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Detecting a local perturbation in a continuous scenery*

Heinrich Matzinger
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332–0160, USA
matzi@math.gatech.edu

Serguei Popov
Instituto de Matemática e Estatística
Universidade de São Paulo
rua do Matão 1010, CEP 05508–090
São Paulo SP, Brasil
popov@ime.usp.br

Abstract

A continuous one-dimensional scenery is a double-infinite sequence of points (thought of as locations of *bells*) in $\mathbb{R}$. Assume that a scenery X is observed along the path of a Brownian motion in the following way: when the Brownian motion encounters a bell different from the last one visited, we hear a ring. The trajectory of the Brownian motion is unknown, whilst the scenery X is known except in some finite interval. We prove that given only the sequence of times of rings, we can a.s. reconstruct the scenery X entirely. For this we take the scenery X to be a local perturbation of a Poisson scenery X'. We present an explicit reconstruction algorithm. This problem is the continuous analog of the "detection of a defect in a discrete scenery". Many of the essential techniques used with discrete sceneries do not work with continuous sceneries.

Key words: Brownian motion, Poisson process, localization test, detecting defects in sceneries seen along random walks.

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1 Introduction

Suppose that countably many bells are placed on $\mathbb{R}$. Start a Brownian motion from 0; each time it hits a bell *different* from the last one visited, we hear a ring. During this process all the bells remain in the same position. The set of locations of the bells in $\mathbb{R}$ is referred to as the *scenery*. Suppose now that we cannot observe the trajectory of the Brownian motion, and that the scenery is not completely known either. On the other hand, let the sequence of time occurrences of the rings be known to us.

The *detection of a local perturbation* problem can be formulated as follows: is it possible to recover the exact scenery a.s. given only the sequence of rings and the scenery up to a local perturbation?

In this paper, we answer this question affirmatively provided that the scenery is a local perturbation of a random realization of a one-dimensional Poisson process with bounded rate. The realization of the one-dimensional Poisson process is known to us but we do not know in which way and where it was perturbed.

This problem is the continuous analog of the problem of detecting a defect in a scenery seen along the path of a random walk. In the discrete case (which is not the case of this paper) one considers a discrete scenery $\xi : \mathbb{Z} \to \{0, 1, \ldots, C-1\}$ and a random walk $\{S_t\}_{t\in\mathbb{N}}$. The discrete scenery is a coloring of the integers with C colors. One observes the discrete scenery seen along the path of the random walk, i.e. the sequence $\chi_0, \chi_1, \ldots$, where $\chi_i := \xi(S_i)$. From this one tries to infer about ξ. For more information about discrete scenery reconstruction and distinguishing see e.g. (1; 4; 5; 9; 10; 11) and references therein.

It is worth noticing that in the case of the present paper, i.e. in the case of a continuous scenery, there are no "colors": all the bells ring in the same way. Hence, we have to use the time length between successive rings to estimate where the Brownian motion is. It turns out that bells close to each other tend to confer a lot of information. In discrete scenery reconstruction it is usually the opposite: long blocks of only one color are the essential "markers".

The continuous case considered here contains one of the major difficulties still open in discrete scenery reconstruction. Roughly speaking, in any part of the scenery one can obtain any finite set of observations in the continuous case. Some finite sets of observations might be untypical but are never impossible. In all the discrete cases, where scenery reconstruction has been proven possible, there exist patterns which can appear in the observations only when the random walk dwells in some specific regions of the scenery. This is one more reason which makes it worthwhile studying the continuous case.

Also, we should mention that one of the main techniques used in discrete reconstruction does not work here. This is the "going in a straight path from x to y" as is used in a majority of discrete reconstruction papers. Instead we use an estimate of the probability to hear a ring a certain amount of time after being at a marker.

There exists one other related continuous problem solved by Burdzy (3). He takes an iterated Brownian motion and shows that the path of the outer one can be a.s. reconstructed. This is the continuous analog of reconstructing a random walk path given an iterated random walk path. Matzinger (9) proved that the reconstruction of a 3-color scenery seen along a simple random walk is equivalent to this problem.

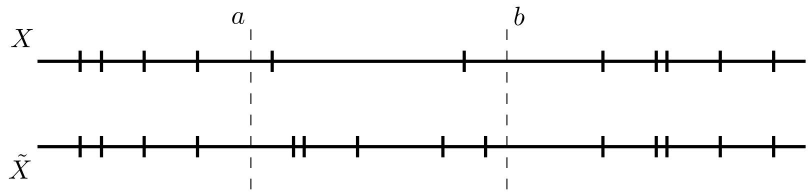

Figure 1: Local perturbation of a scenery

1.1 Notations and the main result

Let us start with the formal definitions used in this paper. A *scenery* is a double infinite sequence $X = (\dots, X_{-1}, X_0, X_1, \dots)$, such that $X_n < X_{n+1}$ for all $n \in \mathbb{Z}$ and $\lim_{n\to-\infty} X_n = -\infty$, $\lim_{n\to+\infty} X_n = +\infty$. The last condition guarantees that the number of points of X in any finite interval is finite.

With some abuse of notation, we denote the set of points in the scenery by the same letter, $X = \{\dots, X_{-1}, X_0, X_1, \dots\}$. Let $\mathcal{M}$ be the set of all such sceneries. Let $\xi(n) := X_n - X_{n-1}$ for all $n \in \mathbb{Z}$. The sequence ξ is thus the sequence of distances between the successive bell-locations.

Definition 1.1. *Scenery $\tilde{X}$ is a local perturbation of X if they coincide everywhere except possibly in a finite interval, i.e., there exist $a, b \in \mathbb{R}$ such that $\tilde{X} \setminus [a, b] = X \setminus [a, b]$ (see Figure 1).*

We emphasize here that all sceneries considered in this paper are locally finite (that is, one is not allowed to perturb a scenery by placing an infinite number of points on a finite interval). So, an equivalent formulation of Definition 1.1 would be: scenery $\tilde{X}$ is a local perturbation of X if $(\tilde{X} \setminus X) \cup (X \setminus \tilde{X})$ is finite. Note also that if $\tilde{X}$ is a local perturbation of X, then X is a local perturbation of $\tilde{X}$.

Let $(W_t, t \geq 0)$ be the standard Brownian motion (starting from 0, unless otherwise indicated). When it is necessary to consider a Brownian motion starting from an arbitrary $x \in \mathbb{R}$, we use the notations $\mathbb{P}^x, \mathbb{E}^x$ for the corresponding probability and expectation. Let $\mathcal{M}^+$ be the set of all infinite sequences $U = (0 = U_0, U_1, U_2 \dots)$, such that $U_n < U_{n+1}$ for all $n \in \mathbb{Z}_+$, and such that $\lim_{n\to+\infty} U_n = +\infty$. Using the scenery X and the trajectory of the Brownian motion W_t, we define the specific sequence of stopping times $Y = (0 = Y_0, Y_1, Y_2 \dots) \in \mathcal{M}^+$ that corresponds to the sequence of ringing-times. More precisely (see Figure 2, the marks on the horizontal line correspond to the bells, the marks on the vertical line correspond to the rings):

$$Y_{n+1} := \inf\{t \geq Y_n : W_t \in X \setminus \{W_{Y_n}\}\}, \qquad n \geq 0$$

(note that the sequence Y always begins with 0, regardless of whether $0 \in X$ or not). From the fact that $X \in \mathcal{M}$ it is elementary to obtain that $Y_n < Y_{n+1} < \infty$ for all $n \in \mathbb{Z}_+$, and that $\lim_{n\to+\infty} Y_n = +\infty$ a.s., so indeed $Y \in \mathcal{M}^+$. Denote by $\chi(n)$, $n = 1, 2, 3, \dots$, the sequence of time lapses between successive rings. Hence, $\chi(n) := Y_n - Y_{n-1}$.

Now, we formulate our main result. Suppose that (the known scenery) X' is a realization of a one-dimensional inhomogeneous Poisson process with intensity bounded away from 0 and $+\infty$. Let us denote by $\mathbf{P}$ the probability measure that refers to X', and by $\mathbf{E}$ the corresponding expectation. The main result of this paper is that $\mathbf{P}$-almost all sceneries have the following property: *every* local perturbation of the scenery is $\mathbb{P}$-a.s. reconstructable by using only the sequence of rings and the unperturbed scenery. More precisely:

