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Stable convergence of generalized L^2 stochastic integrals and the principle of conditioning

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Abstract

We consider generalized adapted stochastic integrals with respect to independently scattered random measures with second moments, and use a decoupling technique, formulated as a "principle of conditioning", to study their stable convergence towards mixtures of infinitely divisible distributions. The goal of this paper is to develop the theory. Our results apply, in particular, to Skorohod integrals on abstract Wiener spaces, and to multiple integrals with respect to independently scattered and finite variance random measures. The first application is discussed in some detail in the final section of the present work, and further extended in a companion paper (Peccati and Taqqu (2006b)). Applications to the stable convergence (in particular, central limit theorems) of multiple Wiener-Itô integrals with respect to independently scattered (and not necessarily Gaussian) random measures are developed in Peccati and Taqqu (2006a, 2007). The present work concludes with an example involving quadratic Brownian functionals.

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1 Introduction

In this paper we establish several criteria, ensuring the stable convergence of sequences of "generalized integrals" with respect to independently scattered random measures over abstract Hilbert spaces. The notion of generalized integral is understood in a very wide sense, and includes for example Skorohod integrals with respect to isonormal Gaussian processes (see e.g. (17)), multiple Wiener-Itô integrals associated to general Poisson measures (see (21), or (13)), or the class of iterated integrals with respect to orthogonalized Teugels martingales introduced in (20). All these random objects can be represented as appropriate generalized "adapted stochastic integrals" with respect to a (possibly infinite) family of Lévy processes, constructed by means of a well-chosen increasing family of orthogonal projections. These adapted integrals are also the limit of sums of arrays of random variables with a special dependence structure. We shall show, in particular, that their asymptotic behavior can be naturally studied by means of a decoupling technique, known as the "principle of conditioning" (see e.g. (12) and (40)), that we develop in the framework of stable convergence (see (11, Chapter 4)).

Our setup is roughly the following. We consider a centered and square integrable random field $X = \{X(h) : h \in \mathfrak{H}\}$, indexed by a separable Hilbert space $\mathfrak{H}$, and verifying the isomorphic relation $\mathbb{E}[X(h)X(h')] = (h,h')_{\mathfrak{H}}$, where $(\cdot,\cdot)_{\mathfrak{H}}$ is the inner product on $\mathfrak{H}$. There is no time involved. To introduce time, we endow the space $\mathfrak{H}$ with an increasing family of orthogonal projections, say π_t, $t \in [0,1]$, such that $\pi_0 = 0$ and $\pi_1 = \text{id.}$ (the identity). Such projections operators induce the (canonical) filtration $\mathcal{F}^\pi = \{\mathcal{F}_t^\pi : t \in [0,1]\}$, where each $\mathcal{F}_t^\pi$ is generated by random variables of the type $X(\pi_t h)$, and one can define (e.g., as in (38) for Gaussian processes) a class of $\mathcal{F}^\pi$-adapted and $\mathfrak{H}$-valued random variables. If for every $h \in \mathfrak{H}$ the application $t \mapsto X(\pi_t h)$ is also a $\mathcal{F}^\pi$-Lévy process, then there exists a natural Itô type stochastic integral, of adapted and $\mathfrak{H}$-valued variables, with respect to the infinite dimensional process $t \mapsto \{X(\pi_t h) : h \in \mathfrak{H}\}$. Denote by $J_X(u)$ the integral of an adapted random variable u with respect to X. As will be made clear in the subsequent discussion, as well as in the companion papers (24) and (23), several random objects appearing in stochastic analysis (such as Skorohod integrals, or the multiple Poisson integrals quoted above) are in fact generalized adapted integrals of the type $J_X(u)$, for some well chosen random field X. Moreover, the definition of $J_X(u)$ mimics in many instances the usual construction of adapted stochastic integrals with respect to real-valued martingales. In particular: (i) each stochastic integral $J_X(u)$ is associated to a $\mathcal{F}^\pi$-martingale, namely the process $t \mapsto J_X(\pi_t u)$ and (ii) $J_X(u)$ is the limit (in L^2) of finite "adapted Riemann sums" of the kind $S(u) = \sum_{j=1,\dots,n} F_j X\left(\left(\pi_{t_{j+1}} - \pi_{t_j}\right)h_j\right)$, where $h_j \in \mathfrak{H}$, $t_n > t_{n-1} > \cdots > t_1$ and $F_j \in \mathcal{F}_{t_j}^\pi$. We show that, by using a decoupling result known as "principle of conditioning" (Theorem 1 in (40) – see Section 2 below for a very general form of such principle), the stable and, in particular, the weak convergence of sequences of sums such as $S(u)$ is completely determined by the asymptotic behavior of random variables of the type

$$\widetilde{S}(u) = \sum_{j=1,\dots,n} F_j \widetilde{X}\left(\left(\pi_{t_{j+1}} - \pi_{t_j}\right)h_j\right),$$

where $\widetilde{X}$ is an independent copy of X. Note that the vector

$$\widetilde{V} = \left(F_1 \widetilde{X}\left(\left(\pi_{t_2} - \pi_{t_1}\right)h_1\right), \dots, F_n \widetilde{X}\left(\left(\pi_{t_{n+1}} - \pi_{t_n}\right)h_n\right)\right),$$

enjoys the specific property of being *decoupled* (i.e., conditionally on the F_j's, its components are independent) and *tangent* to the "original" vector

$$V = \left(F_1 X\left(\left(\pi_{t_2} - \pi_{t_1}\right) h_1\right), ..., F_n X\left(\left(\pi_{t_{n+1}} - \pi_{t_n}\right) h_n\right)\right),$$

in the sense that for every j, and conditionally on the r.v.'s F_k, $k \leq j$, $F_j X\left(\left(\pi_{t_{j+1}} - \pi_{t_j}\right) h_j\right)$ and $F_j \widetilde{X}\left(\left(\pi_{t_{j+1}} - \pi_{t_j}\right) h_j\right)$ have the same law (the reader is referred to (10) or (14) for a discussion of the general theory of tangent processes). The *principle of conditioning* combines "decoupling" and "tangency". The idea is to study the convergence of sequences such as $J_X(u_n)$, $n \geq 1$, where each u_n is adapted, by means of simpler random variables $\widetilde{J}_X(u_n)$, obtained from a decoupled and tangent version of the martingale $t \mapsto J_X(\pi_t u_n)$. In particular (see Theorem 7 below, as well as its consequences) we shall prove that, since such decoupled processes can be shown to have conditionally independent increments, the problem of the stable convergence of $J_X(u_n)$ can be reduced to the study of the convergence in probability of sequences of random Lévy-Khinchine exponents. This represents an extension of the techniques initiated in (19) and (27) where, in a purely Gaussian context, the CLTs for multiple Wiener-Itô integrals are characterized by means of the convergence in probability of the quadratic variation of Brownian martingales. We remark that the extensions of (19) and (27) achieved in this work, and in the three companion papers (24), (23) and (25), go in two directions: (a) we consider general (not necessarily Gaussian) square integrable and independently scattered random measures, (b) we study stable convergence, instead of weak convergence, so that, for instance, our results can be used in the Gaussian case to obtain non-central limit theorems (see e.g. Section 5 below, as well as (24)). The reader is also referred to (22) for an application of the results obtained in (24) to Bayesian survival analysis.

When studying the stable convergence of random variables that are terminal values of continuous-time martingales, one could alternatively use an approach based on the stable convergence of semimartingales, as developed e.g. in (16), (5) or (11, Chapter 4), instead of the above decoupling techniques. However, even in this case the principle of conditioning (which is in some sense the discrete-time skeleton of the general semimartingale results), as formulated in the present paper, often requires less stringent assumptions. For instance, conditions (8) and (38) below are weak versions of the *nesting condition* introduced by Feigin (5).

The main purpose of this paper is to develop a theory of stable convergence of stochastic integrals based on the principle of conditioning. To keep the length of the paper within bounds, we include only a few applications. We focus on the stable convergence of Skorohod integrals to a mixture of Gaussian distributions. We also include an application to the convergence of sequences of Brownian functionals, namely, we show that the (properly normalized) sequence of integrals $\int_0^1 t^{2n}[(W_1^{(n)})^2 - (W_t^{(n)})^2]dt$ converges stably, as $n \to \infty$, to a mixture of Gaussian distributions. In these integrals, the sequence of processes $W^{(n)}$, $n \geq 1$, can be composed, for example, of "flipped" Brownian motions, where the flipping mechanism evolves with n.

Further applications of the theory developed in this paper can be found in the companion papers (24), (23) and (25). In (23) and (25), we study the stable convergence of multiple integrals with respect to non-Gaussian infinitely divisible random measures with finite second moments, with particular focus on double integrals. We provide explicit conditions on the kernels of the integrals for convergence to hold.

In (24), we study the stable convergence of multiple Wiener-Itô integrals by using a martingale approach, and we consider the following application. It is shown in (19) that a sequence of normalized Wiener-Itô integrals converges to a Gaussian distribution if its fourth moments converge to 3. The paper (24) contains an extension of this result to stable convergence towards a mixture of Gaussian distributions.

The paper is organized as follows. In Section 2, we discuss a general version of the principle of conditioning and in Section 3 we present the general setup in which it is applied. The above mentioned convergence results are established in Section 4. In Section 5, our results are applied to study the stable convergence of Skorohod integrals with respect to a general isonormal Gaussian process. Finally, in Section 6 we discuss an application to sequences of quadratic Brownian functionals.

