, $g_{i-1}(w) = i$, which implies $g_i(u) - g_i(w) = i+1-g_{i-1}(w) = 1$. Thus, g_i is indeed a homomorphism. Furthermore, $\min_{u\in B_{r-i}} g_i(u) \geq i$. So g_i satisfies the properties described above. □

2.4 Proof of Theorem 2.1

Fix $n \in \mathbb{N}$. Set $G = G_n$, $f = f_n$, $v_0 = v_n$, $r = r(n)$, and $S = V_n(r)$. We recall the following definitions. For $v \in G$, we denote by $B_r(v)$ the ball of radius r centered at v. We say that $B_r(v)$ *is of exact radius* r, if there exists $u \in B_r(v)$ such that the distance between v and u in G is at least r. The following claim describes the size of a collection of pairwise disjoint balls of exact radius r in G.

Claim 2.5. *Let $V \subseteq G$ be a set of vertices of size $|V| = k$. Then, there exists a set $U \subseteq V$ of size $|U| \geq \lfloor k/S^2 \rfloor$ such that*

1. *For all $u \in U$, the ball $B_r(u)$ is of exact radius r.*
2. *For all $u \neq u' \in U$, we have $B_r(u) \cap B_r(u') = \emptyset$.*

Proof. We prove the claim by induction on the size of V. Induction base: If $|V| < S^2$, then there is nothing to prove. Induction step: Assume $|V| \geq S^2$. If $r = 0$, then set $U = V$, and the claim follows. If $r > 0$, then $S > 1$, which implies $|V| > S$. Then, since S is the size of the maximal ball of radius r in G, there exist $v, v' \in V$ such that the distance between v and v' in G is at least r. Thus, $B_r(v)$ is of exact radius r.

Denote
$$B = \bigcup_{w\in B_r(v)} B_r(w).$$

Then, $|B| \leq S^2$. Denote $V' = V \setminus B$. Then, $|V'| \geq k - S^2$ and $|V'| < |V|$. By induction, there exists a set $U' \subseteq V'$ of size $|U'| \geq \lfloor |V'|/S^2 \rfloor \geq \lfloor k/S^2 \rfloor - 1$ such that

1. For all $u \in U'$, the ball $B_r(u)$ is of exact radius r.
2. For all $u \neq u' \in U'$, we have $B_r(u) \cap B_r(u') = \emptyset$.

Set $U = U' \cup \{v\}$. So, $U \subseteq V$ of size $|U| \geq \lfloor k/S^2 \rfloor$. To complete the proof it remains to show that for all $u \in U'$, we have $B_r(v) \cap B_r(u) = \emptyset$. Indeed, let $u \in U'$. Then, since $u \notin B$, the distance between u and $B_r(v)$ in G is more than r. Thus, $B_r(v) \cap B_r(u) = \emptyset$. $\square$

Returning to the proof of Theorem 2.1: By Claim 2.5, let $k = \lfloor |G|/S^2 \rfloor - 1$, and let $B_1, \ldots, B_k$ be a collection of pairwise disjoint balls of exact radius r in G such that for every $i \in [k]$, we have $v_0 \notin B_i$. Note that every $i \in [k]$ admits $|B_i| \leq S$.

Let $i \in [k]$, and let $g \in \text{Hom}_{v_0}(G \setminus B_i, \mathbb{Z})$ be a homomorphism that can be extended to a homomorphism in $\text{Hom}_{v_0}(G, \mathbb{Z})$. Denote by A_i the event $\{|f(B_i)| \leq r\}$ and by $E_{i,g}$ the event $\left\{ f|_{G \setminus B_i} = g \right\}$. Since g can be extended to a homomorphism in $\text{Hom}_{v_0}(G, \mathbb{Z})$, we have $E_{i,g} \neq \emptyset$. Since there are at most $2^{|B_i|} \leq 2^S$ homomorphisms $f' \in \text{Hom}_{v_0}(G, \mathbb{Z})$ such that f' agrees with g on $G \setminus B_i$, we have

$$|E_{i,g}| \leq 2^S.$$

Since $E_{i,g} \neq \emptyset$, since B_i is of exact radius r, and since $v_0 \notin B_i$, using Lemma 2.4, there exists a homomorphism $h \in \text{Hom}_{v_0}(G, \mathbb{Z})$ such that $|h(B_i)| \geq r + 1$ and h agrees with g on $G \setminus B_i$. That is, $h \notin A_i$ and $h \in E_{i,g}$, which implies $|A_i \cap E_{i,g}| \leq |E_{i,g}| - 1$. Hence,

$$\Pr\left[A_i \;\middle|\; E_{i,g}\right] = \frac{|A_i \cap E_{i,g}|}{|E_{i,g}|} \leq \frac{|E_{i,g}| - 1}{|E_{i,g}|} \leq 1 - 2^{-S}. \tag{2}$$

Note that for $j \neq i$, since $B_j \subseteq G \setminus B_i$, the event $E_{i,g}$ determines A_j; that is, $E_{i,g} \cap A_j$ is either $E_{i,g}$ or empty. Since (2) holds for any g such that $E_{i,g} \neq \emptyset$,

$$\begin{aligned}
\Pr\left[A_i \mid A_j : \ 1 \leq j < i\right] &= \sum_g \Pr\left[A_i \;\middle|\; A_j : \ 1 \leq j < i, \ E_{i,g}\right] \Pr\left[E_{i,g} | A_j : \ 1 \leq j < i\right] \\
&= \sum_g \Pr\left[A_i | E_{i,g}\right] \Pr\left[E_{i,g} | A_j : \ 1 \leq j < i\right] \\
&\leq 1 - 2^{-S}, \qquad (3)
\end{aligned}$$

where the sum is over all homomorphisms $g \in \text{Hom}_{v_0}(G \setminus B_i, \mathbb{Z})$ such that $E_{i,g} \cap \{A_j : 1 \leq j < i\} \neq \emptyset$ (which implies $E_{i,g} \cap \{A_j : 1 \leq j < i\} = E_{i,g}$). Since (3) holds for every $i \in [k]$, and since $S \leq c \log |G|$ (where $c < 1$),

$$\begin{aligned}
\Pr\left[R(f) \leq r\right] &\leq \Pr[\forall\, i \in [k] \ A_i] \\
&= \prod_{i=1}^{k} \Pr\left[A_i \mid A_j : \ 1 \leq j < i\right] \\
&\leq \left(1 - 2^{-S}\right)^k \leq \exp\left(-\frac{k}{2^S}\right) \\
&\leq e^2 \exp\left(-\frac{|G|^{1-c}}{c^2 \log^2 |G|}\right) = o(1),
\end{aligned}$$

where the last equality holds, since $\lim_{n \to \infty} |G_n| = \infty$. $\square$

3 The Cycle - $C_{n,k}$

In this section we study the range of a random homomorphism of the graph $C_{n,k}$, where $n, k \in \mathbb{N}$, and n is even. We consider the graph $C_{n,k}$ mainly for two reasons: First, for logarithmic k, the graph $C_{n,k}$ has almost linear diameter (the diameter of $C_{n,k}$ is $\Omega(n)$, while the size of $C_{n,k}$ is $O(n \log n)$), and still the range of a random homomorphism of $C_{n,k}$ is constant (the range is 3). Second, $C_{n,k}$ is both vertex transitive and edge transitive.

The graph $C_{n,k}$ is a cycle of n layers. Each layer has k vertices, and is connected to both its adjacent layers by a complete bi-partite graph. Thus, the degree of $C_{n,k}$ is $2k$. Formally, the vertex set of $C_{n,k}$ is $\mathbb{Z}_n \times [k]$, and the edge set of $C_{n,k}$ is defined by the relations

$$\forall\, i \in \mathbb{Z}_n \quad s, t \in [k] \quad (i, s) \sim (i + 1, t),$$

where addition is modulo n. ($C_{n,k}$ is also the tensor product of the n-cycle and the complete graph on k vertices with self-loops.) Denote by $\mathcal{H}_{n,k} = \mathrm{Hom}_{(0,1)}(C_{n,k}, \mathbb{Z})$, the set of homomorphisms from $C_{n,k}$ to $\mathbb{Z}$ that map $(0, 1)$ to 0. Since n is even, $C_{n,k}$ is bi-partite, which implies that $\mathcal{H}_{n,k} \neq \emptyset$.