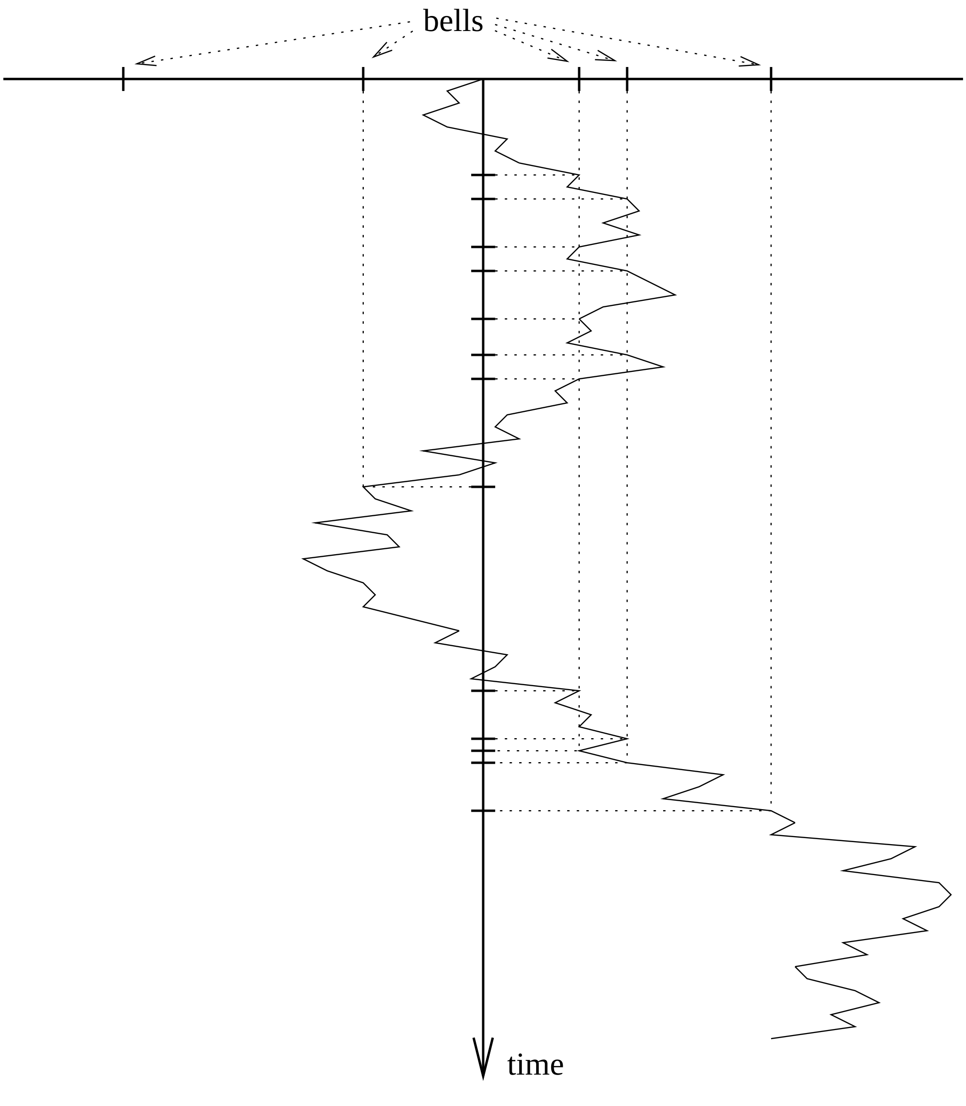

Figure 2: Constructing the sequence of rings

Theorem 1.1. *There exists a measurable function* $\Psi : \mathcal{M} \times \mathcal{M}^+ \mapsto \mathcal{M}$ *such that for* $\mathbf{P}$*-almost all sceneries* X' *we have the following. Let* X *be* any *local perturbation of* X' *and* Y *be the sequence of rings defined above. Then* $\mathbb{P}[\Psi(X', Y) = X] = 1$.

2 Proof of Theorem 1.1

In the proof of this theorem we will suppose for definiteness that X' is a realization of a Poisson process with rate 1, the general case is completely analogous.

The idea of the proof is, roughly speaking, the following: we use couples of bells which are untypically close to each other. The distance to neighbouring bells in the scenery should be much larger. The Brownian motion is likely to produce a long sequence of rings separated by short time intervals when visiting such a couple of bells (as illustrated in Figure 2). In other words, the Brownian motion tends to visit the two bells many times before moving on to another bell in the scenery.

So, when we hear many rings shortly after one another, then this is likely to be caused by two bells at short distance from each other in the scenery. Hence, a sequence of many rings in a short time permits us to estimate the distance between the underlying two bells (provided the sequence was really generated on only two bells close to each other, which is likely). We discuss this in Section 2.1. Then, for a given (large) n, we define a location ζ_n (with a bell there) and construct a sequence of stopping times $\tau_i^{(n)}$ depending only on Y and X' (i.e., on known information) such that, with overwhelming probability $W_{\tau_i^{(n)}} = \zeta_n$, whenever i is not too large. In other words, with large probability we are able to tell whether we are back to the same place. For this we use the information provided by the estimated distances between couple of bells close to each other. This is done in Section 2.2 (see Lemma 2.5). In Section 2.3, we present an algorithm for reconstructing the local perturbation with a high precision, then we consider a sequence of such algorithms which permits us to reconstruct X exactly; however, this is done supposing that the interval where the perturbation took place is known. In Section 2.4 we explain the reconstruction procedure in the case when the interval of perturbation is unknown.

2.1 The main idea: trills and couples

Fix some $\varepsilon_0, \delta_0, \delta_1 > 0$ such that

$$\varepsilon_0 + \delta_0 + \delta_1 < 1/2, \tag{1}$$

$$12\varepsilon_0 < \delta_0. \tag{2}$$

Let z_0 be such that

$$\int_{z_0}^{+\infty} (2\pi u^3)^{-1/2} \exp\Big(-\frac{1}{2u}\Big) du = \frac{1}{2} \tag{3}$$

(z_0 exists and is positive because the above integral taken from 0 to $+\infty$ equals 1, cf. (6) below). Denote also

$$\mathfrak{A}_n^k = \big(z_0^{-1} \mathsf{median}\{\chi(k+1), \dots, \chi(k + \lfloor n^{\delta_0/2} \rfloor)\}\big)^{1/2}.$$

The next two definitions play an important role in our construction.

Definition 2.1. *We say that there is a level-n trill at the mth position of the sequence Y, if* $\chi(m+k) \le n^{-2+2\varepsilon_0+2\delta_0+\delta_1}$ *for all* $k = 1, \dots, \lfloor n^{\delta_0/2} \rfloor$.

Definition 2.2. *Suppose that there is a level-n trill at the mth position of the sequence Y. We say that this trill is good, if* $\mathfrak{A}_n^m \le n^{-1+\varepsilon_0}$.

The main idea is that if there is a good level-n trill in the kth position of the sequence Y, it is very probable that it was produced by the alternating visits of the Brownian motion to some two neighboring points from X that are roughly $\mathfrak{A}_n^k$ away from each other (by alternating visits we mean here that the rings in the piece of the sequence Y under consideration were caused by only two bells). Consider the following

Definition 2.3. *A pair of two consecutive points* (X_k, X_{k+1}) *is called true level-n couple if*

$$\xi(k+1) = X_{k+1} - X_k \le n^{-1+\varepsilon_0}(1 - z_0^{-1} n^{-\delta_0/6}), \tag{4}$$

and

$$\min\{\xi(k), \xi(k+2)\} \ge n^{-1+\varepsilon_0+\delta_0+\delta_1}. \tag{5}$$

It is called almost level-n couple if (5) *holds,* (4) *does not hold, but* $X_{k+1} - X_k \le n^{-1+\varepsilon_0}(1 + z_0^{-1} n^{-\delta_0/6})$. *A pair which is either true level-n couple or almost level-n couple is called level-n couple.*

Let $T_r = \inf\{t \ge 0 : W_t = r\}$ be the hitting time of $r > 0$ by Brownian motion. Then, provided that the Brownian motion starts at 0, the density $f_r(s)$ of T_r is given by (see (2), formula **1**.2.0.2)

$$f_r(s) = r(2\pi s^3)^{-1/2} \exp\Big(-\frac{r^2}{2s}\Big). \tag{6}$$

We recall also the following elementary fact: if $a < b < c$, then (see (2), formula **1**.3.0.4)

$$\mathbb{P}^b[T_a < T_c] = \frac{c-b}{c-a}. \tag{7}$$

Let us consider now a level-n couple (X_k, X_{k+1}). Abbreviate for a moment $a := X_k - n^{-1+\varepsilon_0+\delta_0}$, $b := X_k$, $c := X_{k+1}$, $d := X_{k+1} + n^{-1+\varepsilon_0+\delta_0}$. Note that, by Definition 2.3, it holds that $X_{k-1} < a$ and that $X_{k+2} > d$. By (7), there is $C_1 > 0$ such that

$$\min\{\mathbb{P}^b[T_c < T_a], \mathbb{P}^c[T_b < T_d]\} \ge 1 - C_1 n^{-\delta_0},$$

so for any $x \in \{b, c\} (= \{X_k, X_{k+1}\})$

$$\mathbb{P}[W_{Y_{m+s}} \in \{b, c\} \text{ for any } 1 \le s \le \lfloor n^{\delta_0/2} \rfloor \mid W_{Y_m} = x] \ge 1 - C_1 n^{-\delta_0/2}, \tag{8}$$

i.e., with a large probability the Brownian motion will commute between the points of a level-n couple at least $\lfloor n^{\delta_0/2} \rfloor$ times. Now, it is elementary to see that

$$\mathbb{P}^b[\min\{T_a, T_c\} \le n^{-2+2\varepsilon_0+2\delta_0+\delta_1} \mid T_c < T_a] \ge 1 - \exp(-C_2 n^{\delta_1}) \tag{9}$$

and that the same bound holds if b, a, c are substituted by c, d, b (in this order). Indeed, since the conditional density of $\min\{T_a, T_c\}$ is known (see **1**.3.0.6 of (2)), it is possible to obtain (9) by a direct (although not so simple) computation. It is easier, however, to argue as follows. Using (7), write

$$\mathbb{P}^b[\min\{T_a, T_c\} \le n^{-2+2\varepsilon_0+2\delta_0+\delta_1} \mid T_c < T_a] \le \frac{c-a}{b-a}\,\mathbb{P}^b[\min\{T_a, T_c\} \le n^{-2+2\varepsilon_0+2\delta_0+\delta_1}]. \quad (10)$$

For any starting point within the interval $[a, c]$, the probability that the Brownian motion hits $\{a, c\}$ in time at most $n^{-2+2\varepsilon_0+2\delta_0}$ is bounded away from 0 (say, by a constant κ_1). The time interval $[0, n^{-2+2\varepsilon_0+2\delta_0+\delta_1}]$ contains $\lfloor n^{\delta_1}\rfloor$ non-intersecting intervals of length $n^{-2+2\varepsilon_0+2\delta_0}$, so we have at least $\lfloor n^{\delta_1}\rfloor$ tries to enter $\{a, c\}$:

$$\mathbb{P}^b[\min\{T_a, T_c\} \le n^{-2+2\varepsilon_0+2\delta_0+\delta_1}] \le (1-\kappa_1)^{\lfloor n^{\delta_1}\rfloor}. \quad (11)$$

Since for all n large enough $\frac{c-a}{b-a} \le 2$, we obtain (9) from (10) and (11).