2 Stable convergence and the principle of conditioning

We shall develop a general setting for the *principle of conditioning* (*POC* in the sequel) for arrays of real valued random variables. Our discussion is mainly inspired by a remarkable paper by X.-H. Xue (40), generalizing the classic results by Jakubowski (12) to the framework of stable convergence. Note that the results discussed below refer to a discrete time setting. However, thanks to some density arguments, we will be able to apply most of the POC techniques to general stochastic measures on abstract Hilbert spaces.

Instead of adopting the formalism of (40) we choose, for the sake of clarity, to rely in part on the slightly different language of (6, Ch. 6 and 7). To this end, we shall recall some notions concerning stable convergence, conditional independence and decoupled sequences of random variables. From now on, all random objects are supposed to be defined on an adequate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and all σ-fields introduced below will be tacitly assumed to be complete; $\overset{\mathbb{P}}{\rightarrow}$ means convergence in probability; $\mathbb{R}$ stands for the set of real numbers; $\triangleq$ denotes a new definition.

We start by defining the class $\mathbf{M}$ of random probability measures, and the class $\widehat{\mathbf{M}}$ (resp. $\widehat{\mathbf{M}}_0$) of random (resp. non-vanishing and random) characteristic functions.

Definition A (see e.g. (40)) – Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-field on $\mathbb{R}$.

(A-i) A map $\mu(\cdot,\cdot)$, from $\mathcal{B}(\mathbb{R}) \times \Omega$ to $\mathbb{R}$ is called a *random probability* (on $\mathbb{R}$) if, for every $C \in \mathcal{B}(\mathbb{R})$, $\mu(C,\cdot)$ is a random variable and, for $\mathbb{P}$-a.e. ω, the map $C \mapsto \mu(C,\omega)$, $C \in \mathcal{B}(\mathbb{R})$, defines a probability measure on $\mathbb{R}$. The class of all random probabilities is noted $\mathbf{M}$, and, for $\mu \in \mathbf{M}$, we write $\mathbb{E}\mu(\cdot)$ to indicate the (deterministic) probability measure

$$\mathbb{E}\mu(C) \triangleq \mathbb{E}[\mu(C,\cdot)], \quad C \in \mathcal{B}(\mathbb{R}). \tag{1}$$

(A-ii) For a measurable map $\phi(\cdot,\cdot)$, from $\mathbb{R}\times\Omega$ to $\mathbb{C}$, we write $\phi\in\widehat{\mathbf{M}}$ whenever there exists $\mu\in\mathbf{M}$ such that

$$\phi(\lambda,\omega)=\widehat{\mu}(\lambda)(\omega), \quad \forall\lambda\in\mathbb{R}, \text{ for } \mathbb{P}\text{-a.e. } \omega, \tag{2}$$

where $\widehat{\mu}(\cdot)$ is defined as

$$\widehat{\mu}(\lambda)(\omega)=\begin{cases}\int\exp(i\lambda x)\,\mu(dx,\omega) & \text{if } \mu(\cdot,\omega) \text{ is a probability measure}\\ 1 & \text{otherwise.}\end{cases}, \quad \lambda\in\mathbb{R}. \tag{3}$$

(A-iii) For a given $\phi\in\widehat{\mathbf{M}}$, we write $\phi\in\widehat{\mathbf{M}}_0$ whenever $\mathbb{P}\{\omega:\phi(\lambda,\omega)\neq 0 \ \ \forall\lambda\in\mathbb{R}\}=1$.

When $\mu(\cdot,\omega)$ is not a probability measure, the choice $\widehat{\mu}(\lambda)(\omega)=1$ (i.e. $\mu=$ unit mass at 0) in (3) is arbitrary, and allows $\widehat{\mu}(\lambda)(\omega)$ to be defined for all ω. Observe that, for every $\omega\in\Omega$, $\widehat{\mu}(\lambda)(\omega)$ is a continuous function of λ. The probability $\mathbb{E}\mu(\cdot)=\int_\Omega\mu(\cdot,\omega)\,d\mathbb{P}(\omega)$ defined in (1) is often called a *mixture* of probability measures.

The following definition of *stable convergence* extends the usual notion of convergence in law.

Definition B (see e.g. (11, Chapter 4) or (40)) – Let $\mathcal{F}^*\subseteq\mathcal{F}$ be a σ-field, and let $\mu\in\mathbf{M}$. A sequence of real valued r.v.'s $\{X_n:n\geq 1\}$ is said to *converge* $\mathcal{F}^*$*-stably* to $\mu(\cdot)$, written $X_n\to_{(s,\mathcal{F}^*)}\mu(\cdot)$, if, for every $\lambda\in\mathbb{R}$ and every bounded complex-valued $\mathcal{F}^*$-measurable r.v. Z,

$$\lim_{n\to+\infty}\mathbb{E}\left[Z\times\exp(i\lambda X_n)\right]=\mathbb{E}\left[Z\times\widehat{\mu}(\lambda)\right], \tag{4}$$

where the notation is the same as in (3).

If X_n converges $\mathcal{F}^*$-stably, then the conditional distributions $\mathcal{L}(X_n\mid A)$ converge for any $A\in\mathcal{F}^*$ such that $\mathbb{P}(A)>0$ (see e.g. (11, Section 5, §5c) for further characterizations of stable convergence). The random variable Z in (4) is bounded and complex-valued. By setting $Z=1$, we obtain that if $X_n\to_{(s,\mathcal{F}^*)}\mu(\cdot)$, then the law of the X_n's converges weakly to $\mathbb{E}\mu(\cdot)$. Moreover, by a density argument, $X_n\to_{(s,\mathcal{F}^*)}\mu(\cdot)$ if, and only if, (4) holds for random variables with the form $Z=\exp(i\gamma Y)$, where $\gamma\in\mathbb{R}$ and Y is $\mathcal{F}^*$-measurable. We also note that, if a sequence of random variables $\{U_n:n\geq 0\}$ is such that $(U_n-X_n)\to 0$ in $L^1(\mathbb{P})$ and $X_n\to_{(s,\mathcal{F}^*)}\mu(\cdot)$, then $U_n\to_{(s,\mathcal{F}^*)}\mu(\cdot)$.

The following definition shows how to replace an array $X^{(1)}$ of real-valued random variables by a simpler, *decoupled* array $X^{(2)}$.

Definition C (see (6, Chapter 7)) – Let $\{N_n:n\geq 1\}$ be a sequence of positive natural numbers, and let

$$X^{(i)}\triangleq\left\{X_{n,j}^{(i)}:0\leq j\leq N_n,\ n\geq 1\right\}, \text{ with } X_{n,0}^{(i)}=0,$$

$i=1,2$, be two arrays of real valued r.v.'s, such that, for $i=1,2$ and for each n, the sequence

$$X_n^{(i)}\triangleq\left\{X_{n,j}^{(i)}:0\leq j\leq N_n\right\}$$

is adapted to a discrete filtration $\{\mathcal{F}_{n,j}:0\leq j\leq N_n\}$ (of course, $\mathcal{F}_{n,j}\subseteq\mathcal{F}$). For a given $n\geq 1$, we say that $X_n^{(2)}$ is a *decoupled tangent sequence* to $X_n^{(1)}$ if the following two conditions are verified:

★ (*Tangency*) for each $j = 1, ..., N_n$

$$\mathbb{E}\left[\exp\left(i\lambda X_{n,j}^{(1)}\right) \mid \mathcal{F}_{n,j-1}\right] = \mathbb{E}\left[\exp\left(i\lambda X_{n,j}^{(2)}\right) \mid \mathcal{F}_{n,j-1}\right] \tag{5}$$

for each $\lambda \in \mathbb{R}$, a.s.-$\mathbb{P}$;

★ (*Conditional independence*) there exists a σ-field $\mathcal{G}_n \subseteq \mathcal{F}$ such that, for each $j = 1, ..., N_n$,

$$\mathbb{E}\left[\exp\left(i\lambda X_{n,j}^{(2)}\right) \mid \mathcal{F}_{n,j-1}\right] = \mathbb{E}\left[\exp\left(i\lambda X_{n,j}^{(2)}\right) \mid \mathcal{G}_n\right] \tag{6}$$

for each $\lambda \in \mathbb{R}$, a.s.-$\mathbb{P}$, and the random variables $X_{n,1}^{(2)}, ..., X_{n,N_n}^{(2)}$ are conditionally independent given $\mathcal{G}_n$.

Observe that, in (6), $\mathcal{F}_{n,j-1}$ depends on j, but $\mathcal{G}_n$ does not. The array $X^{(2)}$ is said to be a *decoupled tangent array* to $X^{(1)}$ if $X_n^{(2)}$ is a decoupled tangent sequence to $X_n^{(1)}$ for each $n \geq 1$. Putting (5) and (6) together yields

$$\mathbb{E}\left[\exp\left(i\lambda X_{n,j}^{(1)}\right) \mid \mathcal{F}_{n,j-1}\right] = \mathbb{E}\left[\exp\left(i\lambda X_{n,j}^{(2)}\right) \mid \mathcal{G}_n\right]. \tag{7}$$

We view the *principle of conditioning (POC)* as an approach based on (7). It consists of replacing the marginal distributions of a sequence $\left\{X_{n,j}^{(1)}\right\}_{j=1,...,N_n}$ given its past, by the marginal distributions of a sequence $\left\{X_{n,j}^{(2)}\right\}_{j=1,...,N_n}$ which is "almost independent", more precisely, which is independent given a σ-field $\mathcal{G}_n$ which depends only on n. As n grows, one obtains an array. The goal is to use limit theorems for $\left\{X_{n,j}^{(2)}\right\}_{j=1,...,N_n}$ to derive limit theorems for $\left\{X_{n,j}^{(1)}\right\}_{j=1,...,N_n}$, as $n \to +\infty$.