We show a threshold phenomena (with respect to k) concerning the range of a random homomorphism from $C_{n,k}$ to $\mathbb{Z}$. More precisely, for $k(n) = 2 \log n + \omega(1)$, the range of a random homomorphism is at most 3, with high probability, and on the other hand for $k(n) = 2 \log n - \omega(1)$, the range of a random homomorphism is super-constant, with high probability. The following two theorems make the above statements precise.

Theorem 3.1. *Let $n \in \mathbb{N}$ be even, and let $k = k(n) = 2 \log n + \psi(n)$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is such that $\lim_{n\to\infty} \psi(n) = \infty$. Let $f_n \in_R \mathcal{H}_{n,k}$ be a random homomorphism. Then*

$$\Pr[R(f_n) \leq 3] = 1 - o(1).$$

Theorem 3.2. *Let $n \in \mathbb{N}$ be even, and let $k = k(n) = 2 \log n - \psi(n)$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is monotone and $\lim_{n\to\infty} \psi(n) = \infty$. Let $f_n \in_R \mathcal{H}_{n,k}$ be a random homomorphism. Then*

$$\Pr\left[R(f_n) \geq \frac{2^{\psi(n-2)/4}}{\psi(n)}\right] = 1 - o(1).$$

For the rest of this section we prove the above theorems. The proof of Theorem 3.1 is deferred to Section 3.4, and the proof of Theorem 3.2 is deferred to Section 3.6. We note that for $k = 1$, we have that $C_{n,k}$ is the n-cycle. Thus, $f_n \in_R \mathcal{H}_{n,1}$ is a random walk bridge of length n (a random walk bridge is a random walk conditioned on returning to 0). In this case, Theorem 3.2 gives the bound

$$\Pr\left[R(f_n) \geq \Omega\left(\frac{\sqrt{n}}{\log n}\right)\right] = 1 - o(1).$$

This is consistent with the range of a random walk bridge (see also Remark 3.17).

3.1 Definitions

Let $n, k \in \mathbb{N}$, where n is even. For $i \in \mathbb{Z}_n$, the *i-layer* in $C_{n,k}$ is the set of vertices $\{i\} \times [k]$. Recall that $\mathcal{H}_{n,k} = \mathrm{Hom}_{(0,1)}(C_{n,k}, \mathbb{Z})$ is the set of homomorphisms from $C_{n,k}$ to $\mathbb{Z}$ that map

$(0,1)$ to 0. Denote by $\mathcal{H}^0_{n,k}$ the set of homomorphisms from $C_{n,k}$ to $\mathbb{Z}$ that map the 0-layer to 0; that is,

$$\mathcal{H}^0_{n,k} = \left\{ f \in \mathcal{H}_{n,k} \;\middle|\; f(\{0\} \times [k]) = \{0\} \right\}.$$

When n and k are clear we use $\mathcal{H}^0 = \mathcal{H}^0_{n,k}$.

For $f \in \mathcal{H}_{n,k}$ and $i \in \mathbb{Z}_n$, we say that the i-layer is *constant* in f, if f gets the same value on the entire i-layer; that is, $|f(\{i\} \times [k])| = 1$. We say that the i-layer is *non-constant* in f, if f gets different values on the i-layer; that is, $|f(\{i\} \times [k])| > 1$. Define $\mathcal{NC}(f)$ to be the set of non-constant layers in f; that is,

$$\mathcal{NC}(f) = \left\{ i \in \mathbb{Z}_n \;\middle|\; |f(\{i\} \times [k])| > 1 \right\}.$$

For $\ell \in [0, n]$, define $\mathcal{H}^0(\ell) = \mathcal{H}^0_{n,k}(\ell)$ to be the set of homomorphisms in $\mathcal{H}^0$ that have exactly ℓ non-constant layers.

Loosely speaking, a homomorphism of $C_{n,k}$ corresponds to a path on $\mathbb{Z}$ that starts at 0 and ends at 0. This motivates the following definition. For an even integer $m \in \mathbb{N}$, denote by $\mathcal{P}(m)$ the set of paths of length m on $\mathbb{Z}$ that start at 0 and end at 0; that is,

$$\mathcal{P}(m) = \left\{ (S_0, S_1, \ldots, S_m) \in \mathbb{Z}^m \;\middle|\; \forall\, i \in [m] \quad |S_i - S_{i-1}| = 1 \text{ and } S_0 = S_m = 0 \right\}.$$

Note that $|\mathcal{P}(m)| = \binom{m}{m/2}$.

Consider the values of a homomorphism on the 1-layer. Since all vertices in the 1-layer are connected to a vertex that is mapped to 0, the value of a homomorphism on the 1-layer corresponds to a vector in $\{1, -1\}^k$. In fact, it turns out that the value of a homomorphism on a non-constant layer corresponds to a $\{1, -1\}^k$ non-constant vector. Thus, the following definition will be useful. Define

$$V = V_k = \{1, -1\}^k \setminus \{(1, 1, \ldots, 1), (-1, -1, \ldots, -1)\}.$$

3.2 The Constant Layers

In this section we show some properties of the constant layers. First, we show that homomorphisms in $\mathcal{H}_{n,k}$ do not have two adjacent non-constant layers. Second, we show that if the 0-layer is non-constant in a homomorphism $f \in \mathcal{H}_{n,k}$, then we can think of f as a homomorphism in $\mathcal{H}^0_{n-2,k}$. Third, we show that, conditioned on a specific set of ℓ non-constant layers, a random homomorphism in $\mathcal{H}^0_{n,k}$ corresponds to a random walk bridge of length $n - 2\ell$ (i.e., a random walk of length $n - 2\ell$ on $\mathbb{Z}$ that starts at 0 and ends at 0).

3.2.1 No Two Adjacent Non-constant Layers

Claim 3.3. *Let $f \in \mathcal{H}_{n,k}$ be a homomorphism. Assume that $i \in \mathbb{Z}_n$ is such that the i-layer is non-constant in f. Then there exists $z \in \mathbb{Z}$ such that*

$$f(\{i+1\} \times [k]) = f(\{i-1\} \times [k]) = \{z\}.$$

In particular, both the $(i+1)$-layer and the $(i-1)$-layer are constant in f.

Proof. We prove the claim for the $(i+1)$-layer. The proof for the $(i-1)$-layer is similar. Since the i-layer is non-constant in f, there exist $s, t \in [k]$ such that $f(i,s) < f(i,t)$. Recall that every $q \in [k]$ admits $(i+1,q) \sim (i,s)$ and $(i+1,q) \sim (i,t)$. Thus, every $q \in [k]$ admits $f(i+1,q) = f(i,s)+1 = f(i,t)-1$. Setting $z = f(i,s)+1$, the claim follows. □

3.2.2 If the 0-layer is non-constant in f, we can think of f as a homomorphism of a smaller graph

Let $f \in \mathcal{H}_{n,k} \setminus \mathcal{H}^0_{n,k}$ be a homomorphism. That is, the 0-layer is non-constant in f. Define $f_\downarrow$ as follows:

$$\forall\ i \in \mathbb{Z}_{n-2} \quad s \in [k] \quad f_\downarrow(i,s) = f(i+1,s) - f(1,1).$$

Claim 3.4. *Let $f \in \mathcal{H}_{n,k} \setminus \mathcal{H}^0_{n,k}$ be a homomorphism. Then $f_\downarrow$ is a homomorphism in $\mathcal{H}^0_{n-2,k}$.*

Proof. Since the 0-layer is non-constant in f, by Claim 3.3, there exists $z \in \{1,-1\}$ such that

$$f(\{1\} \times [k]) = f(\{n-1\} \times [k]) = \{z\}\,. \tag{4}$$

First, we show that $f_\downarrow$ is a homomorphism of $C_{n-2,k}$. Indeed, for all $i \in [0, n-4]$ and for all $s, t \in [k]$, we have $f_\downarrow(i+1,s) - f_\downarrow(i,t) = f(i+2,s) - f(i+1,t) \in \{1,-1\}$. Furthermore, for all $s, t \in [k]$, by (4), we have

$$f_\downarrow(0,s) - f_\downarrow(n-3,t) = f(1,s) - f(n-2,t) = f(n-1,1) - f(n-2,t) \in \{1,-1\}\,.$$

Second, for all $s \in [k]$, we have $f_\downarrow(0,s) = f(1,s) - f(1,1) = z - z = 0$. □

In fact, the following claim holds.