Thus, using (9), we obtain that

$$\begin{aligned}\mathbb{P}[\chi(m+s) \le n^{-2+2\varepsilon_0+2\delta_0+\delta_1} \text{ for any } 1 \le s \le \lfloor n^{\delta_0/2}\rfloor \\ \mid W_{Y_m} = x, W_{Y_{m+s}} \in \{b, c\} \text{ for any } 1 \le s \le \lfloor n^{\delta_0/2}\rfloor] \\ \ge\ 1 - n^{\delta_0/2}\exp(-C_2 n^{\delta_1}).\end{aligned} \quad (12)$$

for any $x \in \{b, c\}$. This shows that if the Brownian motion commutes between b and c (without hitting other points of X) at least $\lfloor n^{\delta_0/2}\rfloor$ times, then, with overwhelming probability, we obtain a level-n trill. To show that (for true level-n couples) this trill should normally be good, we have to work a bit more.

First, let us recall Chernoff's bound for the binomial distribution:

Lemma 2.1. *[see e.g. (12), p. 68.] Let $\{\zeta_i, i \ge 1\}$ be i.i.d. random variables with $\mathbb{P}[\zeta_i = 1] = \theta$ and $\mathbb{P}[\zeta_i = 0] = 1-\theta$. Then for any $0 < \theta < \alpha < 1$ and for any $s \ge 1$ we have*

$$\mathbb{P}\Big[\frac{1}{s}\sum_{i=1}^{s}\zeta_i \ge \alpha\Big] \le \exp\{-sH(\alpha,\theta)\}, \quad (13)$$

where

$$H(\alpha,\theta) = \alpha\log\frac{\alpha}{\theta} + (1-\alpha)\log\frac{1-\alpha}{1-\theta} > 0.$$

If $0 < \alpha < \theta < 1$, then (13) holds with $\mathbb{P}[s^{-1}\sum_{i=1}^{s}\zeta_i \le \alpha]$ in the left-hand side.

Now, we define another sequence of stopping times $(Y'_m, m \ge 0)$, constructed in a similar way as the sequence Y, this time supposing, however, that the only bells are in b and c (i.e., in X_k and in X_{k+1}):

$$\begin{aligned} Y'_0 &= 0, \text{ and} \\ Y'_{n+1} &= \inf\big\{t \ge Y'_n : W_t \in \{b, c\} \setminus \{W_{Y'_n}\}\big\}. \end{aligned}$$

Analogously, define $\chi'(i) = Y'_i - Y'_{i-1}$ and

$$\mathfrak{A}'_n = \big(z_0^{-1}\mathsf{median}\{\chi'(1), \ldots, \chi'(\lfloor n^{\delta_0/2}\rfloor)\}\big)^{1/2}.$$

Lemma 2.2. *There is a positive constant γ_1 such that for all n large enough we have*

$$\mathbb{P}\big[\beta^2(z_0 - n^{-\delta_0/6}) \le \mathsf{median}\{\chi'(1),\dots,\chi'(\lfloor n^{\delta_0/2}\rfloor)\} \le \beta^2(z_0 + n^{-\delta_0/6})\big] \ge 1 - \exp(-\gamma_1 n^{-\delta_0/6}) \tag{14}$$

and also

$$\mathbb{P}\big[\mathfrak{A}'_n(1 - z_0^{-1}n^{-\delta_0/6}) \le \beta \le \mathfrak{A}'_n(1 + z_0^{-1}n^{-\delta_0/6})\big] \ge 1 - \exp(-\gamma_1 n^{-\delta_0/6}), \tag{15}$$

where $\beta := c - b = X_{k+1} - X_k$.

Proof. Abbreviate

$$Z := \mathsf{median}\{\chi'(1),\dots,\chi'(\lfloor n^{\delta_0/2}\rfloor)\},$$

and for any $y \in (0,1)$ let $\hat{M}_y$ be such that

$$\int_0^{\hat{M}_y} \beta(2\pi s^3)^{-1/2} \exp\Big(-\frac{\beta^2}{2s}\Big)\, ds = y. \tag{16}$$

Fix a number $p \in (0, 1/2)$ (to be chosen later), and define the random variable $\eta_i = \mathbf{1}\{\chi'(i) \ge \hat{M}_{\frac{1}{2}+p}\}$, so that, by (6), $\mathbb{P}[\eta_i = 1] = 1 - \mathbb{P}[\eta_i = 0] = \frac{1}{2} - p$. Now, we have

$$\mathbb{P}[Z \ge \hat{M}_{\frac{1}{2}+p}] = \mathbb{P}\Big[\lfloor n^{-\delta_0/2}\rfloor \sum_{i=1}^{\lfloor n^{\delta_0/2}\rfloor} \eta_i \ge \frac{1}{2}\Big]. \tag{17}$$

Let us use Lemma 2.1 with $s = \lfloor n^{\delta_0/2}\rfloor$, $\alpha = 1/2$, $\theta = \frac{1}{2} - p$. It holds that

$$\begin{aligned} H(\alpha,\theta) &= \frac{1}{2}\ln\frac{1}{1-2p} + \frac{1}{2}\ln\frac{1}{1+2p} \\ &= \frac{1}{2}\ln\frac{1}{1-4p^2} \\ &\ge p^2 \end{aligned}$$

for all p small enough. So, by (17) and Lemma 2.1 we obtain that

$$\mathbb{P}[Z \ge \hat{M}_{\frac{1}{2}+p}] \le \exp(-p^2\lfloor n^{\delta_0/2}\rfloor).$$

By symmetry, the same estimate holds for $\mathbb{P}[Z \le \hat{M}_{\frac{1}{2}-p}]$, so we obtain

$$\mathbb{P}[\hat{M}_{\frac{1}{2}-p} \le Z \le \hat{M}_{\frac{1}{2}+p}] \ge 1 - 2\exp(-p^2\lfloor n^{\delta_0/2}\rfloor). \tag{18}$$

To proceed, we notice that it is straightforward to obtain from (3) and (6) that $\hat{M}_{1/2} = z_0\beta^2$. Since, by (6), $f_\beta(y)$ is of order β^{-2} when y is of order β^2, there exist positive constants C_4, C_5 such that

$$\hat{M}_{\frac{1}{2}+p} \le z_0\beta^2 + C_4 p\beta^2,$$

$$\hat{M}_{\frac{1}{2}-p} \geq z_0\beta^2 - C_5 p\beta^2,$$

for all p small enough. Now, it remains only to take $p = (\max\{C_4, C_5\})^{-1} n^{-\delta_0/6}$ and use (18) to obtain (14). In order to obtain (15), note that

$$\{\beta^2(z_0 - n^{-\delta_0/6}) \leq Z \leq \beta^2\} = \{\mathfrak{A}'_n(1 + z_0^{-1} n^{-\delta_0/6})^{-1} \leq \beta \leq \mathfrak{A}'_n(1 - z_0^{-1} n^{-\delta_0/6})^{-1}\},$$

and that $(1 + z_0^{-1} n^{-\delta_0/6})^{-1} = 1 - z_0^{-1} n^{-\delta_0/6} + o(n^{-\delta_0/6})$, $(1 - z_0^{-1} n^{-\delta_0/6})^{-1} = 1 + z_0^{-1} n^{-\delta_0/6} + o(n^{-\delta_0/6})$, so (15) can be obtained in exactly the same way as (14). The proof of Lemma 2.2 is completed. □

Consider the events

$$R_{n,m} = \{\chi(m+s) \leq n^{-2+2\varepsilon_0+2\delta_0+\delta_1} \text{ for any } 1 \leq s \leq \lfloor n^{\delta_0/2} \rfloor\}$$

and

$$D_{n,m} = \{\mathfrak{A}^m_n(1 - z_0^{-1} n^{-\delta_0/6}) \leq \beta \leq \mathfrak{A}^m_n(1 + z_0^{-1} n^{-\delta_0/6})\}, \tag{19}$$

where, as before, $\beta := c - b = X_{k+1} - X_k$. We are going to estimate the conditional probability $\mathbb{P}[D_{n,m} \mid R_{n,m}, W_{Y_m} = b]$ from below. To this end, define also the events

$$D'_{n,m} = \{\mathfrak{A}'_n(1 - z_0^{-1} n^{-\delta_0/6}) \leq \beta \leq \mathfrak{A}'_n(1 + z_0^{-1} n^{-\delta_0/6})\},$$

and

$$E_{n,m} = \big\{Y_{m+s} \in \{b, c\} \text{ for all } 0 \leq s \leq \lfloor n^{\delta_0/2} \rfloor\big\}.$$

Write

$$\begin{aligned}
&\mathbb{P}[D_{n,m} \mid R_{n,m}, W_{Y_m} = b] \\
&\quad\geq \mathbb{P}[D_{n,m} E_{n,m} \mid R_{n,m}, W_{Y_m} = b] \\
&\quad= \mathbb{P}[D'_{n,m} E_{n,m} \mid R_{n,m}, W_{Y_m} = b] \\
&\quad\geq 1 - \mathbb{P}[(D'_{n,m})^c \mid R_{n,m}, W_{Y_m} = b] - \mathbb{P}[E^c_{n,m} \mid R_{n,m}, W_{Y_m} = b] \\
&\quad\geq 1 - \frac{\mathbb{P}[(D'_{n,m})^c \mid W_{Y_m} = b]}{\mathbb{P}[R_{n,m} \mid W_{Y_m} = b]} - \mathbb{P}[E^c_{n,m} \mid R_{n,m}, W_{Y_m} = b].
\end{aligned} \tag{20}$$

Recall that $\{b, c\}$ is a level-n couple, so that $\min\{\xi(k), \xi(k+2)\} \geq n^{-1+\varepsilon_0+\delta_0+\delta_1}$. Using (6), we obtain (changing the variables $u = sn^{2-2\varepsilon_0-2\delta_0-2\delta_1}$) that for some $C_6 > 0$ and all n it holds that

$$\begin{aligned}
\mathbb{P}[T_{n^{-1+\varepsilon_0+\delta_0+\delta_1}} \leq n^{-2+2\varepsilon_0+2\delta_0+\delta_1}] &\leq \int_0^{n^{-\delta_1}} (2\pi u^3)^{-1/2} \exp\Big(-\frac{1}{2u}\Big) du \\
&\leq \exp\Big(-\frac{n^{-\delta_1}}{4}\Big) \int_0^{n^{-\delta_1}} (2\pi u^3)^{-1/2} \exp\Big(-\frac{1}{4u}\Big) du \\
&\leq \tilde{\gamma} \exp\Big(-\frac{n^{-\delta_1}}{4}\Big),
\end{aligned} \tag{21}$$

so

$$\mathbb{P}[E^c_{n,m} \mid R_{n,m}, W_{Y_m} = b] \leq \tilde{\gamma} n^{\delta_0/2} \exp\Big(- \frac{n^{-\delta_1}}{4}\Big). \tag{22}$$