Remark – In general, given $X^{(1)}$ as above, there exists a canonical way to construct an array $X^{(2)}$, which is decoupled and tangent to $X^{(1)}$. The reader is referred to (14, Section 2 and 3) for a detailed discussion of this point, as well as other relevant properties of decoupled tangent sequences.

The following result is essentially a reformulation of Theorem 2.1 in (40) into the setting of this section. It is a "stable convergence generalization" of the results obtained by Jakubowski in (12).

Theorem 1 (Xue, 1991)**.** *Let $X^{(2)}$ be a decoupled tangent array to $X^{(1)}$, and let the notation of Definition C prevail (in particular, the collection of σ-fields $\{\mathcal{F}_{n,j}, \mathcal{G}_n : 0 \leq j \leq N_n, n \geq 1\}$ satisfies (5) and (6)). We write, for every n and every $k = 0, ..., N_n$, $S_{n,k}^{(i)} \triangleq \sum_{j=0,...,k} X_{n,j}^{(i)}$, $i = 1, 2$. Suppose that there exists a sequence $\{r_n : n \geq 1\} \subset \mathbb{N}$, and a sequence of σ-fields $\{\mathcal{V}_n : n \geq 1\}$ such that*

$$\mathcal{V}_n \subseteq \mathcal{F} \quad \text{and} \quad \mathcal{V}_n \subseteq \mathcal{V}_{n+1} \cap \mathcal{F}_{n,r_n}, \quad n \geq 1, \tag{8}$$

and, as $n \to +\infty$,

$$\mathbb{E}\left[\exp\left(i\lambda S_{n,r_n \wedge N_n}^{(2)}\right) \mid \mathcal{G}_n\right] \xrightarrow{\mathbb{P}} 1. \tag{9}$$

If moreover

$$\mathbb{E}\left[\exp\left(i\lambda S^{(2)}_{n,N_n}\right) \mid \mathcal{G}_n\right] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda,\omega), \quad \forall \lambda \in \mathbb{R}, \tag{10}$$

where $\phi \in \widehat{\mathbf{M}}_0$ *and,* $\forall \lambda \in \mathbb{R}$, $\phi(\lambda) \in \vee_n \mathcal{V}_n$, *then, as* $n \to +\infty$,

$$\mathbb{E}\left[\exp\left(i\lambda S^{(1)}_{n,N_n}\right) \mid \mathcal{F}_{n,r_n}\right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \tag{11}$$

and

$$S^{(1)}_{n,N_n} \to_{(s,\mathcal{V})} \mu(\cdot), \tag{12}$$

where $\mathcal{V} \triangleq \vee_n \mathcal{V}_n$, *and* $\mu \in \mathbf{M}$ *verifies (2).*

Remarks – (a) Condition (8) says that $\mathcal{V}_n$, $n \geq 1$, must be an increasing sequence of σ-fields, whose nth term is contained in $\mathcal{F}_{n,r_n}$, for every $n \geq 1$. Condition (9) ensures that, for $i = 1, 2$, the sum of the first r_n terms of the vector $X^{(i)}_n$ is asymptotically negligeable (see also (12)).

(b) There are some differences between the statement of Theorem 1 above, and the original result presented in (40). On the one hand, in (40) the sequence $\{N_n : n \geq 1\}$ is such that each N_n is a $\mathcal{F}_{n,\cdot}$-stopping time (but we do not need such a generality). On the other hand, in (40) one considers only the case of the family of σ-fields $\mathcal{V}^*_n = \cap_{j \geq n} \mathcal{F}_{j,r_n}$, $n \geq 1$, where r_n is non decreasing (note that, due to the monotonicity of r_n, the $\mathcal{V}^*_n$'s satisfy automatically (8)). However, by inspection of the proof of (40, Theorem 2.1 and Lemma 2.1), one sees immediately that all is needed to prove Theorem 1 is that the $\mathcal{V}_n$'s verify condition (8). For instance, if r_n is a general sequence of natural numbers such that $\mathcal{F}_{n,r_n} \subseteq \mathcal{F}_{n+1,r_{n+1}}$ for each $n \geq 1$, then the sequence $\mathcal{V}_n = \mathcal{F}_{n,r_n}$, $n \geq 1$, trivially satisfies (8), even if it does not fit Xue's original assumptions.

(c) The main theorem in the paper by Jakubowski (12, Theorem 1.1) (which, to our knowledge, is the first systematic account of the POC) corresponds to the special case $\mathcal{F}_{n,0} = \{\varnothing, \Omega\}$ and $r_n = 0$, $n \geq 1$. Under such assumptions, necessarily $\mathcal{V}_n = \mathcal{F}_{n,0}$, $S^{(i)}_{n,r_n \wedge N_n} = 0$, $i = 1, 2$, and $\phi(\lambda)$, which is $\vee_n \mathcal{V}_n = \{\varnothing, \Omega\}$ – measurable, is deterministic for every λ. In particular, relations (8) and (9) become immaterial. See also (15, Theorem 5.8.3) and (6, Theorem 7.1.4) for some detailed discussions of the POC in this setting.

(d) For the case $r_n = 0$ and $\mathcal{F}_{n,0} = \mathcal{A}$ ($n \geq 1$), where $\mathcal{A}$ is not trivial, see also (9, Section (1.c)).

The next proposition is used in Section 5 and (24).

Proposition 2. *Let the notation of Theorem 1 prevail, suppose that the sequence* $S^{(1)}_{n,N_n}$ *verifies (11) for some* $\phi \in \widehat{\mathbf{M}}_0$, *and assume moreover that there exists a finite random variable* $C(\omega) > 0$ *such that, for some* $\eta > 0$,

$$\mathbb{E}\left[\left|S^{(1)}_{n,N_n}\right|^\eta \mid \mathcal{F}_{n,r_n}\right] < C(\omega), \ \forall n \geq 1, \ \ a.s.\text{-}\mathbb{P}. \tag{13}$$

Then, there exists a subsequence $\{n(k) : k \geq 1\}$ *such that, a.s. -* $\mathbb{P}$,

$$\mathbb{E}\left[\exp\left(i\lambda S^{(1)}_{n(k),N_{n(k)}}\right) \mid \mathcal{F}_{n(k),r_{n(k)}}\right] \underset{k \to +\infty}{\to} \phi(\lambda) \tag{14}$$

for every real λ.

Proof. Combining (11) and (13), we deduce the existence of a set Ω^* of probability one, as well as of a subsequence $n(k)$, such that, for every $\omega \in \Omega^*$, relation (13) is satisfied and (14) holds for every rational λ. We now fix $\omega \in \Omega^*$, and show that (14) holds for all real λ. Relations (11) and (13) also imply that

$$\mathbb{P}_k^{\omega}[\cdot] = \mathbb{P}\left[S^{(1)}_{n(k),N_{n(k)}} \in \cdot \mid \mathcal{F}_{n(k),r_{n(k)}}\right](\omega), \quad k \geq 1,$$

is tight and hence relatively compact: every sequence of $n(k)$ has a further subsequence $\{n(k_r) : r \geq 1\}$ such that $\mathbb{P}_{k_r}^{\omega}[\cdot]$ is weakly convergent, so that the corresponding characteristic function converges. In view of (14), such characteristic function must also satisfy the asymptotic relation

$$\mathbb{E}\left[\exp\left(i\lambda S^{(1)}_{n(k_r),N_{n(k_r)}}\right) \mid \mathcal{F}_{n(k_r),r_{n(k_r)}}\right](\omega) \underset{r\to+\infty}{\to} \phi(\lambda)(\omega)$$

for every rational λ, hence for every real λ, because $\phi(\lambda)(\omega)$ is continuous in λ. □

3 General framework for applications of the POC

We now present a general framework in which the POC techniques discussed in the previous paragraph can be applied. The main result of this section turns out to be the key tool to obtain stable convergence results for multiple stochastic integrals with respect to independently scattered random measures.

Our first goal is to define an Itô type stochastic integral with respect to a real valued and square integrable stochastic process X (not necessarily Gaussian) verifying the following three conditions: (i) X is indexed by the elements f of a real separable Hilbert space $\mathfrak{H}$, (ii) X satisfies the isomorphic relation

$$\mathbb{E}[X(f)X(g)] = (f,g)_{\mathfrak{H}}, \quad \forall f,g \in \mathfrak{H}, \tag{15}$$

and (iii) X has independent increments (the notion of "increment", in this context, is defined through orthogonal projections–see below). We shall then show that the asymptotic behavior of such integrals can be studied by means of arrays of random variables, to which the POC applies quite naturally. Note that the elements of $\mathfrak{H}$ need not be functions – they may be e.g. distributions on $\mathbb{R}^d$, $d \geq 1$. Our construction is inspired by the theory developed by L. Wu (see (39)) and A.S. Üstünel and M. Zakai (see (38)), concerning Skorohod integrals and filtrations on abstract Wiener spaces. These author have introduced the notion of time in the context of abstract Wiener spaces by using resolutions of the identity.