Claim 3.5. *Let $f \in_R \mathcal{H}_{n,k} \setminus \mathcal{H}^0_{n,k}$ be a random homomorphism. Then $f_\downarrow$ is uniformly distributed in $\mathcal{H}^0_{n-2,k}$.*

Proof. We will show that the mapping

$$\text{from} \quad \mathcal{H}_{n,k} \setminus \mathcal{H}^0_{n,k} \quad \text{to} \quad \mathcal{H}^0_{n-2,k} \times \{1,-1\} \times \Big(\{0,2\}^{k-1} \setminus \{(0,\dots,0)\}\Big)$$

defined by

$$f \mapsto (f_\downarrow, f(1,1), f(1,1)\cdot f(0,2), f(1,1)\cdot f(0,3), \dots, f(1,1)\cdot f(0,k))$$

is a bijection, where $f_\downarrow \in \mathcal{H}^0_{n-2,k}$ is the homomorphism defined above, $f(1,1) \in \{1,-1\}$, and $(f(1,1)\cdot f(0,2), f(1,1)\cdot f(0,3), \dots, f(1,1)\cdot f(0,k)) \in \{0,2\}^{k-1}$ is a non-zero vector.

First, the mapping is injective. Indeed, let $f^1 \neq f^2$ be two homomorphisms in $\mathcal{H}_{n,k} \setminus \mathcal{H}^0_{n,k}$. If $f^1(1,1) \neq f^2(1,1)$, then the images of f^1 and f^2 are different (in the second coordinate). Otherwise, assume that

$$f^1(1,1) = f^2(1,1). \tag{5}$$

There exist $i \in \mathbb{Z}_n$ and $s \in [k]$ such that $f^1(i,s) \neq f^2(i,s)$. If either $i = 1$ or $i = n-1$, since the 0-layer is non-constant in both f^1 and f^2, using Claim 3.3, then $f^1(1,1) = f^1(i,s) \neq f^2(i,s) =$

$f^2(1,1)$ (contradicting (5)). Otherwise, if $i=0$, then $f^1(1,1)\cdot f^1(0,s)\neq f^2(1,1)\cdot f^2(0,s)$, for $s>1$ (since $f^1(0,1)=f^2(0,1)=0$), implying that the images of f^1 and f^2 are different. Finally, if $i\in[2,n-2]$, we have $f^1_\downarrow(i-1,s)=f^1(i,s)-f^1(1,1)\neq f^2(i,s)-f^2(1,1)=f^2_\downarrow(i-1,s)$, so the images of f^1 and f^2 are different (in the first coordinate).

Second, the mapping is surjective. Indeed, given a homomorphism $g\in\mathcal{H}^0_{n-2,k}$, an integer $z\in\{1,-1\}$, and a non-zero vector $(v_2,\ldots,v_k)\in\{0,2\}^{k-1}$, define

$$\forall\ i\in\mathbb{Z}_n\ s\in[k]\quad f(i,s)=\begin{cases}0 & i=0,s=1\\ z\cdot v_s & i=0,s\neq 1\\ z & i=n-1\\ g(i-1,s)+z & i\in[1,n-2].\end{cases}$$

Thus, for every $i\in\mathbb{Z}_n$ and $s,t\in[k]$, (recall that $g(\{0\}\times[k])=\{0\}$),

$$f(i+1,s)-f(i,t)=\begin{cases}g(0,s)+z-0=z\in\{1,-1\} & i=0,t=1\\ g(0,s)+z-z\cdot v_t\in\{1,-1\} & i=0,t\neq 1\\ 0-z\in\{1,-1\} & i=n-1,s=1\\ z\cdot v_s-z\in\{1,-1\} & i=n-1,s\neq 1\\ z-(g(n-3,t)+z)\in\{1,-1\} & i=n-2\\ g(i,s)+z-(g(i-1,t)+z)\in\{1,-1\} & i\in[1,n-3],\end{cases}$$

which implies $f\in\mathcal{H}_{n,k}$. Furthermore, since $(v_2,\ldots,v_k)$ is a non-zero vector, it follows that $f(\{0\}\times[k])=\{0,2z\}$, which implies $f\notin\mathcal{H}^0_{n,k}$. Finally, we will show that $f\mapsto(g,z,v_2,\ldots,v_k)$. Indeed, for all $i\in\mathbb{Z}_{n-2}$ and $s\in[k]$, we have $f_\downarrow(i,s)=f(i+1,s)-f(1,1)=g(i,s)+z-(g(0,1)+z)=g(i,s)$. Also $f(1,1)=g(0,1)+z=z$, and for all $s\in[2,k]$, we have $f(1,1)\cdot f(0,s)=z\cdot z\cdot v_s=v_s$.

The size of the range of the above defined mapping is $\left|\mathcal{H}^0_{n-2,k}\right|\cdot 2\cdot(2^{k-1}-1)$. Therefore, for every $g\in\mathcal{H}^0_{n-2,k}$,

$$\Pr\left[f_\downarrow=g\right]=\frac{2\cdot(2^{k-1}-1)}{\left|\mathcal{H}_{n,k}\setminus\mathcal{H}^0_{n,k}\right|}=\frac{1}{\left|\mathcal{H}^0_{n-2,k}\right|}.$$

□

3.2.3 Conditioned on the Set of Non-constant Layers, a Random Homomorphism is a Random Walk Bridge

Let

$$I=\{i_1<\cdots<i_\ell\}\subseteq[n-1]$$

be a set of size ℓ such that for every $i\in[n-2]$, either $i\notin I$ or $i+1\notin I$ (or both). We think of I as a set of non-constant layers (recall Claim 3.3).

Denote by $\mathcal{H}(I,n)$ the set of homomorphisms f in $\mathcal{H}^0_{n,k}$ such that I is the set of non-constant layers in f (we think of k as fixed). Recall that $\mathcal{P}(n-2\ell)$ is the set of paths on $\mathbb{Z}$ of length $n-2\ell$ that start at 0 and end at 0, and recall that

$$V=\{1,-1\}^k\setminus\{(1,1,\ldots,1),(-1,-1,\ldots,-1)\}.$$

For a homomorphism $f \in \mathcal{H}_{n,k}$, define the *range of the constant layers in* f to be

$$\mathbf{RC}(f) = \left\{ f(i,1) \;\middle|\; i \in \mathbb{Z}_n \text{ is such that the } i\text{-layer is constant in } f \right\}.$$

For a path $(S_0, S_1, \ldots, S_{n-2\ell})$ in $P(n-2\ell)$, define the *range of the path* to be

$$\mathbf{Rng}(S_0, S_1, \ldots, S_{n-2\ell}) = \left\{ S_i \;\middle|\; 0 \le i \le n-2\ell \right\}.$$

Loosely speaking, the following proposition shows that, conditioned on the set of non-constant layers, a random homomorphism in $\mathcal{H}^0$ is a random walk bridge.

Proposition 3.6. *Let $I = \{i_1 < i_2 < \cdots < i_\ell\} \subseteq [n-1]$ be a set of size ℓ such that for all $i \in [n-2]$, either $i \notin I$ or $i+1 \notin I$. Then there exists a bijection φ between $\mathcal{H}(I,n)$ and $P(n-2\ell) \times V^\ell$. Furthermore, denote $\varphi = (\varphi^1, \varphi^2)$. Then for all $f \in \mathcal{H}(I,n)$,*

$$\mathbf{RC}(f) = \mathbf{Rng}(\varphi^1(f)).$$

For the rest of this section we prove Proposition 3.6.

3.2.4 Proof of Proposition 3.6

We prove the proposition by induction on ℓ. The induction step is based on the following claim. The claim shows that given a non-constant layer in a homomorphism f of $C_{n,k}$, we can think of f as a homomorphism of $C_{n-2,k}$.

In what follows, for simplicity, we use the following convention: For a homomorphism $f \in \mathcal{H}_{n,k}$, and integers $i \in \mathbb{N}$ and $s \in [k]$, we define $f(i,s) = f(i \pmod n, s)$.

Claim 3.7. *Let $I = \{i_1 < \cdots < i_\ell\} \subseteq [n-1]$ be a set of size ℓ such that for every $i \in [n-2]$, either $i \notin I$ or $i+1 \notin I$. Let $f \in \mathcal{H}(I,n)$ be a homomorphism. Define $f' \in \mathcal{H}(I \setminus \{i_\ell\}, n-2)$ by*

$$\forall\, i \in [0, n-3]\ s \in [k] \quad f'(i,s) = \begin{cases} f(i,s) & i < i_\ell \\ f(i+2,s) & i \ge i_\ell. \end{cases}$$

Define $v_f \in V$ by

$$\forall\, s \in [k] \quad v_f(s) = f(i_\ell, s) - f(i_\ell - 1, s).$$

Then the map $\Phi : \mathcal{H}(I,n) \to \mathcal{H}(I \setminus \{i_\ell\}, n-2) \times V$ defined by $\Phi(f) = (f', v_f)$ is a bijection.