By (12), $\mathbb{P}[R_{n,m} \mid W_{Y_m} = b] \geq 1/2$ for all n large enough, and we can bound $\mathbb{P}[(D'_{n,m})^c \mid W_{Y_m} = b]$ from above by using Lemma 2.2. So, using (20) and (22), we obtain

$$\mathbb{P}[D_{n,m} \mid R_{n,m}, W_{Y_m} = b] \geq 1 - 2\exp(-\gamma_1 n^{-\delta_0/6}) - \tilde{\gamma} n^{\delta_0/2} \exp\Big(- \frac{n^{-\delta_1}}{4}\Big) \tag{23}$$

(clearly, the same estimate is valid if we substitute "$W_{Y_m} = b$" by "$W_{Y_m} = c$" in the above calculations). In words, the above equation shows that if a true level-n couple causes a level-n trill, then, with a very high probability, that trill will be good and that one will be able to obtain the distance between the points in the couple with a high precision. Also, by (8) and (12), we obtain that

$$\mathbb{P}[R_{n,m} \mid H^*_m] \geq 1 - C_6 n^{-\delta_0/2}, \tag{24}$$

where the event H^*_m is defined by

$$H^*_m = \{W_{Y_m} \text{ is a point of some level-}n \text{ couple}\}.$$

Now, we have to figure out how likely it is to produce good level-n trills elsewhere, not in level-n couples. First, we observe that, since the interval between any two consecutive rings in a level-n trill are at most $n^{-2+2\varepsilon_0+2\delta_0+\delta_1}$, the bells where the rings were produced should not be at distance more than $n^{-1+\varepsilon_0+\delta_0+\delta_1}$ from each other (otherwise the probability of producing such closely placed rings would be at least stretched-exponentially small). On the other hand, if we have three or more close bells (with distance of order $n^{-1+\varepsilon_0}$ from each other), then such a group of bells is, in principle, capable to produce a good level-n trill as well.

Suppose, however, that we know that we are in some region where there are no triples of close points (bells). More precisely, suppose that there are bells in points $a, b, c, d \in \mathbb{R}$, and $|b - c| < n^{-1+\varepsilon_0+\delta_0+\delta_1}$, while $\min\{|a - b|, |c - d|\} > n^{-1+\varepsilon_0+\delta_0+\delta_1}$; however, b is not close enough to c to form a level-n couple. Then, one can obtain that

$$\mathbb{P}[\text{there is a good level-}n \text{ trill at } m \,|\, H^*_m(b)] \leq \exp(-\gamma_1 n^{-\delta_0/6}) + \tilde{\gamma} n^{\delta_0/2} \exp\Big(- \frac{n^{-\delta_1}}{4}\Big), \tag{25}$$

where $H^*_m(b) = \{W_{Y_m} = b\}$. Indeed, let H' be the event that the good level-n trill in m was produced only in $\{b, c\}$. Then, as in Lemma 2.2, we show that

$$\mathbb{P}[\{\text{there is a good level-}n \text{ trill at } m\} \cap H' \,|\, H^*_m(b)] \leq \exp(-\gamma_1 n^{-\delta_0/6}).$$

On the other hand, if the event H' does not occur, this means that, at some stage during this trill, the particle should cover the distance at least $n^{-1+\varepsilon_0+\delta_0+\delta_1}$ in time at most $n^{-2+2\varepsilon_0+2\delta_0+\delta_1}$. Applying (21), we obtain

$$\mathbb{P}[\{\text{there is a good level-}n \text{ trill at } m\} \cap (H')^c \,|\, H^*_m(b)] \leq \tilde{\gamma} n^{\delta_0/2} \exp\Big(- \frac{n^{-\delta_1}}{4}\Big).$$

Now, for the sake of convenience we introduce some definitions concerning trills and couples:

Definition 2.4. *A level-n trill is compatible with a level-n couple with the distance β between the points, if (supposing for definiteness that the trill begins at the mth position of the sequence Y) the event $D_{n,m}$, defined in (19), occurs.*

Definition 2.5. *We say that a level-n trill was produced by a level-n couple, if all the rings of the trill occurred in the bells of the couple.*

For what follows (abbreviating for a moment $v_n := z_0^{-1} n^{-\delta_0/6}$), we suppose that n is such that

$$\max\Big\{-\frac{1-v_n}{1+v_n}+1, \frac{1+v_n}{1-v_n}-1\Big\} < 3v_n, \tag{26}$$

and

$$\min\Big\{\frac{(1+6v_n)(1-v_n)}{1+v_n}-1, -\frac{1+v_n}{(1+6v_n)(1-v_n)}+1,$$
$$-\frac{(1-6v_n)(1+v_n)}{1-v_n}+1, \frac{1-v_n}{(1-6v_n)(1+v_n)}-1\Big\} > 3v_n \tag{27}$$

(clearly, this holds for all but finitely many positive integers n).

Definition 2.6. *(i) Two level-n couples with the distances between their points being respectively β_1, β_2 are called n-similar if*

$$\min\{|\beta_1\beta_2^{-1}-1|, |\beta_1^{-1}\beta_2-1|\} \leq 6z_0^{-1}n^{-\delta_0/6}. \tag{28}$$

(ii) Two level-n trills (in positions m_1, m_2) are called n-similar if

$$\min\{|\mathfrak{A}_n^{m_1}(\mathfrak{A}_n^{m_2})^{-1}-1|, |(\mathfrak{A}_n^{m_1})^{-1}\mathfrak{A}_n^{m_2}-1|\} \leq 3z_0^{-1}n^{-\delta_0/6}.$$

Two level-n couples (trills) are called n-different, if they are not n-similar.

From (26) it is straightforward to obtain that if two level-n trills are both compatible with a level-n couple, then they are n-similar, and also if two level-n couples are compatible with a level-n trill, then they are n-similar. Also, two almost level-n couples are always n-similar. Using the above definition, we summarize the results of this section in the following

Lemma 2.3. *There is a positive constant γ_2 such that:*

(i) With probability at least $1-2\exp(-\gamma_1 n^{-\delta_0/6}) - \tilde{\gamma} n^{\delta_0/2}\exp(-\frac{n^{-\delta_1}}{4})$, given that a level-n couple produces a level-n trill, the former will be compatible with the latter.

(ii) With probability at least $1-4\exp(-\gamma_1 n^{-\delta_0/6}) - 2\tilde{\gamma} n^{\delta_0/2}\exp(-\frac{n^{-\delta_1}}{4})$, n-different couples produce n-different trills.

(iii) Suppose that $W_{Y_m} = b$, where b is not from a level-n couple, and in the interval $[b - 2n^{-1+\varepsilon_0+\delta_0+\delta_1}, b + 2n^{-1+\varepsilon_0+\delta_0+\delta_1}]$ there are at most two bells (including the one in b). Then, with probability at least $1-\exp(-\gamma_1 n^{-\delta_0/6}) + \tilde{\gamma} n^{\delta_0/2}\exp\Big(-\frac{n^{-\delta_1}}{4}\Big)$, there is no good level-n trill at the mth position of the sequence Y.

Proof. Item (i) follows from (23). By Definition 2.6 and (27), if two level-n couples are n-different and two level-n trills are compatible with the first and the second couple correspondingly, then the trills are n-different. So, (ii) follows from (i).

To see that (iii) holds, consider two cases. First, suppose that b is the only bell in the interval $[b-n^{-1+\varepsilon_0+\delta_0+\delta_1}, b+n^{-1+\varepsilon_0+\delta_0+\delta_1}]$. In this case, even the time interval $\chi(m+1)$ will be greater than $n^{-2+2\varepsilon_0+2\delta_0+\delta_1}$ with probability at least $\tilde{\gamma}\exp(-\frac{n^{-\delta_1}}{4})$ (this is by (21)), so that there will be no level-n trill at the mth position. On the other hand, if there is another bell in the interval $[b-n^{-1+\varepsilon_0+\delta_0+\delta_1}, b+n^{-1+\varepsilon_0+\delta_0+\delta_1}]$, then (iii) follows from (25). □

2.2 Localization test

The purpose of this section is to construct a test which, with high probability, is able to tell us if the Brownian motion is back to the same place.

Suppose that the local perturbation of the scenery X' was made in the interval $[-\ell,\ell]$, in other words, the "real" scenery X is known precisely in $\mathbb{R}\setminus[-\ell,\ell]$. We construct now a localization test depending on parameters n and ℓ. Define the events

$$\begin{aligned} G_{i,1}^{(n)} &= \Big\{\text{in the interval } [in^{1-\frac{\varepsilon_0}{2}}, (i+1)n^{1-\frac{\varepsilon_0}{2}}) \text{ there are at most } n^{\frac{3\varepsilon_0}{4}} \\ &\qquad \text{pairs } X_k, X_{k+1} \text{ such that } X_{k+1}-X_k \le n^{-1+\varepsilon_0}(1+z_0^{-1}n^{-\delta_0/6})\Big\}, \\ G_{i,2}^{(n)} &= \Big\{\text{in the interval } [in^{1-\frac{\varepsilon_0}{2}}, (i+1)n^{1-\frac{\varepsilon_0}{2}}) \text{ there are at least } n^{\frac{\varepsilon_0}{4}} \text{ true level-}n \\ &\qquad \text{couples which are } n\text{-different from all the level-}n \text{ couples in } [\ell, 5n]\Big\}, \end{aligned}$$

and let $G_i^{(n)} = G_{i,1}^{(n)} \cap G_{i,2}^{(n)}$.

Now, we define the values of n for which the localization test will be constructed.