Definition D (see e.g. (2), (41) and (38)) – Let $\mathfrak{H}$ be a separable real Hilbert space, endowed with an inner product $(\cdot,\cdot)_{\mathfrak{H}}$ ($\|\cdot\|_{\mathfrak{H}}$ is the corresponding norm). A (continuous) *resolution of the identity*, is a family $\pi = \{\pi_t : t \in [0,1]\}$ of orthogonal projections satisfying:

(D-i) $\pi_0 = 0$, and $\pi_1 = id.$;

(D-ii) $\forall 0 \leq s < t \leq 1$, $\pi_s\mathfrak{H} \subseteq \pi_t\mathfrak{H}$;

(D-iii) $\forall 0 \leq t_0 \leq 1$, $\forall h \in \mathfrak{H}$, $\lim_{t\to t_0} \|\pi_t h - \pi_{t_0} h\|_{\mathfrak{H}} = 0$.

The class of all resolutions of the identity satisfying conditions (**D-i**)–(**D-iii**) is denoted $\mathcal{R}(\mathfrak{H})$. A subset F (not necessarily closed, nor linear) of $\mathfrak{H}$ is said to be π-*reproducing* if the linear span of the set $\{\pi_t f : f \in F,\ t \in [0,1]\}$ is dense in $\mathfrak{H}$ (in which case we say that such a set is *total* in $\mathfrak{H}$). For a given $\pi \in \mathcal{R}(\mathfrak{H})$, the class of all π-reproducing subsets $F \subset \mathfrak{H}$ is noted $\mathbf{R}(\pi)$. The *rank* of π is the smallest of the dimensions of all the closed subspaces generated by the sets $F \in \mathbf{R}(\pi)$. A set $F \in \mathbf{R}(\pi)$ is called *fully orthogonal* if $(\pi_t f, g)_{\mathfrak{H}} = 0$ for every $t \in [0,1]$ and every $f, g \in F$ such that $f \neq g$.

Remarks – (a) Since $\mathfrak{H}$ is separable, for every resolution of the identity π there always exists a countable π-reproducing subset of $\mathfrak{H}$.

(b) Let π be a resolution of the identity, and note $\overline{\text{v.s.}}(A)$ the closure of the vector space generated by some $A \subseteq \mathfrak{H}$. By a standard Gram-Schmidt orthogonalization procedure, it is easy to prove that, for every π-reproducing subset $F \in \mathbf{R}(\pi)$ such that $\dim(\overline{\text{v.s.}}(F)) = rank(\pi)$, there exists a *fully orthogonal* subset $F' \in \mathbf{R}(\pi)$, such that $\dim(\overline{\text{v.s.}}(F')) = \dim(\overline{\text{v.s.}}(F))$ (see e.g. (2, Lemma 23.2), or (38, p. 27)).

Examples – The following examples are related to the content of Section 5 and Section 6.

(a) Take $\mathfrak{H} = L^2([0,1], dx)$, i.e. the space of square integrable functions on $[0,1]$. Then, a family of projection operators naturally associated to $\mathfrak{H}$ can be as follows: for every $t \in [0,1]$ and every $f \in \mathfrak{H}$,

$$\pi_t f(x) = f(x)\,\mathbf{1}_{[0,t]}(x). \tag{16}$$

It is easily seen that this family $\pi = \{\pi_t : t \in [0,1]\}$ is a resolution of the identity verifying conditions **(Di)**–**(Diii)** in Definition D. Also, $rank(\pi) = 1$, since the linear span of the projections of the function $f(x) \equiv 1$ generates $\mathfrak{H}$.

(b) If $\mathfrak{H} = L^2\left([0,1]^2, dxdy\right)$, we define: for every $t \in [0,1]$ and every $f \in \mathfrak{H}$,

$$\pi_t f(x,y) = f(x,y)\,\mathbf{1}_{[0,t]^2}(x,y). \tag{17}$$

The family $\pi = \{\pi_t : t \in [0,1]\}$ appearing in (17) is a resolution of the identity as in Definition D. However, in this case $rank(\pi) = +\infty$. Other choices of π_t are also possible, for instance

$$\pi_t f(x,y) = f(x,y)\,\mathbf{1}_{\left[\frac{1}{2}-\frac{t}{2},\frac{1}{2}+\frac{t}{2}\right]^2}(x,y),$$

which expands from the center of the square $[0,1]^2$.

3.1 The class $\mathcal{R}_X(\mathfrak{H})$ of resolutions

Now fix a real separable Hilbert space $\mathfrak{H}$, as well as a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In what follows, we will write

$$X = X(\mathfrak{H}) = \{X(f) : f \in \mathfrak{H}\} \tag{18}$$

to denote a collection of centered random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by the elements of $\mathfrak{H}$ and satisfying the isomorphic relation (15) (we use the notation $X(\mathfrak{H})$ when the role of the space $\mathfrak{H}$ is relevant to the discussion). Note that relation (15) implies that, for every $f, g \in \mathfrak{H}$, $X(f+g) = X(f) + X(g)$, a.s.-$\mathbb{P}$.

Let $X(\mathfrak{H})$ be defined as in (18). Then, for every resolution $\pi = \{\pi_t : t \in [0,1]\} \in \mathcal{R}(\mathfrak{H})$, the following property is verified: $\forall m \geq 2$, $\forall h_1, ..., h_m \in \mathfrak{H}$ and $\forall 0 \leq t_0 < t_1 < ... < t_m \leq 1$, the vector

$$\left(X\left(\left(\pi_{t_1} - \pi_{t_0}\right) h_1\right), X\left(\left(\pi_{t_2} - \pi_{t_1}\right) h_2\right) ..., X\left(\left(\pi_{t_m} - \pi_{t_{m-1}}\right) h_m\right)\right) \tag{19}$$

is composed of uncorrelated random variables, because the π_t's are orthogonal projections. We stress that the class $\mathcal{R}(\mathfrak{H})$ depends only on the Hilbert space $\mathfrak{H}$, and not on X. Now define $\mathcal{R}_X(\mathfrak{H})$ to be the subset of $\mathcal{R}(\mathfrak{H})$ containing those π such that the vector (19) is composed of jointly independent random variables, for any choice of $m \geq 2$, $h_1, ..., h_m \in \mathfrak{H}$ and $0 \leq t_0 < t_1 < ... < t_m \leq 1$. The set $\mathcal{R}_X(\mathfrak{H})$ depends in general of X. Note that, if $X(\mathfrak{H})$ is a Gaussian family, then $\mathcal{R}_X(\mathfrak{H}) = \mathcal{R}(\mathfrak{H})$ (see Section 3 below). To every $\pi \in \mathcal{R}_X(\mathfrak{H})$ we associate the filtration

$$\mathcal{F}_t^{\pi}(X) = \sigma\{X(\pi_t f) : f \in \mathfrak{H}\}, \quad t \in [0,1], \tag{20}$$

so that, for instance, $\mathcal{F}_1^{\pi}(X) = \sigma(X)$.

Remark – Note that, for every $h \in \mathfrak{H}$ and every $\pi \in \mathcal{R}_X(\mathfrak{H})$, the stochastic process $t \mapsto X(\pi_t h)$ is a centered, square integrable $\mathcal{F}_t^{\pi}(X)$-martingale with independent increments. Moreover, since π is continuous and (15) holds, $X(\pi_s h) \xrightarrow{\mathbb{P}} X(\pi_t h)$ whenever $s \to t$. In the terminology of (34, p. 3), this implies that $\{X(\pi_t h) : t \in [0,1]\}$ is an *additive process in law*. In particular, if $\mathcal{R}_X(\mathfrak{H})$ is not empty, for every $h \in \mathfrak{H}$ the law of $X(\pi_1 h) = X(h)$ is infinitely divisible (see e.g. (34, Theorem 9.1)). As a consequence (see (34, Theorem 8.1 and formula (8.8), p. 39)), for every $h \in \mathfrak{H}$ there exists a unique pair $\left(c^2(h), \nu_h\right)$ such that $c^2(h) \in [0, +\infty)$ and ν_h is a measure on $\mathbb{R}$ satisfying

$$\nu_h(\{0\}) = 0, \qquad \int_{\mathbb{R}} x^2 \nu_h(dx) < +\infty, \tag{21}$$

and moreover, for every $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(i\lambda X(h))] = \exp\left[-\frac{c^2(h)\lambda^2}{2} + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda x)\,\nu_h(dx)\right]. \tag{22}$$

Observe that, since the Lévy-Khintchine representation of an infinitely divisible distribution is unique, the pair $\left(c^2(h), \nu_h\right)$ does not depend on the choice of $\pi \in \mathcal{R}_X(\mathfrak{H})$. In what follows, when $\mathcal{R}_X(\mathfrak{H}) \neq \varnothing$, we will use the notation: for every $\lambda \in \mathbb{R}$ and every $h \in \mathfrak{H}$,

$$\psi_{\mathfrak{H}}(h; \lambda) \triangleq -\frac{c^2(h)\lambda^2}{2} + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda x)\,\nu_h(dx), \tag{23}$$

where the pair $\left(c^2(h), \nu_h\right)$, characterizing the law of the random variable $X(h)$, is given by (22). Note that, if $h_n \to h$ in $\mathfrak{H}$, then $X(h_n) \to X(h)$ in $L^2(\mathbb{P})$, and therefore $\psi_{\mathfrak{H}}(h_n; \lambda) \to \psi_{\mathfrak{H}}(h; \lambda)$ for every $\lambda \in \mathbb{R}$ (uniformly on compacts). We shall always endow $\mathfrak{H}$ with the σ-field $\mathcal{B}(\mathfrak{H})$, generated by the open sets with respect to the distance induced by the norm $\|\cdot\|_{\mathfrak{H}}$. Since, for every real λ, the complex-valued application $h \mapsto \psi_{\mathfrak{H}}(h; \lambda)$ is continuous, it is also $\mathcal{B}(\mathfrak{H})$-measurable.