Proof. First we show that f' and v_f are well defined:

For f', choose some $i \in [0, n-3]$ and $s, t \in [k]$. If $i+1 < i_\ell$, then

$$f'(i+1,s) - f'(i,t) = f(i+1,s) - f(i,t) \in \{1,-1\}.$$

If $i \ge i_\ell$ then

$$f'(i+1,s) - f'(i,t) = f(i+3,s) - f(i+2,t) \in \{1,-1\}.$$

We are left with the case $i = i_\ell - 1$. Since the i_ℓ-layer is non-constant in f, by Claim 3.3, $f(i+2,t) = f(i,t)$. Thus,

$$f'(i+1,s) - f'(i,t) = f(i+3,s) - f(i,t) = f(i+3,s) - f(i+2,t) \in \{1,-1\}.$$

So $f' \in \mathcal{H}_{n-2,k}$. Since $i_\ell > 0$, for all $s \in [k]$, we have $f'(0,s) = f(0,s) = 0$. Thus, $f' \in \mathcal{H}^0_{n-2,k}$. Also, for any layer $i < i_\ell$, since $f'(\{i\} \times [k]) = f(\{i\} \times [k])$, we get that for any $i < i_\ell$,

$$\text{the } i\text{-layer is constant in } f' \quad \Leftrightarrow \quad \text{the } i\text{-layer is constant in } f.$$

For any $i \geq i_\ell$, we have that $f'(\{i\} \times [k]) = f(\{i+2\} \times [k])$. Since $i+2 > i_\ell$, the $(i+2)$-layer is constant in f, which implies that the i-layer is constant in f'. So, the set of non-constant layers in f' is the set $I \setminus \{i_\ell\}$, and $f' \in \mathcal{H}(I \setminus \{i_\ell\}, n-2)$.

Now we show that v_f is well defined: Since the i_ℓ-layer is non-constant in f, using Claim 3.3, there exist $s, t \in [k]$ such that $v_f(s) \neq v_f(t)$, so v_f is in V.

To show that Φ is a bijection, we provide the inverse map $\Psi = \Phi^{-1}$. Define

$$\Psi : \mathcal{H}(I \setminus \{i_\ell\}, n-2) \times V \to \mathcal{H}(I, n)$$

as follows: For a pair $g' \in \mathcal{H}(I \setminus \{i_\ell\}, n-2)$ and $v \in V$, define $g \in \mathcal{H}(I, n)$ by

$$\forall\, i \in [0, n-1]\ s \in [k] \quad g(i,s) = \begin{cases} g'(i,s) & i < i_\ell \\ g'(i-1,s) + v(s) & i = i_\ell \\ g'(i-2,s) & i \geq i_\ell + 1, \end{cases}$$

and define $\Psi(g', v) = g$.

Claim. $g \in \mathcal{H}(I, n)$.

Proof. Choose some $i \in [0, n-1]$ and $s, t \in [k]$. If $i + 1 < i_\ell$, then

$$g(i+1, s) - g(i,t) = g'(i+1, s) - g'(i,t) \in \{1, -1\}.$$

If $i \geq i_\ell + 1$, then

$$g(i+1, s) - g(i,t) = g'(i-1, s) - g'(i-2,t) \in \{1, -1\}.$$

If $i = i_\ell - 1$, then, using Claim 3.3,

$$g(i+1, s) - g(i,t) = g'(i, s) + v(s) - g'(i,t) = v(s) \in \{1, -1\}.$$

If $i = i_\ell$, then, using Claim 3.3,

$$g(i+1, s) - g(i,t) = g'(i-1, s) - (g'(i-1,t) + v(t)) \in \{1, -1\}.$$

So $g \in \mathcal{H}_{n,k}$.

Now, since $i_\ell > 0$, we have that $g(\{0\} \times [k]) = g'(\{0\} \times [k]) = \{0\}$. So $g \in \mathcal{H}^0_{n,k}$.

Finally, for $i < i_\ell$, we have that $g(\{i\} \times [k]) = g'(\{i\} \times [k])$. Thus, for $i < i_\ell$, we have that

$$\text{the } i\text{-layer is constant in } g \quad \Leftrightarrow \quad \text{the } i\text{-layer is constant in } g'.$$

If $i > i_\ell$, then $g(\{i\} \times [k]) = g'(\{i-2\} \times [k])$. Since $i - 2 > i_\ell - 2$, we have $i - 2 \notin I \setminus \{i_\ell\}$. Hence, the $(i-2)$-layer is constant in g', and so the i-layer is constant in g. If $i = i_\ell$, then, since $i - 1 \notin I \setminus \{i_\ell\}$, the $(i-1)$-layer is constant in g'. Also

$$v \notin \{(1, 1, \ldots, 1), (-1, -1, \ldots, -1)\}.$$

So we have that the i-layer is non-constant in g (since $g(\{i\} \times [k]) = \{g'(i-1,1)-1, g'(i-1,1)+1\}$).

Thus, the set of non-constant layers of g is

$$(I \setminus \{i_\ell\}) \cup \{i_\ell\} = I.$$

So $g \in \mathcal{H}(I, n)$, proving the claim. □

Now, let $f \in \mathcal{H}(I, n)$ and let $g = \Psi(\Phi(f))$. Let $i \in [0, n-1]$ and let $s \in [k]$. If $i - 2 = i_\ell - 1$, then since the i_ℓ-layer is non-constant in f, using Claim 3.3,

$$f'(i-2, s) = f(i_\ell - 1, s) = f(i_\ell + 1, s) = f(i, s).$$

If $i - 2 > i_\ell - 1$, then $f'(i-2, s) = f(i, s)$. So, if $i - 2 \geq i_\ell - 1$, we have that $f'(i-2, s) = f(i, s)$. Thus, for all $i \in [0, n-1]$ and $s \in [k]$,

$$g(i,s) = \begin{cases} f'(i,s) = f(i,s) & i < i_\ell \\ f'(i-1,s) + v_f(s) = f(i-1,s) + f(i_\ell, s) - f(i_\ell - 1, s) & i = i_\ell \\ f'(i-2,s) = f(i,s) & i \geq i_\ell + 1. \end{cases}$$

So $g \equiv f$, and $\Psi = \Phi^{-1}$. Thus, Φ is a bijection as claimed. □

Claim 3.8. *Let $\Phi : \mathcal{H}(I, n) \to \mathcal{H}(I \setminus \{i_\ell\}, n-2) \times V$ be the bijection defined in Claim 3.7. Let $f \in \mathcal{H}(I, n)$ be a homomorphism, and let $(f', v_f) = \Phi(f)$. Then*

$$\mathbf{RC}(f) = \mathbf{RC}(f').$$

Proof. Let $z \in \mathbf{RC}(f)$. So there exists $0 \leq i \leq n-1$ such that $f(i, 1) = z$ and the i-layer is constant in f.

If $i < i_\ell$, then $f'(\{i\} \times [k]) = f(\{i\} \times [k]) = \{z\}$, and so the i-layer is constant in f'. Thus, $z = f'(i, 1) \in \mathbf{RC}(f')$.

We can exclude the case $i = i_\ell$, because the i_ℓ-layer is non-constant in f.

If $i = i_\ell + 1$, then using Claim 3.3, for all $s \in [k]$, since the i-layer is constant in f,

$$f'(i-2, s) = f(i_\ell - 1, s) = f(i_\ell + 1, s) = f(i, s) = z.$$

So, the $(i-2)$-layer is constant in f', and $z = f'(i-2, 1) \in \mathbf{RC}(f')$.

If $i > i_\ell + 1$, then $f'(\{i-2\} \times [k]) = f(\{i\} \times [k]) = \{z\}$. So, the $(i-2)$-layer is constant in f', and $z = f'(i-2, 1) \in \mathbf{RC}(f')$.

This establishes $\mathbf{RC}(f) \subseteq \mathbf{RC}(f')$.

Now let $z \in \mathbf{RC}(f')$. So, there exists $0 \leq i \leq n-3$ such that $f'(i, 1) = z$ and the i-layer is constant in f'.

If $i < i_\ell$, then $f(\{i\} \times [k]) = f'(\{i\} \times [k]) = \{z\}$. So, the i-layer is constant in f, and $z = f(i, 1) \in \mathbf{RC}(f)$.