Definition 2.7. *We say that $n > 2\ell$ is good, if:*

(i) On the interval $[n/2, \pi n]$ there are at least $n^{\varepsilon_0}/3$ true level-n couples, and the same holds for the interval $[\pi n, 5n]$.

(ii) All the level-n couples on the interval $[\ell, 5n]$ are n-different true level-n couples.

(iii) Any subinterval of $[\ell, 5n]$ of length $4n^{-1+\varepsilon_0+\delta_0+\delta_1}$ contains at most two bells. Note that this implies that any pair of consecutive bells X_k, X_{k+1} such that $X_{k+1}-X_k \le n^{-1+\varepsilon_0}(1-z_0^{-1}n^{-\delta_0/6})$ and $\{X_k, X_{k+1}\} \subset [\ell, 5n]$ is a level-n couple.

(iv) for any $i \in \mathbb{Z}$ such that $[in^{1-\frac{\varepsilon_0}{2}}, (i+1)n^{1-\frac{\varepsilon_0}{2}}) \cap [\ell, \pi n] = \emptyset$ and that $|i| < \exp(n^{\frac{\varepsilon_0}{8}})$ the event $G_i^{(n)}$ holds.

(v) On any interval of length $n^{1-\frac{\varepsilon_0}{2}}$, which is contained in $[\ell, 5n]$, there are at least $n^{\frac{\varepsilon_0}{4}}$ true level-n couples.

(vi) In the set $[-n^2, n^2] \setminus [-\ell, \ell]$, the minimal distance between two neighboring bells is at least n^{-3}.

Here we have chosen π just for definiteness, it could be another transcendental number which is between 1/2 and 5. The reason why we need a transcendental number there will become clear in Section 2.3, see the argument between (52) and (53). The following lemma ensures that there is an infinite sequence of good ns:

Lemma 2.4. *There exists $C > 0$ such that $\mathbf{P}[n$ is good$] \geq 1 - n^{-C}$ for all n large enough.*

Proof. We obtain lower bounds on the probabilities of the events described in items (i)–(vi) of Definition 2.7.

Item (i). If $j > \ell$ then for all large enough n

$$\begin{aligned}\mathbf{P}[&\text{there exists } k \text{ such that } X_{k-1} < j, \{X_k, X_{k+1}\} \subset (j, j+1), X_{k+1} > j+1,\\ &\min\{X_k - j, j + 1 - X_{k+1}\} \geq n^{-1+\varepsilon_0+\delta_0+\delta_1}, \{X_k, X_{k+1}\} \text{ is a true level-}n \text{ couple}]\\ &\qquad = e^{-1}n^{-1+\varepsilon_0}(1 - z_0^{-1}n^{-\delta_0/6})(1 - 2n^{-1+\varepsilon_0+\delta_0+\delta_1}) \end{aligned} \tag{29}$$

(note that the probability that there are exactly two points of X' on a unit interval is $(2e)^{-1}$). Since each of the intervals $[n/2, \pi n]$ and $[\pi n, 5n]$ contains more than n nonintersecting subintervals of length 1, using Lemma 2.1, we obtain that

$$\mathbf{P}[\text{event in (i) occurs}] \geq 1 - \exp(-L_1 n^{\varepsilon_0}) \tag{30}$$

for some L_1.

Item (ii). First, let us prove that, with large probability, the total number of level-n couples on the interval $[\ell, 5n]$ is $O(n^{\varepsilon_0})$. Let $k_0 = \min\{k : X_k > \ell\}$, and define $V_1 = X_{k_0} - \ell$, $V_i = \xi(k_0 + j - 1)$, $i \geq 2$. The random variables $V_i, i \geq 1$ are i.i.d. exponentials with parameter 1. Note that there exists $L_2 > 0$ such that (since $\mathbf{E}V_i = 1$)

$$\mathbf{P}\Big[\sum_{i=1}^{6n} V_i \leq 5n - \ell\Big] \leq \exp(-L_2 n), \tag{31}$$

so that with a very large probability the scenery in $[\ell, 5n]$ is determined by $(V_i, i = 1, \ldots, 6n)$. Then, since $\mathbf{P}[V_i \leq n^{-1+\varepsilon_0}] = O(n^{-1+\varepsilon_0})$, using Lemma 2.1, we obtain that for large enough L_3 and some $L_4 > 0$

$$\mathbf{P}\Big[\sum_{i=1}^{6n} \mathbf{1}\{V_i \leq 2n^{-1+\varepsilon_0}\} \geq L_3 n^{\varepsilon_0}\Big] \leq \exp(-L_4 n^{\varepsilon_0}) \tag{32}$$

for all n.

Let us call two *numbers* β_1, β_2 n-similar, if (28) holds. There exists L_5 such that for any $\beta_1 \leq 2n^{-1+\varepsilon_0}$

$$\big|\{\beta_2 \in \mathbb{R} : \min\{|\beta_1\beta_2^{-1} - 1|, |\beta_1^{-1}\beta_2 - 1|\} \leq 6z_0^{-1}n^{-\delta_0/6}\}\big| \leq L_5 n^{-1+\varepsilon_0-\frac{\delta_0}{6}}.$$

So, there exists L_6 such that

$$\mathbf{P}[\text{there exists } j < i \text{ such that } V_i \text{ is } n\text{-similar with } V_j \mid V_1, \ldots, V_{i-1}]$$

$$\leq \quad L_5 L_6 n^{-1+\varepsilon_0-\frac{\delta_0}{6}} \sum_{j=1}^{i} \mathbf{1}\{V_i \leq 2n^{-1+\varepsilon_0}\}. \tag{33}$$

Now, using (33) together with (31), (32), we obtain that (recall (2))

$$\mathbf{P}[\text{event in (ii) occurs}] \geq 1 - L_7 n^{-(\frac{\delta_0}{6}-2\varepsilon_0)}. \tag{34}$$

Item (iii). One can construct a collection of (at most) $2n^{2-\varepsilon_0-\delta_0-\delta_1}$ intervals of length $8n^{-1+\varepsilon_0+\delta_0+\delta_1}$ such that any subinterval of $[\ell, 5n]$ of length $4n^{-1+\varepsilon_0+\delta_0+\delta_1}$ is fully contained in at least one of the intervals of that collection. The probability that there exist more than two bells in an interval of length $8n^{-1+\varepsilon_0+\delta_0+\delta_1}$ is $O(n^{-3+3\varepsilon_0+3\delta_0+3\delta_1})$, so (recall (1))

$$\mathbf{P}[\text{event in (iii) occurs}] \geq 1 - L_8 n^{-1+2\varepsilon_0+2\delta_0+2\delta_1}. \tag{35}$$

Item (iv). Analogously to item (i), one can show that for all n large enough

$$\mathbf{P}[G^{(n)}_{i,1}] \geq 1 - \exp(-L_9 n^{\frac{\varepsilon_0}{2}}). \tag{36}$$

To estimate the probability of $G^{(n)}_{i,2}$, we first argue, as in (32) and item (i), that with large (as in the right-hand side of (36)) probability there are $O(n^{\frac{\varepsilon_0}{2}})$ true level-n couples. Similarly to (33), one may obtain that each of these true level-n couples has a good (bounded away from 0) chance to be different from all those in the interval $[\ell, \pi n]$. This shows that

$$\mathbf{P}[G^{(n)}_{i,2}] \geq 1 - \exp(-L_{10} n^{\frac{\varepsilon_0}{2}}), \tag{37}$$

and so

$$\mathbf{P}[\text{event in (iv) occurs}] \geq 1 - 2\exp(n^{\frac{\varepsilon_0}{8}})(\exp(-L_9 n^{\frac{\varepsilon_0}{2}}) + \exp(-L_{10} n^{\frac{\varepsilon_0}{2}})). \tag{38}$$

Item (v). As in item (i), one can see that, on a fixed interval of length $n^{1-\frac{\varepsilon_0}{2}}$ there are at least $n^{\frac{\varepsilon_0}{4}}$ true level-n couples with probability at least $1 - \exp(-L_{11} n^{\frac{\varepsilon_0}{2}})$. So,

$$\mathbf{P}[\text{event in (v) occurs}] \geq 1 - L_{12} n^{\frac{\varepsilon_0}{2}} \exp(-L_{11} n^{\frac{\varepsilon_0}{2}}). \tag{39}$$

Item (vi). Again, one can consider a collection of n^5 intervals of length $2n^{-3}$ such that any two points within $[-n^2, n^2]$ which are at most n^{-3} away from each other belong to (at least) one of those intervals. Since the probability of having at least two bells in an interval of length $2n^{-3}$ is $O(n^{-6})$, we obtain

$$\mathbf{P}[\text{event in (vi) occurs}] \geq 1 - \frac{L_{13}}{n}. \tag{40}$$

Lemma 2.4 now follows from (30), (34), (35), (38), (39), (40). □

Now, we construct the localization test. Suppose that n is good and consider all the level-n couples in the interval $[n/2, \pi n]$. Let $(\zeta'_n, \zeta'_n + \Delta'_n)$ be the leftmost (true) level-n couple on that interval, $(\zeta_n, \zeta_n + \Delta_n)$ the rightmost one, and let $\psi(n)$ be the number of other level-n couples there (note that, by (i) of Definition 2.7, $\psi(n) \geq n^{\varepsilon_0}/3 - 2$).

Define $\tau_0^{(n)} = 0$ and, for $i \geq 1$,

$$\tau_i^{(n)} = \inf\{t \geq \tau_{i-1}^{(n)} + 3n^2 : t \text{ satisfies (A), (B), (C), (D) below}\}, \tag{41}$$

where

(A) there exists $s \in [t - n^2, t)$ and $m_1 \in \mathbb{Z}_+$ such that $Y_{m_1} = s$ and there is a good level-n trill in m_1 compatible with the couple $(\zeta_n', \zeta_n' + \Delta_n')$;

(B) the number of n-different good level-n trills on the time interval $[t - n^2, t)$ is at least $\frac{\psi(n)}{2}$;

(C) for any good level-n trill from that interval there exists a level-n couple on $[n/2, \pi n]$ which is compatible to that trill;

(D) (suppose without restriction of generality that $\lfloor n^{\delta_0/2} \rfloor$ is even) for some $m_2 \in \mathbb{Z}_+$ there is a good level-n trill in m_2 such that it is compatible with the couple $(\zeta_n, \zeta_n + \Delta_n)$ and $Y_{m_2 + \lfloor n^{\delta_0/2} \rfloor} = t$.