Examples – (a) Take $\mathfrak{H} = L^2([0,1], dx)$, suppose that $X(\mathfrak{H}) = \{X(h) : h \in \mathfrak{H}\}$ is a centered Gaussian family verifying (15), and define the resolution of the identity $\pi = \{\pi_t : t \in [0,1]\}$ according to (16). Then, if $\mathbf{1}$ indicates the function which is constantly equal to one, the process

$$W_t \triangleq X(\pi_t \mathbf{1}), \quad t \in [0,1], \tag{24}$$

is a standard Brownian motion started from zero,

$$\mathcal{F}_t^\pi(X) = \sigma\{W_s : s \leq t\}, \quad \forall t \in [0,1],$$

and, for every $f \in \mathfrak{H}$,

$$X(\pi_t f) = \int_0^t f(s)\, dW_s,$$

where the stochastic integration is in the usual Wiener-Itô sense. Of course, $X(\pi_t f)$ is a Gaussian $\mathcal{F}_t^\pi(X)$ - martingale with independent increments, and also, by using the notation (23), for every $f \in L^2([0,1], dx)$ and $\lambda \in \mathbb{R}$, $\psi_{\mathfrak{H}}(f;\lambda) = -\left(\lambda^2/2\right)\int_0^1 f(x)^2\, dx$.

(b) Take $\mathfrak{H} = L^2\left([0,1]^2, dxdy\right)$ and define the resolution $\pi = \{\pi_t : t \in [0,1]\}$ as in (17). We consider a compensated Poisson measure $\widehat{N} = \left\{\widehat{N}(C) : C \in \mathcal{B}\left([0,1]^2\right)\right\}$ over $[0,1]^2$. This means that (1) for every $C \in \mathcal{B}\left([0,1]^2\right)$,

$$\widehat{N}(C) \overset{\text{law}}{=} N(C) - \mathbb{E}(N(C))$$

where $N(C)$ is a Poisson random variable with parameter $Leb(C)$ (i.e., the Lebesgue measure of C), and (2) $\widehat{N}(C_1)$ and $\widehat{N}(C_2)$ are stochastically independent whenever $C_1 \cap C_2 = \varnothing$. Then, the family $X(\mathfrak{H}) = \{X(h) : h \in \mathfrak{H}\}$, defined by

$$X(h) = \int_{[0,1]^2} h(x,y)\, \widehat{N}(dx,dy), \quad h \in \mathfrak{H},$$

satisfies the isomorphic relation (15). Moreover

$$\mathcal{F}_t^\pi(X) = \sigma\left\{\widehat{N}([0,s] \times [0,u]) : s \vee u \leq t\right\}, \quad \forall t \in [0,1],$$

and for every $h \in \mathfrak{H}$, the process

$$X(\pi_t h) = \int_{[0,t]^2} h(x,y)\, \widehat{N}(dx,dy), \quad t \in [0,1],$$

is a $\mathcal{F}_t^\pi(X)$ – martingale with independent increments, and hence $\pi \in \mathcal{R}_X(\mathfrak{H})$. Moreover, for every $h \in L^2\left([0,1]^2, dxdy\right)$ and $\lambda \in \mathbb{R}$ the exponent $\psi_{\mathfrak{H}}(h;\lambda)$ in (23) verifies the relation (see e.g. (34, Proposition 19.5))

$$\psi_{\mathfrak{H}}(h;\lambda) = \int_0^1 \int_0^1 \left[\exp(i\lambda h(x,y)) - 1 - i\lambda h(x,y)\right] dxdy.$$

3.2 The stochastic integrals J_X^π and $J_{\tilde{X}}^\pi$

We now want to consider random variables with values in $\mathfrak{H}$, and define an Itô type stochastic integral with respect to X. To do so, we let $L^2(\mathbb{P},\mathfrak{H},X) = L^2(\mathfrak{H},X)$ be the space of $\sigma(X)$-measurable and $\mathfrak{H}$-valued random variables Y satisfying $\mathbb{E}\left[\|Y\|_{\mathfrak{H}}^2\right] < +\infty$ (note that $L^2(\mathfrak{H},X)$ is a Hilbert space, with inner product $(Y,Z)_{L^2(\mathfrak{H},X)} = \mathbb{E}\left[(Y,Z)_{\mathfrak{H}}\right]$). Following for instance (38) (which concerns uniquely the Gaussian case), we associate to every $\pi \in \mathcal{R}_X(\mathfrak{H})$ the subspace $L_\pi^2(\mathfrak{H},X)$ of the π-*adapted* elements of $L^2(\mathfrak{H},X)$, that is: $Y \in L_\pi^2(\mathfrak{H},X)$ if, and only if, $Y \in L^2(\mathfrak{H},X)$ and, for every $t \in [0,1]$

$$\pi_t Y \in \mathcal{F}_t^\pi(X) \quad \text{or, equivalently,} \quad (Y,\pi_t h)_{\mathfrak{H}} \in \mathcal{F}_t^\pi(X) \quad \forall h \in \mathfrak{H}. \tag{25}$$

For any resolution $\pi \in \mathcal{R}_X(\mathfrak{H})$, $L_\pi^2(\mathfrak{H},X)$ is a closed subspace of $L^2(\mathfrak{H},X)$. Indeed, if $Y_n \in L_\pi^2(\mathfrak{H},X)$ and $Y_n \to Y$ in $L^2(\mathfrak{H},X)$, then necessarily $(Y_n,\pi_t h)_{\mathfrak{H}} \overset{\mathbb{P}}{\to} (Y,\pi_t h)_{\mathfrak{H}}$ $\forall t \in [0,1]$ and every $h \in \mathfrak{H}$, thus yielding $Y \in L_\pi^2(\mathfrak{H},X)$. We will occasionally write $(u,z)_{L_\pi^2(\mathfrak{H})}$ instead of $(u,z)_{L^2(\mathfrak{H})}$, when both u and z are in $L_\pi^2(\mathfrak{H},X)$. Now define, for $\pi \in \mathcal{R}_X(\mathfrak{H})$, $\mathcal{E}_\pi(\mathfrak{H},X)$ to be the space of (π-adapted) *elementary* elements of $L_\pi^2(\mathfrak{H},X)$, that is, $\mathcal{E}_\pi(\mathfrak{H},X)$ is the collection of those elements of $L_\pi^2(\mathfrak{H},X)$ that are linear combinations of $\mathfrak{H}$-valued random variables of the type

$$h = \Phi(t_1)(\pi_{t_2} - \pi_{t_1}) f, \tag{26}$$

where $t_2 > t_1$, $f \in \mathfrak{H}$ and $\Phi(t_1)$ is a random variable which is square-integrable and $\mathcal{F}_{t_1}^\pi(X)$ - measurable.

Lemma 3. *For every $\pi \in \mathcal{R}_X(\mathfrak{H})$, the set $\mathcal{E}_\pi(\mathfrak{H},X)$, of adapted elementary elements, is total (i.e., its span is dense) in $L_\pi^2(\mathfrak{H},X)$.*

Proof. The proof is similar to (38, Lemma 2.2). Suppose $u \in L_\pi^2(\mathfrak{H},X)$ and $(u,g)_{L^2(\mathfrak{H},X)} = 0$ for every $g \in \mathcal{E}_\pi(\mathfrak{H},X)$. We shall show that $u = 0$, a.s. - $\mathbb{P}$. For every $t_{i+1} > t_i$, every bounded and $\mathcal{F}_{t_i}^\pi(X)$-measurable r.v. $\Phi(t_i)$, and every $f \in \mathfrak{H}$

$$\mathbb{E}\left[\left(\Phi(t_i)\left(\pi_{t_{i+1}} - \pi_{t_i}\right) f, u\right)_{\mathfrak{H}}\right] = 0,$$

and therefore $t \mapsto (\pi_t f, u)_{\mathfrak{H}}$ is a continuous (since π is continuous) $\mathcal{F}_t^\pi(X)$ - martingale starting from zero. Moreover, for every $0 = t_0 < \cdots < t_n = 1$

$$\sum_{i=0}^{n-1} \left|\left(f, \left(\pi_{t_{i+1}} - \pi_{t_i}\right) u\right)_{\mathfrak{H}}\right| \leq \|u\|_{\mathfrak{H}} \|f\|_{\mathfrak{H}} \underset{\text{a.s.-}\mathbb{P}}{<} +\infty,$$

which implies that the continuous martingale $t \mapsto (\pi_t f, u)_{\mathfrak{H}}$ has also (a.s.-$\mathbb{P}$) bounded variation. It is therefore constant and hence equal to zero (see e.g. (31, Proposition 1.2)). It follows that, a.s.-$\mathbb{P}$, $(f,u)_{\mathfrak{H}} = (\pi_1 f, u)_{\mathfrak{H}} = 0$ for every $f \in \mathfrak{H}$, and consequently $u = 0$, a.s.-$\mathbb{P}$. □