If $i \geq i_\ell$, then $f(\{i+2\} \times [k]) = f'(\{i\} \times [k]) = \{z\}$. So, the $(i+2)$-layer is constant in f and $z = f(i+2, 1) \in \mathbf{RC}(f)$.

So, $\mathbf{RC}(f') \subseteq \mathbf{RC}(f)$, which implies $\mathbf{RC}(f) = \mathbf{RC}(f')$. □

Back to the proof of Proposition 3.6. Loosely speaking, we define φ to be ℓ compositions of Φ, where Φ is the map defined in Claim 3.7.

Let $f \in \mathcal{H}(I,n)$ be a homomorphism. For an integer $j \in [1,\ell]$, define the set

$$I_j = \{i_1 < i_2 < \cdots < i_{\ell-j}\}\,.$$

Define ℓ homomorphisms $f_1,\ldots,f_\ell$ and ℓ vectors $v_1,\ldots,v_\ell$ inductively as follows: $(f_1,v_1) = \Phi(f)$, and $(f_j,v_j) = \Phi(f_{j-1})$. By Claim 3.7, for every $j \in [1,\ell]$, we have $f_j \in \mathcal{H}(I_j, n-2j)$. Since $I_\ell = \emptyset$, we have $f_\ell \in \mathcal{H}(\emptyset, n-2\ell)$. By Claims 3.7 and 3.8, the map

$$f \mapsto (f_\ell, v_1, \ldots, v_\ell)$$

is a bijection, and

$$\mathbf{RC}(f) = \mathbf{RC}(f_1) = \cdots = \mathbf{RC}(f_\ell). \tag{6}$$

We claim that there exists a bijection $\psi : \mathcal{H}(\emptyset, n-2\ell) \to P(n-2\ell)$ such that for all $g \in \mathcal{H}(\emptyset, n-2\ell)$,

$$\mathbf{RC}(g) = \mathbf{Rng}(\psi(g)). \tag{7}$$

Indeed, let $g \in \mathcal{H}(\emptyset, n-2\ell)$. Then, $\psi(g) = (g(0,1), g(1,1), \ldots, g(n-2\ell,1))$ is a path of length $n-2\ell$ in $\mathbb{Z}$, starting at 0 and ending at 0. Furthermore, (7) holds. Note that homomorphisms in $\mathcal{H}(\emptyset, n-2\ell)$ do not have non-constant layers. So given a path $(S_0, S_1, \ldots, S_{n-2\ell})$ in $P(n-2\ell)$, we can define $g' \in \mathcal{H}(\emptyset, n-2\ell)$ by

$$\forall\, i \in \mathbb{Z}_{n-2\ell}\ s \in [k] \quad g'(i,s) = S_i.$$

Since $\psi(g') = (S_0, S_1, \ldots, S_{n-2\ell})$, it follows that ψ is a bijection.

Finally, define φ as follows:

$$\varphi : \mathcal{H}(I,n) \to P(n-2\ell) \times V^\ell \qquad \varphi(f) = ((S_0,\ldots,S_{n-2\ell}), (v_1,\ldots,v_\ell)),$$

where $(S_0,\ldots,S_{n-2\ell}) = \psi(f_\ell)$. By (6) and (7), we have $\mathbf{RC}(f) = \mathbf{Rng}(S_0,\ldots,S_{n-2\ell})$. □

3.3 The Size of $\mathcal{H}^0$

Fix $n,k \in \mathbb{N}$ (n is even) for the rest of this section. We consider $\mathcal{H}^0 = \mathcal{H}^0_{n,k}$. Note that, by Claim 3.3, every $f \in \mathcal{H}^0$ has at most $n/2$ non-constant layers. Thus,

$$\left|\mathcal{H}^0\right| = \sum_{\ell=0}^{n/2} \left|\mathcal{H}^0(\ell)\right|, \tag{8}$$

where $\mathcal{H}^0(\ell)$ is the set of homomorphisms of $C_{n,k}$ that have exactly ℓ non-constant layers.

The following lemma gives a formula for the size of $\mathcal{H}^0(\ell)$.

Lemma 3.9. *Let $\ell \in [0, n/2]$. Then*

$$\left|\mathcal{H}^0(\ell)\right| = \binom{n-\ell}{\ell}\binom{n-2\ell}{n/2-\ell}(2^k-2)^\ell.$$

Proof. If $k = 1$, then

$$\mathcal{H}^0(\ell) = \begin{cases} \binom{n}{n/2} & \ell = 0 \\ 0 & \ell > 0, \end{cases}$$

proving the lemma. So assume that $k > 1$. Define a family of sets

$$\mathcal{I} = \left\{ I \subseteq [n-1] \;\middle|\; |I| = \ell \text{ and for every } i \in [n-2], \text{ either } i \notin I \text{ or } i+1 \notin I \right\}.$$

Let $f \in \mathcal{H}^0(\ell)$ be a homomorphism. Recall that $\mathcal{NC}(f)$ is the set of non-constant layers of f. Since $f \in \mathcal{H}^0$, we have $0 \notin \mathcal{NC}(f)$, which implies that $\mathcal{NC}(f) \subseteq [n-1]$. Since $f \in \mathcal{H}^0(\ell)$, we have $|\mathcal{NC}(f)| = \ell$. Using Claim 3.3, for every $i \in [n-2]$, either $i \notin \mathcal{NC}(f)$ or $i+1 \notin \mathcal{NC}(f)$. Therefore, $\mathcal{NC}(f) \in \mathcal{I}$. Furthermore, for every set $I \in \mathcal{I}$, there exists a homomorphism $g \in \mathcal{H}^0(\ell)$ such that $\mathcal{NC}(g) = I$ (since $k > 1$). Hence,

$$\mathcal{I} = \left\{ \mathcal{NC}(f) \;\middle|\; f \in \mathcal{H}^0(\ell) \right\}. \tag{9}$$

Define a map $\rho : \mathcal{I} \to \binom{[n-\ell]}{\ell}$ (where $\binom{[n-\ell]}{\ell}$ is the family of subsets of $[n-\ell]$ of size ℓ) as follows:

$$\forall\, I = \{i_1 < \cdots < i_\ell\} \in \mathcal{I} \quad \rho(I) = \{i_1 < i_2 - 1 < i_3 - 2 < \cdots < i_\ell - (\ell - 1)\}.$$

For $I = \{i_1 < \cdots < i_\ell\} \in \mathcal{I}$, since $|I| = \ell$ and since for every $i \in [n-2]$, either $i \notin I$ or $i+1 \notin I$, the set $\rho(I)$ is a subset of $[n-\ell]$ of size ℓ. So ρ is well defined.

Claim 3.10. *The map ρ is a bijection between $\mathcal{I}$ and $\binom{[n-\ell]}{\ell}$.*

Proof. For a set $J = \{j_1 < \cdots < j_\ell\} \subseteq [n-\ell]$, define the map ρ^{-1} by

$$\rho^{-1}(J) = \{j_1 < j_2 + 1 < \cdots < j_\ell + (\ell - 1)\}.$$

Thus, $\rho^{-1}(J)$ is of size ℓ and for every $i \in [n-2]$, either $i \notin \rho^{-1}(J)$ or $i+1 \notin \rho^{-1}(J)$, which implies $\rho^{-1}(J) \in \mathcal{I}$. Since every $I \in \mathcal{I}$ admits $\rho^{-1}(\rho(I)) = I$, it follows that ρ is a bijection. □

Back to the proof of Lemma 3.9. By (9),

$$\left|\mathcal{H}^0(\ell)\right| = \sum_{I \in \mathcal{I}} |\mathcal{H}(I, n)|,$$

where $\mathcal{H}(I,n)$ is the set of homomorphisms $f \in \mathcal{H}^0$ such that $\mathcal{NC}(f) = I$. By Proposition 3.6, for every $I \in \mathcal{I}$,

$$|\mathcal{H}(I,n)| = |\mathcal{P}(n-2\ell)|\,|V|^\ell = \binom{n-2\ell}{n/2-\ell}(2^k-2)^\ell.$$

By Claim 3.10,

$$|\mathcal{I}| = \binom{n-\ell}{\ell}.$$

So the lemma follows. □

3.4 Upper Bound for $k = 2\log n + \omega(1)$ - Theorem 3.1

In this part we show that for $k = 2\log n + \omega(1)$, the range of a random homomorphism from $C_{n,k}$ to $\mathbb{Z}$ is 3, with high probability. We use the formula for $\left|\mathcal{H}^0(\ell)\right|$ to conclude that f has $n/2$ non-constant layers, with high probability. Which implies that f is "almost" constant.