In words, the above (A)–(D) are what we typically observe when the Brownian motion crosses the interval $[n/2, \pi n]$ (see Figure 3).

The main result of this section is the following

Lemma 2.5. *There exist $\delta_2, \delta_3 > 0$ such that*

$$\mathbb{P}[W_{\tau_i^{(n)}} = \zeta_n \text{ for all } i = 1, \dots, \lfloor \exp(n^{\delta_2}) \rfloor] \geq 1 - \exp(-n^{\delta_3}). \tag{42}$$

Proof. Choose a number $\delta_2 > 0$ such that

$$\delta_2 < \min\Big\{\frac{\delta_0}{6}, \delta_1, \frac{\varepsilon_0}{8}\Big\} \tag{43}$$

(in fact, due to (2), in the above display $\frac{\delta_0}{6}$ is redundant; it is included to make it more clear that δ_2 should be less than $\frac{\delta_0}{6}$).

Let us say that a time interval $[t_1, t_2]$ is a crossing of the interval $[a, b]$ by the Brownian motion, if $W_{t_1} = a$, $W_{t_2} = b$, and $W_s \notin \{a, b\}$ for $s \in (t_1, t_2)$. We say that a crossing $[t_1, t_2]$ of the interval $[n/2, \pi n]$ by the Brownian motion is *good*, if $t_2 - t_1 \leq n^2$, and there is j_0 such that $\tau_{j_0}^{(n)} \in [t_1, t_2]$ (see (A)–(D) above). Define the events

$$U_1^{(n)} = \Big\{\text{up to time } \exp(3n^{\delta_2}), \text{ there are at least } \lfloor \exp(n^{\delta_2}) \rfloor \text{ good crossings of the interval } [n/2, \pi n]\Big\},$$

$$U_2^{(n)} = \Big\{\text{up to time } \exp(3n^{\delta_2}), \text{ all the good level-}n\text{ trills produced when the Brownian motion was in the interval } [\ell, 5n] \text{ correspond to level-}n\text{ couples compatible with those trills}\Big\},$$

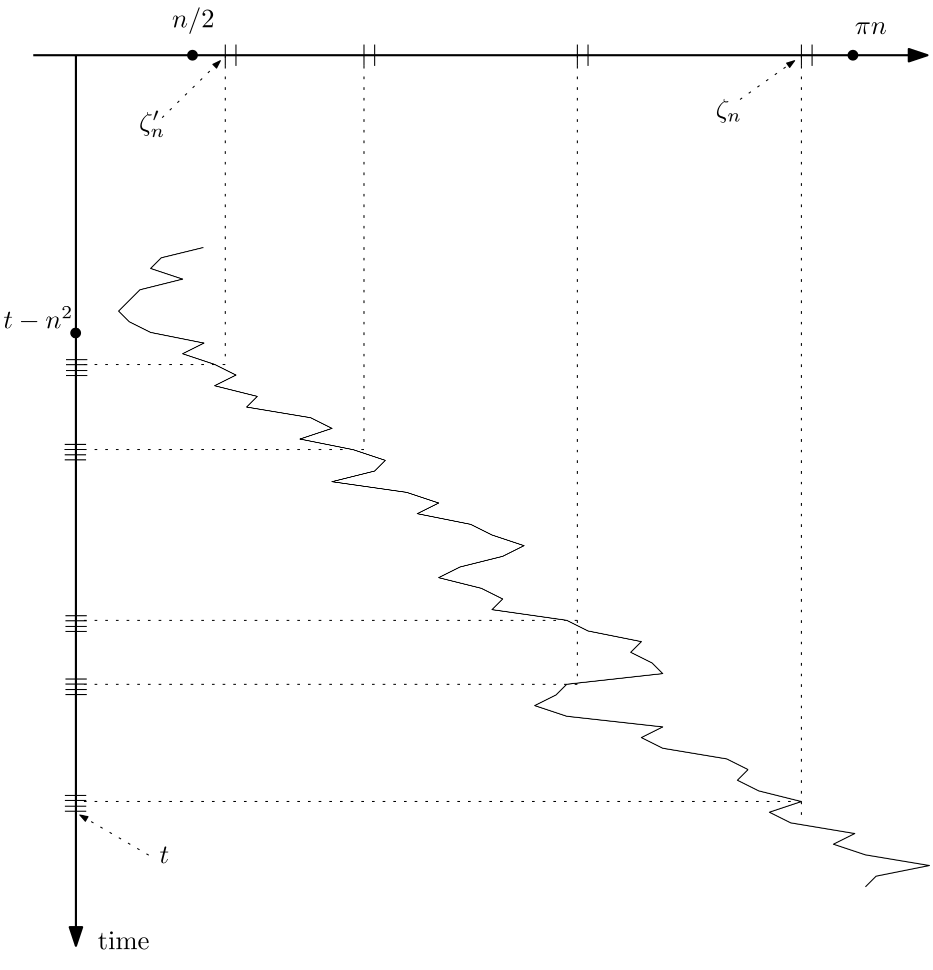

Figure 3: A typical crossing of the interval $[n/2, \pi n]$; only the level-n couples and the level-n trills are marked

$$
\begin{aligned}
U_3^{(n)} \;=\; & \Big\{\text{up to time } \exp(3n^{\delta_2}), \text{ on any time interval } I \text{ of length at least} \\
& n^{2-\frac{\varepsilon_0}{2}} \text{ and such that } \{W_s, s\in I\}\cap[n/2,\pi n]=\emptyset, \text{ one finds} \\
& \text{at least } n^{\frac{\varepsilon_0}{4}} \text{ good level-}n \text{ trills and at least } \tfrac{1}{2}n^{\frac{\varepsilon_0}{4}} \text{ of those trills} \\
& \text{are not compatible with any of the couples from } [n/2,\pi n]\Big\}, \\
U_4^{(n)} \;=\; & \Big\{\text{up to time } \exp(3n^{\delta_2}), \text{ on any time interval } I \text{ of length at least} \\
& n^{2-\frac{\varepsilon_0}{2}} \text{ the range of the Brownian motion is at most } n^{1-\frac{\varepsilon_0}{8}}\Big\},
\end{aligned}
$$

where the range of the Brownian motion on a time interval is the difference between the maximum and the minimum of the Brownian motion on that interval.

First, let us show that on $U_1^{(n)} \cap U_2^{(n)} \cap U_3^{(n)} \cap U_4^{(n)}$ the event $\{W_{\tau_i^{(n)}} = \zeta_n$ for all $i = 1, \dots, \lfloor \exp(n^{\delta_2}) \rfloor\}$ occurs. Since each good crossing corresponds to (at least) one occurrence of $\tau_\cdot^{(n)}$, on $U_1^{(n)}$ we have that $\tau^{(n)}_{\lfloor \exp(n^{\delta_2})\rfloor} \leq \exp(3n^{\delta_2})$. Now, let us suppose that there exists $i_0 \leq \lfloor \exp(n^{\delta_2}) \rfloor$ such that $a_0 := W_{\tau_{i_0}^{(n)}} \neq \zeta_n$. Consider the two possible cases: $a_0 \in [\ell, 5n]$, or $a_0 \notin [\ell, 5n]$. We know that $\tau_{i_0}^{(n)}$ is at the end of a level-n trill compatible with the level-n couple $(\zeta_n, \zeta_n + \Delta_n)$, so on the event $U_2^{(n)}$ it is impossible to have $a_0 \in [\ell, 5n]$. On the other hand, if $a_0 \notin [\ell, 5n]$, then (since $(5-\pi)n > n^{1-\frac{\varepsilon_0}{2}}$) on the event $U_4^{(n)}$ we have that $W_s \notin [n/2, \pi n]$ for all $s \in [\tau_{i_0}^{(n)} - n^{2-\frac{\varepsilon_0}{2}}, \tau_{i_0}^{(n)}]$. Thus, on $U_3^{(n)}$ we have that on the time interval $[\tau_{i_0}^{(n)} - n^{2-\frac{\varepsilon_0}{2}}, \tau_{i_0}^{(n)}]$ there will be good level-n trills which are not compatible with any of the level-n couples from $[n/2, \pi n]$; clearly, this contradicts (41).

Now let us estimate the probabilities of the events $U_i^{(n)}$, $i = 1, 2, 3, 4$.

First, we deal with $U_2^{(n)}$. Recall that, by Definition 2.7 (vi), the minimal distance between the bells in $[\ell, 5n]$ is at least n^{-3}. So, given that the particle is in some bell there, the time until the next ring will be greater than n^{-7} with probability at least $1 - \exp(-C_1 n^{1/2})$ for some $C_1 > 0$. Thus, up to time $\exp(3n^{\delta_2})$ we will have at most $n^7 \exp(3n^{\delta_2})$ rings produced by the bells in $[\ell, 5n]$, with probability at least

$$
1 - n^7 \exp(-C_1 n^{1/2} + 3n^{\delta_2})
$$

(recall that $\delta_2 < 1/2$ by e.g. (1)). Using Lemma 2.3 and (22), one obtains

$$
\mathbb{P}[U_2^{(n)}] \geq 1 - n^7 \exp(3n^{\delta_2})\big(\exp(-C_1 n^{1/2}) + \exp(-\gamma_2 n^{\frac{\delta_0}{6}}) + \tilde{\gamma}\exp(-n^{\delta_1}/4)\big). \tag{44}
$$

To estimate the probability of $U_1^{(n)}$, we note that by (24) and Lemma 2.3, the probability that a crossing of the interval $[n/2, \pi n]$ is good, is bounded away from 0 by some constant C_2. Also, with probability at least $1 - C_3 \exp(-n^{\delta_2})$ up to time $\exp(3n^{\delta_2})$ there will be at least $2C_2^{-1}\exp(n^{\delta_2})$ crossings of that interval. So,

$$
\mathbb{P}[U_1^{(n)}] \geq 1 - C_4 \exp(-n^{\delta_2}) \tag{45}
$$

for some $C_4 > 0$.