We now want to introduce, for every $\pi \in \mathcal{R}_X(\mathfrak{H})$, an Itô type stochastic integral with respect to X. To this end, we fix $\pi \in \mathcal{R}_X(\mathfrak{H})$ and first consider simple integrands of the form $h = \sum_{i=1}^n \lambda_i h_i \in \mathcal{E}_\pi(\mathfrak{H},X)$, where $\lambda_i \in \mathbb{R}$, $n \geq 1$, and h_i is as in (26), i.e.

$$h_i = \Phi_i\left(t_1^{(i)}\right)\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}}\right) f_i, \quad f_i \in \mathfrak{H}, \quad i = 1,...,n, \tag{27}$$

with $t_2^{(i)} > t_1^{(i)}$, and $\Phi_i\left(t_1^{(i)}\right) \in \mathcal{F}^\pi_{t_1^{(i)}}(X)$ and square integrable. Then, the stochastic integral of such a h with respect to X and π, is defined as

$$J_X^\pi(h) = \sum_{i=1}^n \lambda_i J_X^\pi(h_i) = \sum_{i=1}^n \lambda_i \Phi_i\left(t_1^{(i)}\right) X\left(\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}}\right) f_i\right). \tag{28}$$

Observe that the $\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}}\right) f_i$ in (27) becomes the argument of X in (28). Note also that, although X has π-independent increments, there may be a very complex dependence structure between the random variables

$$J_X^\pi(h_i) = \Phi_i\left(t_1^{(i)}\right) X\left(\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}}\right) f_i\right), \quad i = 1, ..., n,$$

since the Φ_i's are non-trivial functionals of X. We therefore introduce a "decoupled" version of the integral $J_X^\pi(h)$, by considering an independent copy of X, noted $\widetilde{X}$, and by substituting X with $\widetilde{X}$ in formula (28). That is, for every $h \in \mathcal{E}_\pi(\mathfrak{H}, X)$ as in (27) we define

$$J_{\widetilde{X}}^\pi(h) = \sum_{i=1}^n \lambda_i \Phi_i\left(t_1^{(i)}\right) \widetilde{X}\left(\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}}\right) f_i\right). \tag{29}$$

Note that if $h \in \mathcal{E}_\pi(\mathfrak{H}, X)$ is non random, i.e. $h(\omega) = h^* \in \mathfrak{H}$, a.s.-$\mathbb{P}(d\omega)$, then the integrals $J_X^\pi(h) = X(h^*)$ and $J_{\widetilde{X}}^\pi(h) = \widetilde{X}(h^*)$ are independent copies of each other.

Proposition 4. *Fix $\pi \in \mathcal{R}_X(\mathfrak{H})$. Then, for every $h, h' \in \mathcal{E}_\pi(\mathfrak{H}, X)$,*

$$\begin{aligned} \mathbb{E}\left(J_X^\pi(h) J_X^\pi(h')\right) &= \left(h, h'\right)_{L_\pi^2(\mathfrak{H})} \\ \mathbb{E}\left(J_{\widetilde{X}}^\pi(h) J_{\widetilde{X}}^\pi(h')\right) &= \left(h, h'\right)_{L_\pi^2(\mathfrak{H})}. \end{aligned} \tag{30}$$

As a consequence, there exist two linear extensions of J_X^π and $J_{\widetilde{X}}^\pi$ to $L_\pi^2(\mathfrak{H}, X)$ satisfying the following two conditions:

1. *if h_n converges to h in $L_\pi^2(\mathfrak{H}, X)$, then*

$$\lim_{n\to+\infty} \mathbb{E}\left[\left(J_X^\pi(h_n) - J_X^\pi(h)\right)^2\right] = \lim_{n\to+\infty} \mathbb{E}\left[\left(J_{\widetilde{X}}^\pi(h_n) - J_{\widetilde{X}}^\pi(h)\right)^2\right] = 0;$$

2. *for every $h, h' \in L_\pi^2(\mathfrak{H}, X)$*

$$\mathbb{E}\left(J_X^\pi(h) J_X^\pi(h')\right) = \mathbb{E}\left(J_{\widetilde{X}}^\pi(h) J_{\widetilde{X}}^\pi(h')\right) = \left(h, h'\right)_{L_\pi^2(\mathfrak{H})}. \tag{31}$$

The two extensions J_X^π and $J_{\widetilde{X}}^\pi$ are unique, in the sense that if $\widehat{J}_X^\pi$ and $\widehat{J}_{\widetilde{X}}^\pi$ are two other extensions satisfying properties 1 and 2 above, then necessarily, a.s.-$\mathbb{P}$,

$$J_X^\pi(h) = \widehat{J}_X^\pi(h) \quad \text{and} \quad J_{\widetilde{X}}^\pi(h) = \widehat{J}_{\widetilde{X}}^\pi(h)$$

for every $h \in L_\pi^2(\mathfrak{H}, X)$.

Proof. It is sufficient to prove (30) when h and h' are simple adapted elements of the kind (26), and in this case the result follows from elementary computations. Since, according to Lemma 3, $\mathcal{E}_{\pi}(\mathfrak{H}, X)$ is dense in $L^2_{\pi}(\mathfrak{H}, X)$, the result is obtained from a standard density argument. $\square$

The following property, which is a consequence of the above discussion, follows immediately.

Corollary 5. *For every $f \in L^2_{\pi}(\mathfrak{H}, X)$, the process*

$$t \mapsto J^{\pi}_{X}(\pi_t f), \quad t \in [0,1]$$

is a real valued $\mathcal{F}^{\pi}_t$ - martingale initialized at zero.

Observe that the process $t \mapsto J^{\pi}_{X}(\pi_t f)$, $t \in [0,1]$, need not have independent (nor conditionally independent) increments. On the other hand, due to the independence between X and $\widetilde{X}$, and to (19), *conditionally* on the σ-field $\sigma(X)$, the increments of the process $t \mapsto J^{\pi}_{\widetilde{X}}(\pi_t f)$ are independent (to see this, just consider the process $J^{\pi}_{\widetilde{X}}(\pi_t f)$ for an elementary f as in (29), and observe that, in this case, conditioning on $\sigma(X)$ is equivalent to conditioning on the Φ_i's; the general case is obtained once again by a density argument). It follows that the random process $J^{\pi}_{\widetilde{X}}(\pi.f)$ can be regarded as being *decoupled and tangent* to $J^{\pi}_{X}(\pi.f)$, in a spirit similar to (14, Definition 4.1), (8) or (7). We stress, however, that $J^{\pi}_{\widetilde{X}}(\pi.f)$ need not meet the definition of a tangent process given in such references, which is based on a notion of convergence in the Skorohod topology, rather than on the L^2-convergence adopted in the present paper. The reader is referred to (8) for an exhaustive characterization of processes with conditionally independent increments.

3.3 Conditional distributions

Now, for $h \in \mathfrak{H}$ and $\lambda \in \mathbb{R}$, define the exponent $\psi_{\mathfrak{H}}(h;\lambda)$ according to (23), and observe that every $f \in L^2_{\pi}(\mathfrak{H}, X)$ is a random element with values in $\mathfrak{H}$. It follows that the quantity $\psi_{\mathfrak{H}}(f(\omega);\lambda)$ is well defined for every $\omega \in \Omega$ and every $\lambda \in \mathbb{R}$, and moreover, since $\psi_{\mathfrak{H}}(\cdot;\lambda)$ is $\mathcal{B}(\mathfrak{H})$-measurable, for every $f \in L^2_{\pi}(\mathfrak{H}, X)$ and every $\lambda \in \mathbb{R}$, the complex-valued application $\omega \mapsto \psi_{\mathfrak{H}}(f(\omega);\lambda)$ is $\mathcal{F}$-measurable.

Proposition 6. *For every $\lambda \in \mathbb{R}$ and every $f \in L^2_{\pi}(\mathfrak{H}, X)$,*

$$\mathbb{E}\left[\exp\left(i\lambda J^{\pi}_{\widetilde{X}}(f)\right) \mid \sigma(X)\right] = \exp\left[\psi_{\mathfrak{H}}(f;\lambda)\right], \quad a.s.\text{-}\mathbb{P}. \tag{32}$$

Proof. For $f \in \mathcal{E}_{\pi}(\mathfrak{H}, X)$, formula (32) follows immediately from the independence of X and $\widetilde{X}$. Now fix $f \in L^2_{\pi}(\mathfrak{H}, X)$, and select a sequence $(f_n) \subset \mathcal{E}_{\pi}(\mathfrak{H}, X)$ such that

$$\mathbb{E}\left[\|f_n - f\|^2_{\mathfrak{H}}\right] \to 0 \tag{33}$$

(such a sequence f_n always exists, due to Lemma 3). Since (33) implies that $\|f_n - f\|_{\mathfrak{H}} \xrightarrow{\mathbb{P}} 0$, for every subsequence n_k there exists a further subsequence $n_{k(r)}$ such that $\left\|f_{n_{k(r)}} - f\right\|_{\mathfrak{H}} \to 0$, a.s.