3.4.1 Many Non-constant Layers

To prove Theorem 3.1, we use the following lemma, which states that there are $n/2$ non-constant layers in a random homomorphism of $C_{n,k}$. Note that, by Claim 3.3, the maximal number of non-constant layers in every homomorphism of $C_{n,k}$ is $n/2$.

Lemma 3.11. *Let $n \in \mathbb{N}$ be even, and let $k = k(n) \geq 2\log n + \psi(n)$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is such that $\lim_{n\to\infty} \psi(n) = \infty$. Let $f_n \in_R \mathcal{H}^0_{n,k}$ be a random homomorphism. Then*

$$\Pr[|\mathcal{NC}(f_n)| = n/2] = 1 - o(1).$$

Proof. Fix $n \in \mathbb{N}$, let $\mathcal{H}^0 = \mathcal{H}^0_{n,k(n)}$, and let $f = f_n \in_R \mathcal{H}^0$ be a random homomorphism. For every $\ell \in [0, n/2]$, denote $h^0(\ell) = \left|\mathcal{H}^0(\ell)\right|$. By Lemma 3.9, every $\ell \in [0, n/2]$ admits

$$h^0(\ell) = \binom{n-\ell}{\ell}\binom{n-2\ell}{n/2-\ell}(2^k-2)^\ell,$$

which implies

$$\frac{h^0(\ell+1)}{h^0(\ell)} = \frac{(n-2\ell)(n-2\ell-1)(n/2-\ell)^2(2^k-2)}{(n-\ell)(\ell+1)(n-2\ell)(n-2\ell-1)} = \frac{(n/2-\ell)^2(2^k-2)}{(n-\ell)(\ell+1)}.$$

Thus, since $k \geq 2\log n + \psi(n)$ and since $n \geq 2$, every $0 \leq \ell < n/2$ admits

$$\frac{h^0(\ell+1)}{h^0(\ell)} \geq 2^{\psi(n)}.$$

Thus, every $0 \leq \ell < n/2$ admits

$$\frac{h^0(n/2)}{h^0(\ell)} \geq (2^{\psi(n)})^{n/2-\ell},$$

which implies

$$\sum_{0\leq\ell<n/2} h^0(\ell) = \sum_{0\leq\ell<n/2} h^0(n/2)\frac{h^0(\ell)}{h^0(n/2)} \leq h^0(n/2)\sum_{0\leq\ell<n/2}(2^{\psi(n)})^{\ell-n/2} = o(h^0(n/2)),$$

where the last equality holds, since $\lim_{n\to\infty}\psi(n) = \infty$. Thus,

$$\Pr[|\mathcal{NC}(f)| = n/2] = \frac{h^0(n/2)}{\sum_{0\leq\ell\leq n/2} h^0(\ell)} = 1 - o(1).$$

□

3.4.2 Proof of Theorem 3.1

Denote $f = f_n$, and consider the following two cases:

Case one: Assume that $f \in_R \mathcal{H}^0_{n,k}$. By Lemma 3.11, with probability $1 - o(1)$, there are $n/2$ non-constant layers in f. Thus, since n is even and since the 0-layer is constant in f, using

Claim 3.3, all the odd layers are non-constant in f. Therefore, all the even layers are mapped to 0. Hence, with probability $1-o(1)$, we have $f(C_{n,k}) \subseteq \{-1,0,1\}$, and $R(f) \leq 3$.

Case two: Assume that $f \in_R \mathcal{H}_{n,k} \setminus \mathcal{H}^0_{n,k}$. By Claim 3.5, we have that $f_\downarrow$ is uniformly distributed in $\mathcal{H}^0_{n-2,k}$. Hence, by Lemma 3.11, with probability $1-o(1)$, there are $n/2-1$ non-constant layers in $f_\downarrow$. Thus, by definition of $f_\downarrow$, including the 0-layer, there are $n/2$ non-constant layers in f. Hence, by Claim 3.3, all the even layers are non-constant in f, and all the odd layers are constant in f. Hence, with probability $1-o(1)$, either $f(C_{n,k}) \subseteq \{0,1,2\}$ or $f(C_{n,k}) \subseteq \{0,-1,-2\}$. Therefore, with probability $1-o(1)$, we have $R(f) \leq 3$. □

3.5 The Number of Non-constant Layers Determines the Range

In the previous section we have seen that if the number of non-constant layers is large, then the range is small. In this section we show that, in fact, the number of non-constant layers determines the range of a random homomorphism. The following lemma gives a lower bound on the range of a random homomorphism of $C_{n,k}$ with exactly ℓ non-constant layers. The lower bound is determined by ℓ. We note that a similar upper bound can be proven.

Lemma 3.12. *Let $n \in \mathbb{N}$ be even, and let $k = k(n) \in \mathbb{N}$. Let $\ell = \ell(n) \in \mathbb{N}$ be such that $\lim_{n\to\infty}(n-2\ell) = \infty$. Let $f_n \in_R \mathcal{H}^0_{n,k}(\ell)$ be a random homomorphism from $C_{n,k}$ to $\mathbb{Z}$ with exactly ℓ non-constant layers such that $f(\{0\} \times [k]) = \{0\}$. Then for every $\alpha > 0$,*

$$\Pr\left[R(f_n) \geq \alpha\sqrt{n-2\ell}\right] \geq (1-o(1))(1-2\alpha^2),$$

where the $o(1)$ term is as n tends to infinity, and is independent of α.

Loosely speaking, the proof of the lemma is as follows. Conditioned on f having ℓ non-constant layers, f corresponds to a random walk bridge of length $n-2\ell$. Since the range of such a walk is roughly $\sqrt{n-2\ell}$, the range of f is roughly $\sqrt{n-2\ell}$.

3.5.1 Proof of Lemma 3.12

Recall that for an even $m \in \mathbb{N}$, we defined $\mathcal{P}(m)$ to be the set of paths on $\mathbb{Z}$ of length m that start at 0 and end at 0. The following proposition shows that, with high probability, the range of a random walk bridge of length m is at least $\Omega(\sqrt{m})$.

Proposition 3.13. *Let $m \in \mathbb{N}$ be even, and let $(S_0, S_1, \ldots, S_m) \in_R \mathcal{P}(m)$ be a random walk bridge of length m. Then for every $\alpha > 0$,*

$$\Pr[|\{S_0, S_1, \ldots, S_m\}| \geq \alpha\sqrt{m}] \geq (1-o(1))(1-2\alpha^2),$$

where the $o(1)$ term is as m tends to infinity, and is independent of α.

First we use the proposition to prove the lemma. Fix $n \in \mathbb{N}$. Partition $\mathcal{H}^0(\ell)$ as follows:

$$\mathcal{H}^0(\ell) = \bigcup_I \mathcal{H}(I,n),$$

where $I \subseteq [n-1]$, and $\mathcal{H}(I,n)$ is the set of homomorphisms $f \in \mathcal{H}^0(\ell)$ such that $\mathcal{NC}(f) = I$. Note that by Claim 3.3, if $\mathcal{H}(I,n) \neq \emptyset$, then $|I| = \ell$ and for all $i \in [n-2]$, either $i \notin I$ or $i+1 \notin I$.

Fix I such that $\mathcal{H}(I,n) \neq \emptyset$, and let $g \in_R \mathcal{H}(I,n)$. Denote

$$((S_0, S_1, \ldots, S_{n-2\ell}), (v_1, \ldots, v_\ell)) = \varphi(g),$$

where φ is the bijection given by Proposition 3.6. Thus, by Proposition 3.6, we have $R(g) \geq |\{S_0, S_1, \ldots, S_{n-2\ell}\}|$, and $(S_0, S_1, \ldots, S_{n-2\ell})$ is uniformly distributed in $\mathcal{P}(n-2\ell)$.