Now, note that the event $U_4^{(n)}$ occurs if on each of the intervals (of length $\frac{1}{2}n^{2-\frac{\varepsilon_0}{2}}$) $[(i-1)n^{2-\frac{\varepsilon_0}{2}}, in^{2-\frac{\varepsilon_0}{2}}]$, $i=1,\dots,\lfloor 2n^{-2+\frac{\varepsilon_0}{2}}\exp(3n^{\delta_2})\rfloor$, the range of the Brownian motion is at most $n^{1-\frac{\varepsilon_0}{8}}$. So, since for each i that happens with probability at least $1-\exp(-n^{\frac{\varepsilon_0}{8}})$, we obtain

$$\mathbb{P}[U_4^{(n)}] \geq 1 - 2n^{-2+\frac{\varepsilon_0}{2}}\exp(-n^{\frac{\varepsilon_0}{8}} + 3n^{\delta_2}). \tag{46}$$

The probability of the event $U_3^{(n)}$ can be bounded from below in the following way. Note that for each time interval of length $n^{2-\frac{\varepsilon_0}{2}}$ the range of the Brownian motion on that interval is greater than $2n^{1-\frac{\varepsilon_0}{2}}$ with probability at least $1-\exp(-C_5 n^{\frac{\varepsilon_0}{4}})$. Note also that

$$\mathbb{P}\Big[\max_{s\leq \exp(3n^{\delta_2})} |W_s| \leq \exp(n^{\frac{\varepsilon_0}{8}})\Big] \geq 1 - \exp\Big(-n^{\frac{\varepsilon_0}{8}} - \frac{3}{2}n^{\delta_2}\Big).$$

Then we use Definition 2.7 (iv) and (v) and Lemma 2.3 to obtain that

$$\mathbb{P}[U_3^{(n)}] \geq 1 - \exp(-C_6 n^{\frac{\varepsilon_0}{8}}) \tag{47}$$

for some $C_6 > 0$.

Using (44)–(47) it is straightforward to obtain (42), thus finishing the proof of Lemma 2.5. □

2.3 Reconstruction algorithm for the case when the interval of perturbation is known

In this section we describe the algorithm that reconstructs the local perturbation using the localization test of Section 2.2. As in the previous section, we assume here that it is known that the perturbation took place on the interval $[-\ell,\ell]$.

Let $k_1 = \min\{k : X_k \in [-\ell,\ell]\}$, $k_2 = \max\{k : X_k \in [-\ell,\ell]\}$. Denote by $m = k_2 - k_1 + 1$ the number of points of the (true) scenery in the interval $[-\ell,\ell]$, and abbreviate by $a_i = X_{k_1+i-1}+\ell$ the distance from the left end of the interval to the ith point of the scenery there, $i=1,\dots,m$. Moreover, for $i=1,2,3,\dots$ denote $A_i = a_1^i + \dots + a_m^i$.

Before plunging into technical details, we give a heuristical explanation of what is going to happen in this section. By Lemma 2.5, there is a sequence of stopping times $\tau_i^{(n)}$ such that at those times the Brownian motion is in some known location ζ_n (which is at distance $O(n)$ from the interval $[-\ell,\ell]$) for $i=1,\dots,\lfloor\exp(n^{\delta_2})\rfloor$, with large probability. So, one can obtain a good empirical approximation $Z^{(n)}$ for $\mathbb{P}^{\zeta_n}$[there is a ring in $[n^2, n^2+\theta_n]$], where θ_n is some suitably defined small number. We supposed that outside $[-\ell,\ell]$ the scenery is known, so we have also an empirical approximation $\hat{Z}^{(n)}$ for

$$\mathbb{P}^{\zeta_n}[\text{there is a ring in } [n^2, n^2+\theta_n] \text{ caused inside } [-\ell,\ell]].$$

Consider the quantity $B^{(n)}$ defined in (50), note that it is expressed in terms of the unknown numbers $m, a_1, \dots, a_m$. After some technical work (one has to show that it is unlikely that there are two or more rings in the time interval $[n^2, n^2+\theta_n]$), with the help of Lemma 2.6 it turns out that $B^{(n)}$ is (up to smaller terms) the analitical expression for the probability in the above display; also, $\hat{Z}^{(n)}$ and $B^{(n)}$ are typically very close (formula (52)). Then, we represent $B^{(n)}$ in terms of the quantities $m, A_1, A_2, A_3, \dots$ (formula (53)). Analysing (53), one sees that, dividing $B^{(n)}$ by some (known) quantity, we obtain the expression of the form

$$m + \text{const}\frac{A_1}{n} + \text{const}'\frac{A_2}{n^2} + \cdots$$

(the constants are known as well). Now, the idea is to reconstruct first the quantity m; then, given m, reconstruct A_1; then, given m and A_1, reconstruct A_2, and so on. As the last step, we recover the values of $a_1, \dots, a_m$ from $A_1, \dots, A_m$.

We need the following technical fact:

Lemma 2.6. *Suppose that $\theta = o(n^{-3})$ and $x = O(n)$. Then*

$$\begin{aligned} &\mathbb{P}\big[W_t = x \text{ for some } t \in [n^2, n^2 + \theta]\big] \\ &\quad = \frac{1}{\sqrt{2\pi} n} \exp\Big(-\frac{x^2}{2n^2}\Big)\Big(\frac{2\sqrt{2}}{\sqrt{\pi}}\theta^{1/2} + O(n^{-2}\theta^{3/2})\Big). \end{aligned} \tag{48}$$

Proof. By (6) and conditioning on W_{n^2}, the left-hand side of (48) can be written as follows:

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} n} \exp\Big(-\frac{(y-x)^2}{2n^2}\Big) \int_0^\theta \frac{|y|}{\sqrt{2\pi} s^{3/2}} \exp\Big(-\frac{y^2}{2s}\Big) ds\, dy \\ &\quad = \frac{1}{\sqrt{2\pi} n} \exp\Big(-\frac{x^2}{2n^2}\Big) \int_0^\theta \frac{1}{\sqrt{2\pi} s^{3/2}} \int_{-\infty}^{+\infty} |y| \exp\Big(-\frac{y^2}{2n^2} + \frac{xy}{n^2} - \frac{y^2}{2s}\Big) dy\, ds. \end{aligned}$$

Then, in the last integral we change the variables $u := \frac{y^2}{2s}$:

$$\begin{aligned} &\int_0^\theta \frac{1}{\sqrt{2\pi} s^{3/2}} \int_{-\infty}^{+\infty} |y| \exp\Big(-\frac{y^2}{2n^2} + \frac{xy}{n^2} - \frac{y^2}{2s}\Big) dy\, ds \\ &\quad = \int_0^\theta \frac{1}{\sqrt{2\pi s}} \int_0^{+\infty} \Big(\exp\Big(-\frac{su}{n^2} + \frac{x\sqrt{2su}}{n^2} - u\Big) + \exp\Big(-\frac{su}{n^2} - \frac{x\sqrt{2su}}{n^2} - u\Big)\Big) du\, ds \\ &\quad = \int_0^\theta \frac{1}{\sqrt{2\pi s}} (2 + sO(n^{-2}))\, ds, \end{aligned}$$

and we arrive at (48). □

Define $\theta_n = \exp(-n^{\delta_2/2})$. Let

$$Z_i^{(n)} = \mathbf{1}\{\text{there is a ring in the interval } [\tau_i^{(n)} + n^2, \tau_i^{(n)} + n^2 + \theta_n]\},$$

and let

$$Z^{(n)} = \exp(-n^{\delta_2}) \sum_{i=1}^{\lfloor \exp(n^{\delta_2}) \rfloor} Z_i^{(n)}.$$

Let

$$h^{(n)} = \mathbb{P}^{\zeta_n}[\text{there is a ring in the interval } [n^2, n^2+\theta_n]].$$

By Lemma 2.5 and usual large deviation techniques (use e.g. Lemma 2.1), we obtain that

$$\mathbb{P}\Big[|Z^{(n)} - h^{(n)}| > \exp\Big(-\frac{n^{\delta_2}}{2}\Big)\Big] \le \exp\Big(-\frac{n^{\delta_2}}{4}\Big). \tag{49}$$

Define

$$\mu^{(n)} = \mathbb{P}^{\zeta_n}[\text{there is a ring in the interval } [n^2, n^2+\theta_n] \text{ caused by a bell outside } [-\ell,\ell]].$$

Note that, since the scenery outside $[-\ell,\ell]$ is completely known to us, $\mu^{(n)}$ is known as well. Let $\hat{Z}^{(n)} = Z^{(n)} - \mu^{(n)}$ and abbreviate also $b_n = (\zeta_n + \ell)/n$. Let

$$B^{(n)} = \frac{2\theta_n^{1/2}}{\pi n}\exp\Big(-\frac{b_n^2}{2}\Big)\sum_{i=1}^m \exp\Big(-\frac{a_i^2}{2n^2} + \frac{b_n a_i}{n}\Big). \tag{50}$$

By Definition 2.7 (vi), if n is large enough, on the interval $[-n^2, n^2]$ the minimal distance between any two bells is at least n^{-3}, so

$$\begin{aligned}
&\mathbb{P}^{\zeta_n}[\text{there are at least two rings in the interval } [n^2, n^2+\theta_n]] \\
&\quad\le \mathbb{P}^{\zeta_n}[|W_{n^2}| > n^2/2] + \mathbb{P}^0[\max_{s\le\theta_n}|W_s| \ge n^{-3}] \\
&\quad\le e^{-n}
\end{aligned} \tag{51}$$

for all n large enough. Let S be the number of rings in the time interval $[n^2, n^2+\theta_n]$, and define the events

$$\begin{aligned}
\tilde{H} &= \{\text{all the rings in } [n^2, n^2+\theta_n] \text{ were caused by the bells inside } [-\ell,\ell]\}, \\
\tilde{H}_i &= \{\text{there is a ring in } [n^2, n^2+\theta_n] \text{ caused by the bell in } a_i - \ell\},
\end{aligned}$$