- $\mathbb{P}$, thus implying $\psi_{\mathfrak{H}}\left(f_{n_{k(r)}};\lambda\right) \to \psi_{\mathfrak{H}}(f;\lambda)$ for every $\lambda \in \mathbb{R}$, a.s. - $\mathbb{P}$. Then, for every $\lambda \in \mathbb{R}$, $\psi_{\mathfrak{H}}(f_n;\lambda) \xrightarrow{\mathbb{P}} \psi_{\mathfrak{H}}(f;\lambda)$, and therefore $\exp[\psi_{\mathfrak{H}}(f_n;\lambda)] \xrightarrow{\mathbb{P}} \exp[\psi_{\mathfrak{H}}(f;\lambda)]$. On the other hand,

$$\begin{aligned}\mathbb{E}\left|\mathbb{E}\left[\exp\left(i\lambda J^{\pi}_{\widetilde{X}}(f_n)\right) - \exp\left(i\lambda J^{\pi}_{\widetilde{X}}(f)\right) \mid \sigma(X)\right]\right| &\leq |\lambda|\,\mathbb{E}\left|J^{\pi}_{\widetilde{X}}(f_n) - J^{\pi}_{\widetilde{X}}(f)\right| \\ &\leq |\lambda|\,\mathbb{E}\left[\left(J^{\pi}_{\widetilde{X}}(f_n) - J^{\pi}_{\widetilde{X}}(f)\right)^2\right]^{\frac{1}{2}} \\ &= |\lambda|\,\mathbb{E}\left[\|f_n - f\|^2_{\mathfrak{H}}\right]^{\frac{1}{2}} \to 0,\end{aligned}$$

where the equality follows from (31), thus yielding

$$\exp[\psi_{\mathfrak{H}}(f_n;\lambda)] = \mathbb{E}\left[\exp\left(i\lambda J^{\pi}_{\widetilde{X}}(f_n)\right) \mid \sigma(X)\right] \xrightarrow{\mathbb{P}} \mathbb{E}\left[\exp\left(i\lambda J^{\pi}_{\widetilde{X}}(f)\right) \mid \sigma(X)\right],$$

and the desired conclusion is therefore obtained. □

Examples – (a) Take $\mathfrak{H} = L^2([0,1], dx)$ and suppose that $X(\mathfrak{H}) = \{X(h) : h \in \mathfrak{H}\}$ is a centered Gaussian family verifying (15). Define also $\pi = \{\pi_t : t \in [0,1]\} \in \mathcal{R}(\mathfrak{H})$ according to (16), and write W to denote the Brownian motion introduced in (24). The subsequent discussion will make clear that $L^2_\pi(\mathfrak{H}, X)$ is, in this case, the space of square integrable processes that are adapted to the Brownian filtration $\sigma\{W_u : u \leq t\}$, $t \in [0,1]$. Moreover, for every $t \in [0,1]$ and $u \in L^2_\pi(\mathfrak{H}, X)$

$$J^{\pi}_X(\pi_t u) = \int_0^t u(s)\,dW_s \quad \text{and} \quad J^{\pi}_{\widetilde{X}}(\pi_t u) = \int_0^t u(s)\,d\widetilde{W}_s,$$

where the stochastic integration is in the Itô sense, and $\widetilde{W}_t \triangleq \widetilde{X}\left(\mathbf{1}_{[0,t]}\right)$ is a standard Brownian motion independent of X.

(b) (*Orthogonalized Teugels martingales*, see (20)) Let $Z = \{Z_t : t \in [0,1]\}$ be a real-valued and centered Lévy process, initialized at zero and endowed with a Lévy measure ν satisfying the condition: for some $\varepsilon, \lambda > 0$

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\,\nu(dx) < +\infty.$$

Then, for every $i \geq 2$, $\int_{\mathbb{R}} |x|^i\,\nu(dx) < +\infty$, and Z_t has moments of all orders. Starting from Z, for every $i \geq 1$ one can therefore define the *compensated power jump process* (or *Teugel martingale*) of order i, noted $Y^{(i)}$, as $Y^{(1)}_t = Z_t$ for $t \in [0,1]$, and, for $i \geq 2$ and $t \in [0,1]$,

$$Y^{(i)}_t = \sum_{0<s\leq t} (\Delta Z_t)^i - \mathbb{E}\sum_{0<s\leq t} (\Delta Z_t)^i = \sum_{0<s\leq t} (\Delta Z_t)^i - t\int_{\mathbb{R}} x^i \nu(dx).$$

Plainly, each $Y^{(i)}$ is a centered Lévy process. Moreover, according to (20, pp. 111-112), for every $i \geq 1$ it is possible to find (unique) real coefficients $a_{i,1}, ..., a_{i,i}$, such that $a_{i,i} = 1$ and the stochastic processes

$$H^{(i)}_t = Y^{(i)}_t + a_{i,i-1}Y^{(i-1)}_t + \cdots + a_{i,1}Y^{(1)}_t, \quad t \in [0,1], \quad i \geq 1,$$

are strongly orthogonal centered martingales (in the sense of (29, p.148)), also verifying $\mathbb{E}\left[H_t^{(i)}H_s^{(j)}\right] = \delta_{ij}\,(t \wedge s)$, where δ_{ij} is the Kronecker symbol. Observe that $H^{(i)}$ is again a Lévy process, and that, for every deterministic $g, f \in L^2([0,1], ds)$, the integrals $\int_0^1 f(s)\,dH_s^{(i)}$ and $\int_0^1 g(s)\,dH_s^{(j)}$ are well defined and such that

$$\mathbb{E}\left[\int_0^1 f(s)\,dH_s^{(i)} \int_0^1 g(s)\,dH_s^{(j)}\right] = \delta_{ij}\int_0^1 g(s)\,f(s)\,ds. \tag{34}$$

Now define $\mathfrak{H} = L^2(\mathbb{N}\times[0,1], \kappa(dm)\times ds)$, where $\kappa(dm)$ is the counting measure, and define, for $h(\cdot,\cdot) \in \mathfrak{H}$, $t \in [0,1]$, and $(m,s) \in \mathbb{N}\times[0,1]$,

$$\pi_t h(m,s) = h(m,s)\,\mathbf{1}_{[0,t]}(s).$$

It is clear that $\pi = \{\pi_t : t \in [0,1]\} \in \mathcal{R}(\mathfrak{H})$. Moreover, for every $h(\cdot,\cdot) \in \mathfrak{H}$, we define

$$X(h) = \sum_{m=1}^{\infty}\int_0^1 h(m,s)\,dH_s^{(m)},$$

where the series is convergent in $L^2(\mathbb{P})$, since $\mathbb{E}X(h)^2 = \sum\int_0^1 h(m,s)^2\,ds < +\infty$, due to (34) and the fact that $h \in \mathfrak{H}$. Since the $H^{(m)}$ are strongly orthogonal and (34) holds, one sees immediately that, for every $h, h' \in \mathfrak{H}$, $\mathbb{E}[X(h)X(h')] = (h,h')_{\mathfrak{H}}$, and moreover, since for every m and every h the process $t \mapsto \int_0^1 \pi_t h(m,s)\,dH_s^{(m)} = \int_0^t h(m,s)\,dH_s^{(m)}$ has independent increments, $\pi \in \mathcal{R}_X(\mathfrak{H})$. We can also consider random h, and, by using (20), give the following characterization of random variables $h \in L^2_\pi(\mathfrak{H}, X)$, and the corresponding integrals $J_X^\pi(h)$ and $J_{\widetilde{X}}^\pi(h)$: (i) for every random element $h \in L^2_\pi(\mathfrak{H}, X)$ there exists a family $\left\{\phi_{m,t}^{(h)} : t \in [0,1], m \geq 1\right\}$ of real-valued and $\mathcal{F}_t^\pi$-predictable processes such that for every fixed m, the process $t \mapsto \phi_{m,t}^{(h)}$ is a modification of $t \mapsto h(m,t)$; (ii) for every $h \in L^2_\pi(\mathfrak{H}, X)$,

$$J_X^\pi(h) = \sum_{m=1}^{\infty}\int_0^1 \phi_{m,t}^{(h)}\,dH_t^{(m)}, \tag{35}$$

where the series is convergent in $L^2(\mathbb{P})$; (iii) for every $h \in L^2_\pi(\mathfrak{H}, X)$,

$$J_{\widetilde{X}}^\pi(h) = \sum_{m=1}^{\infty}\int_0^1 \phi_{m,t}^{(h)}\,d\widetilde{H}_t^{(m)}, \tag{36}$$

where the series is convergent in $L^2(\mathbb{P})$, and the sequence $\left\{\widetilde{H}^{(m)} : m \geq 1\right\}$ is an independent copy of $\left\{H^{(m)} : m \geq 1\right\}$. Note that by using (20, Theorem 1), one would obtain an analogous characterization in terms of iterated stochastic integrals of deterministic kernels.