Let $f = f_n \in_R \mathcal{H}^0(\ell)$ be a random homomorphism. Then, for all I such that $\mathcal{H}(I,n) \neq \emptyset$,

$$\Pr\left[R(f) \geq \alpha\sqrt{n-2\ell} \,\middle|\, f \in \mathcal{H}(I,n)\right] \geq \Pr\left[|\{S_0, S_1, \ldots, S_{n-2\ell}\}| \geq \alpha\sqrt{n-2\ell}\right],$$

where $(S_0, , \ldots, S_{n-2\ell}) \in_R \mathcal{P}(n-2\ell)$ is a random walk bridge. Thus, we have

$$\begin{aligned}\Pr\left[R(f) \geq \alpha\sqrt{n-2\ell}\right] &= \sum_I \Pr\left[R(f) \geq \alpha\sqrt{n-2\ell} \,\middle|\, f \in \mathcal{H}(I,n)\right] \Pr[f \in \mathcal{H}(I,n)] \\ &\geq \Pr\left[|\{S_0, S_1, \ldots, S_{n-2\ell}\}| \geq \alpha\sqrt{n-2\ell}\right], \qquad (10)\end{aligned}$$

where the sum is over all sets $I \subseteq [n-1]$ such that $\mathcal{H}(I,n) \neq \emptyset$. Thus, by Proposition 3.13, since $\lim_{n\to\infty}(n-2\ell) = \infty$,

$$(10) \geq (1 - o(1))\left(1 - 2\alpha^2\right),$$

and the $o(1)$ term is as n tends to infinity, and is independent of α. □

3.5.2 Proof of Proposition 3.13

If $\alpha \geq 1$, then the Proposition follows. Thus, assume $\alpha < 1$.

Let $T \in [m]$. Before proving the proposition we show that a path in $\mathbb{Z}$ of length m from 0 to 0 that passes through T corresponds to a path in $\mathbb{Z}$ of length m from 0 to $2T$. Formally,

Claim 3.14. *There exists a bijection between paths in $\mathcal{P}(m)$ that pass through T and paths in $\mathbb{Z}$ of length m that start at 0 and end at $2T$.*

Proof. Let $(S_0, S_1, \ldots, S_m) \in \mathcal{P}(m)$ be such that there exists $j \in [m]$ that admits $S_j = T$. Let $j^* = \min\{j \in [m] \mid S_j = T\}$. The bijection is reflecting the path around T from j^* onwards. That is, for $j \in [0, j^*]$ set $S'_j = S_j$, and for $j \in [j^*+1, m]$ set $S'_j = 2T - S_j$. Thus, $S'_0 = 0$, $S'_{j^*} = T$ and $S'_m = 2T$. Furthermore, $(S'_0, S'_1, \ldots, S'_m)$ is a path in $\mathbb{Z}$ of length m such that $S'_0 = 0$ and $S'_m = 2T$. Note that j^* is also the first time that S' passes through T.

Since every path in $\mathbb{Z}$ of length m that starts at 0 and ends at $2T$ passes through T, the above defined map is a bijection. Indeed, we show how to invert the above defined map. Let $0 = S'_0, \ldots, S'_m = 2T$ be a path in $\mathbb{Z}$ of length m. Let $j^* = \min\left\{j \in [m] \mid S'_j = T\right\}$. For $j \in [0, j^*]$ set $S_j = S'_j$, and for $j \in [j^*+1, m]$ set $S_j = 2T - S'_j$. □

Since there are $\binom{m}{m/2-T}$ paths in $\mathbb{Z}$ of length m that start at 0 and end at $2T$, using Claim 3.14,

$$\Pr[|\{S_0, S_1, \ldots, S_m\}| \geq T] \geq \Pr[\exists\, j \in [m] \;:\; S_j = T] = \frac{\binom{m}{m/2-T}}{\binom{m}{m/2}}. \qquad (11)$$

Using Stirling's formula (recall that for $x \geq 0$, we have $1-x \leq e^{-x}$ and $1+x \leq e^{x}$), substituting $T = \alpha\sqrt{m}$, we have

$$\begin{aligned}
(11) &= \frac{(m/2)!(m/2)!}{(m/2-T)!(m/2+T)!} \\
&= (1-o(1))\left(1-\frac{2T}{m}\right)^{-m/2+T}\left(1+\frac{2T}{m}\right)^{-m/2-T} \\
&= (1-o(1))\left(1-\frac{4T^2}{m^2}\right)^{-m/2+T}\left(1+\frac{2T}{m}\right)^{-2T} \\
&\geq (1-o(1))e^{\frac{4T^2}{m^2}(m/2-T)-\frac{4T^2}{m}} \\
&\geq (1-o(1))\left(1-\frac{2T^2}{m}\right) \\
&= (1-o(1))(1-2\alpha^2),
\end{aligned}$$

where the $o(1)$ term is as m tends to infinity, and is independent of α, since $\alpha < 1$. □

3.6 A Lower Bound for $k = 2\log n - \omega(1)$ - Theorem 3.2

In this part we show that for $k = 2\log n - \omega(1)$, the range of a random homomorphism from $C_{n,k}$ to $\mathbb{Z}$ is super-constant, with high probability. The proof plan is as follows: First, we prove that there are many constant layers in a random homomorphism f. Second, using Lemma 3.12, we conclude that the range of f is large.

3.6.1 Many Constant Layers

The following lemma shows that a random homomorphism of $C_{n,k}$ has many constant layers.

Lemma 3.15. *Let $n \in \mathbb{N}$ be even, and let $k = k(n) = 2\log n - \psi(n)$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is monotone and $\lim_{n\to\infty} \psi(n) = \infty$. Let $f_n \in_R \mathcal{H}_{n,k}$ be a random homomorphism. Let $\beta : \mathbb{N} \to \mathbb{R}^+$ be such that for large enough $n \in \mathbb{N}$, we have $\beta(n) \leq n/4$. Then for large enough $n \in \mathbb{N}$, we have*

$$\Pr\left[|\mathcal{NC}(f_n)| > n/2 - \beta\right] \leq 16\beta^2 2^{-\psi(n)}.$$

Proof. Fix some large enough $n \in \mathbb{N}$ such that $\beta \leq n/4$. For $\ell \in [0, n/2]$, set $h^0(\ell) = \left|\mathcal{H}^0_{n,k}(\ell)\right|$. Let $\ell \in [n/2-\beta, n/2-1]$, then by Lemma 3.9, since $\beta \leq n/4$,

$$\frac{h^0(\ell+1)}{h^0(\ell)} = \frac{(n/2-\ell)^2(2^k-2)}{(n-\ell)(\ell+1)} \leq \frac{4\beta^2 2^k}{n(n-2\beta)} \leq 8\beta^2 2^{-\psi(n)}.$$

Hence, setting $\gamma = 8\beta^2 2^{-\psi(n)}$, every $\ell \in [n/2 - \beta, n/2]$ admits

$$\frac{h^0(\ell)}{h^0(n/2-\beta)} \leq \gamma^{\ell-(n/2-\beta)}.$$

Thus, (if $\gamma \geq 1/2$, the claim follows) since $\gamma < 1/2$,

$$\Pr\left[|\mathcal{NC}(f_n)| > n/2 - \beta\right] \leq \frac{1}{h^0(n/2-\beta)} \sum_{\ell=n/2-\beta+1}^{n/2} h^0(\ell) \leq \sum_{i=1}^{\infty} \gamma^i \leq 2\gamma.$$

□

3.6.2 Proof of Theorem 3.2

First we consider homomorphisms in $\mathcal{H}^0$.

Claim 3.16. *Let $n \in \mathbb{N}$ be even, and let $k = k(n) = 2\log n - \psi(n)$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is monotone and $\lim_{n\to\infty} \psi(n) = \infty$. Let $f_n \in_R \mathcal{H}^0_{n,k}$ be a random homomorphism. Let $\varepsilon : \mathbb{N} \to \mathbb{R}^+$ be such that for large enough $n \in \mathbb{N}$, we have $\varepsilon(n) \leq 1/8$ and $\lim_{n\to\infty} \varepsilon(n) 2^{\psi(n)/2} = \infty$. Then*

$$\Pr\left[R(f_n) \geq \sqrt{2}\ \varepsilon 2^{\psi(n)/4}\right] \geq (1 - o(1))(1 - 2\varepsilon)^2.$$

Proof. Consider large enough $n \in \mathbb{N}$ such that $\varepsilon(n) \leq 1/8$. Set $\beta = \beta(n) = \varepsilon(n) 2^{\psi(n)/2}$. Note that $\lim_{n\to\infty} \beta(n) = \infty$. Since $1 \leq k = 2\log n - \psi(n)$ and $\varepsilon < 1/4$, we have $\beta \leq n/4$. Let $\ell = \ell(n) \in [0, n/2 - \beta]$, then $\lim_{n\to\infty}(n - 2\ell) \geq \lim_{n\to\infty} 2\beta(n) = \infty$. Thus, by Lemma 3.12, for any $\alpha > 0$,

$$\Pr\left[R(f_n) \geq \alpha\sqrt{2\beta} \mid f_n \in \mathcal{H}^0(\ell)\right] \geq \Pr\left[R(f_n) \geq \alpha\sqrt{n - 2\ell} \mid f_n \in \mathcal{H}^0(\ell)\right] \geq (1 - o(1))(1 - 2\alpha^2).$$