$i = 1,\dots,m$. We have

$$h^{(n)} = \mathbb{P}[S \ge 1] = \mu^{(n)} + \mathbb{P}[S \ge 1, \tilde{H}],$$

so $Z^{(n)} - h^{(n)} = \hat{Z}^{(n)} - \mathbb{P}[S \ge 1, \tilde{H}]$. Then, we can write

$$\mathbb{P}[S \ge 1, \tilde{H}] \le \sum_{i=1}^m \mathbb{P}[\tilde{H}_i],$$

and

$$\mathbb{P}[S \ge 1, \tilde{H}] \ge \sum_{i=1}^m \mathbb{P}[\tilde{H}_i] - m\mathbb{P}[S \ge 2].$$

By Lemma 2.6 and (49)–(51), we can write for all n large enough

$$\Big|B^{(n)} - \sum_{i=1}^m \mathbb{P}[\tilde{H}_i]\Big| \le \tilde{C} m n^{-3}\exp\Big(-3n^{\delta_2/2}2\Big).$$

Thus, if n is large enough, we have

$$\mathbb{P}\Big[\Big|\hat{Z}^{(n)} - B^{(n)}\Big| > 2\exp(-n^{\delta_2/2})\Big] \le \exp\Big(-\frac{n^{\delta_2}}{4}\Big). \tag{52}$$

Consider now the function $\varphi_b(x) = \exp\big(-\frac{x^2}{2} + bx\big)$ and its Taylor series in $x = 0$:

$$\varphi_b(x) = \exp\Big(-\frac{x^2}{2} + bx\Big) = 1 + \sum_{k=1}^{\infty} M_k(b)x^k.$$

It is easy to see that $M_k(b)$ is a polynomial of kth degree of b, so if b is a transcendental number, $M_k(b) \neq 0$ for all k. By Definition 2.7 (v), we have that $b_n \to \pi$ as $n \to \infty$, so if n is large enough, then we have $M_i(b_n) \neq 0$ for all $i \le m$.

Now, we can write

$$B^{(n)} = \frac{2\theta_n^{1/2}}{\pi n}\exp\Big(-\frac{b_n^2}{2}\Big)\Big(m + \frac{M_1(b_n)A_1}{n} + \frac{M_2(b_n)A_2}{n^2} + \cdots\Big). \tag{53}$$

Let us define the estimator for m (recall that m is the number of points of the scenery X in $[-\ell,\ell]$):

$$\hat{m}(n) = \Big[\hat{Z}^{(n)}\exp\Big(\frac{b_n^2}{2}\Big)\frac{\pi n}{2\theta_n^{1/2}}\Big]; \tag{54}$$

here $[y]$ stands for the integer part of $y + \frac{1}{2}$, i.e., $[y]$ is the integer closest to y.

Given m, define the estimator for A_1 (cf. (53)):

$$\hat{A}_1(n;m) = \Big(\hat{Z}^{(n)}\exp\Big(\frac{b_n^2}{2}\Big)\frac{\pi n}{2\theta_n^{1/2}} - m\Big)\frac{n}{M_1(b_n)},$$

and, for all $i \ge 2$, given m and $A_1,\dots,A_{i-1}$, define the estimator for A_i:

$$\hat{A}_i(n;m,A_1,\dots,A_{i-1}) = \Big(\hat{Z}^{(n)}\exp\Big(\frac{b_n^2}{2}\Big)\frac{\pi n}{2\theta_n^{1/2}} - m - \sum_{j=1}^{i-1}\frac{M_j(b_n)A_j}{n^j}\Big)\frac{n^i}{M_i(b_n)}.$$

Since, up to polynomial terms, $B^{(n)}$ is of order $\exp(-\frac{n^{\delta_2/2}}{2})$, if the event $\{|\hat{Z}^{(n)} - B^{(n)}| \le 2\exp(-n^{\delta_2/2})\}$ occurs, then for n large enough it holds that $\hat{m}(n) = m$. So, using (52), one can observe that

$$\mathbb{P}[\hat{m}(n) \neq m] \le \exp\Big(-\frac{n^{\delta_2}}{4}\Big) \tag{55}$$

and, using the abbreviation $C = 2\pi e^{\pi^2/2}|M_i(\pi)|^{-1}$ (note that $|M_i(b_n)| \ge |M_i(\pi)|/2$ for all large enough n)

$$\mathbb{P}\Big[|\hat{A}_i(n;m,A_1,\dots,A_{i-1}) - A_i| \ge Cn^{i+1}\exp\Big(-\frac{n^{\delta_2/2}}{2}\Big)\Big] \le \exp\Big(-\frac{n^{\delta_2}}{4}\Big). \tag{56}$$

Now, informally, the idea is the following: take a sequence of ns going fast to infinity, reconstruct m (using also the Borel-Cantelli lemma), then reconstruct A_1, and so on. Formally, consider the sequence $n_k = 2^k$, $k = 1, 2, 3, \ldots$. Then, by Lemma 2.4, n_k will be good for all but finitely many k. Using (55) and Borel-Cantelli lemma, we obtain that there is k_0 such that

$$\hat{m}(n_k) = m \qquad \text{for all } k \geq k_0. \tag{57}$$

Then, given m, we are able to determine A_1 in the following way: by (56),

$$\lim_{k\to\infty} \hat{A}_1(n_k, m) = A_1 \qquad \text{a.s.} \tag{58}$$

Inductively, given m and $A_1, \ldots A_{i-1}$, we determine A_i by

$$\lim_{k\to\infty} \hat{A}_i(n_k, m, A_1, \ldots, A_{i-1}) = A_i \qquad \text{a.s.,} \tag{59}$$

for all $i \leq m$.

At this point we need the following elementary fact:

Lemma 2.7. *Suppose that $a_1, \ldots, a_m$ are positive numbers satisfying the following system of algebraic equations*

$$\begin{cases} a_1 + \cdots + a_m & = & d_1 \\ & \cdots & \\ a_1^m + \cdots + a_m^m & = & d_m \end{cases} \tag{60}$$

Suppose also that $(a_1', \ldots, a_m')$ is another solution of the system (60). Then, it holds that $\{a_1, \ldots, a_m\} = \{a_1', \ldots, a_m'\}$, i.e., $a_1', \ldots, a_m'$ is simply a reordering of the collection $a_1, \ldots, a_m$.

Proof. This is an easy consequence of Newton's and Vieta's formulas (see e.g. Chapter 6.2 of (7)). □

To conclude this section, it remains to note that, by Lemma 2.7, one can uniquely determine $a_1, \ldots, a_m$ from $A_1, \ldots, A_m$.

2.4 Reconstruction algorithm for the general case

Now, suppose that we do not know about where the perturbation took place, and that we only know it is local in the sense of Definition 1.1. This means that there exists N_0 (which is not known to us) such that the interval of perturbation is fully inside $[-N, N]$ for all $N \geq N_0$. Denote by $\tilde{X}^{(N)}$ the result of application of the reconstruction algorithm of Section 2.3 with $\ell := N$. Note, however, that it is not clear if the algorithm of Section 2.3 produces any result (i.e., (57), (59) hold) when the perturbation is not limited to $[-N, N]$. When the algorithm does not produce the result, we formally define $\tilde{X}^{(N)} := \emptyset$.

Then, it is clear that the true scenery X can be obtained as

$$X = \lim_{N\to\infty} \tilde{X}^{(N)},$$

where the limit can be formally defined in any reasonable sense, since a.s. $\tilde{X}^{(N)} = X$ for all $N \geq N_0$. This concludes the proof of Theorem 1.1. □

3 Final remarks and open problems

- In Theorem 1.1, we can suppose that X is equivalent to some local perturbation of X', where "equivalent" means "can be obtained by shift and (possibly) reflection". In this case, we can reconstruct X up to equivalence, i.e., the result of the application of the reconstruction algorithm will be a.s. equivalent to X.

- If $X^{(1)}$ is *any* scenery, and $X^{(2)}$ is a random realization (independent of $X^{(1)}$) of a one-dimensional Poisson process, then a.s. $X^{(1)}$ and $X^{(2)}$ are distinguishable. We do not describe the distinguishing algorithm in detail here (since the sceneries are "globally" different, it is much easier to distinguish them), it is possible to construct this algorithm using e.g. the following idea: the localization test build upon $X^{(2)}$ (as in Section 2.2) will typically fail for $X^{(1)}$. This shows that almost every two sceneries are distinguishable.

- However, the method of this paper is not applicable to periodic sceneries (this includes also the problem of reconstructing a scenery on a circle), so we cannot answer the question whether one can distinguish between any two periodic sceneries, or reconstruct a single defect in a periodic scenery. This is because the main idea of the present paper is that one can find pairs of bells that are arbitrarily close to each other, and then one can build a localization test based upon those pairs; in a periodic scenery, this is not possible.

- The question whether there are indistinguishable sceneries is open as well (in the discrete case such sceneries do exist, see (8)).

- Another open question is whether one can reconstruct a completely unknown scenery (supposing, say, that it is a random realization of a Poisson process), up to equivalence. For now, it seems to be a difficult problem. The reason is that, as mentioned before, many methods used in discrete scenery reconstruction do not work here. Specifically, it is possible to construct a localization test even for a completely unknown scenery (obtained as a realization of a Poisson process) in roughly the same way as in this paper, but then it is not clear how to reconstruct the scenery outside the close pairs, since the "straight crossing" method of discrete scenery reconstruction does not work here. Perhaps, a first step in this direction would be building a reconstruction algorithm which works on the sceneries produced not by a Poisson process, but by some process that "favors" more the close pairs of bells.

- The question of how much information about the true scenery can be extracted from a finite piece of observations (say, up to time t) seems tractable, but is left unaddressed in this paper. For now, we can conjecture (but not yet prove) that if the interval of perturbation is known, then in time t one can reconstruct the scenery there with precision t^{-const} and with confidence $1-t^{-const'}$. However, it is not clear to us with what confidence one can localise the interval of perturbation up to time t, in the case when that interval is unknown.

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