Thus, for any $\alpha > 0$,

$$\begin{aligned}\Pr\left[R(f_n) \geq \alpha\sqrt{2\beta}\right] &\geq \sum_{\ell=0}^{n/2-\beta} \Pr\left[R(f_n) \geq \alpha\sqrt{2\beta} \mid f_n \in \mathcal{H}^0(\ell)\right] \Pr\left[f_n \in \mathcal{H}^0(\ell)\right] \\ &\geq (1 - o(1))(1 - 2\alpha^2) \Pr\left[|\mathcal{NC}(f_n)| \leq n/2 - \beta\right]. \qquad (12)\end{aligned}$$

By Lemma 3.15,

$$(12) \geq (1 - o(1))(1 - 2\alpha^2)(1 - 16\beta^2 2^{-\psi(n)}) = (1 - o(1))(1 - 2\alpha^2)(1 - 16\varepsilon^2).$$

Taking $\alpha = \sqrt{\varepsilon}$, since $\varepsilon \leq 1/8$, we have

$$\Pr\left[R(f_n) \geq \sqrt{2}\ \varepsilon 2^{\psi(n)/4}\right] \geq (1 - o(1))(1 - 2\varepsilon)(1 - 16\varepsilon^2) \geq (1 - o(1))(1 - 2\varepsilon)^2.$$

□

Remark 3.17. If $k = 1$, then $f_n \in_R \mathcal{H}^0_{n,k}$ is a random walk bridge of length n. In this case, Claim 3.16 gives the bound

$$\Pr\left[R(f_n) \geq 2^{1/4}\varepsilon\sqrt{n}\right] \geq (1 - o(1))(1 - 2\varepsilon)^2.$$

Back to the proof of Theorem 3.2. Set $\varepsilon = 2^{-1/2}\psi(n)^{-1}$. Note that $\lim_{n\to\infty}\varepsilon(n) = 0$, and $\lim_{n\to\infty}\varepsilon(n)2^{\psi(n)/2} = \infty$. We denote $f = f_n \in_R \mathcal{H}_{n,k}$. Consider the following two cases:

Case one: Assume that $f \in_R \mathcal{H}^0_{n,k}$. By Claim 3.16, with probability at least

$$(1 - o(1))(1 - 2\varepsilon)^2 = 1 - o(1),$$

the range of f is at least

$$\psi(n)^{-1}2^{\psi(n)/4}.$$

Case two: Assume that $f \in_R \mathcal{H}_{n,k}\backslash\mathcal{H}^0_{n,k}$. By Claim 3.5, we have that $f_\downarrow$ is uniformly distributed in $\mathcal{H}^0_{n-2,k}$. Thus, by Claim 3.16, with probability at least

$$(1 - o(1))(1 - 2\varepsilon)^2 = 1 - o(1),$$

the range of $f_\downarrow$ is at least

$$\psi(n-2)^{-1}2^{\psi(n-2)/4}.$$

By definition of $f_\downarrow$, the size of the range of f is at least the size of the range of $f_\downarrow$. So, with probability $1-o(1)$, the range of f is at least $\psi(n)^{-1}2^{\psi(n-2)/4}$ (recall that $\psi(n)$ is monotone). □

4 Further Research

We list some possible further research directions regarding random homomorphisms of graphs:

1. Let G be a graph. A function $f : G \to \mathbb{Z}$ is called *Lipschitz* if it satisfies

   $$\forall\ v \sim u \in G \quad |f(u) - f(v)| \le 1.$$

   Note that a homomorphism is always Lipschitz, but typically the set of homomorphisms is much smaller than the set of Lipschitz functions. For a graph G and a vertex $v \in G$, define $\mathrm{Lip}^0_v(G, \mathbb{Z})$ to be the set of all Lipschitz functions from G to $\mathbb{Z}$ that map v to 0.

   Conjecture. *Let $\{G_n\}$ be a family of bi-partite graphs with $\lim_{n\to\infty}|G_n| = \infty$. Assume that for all n, G_n has maximal degree d (d independent of n). Let $f_n \in_R \mathrm{Hom}^0_{v_n}(G_n, \mathbb{Z})$ be a random homomorphism, and let $g_n \in_R \mathrm{Lip}^0_{v_n}(G_n, \mathbb{Z})$ be a random Lipschitz function. Then,*

   $$\frac{\mathbf{E}\left[R(f_n)\right]}{\mathbf{E}\left[R(g_n)\right]} = \Theta(1),$$

   where $\Theta(\cdot)$ depends on d.

2. Let G be a bi-partite graph, and let Δ be the diameter of G. For any homomorphism $f \in \mathrm{Hom}^0_v(G, \mathbb{Z})$, we have that $R(f) = O(\Delta)$. But this naive bound should not be the typical bound, at least not for symmetric graphs.

   Conjecture. *Let $\{G_n\}$ be a family of vertex transitive bi-partite graphs with $\lim_{n\to\infty}|G_n| = \infty$. Let Δ_n be the diameter of G_n, and let $f_n \in_R \mathrm{Hom}^0_{v_n}(G_n, \mathbb{Z})$ be a random homomorphism. Then,*

   $$\mathbf{E}\left[R(f_n)\right] = O\left(\sqrt{\Delta_n}\right).$$

Note that the conjecture is false if the assumption of vertex transitivity is dropped. Consider, for example, the star graph with Δ arms of length Δ. This graph has expected range of $\Theta(\Delta)$.

3. For $d \in \mathbb{N}$, let $\mathcal{O}(d)$ be the set of all *odd* positive integers at most d. For an even integer $n \in \mathbb{N}$, let $G_{n,d}$ be the graph whose vertices are $\mathbb{Z}_n$, and edges are defined by the relations

$$i \sim j \quad \Longleftrightarrow \quad \exists\, d' \in \mathcal{O}(d) \;\; i = j + d' \pmod{n}.$$

Note that $G_{n,d}$ is vertex transitive and bi-partite (for all n and d). Note also that the diameter of $G_{n,d}$ is $\Theta(n/d)$, and the degree of $G_{n,d}$ is $\Theta(d)$.

Conjecture. *There exists a constant $c > 0$ such that*

(a) If $d = c\log(n) - \omega(1)$,

$$\Pr\left[R(f_n) \geq \omega(1)\right] = 1 - o(1),$$

where $f_n \in_R \mathrm{Hom}_0^0(G_{n,d}, \mathbb{Z})$ is a random homomorphism.

(b) There exists a constant $b \in \mathbb{N}$ such that if $d = c\log(n) + \omega(1)$, then

$$\Pr\left[R(f_n) \leq b\right] = 1 - o(1),$$

where $f_n \in_R \mathrm{Hom}_0^0(G_{n,d}, \mathbb{Z})$ is a random homomorphism.

4. In this paper we only consider homomorphisms of graphs into $\mathbb{Z}$. Instead of $\mathbb{Z}$ consider the infinite star with k arms. That is, the graph S_k, whose vertices are

$$V(S_k) = \left\{(i, s) \;\middle|\; 0 < i \in \mathbb{N}\;,\; 1 \leq s \leq k\right\} \cup \{0\},$$

and edges are $(1, s) \sim 0$ for all $1 \leq s \leq k$, and

$$\forall\, 1 < i \in \mathbb{N}\;,\; 1 \leq s \leq k\;, \quad (i, s) \sim (i \pm 1, s).$$

It can be shown (5) that a random homomorphism from the interval $[n]$ to S_3, has range $O(\log(n))$ (as opposed to $\Theta(\sqrt{n})$ when the homomorphism is into $\mathbb{Z} = S_2$). Is this the case for all bi-partite graphs G? That is, let $\{G_n\}$ be a family of bi-partite graphs such that $\lim_{n\to\infty} |G_n| = \infty$. Let $f_n \in_R \mathrm{Hom}_{v_n}^0(G_n, S_3)$. Is it true that

$$\mathbf{E}\left[R(f_n)\right] = O\left(\log |G_n|\right)?$$

Or maybe even

$$\lim_{n\to\infty} \Pr\left[R(f_n) = O\left(\log |G_n|\right)\right] = 1?$$

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