4 Stable convergence of stochastic integrals

We shall now apply Theorem 1 to the setup outlined in the previous paragraph. Let $\mathfrak{H}_n$, $n \geq 1$, be a sequence of real separable Hilbert spaces, and, for each $n \geq 1$, let

$$X_n = X_n(\mathfrak{H}_n) = \{X_n(g) : g \in \mathfrak{H}_n\}, \tag{37}$$

be a centered, real-valued stochastic process, indexed by the elements of $\mathfrak{H}_n$ and such that $\mathbb{E}[X_n(f)X_n(g)] = (f,g)_{\mathfrak{H}_n}$. The processes X_n are not necessarily Gaussian. As before, $\widetilde{X}_n$ indicates an independent copy of X_n, for every $n \geq 1$.

Theorem 7. *Let the previous notation prevail, and suppose that the processes X_n, $n \geq 1$, appearing in (37) (along with the independent copies $\widetilde{X}_n$) are all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $n \geq 1$, let $\pi^{(n)} \in \mathcal{R}_{X_n}(\mathfrak{H}_n)$ and $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$. Suppose also that there exists a sequence $\{t_n : n \geq 1\} \subset [0,1]$ and a collection of σ-fields $\{\mathcal{U}_n : n \geq 1\}$, such that*

$$\lim_{n\to+\infty} \mathbb{E}\left[\left\|\pi^{(n)}_{t_n} u_n\right\|^2_{\mathfrak{H}_n}\right] = 0$$

and

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}^{\pi^{(n)}}_{t_n}(X_n). \tag{38}$$

If

$$\exp\left[\psi_{\mathfrak{H}_n}(u_n;\lambda)\right] = \mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{\widetilde{X}_n}(u_n)\right) \mid \sigma(X_n)\right] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda,\omega), \quad \forall\lambda\in\mathbb{R}, \tag{39}$$

where $\psi_{\mathfrak{H}_n}(u_n;\lambda)$ is defined according to (23), $\phi \in \widehat{\mathbf{M}}_0$ and, $\forall\lambda \in \mathbb{R}$,

$$\phi(\lambda) \in \vee_n \mathcal{U}_n \triangleq \mathcal{U}^*,$$

then, as $n \to +\infty$,

$$\mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u_n)\right) \mid \mathcal{F}^{\pi^{(n)}}_{t_n}(X_n)\right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall\lambda\in\mathbb{R}, \tag{40}$$

and

$$J^{\pi^{(n)}}_{X_n}(u_n) \to_{(s,\mathcal{U}^*)} \mu(\cdot), \tag{41}$$

where $\mu \in \mathbf{M}$ verifies (2).

Remarks – (1) The first equality in (39) follows from Proposition 6.

(2) The proof of Theorem 7 uses Theorem 1, which assumes $\phi \in \widehat{\mathbf{M}}_0$, that is, ϕ is non-vanishing. If $\phi \in \widehat{\mathbf{M}}$ (instead of $\widehat{\mathbf{M}}_0$) and if, for example, there exists a subsequence n_k such that,

$$\mathbb{P}\left\{\omega : \exp\left[\psi_{\mathfrak{H}_{n_k}}(u_{n_k}(\omega);\lambda)\right] \to \phi(\lambda,\omega), \quad \forall\lambda\in\mathbb{R}\right\} = 1,$$

then, given the nature of $\psi_{\mathfrak{H}_{n_k}}$, $\phi(\lambda,\omega)$ is necessarily, for $\mathbb{P}$-a.e. ω, the Fourier transform of an infinitely divisible distribution (see e.g. (34, Lemma 7.5)), and therefore $\phi \in \widehat{\mathbf{M}}_0$. A similar remark applies to Theorem 12 below.

(3) For $n \geq 1$, the process $t \mapsto J^{\pi^{(n)}}_{X_n}\left(\pi^{(n)}_t u_n\right)$ is a martingale and hence admits a càdlàg modification. Then, an alternative approach to obtain results for stable convergence is to use the well-known criteria for the stable convergence of continuous-time càdlàg semimartingales, as stated e.g. in (5, Proposition 1 and Theorems 1 and 2) or (11, Chapter 4). However, the formulation in terms of "principle of conditioning" yields, in our setting, more precise results, by using less stringent assumptions. For instance, (38) can be regarded as a weak version of the

“nesting condition” used in (5, p. 126), whereas (40) is a refinement of the conclusions that can be obtained by means of (5, Proposition 1).

(4) Suppose that, under the assumptions of Theorem 7, there exists a càdlàg process $Y = \{Y_t : t \in [0,1]\}$ such that, conditionally on $\mathcal{U}^*$, Y has independent increments and $\phi(\lambda) = \mathbb{E}[\exp(i\lambda Y_1) \mid \mathcal{U}^*]$. In this case, formula (41) is equivalent to saying that $J^{\pi^{(n)}}_{X_n}(u_n)$ converges $\mathcal{U}^*$-stably to Y_1. See (8, Section 4) for several results concerning the stable convergence (for instance, in the sense of finite dimensional distributions) of semimartingales towards processes with conditionally independent increments.

Before proving Theorem 7, we consider the important case of a *nested* sequence of resolutions. More precisely, assume that $\mathfrak{H}_n = \mathfrak{H}$, $X_n = X$, for every $n \geq 1$, and that the sequence $\pi^{(n)} \in \mathcal{R}_X(\mathfrak{H})$, $n \geq 1$, is nested in the following sense: for every $t \in [0,1]$ and every $n \geq 1$,

$$\pi^{(n)}_t \mathfrak{H} \subseteq \pi^{(n+1)}_t \mathfrak{H} \tag{42}$$

(note that if $\pi^{(n)} = \pi$ for every n, then (42) is trivially satisfied); in this case, if t_n is non decreasing, the sequence $\mathcal{U}_n = \mathcal{F}^{\pi^{(n)}}_{t_n}(X)$, $n \geq 1$, automatically satisfies (38). We therefore have the following consequence of Theorem 7.

Corollary 8. *Under the above notation and assumptions, suppose that the sequence $\pi^{(n)} \in \mathcal{R}_X(\mathfrak{H})$, $n \geq 1$, is nested in the sense of (42), and let $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}, X)$, $n \geq 1$. Suppose also that there exists a non-decreasing sequence $\{t_n : n \geq 1\} \subset [0,1]$ s.t.*

$$\lim_{n \to +\infty} \mathbb{E}\left[\left\|\pi^{(n)}_{t_n} u_n\right\|^2_{\mathfrak{H}}\right] = 0. \tag{43}$$

If

$$\exp[\psi_{\mathfrak{H}}(u_n;\lambda)] = \mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{\widetilde{X}_n}(u_n)\right) \mid \sigma(X_n)\right] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda,\omega), \quad \forall \lambda \in \mathbb{R},$$

where $\phi \in \widehat{\mathbf{M}}_0$ and, $\forall \lambda \in \mathbb{R}$, $\phi(\lambda) \in \vee_n \mathcal{F}^{\pi^{(n)}}_{t_n}(X) \triangleq \mathcal{F}_*$, then, as $n \to +\infty$,

$$\mathbb{E}\left[\exp(i\lambda J_X(u_n)) \mid \mathcal{F}^{\pi^{(n)}}_{t_n}(X)\right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R},$$

and

$$J_X(u_n) \to_{(s,\mathcal{F}_*)} \mu(\cdot),$$

where $\mu \in \mathbf{M}$ verifies (2).

In the next result $\{u_n\}$ may still be random, but $\phi(\lambda)$ is non-random. It follows from Corollary 8 by taking $t_n = 0$ for every n, so that (43) is immaterial, and $\mathcal{F}_*$ becomes the trivial σ-field.

Corollary 9. *Keep the notation of Corollary 8, and consider a (not necessarily nested) sequence $\pi^{(n)} \in \mathcal{R}_X(\mathfrak{H})$, $n \geq 1$. If*

$$\exp[\psi_{\mathfrak{H}}(u_n;\lambda)] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R},$$

where ϕ is the Fourier transform of some non-random measure μ such that $\phi(\lambda) \neq 0$ for every $\lambda \in \mathbb{R}$, then, as $n \to +\infty$,

$$\mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_X(u_n)\right)\right] \to \phi(\lambda), \quad \forall \lambda \in \mathbb{R},$$

that is, the law of $J^{\pi^{(n)}}_X(u_n)$ converges weakly to μ.

Proof of Theorem 7 – Since $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$, there exists, thanks to Lemma 3 a sequence $u_n^e \in \mathcal{E}_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$, $n \geq 1$, such that (by using the isometry properties of $J^{\pi^{(n)}}_{\widetilde{X}_n}$ and $J^{\pi^{(n)}}_{X_n}$, as stated in Proposition 4)

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \mathbb{E}\left[\|u_n - u_n^e\|^2_{\mathfrak{H}_n}\right] = \lim_{n\to+\infty} \mathbb{E}\left[\left(J^{\pi^{(n)}}_{\widetilde{X}_n}(u_n) - J^{\pi^{(n)}}_{\widetilde{X}_n}(u_n^e)\right)^2\right] \\ &= \lim_{n\to+\infty} \mathbb{E}\left[\left(J^{\pi^{(n)}}_{X_n}(u_n) - J^{\pi^{(n)}}_{X_n}(u_n^e)\right)^2\right] \end{aligned} \tag{44}$$

and

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \mathbb{E}\left[\left\|\pi^{(n)}_{t_n} u_n^e\right\|^2_{\mathfrak{H}_n}\right] = \lim_{n\to+\infty} \mathbb{E}\left[\left(J^{\pi^{(n)}}_{\widetilde{X}_n}\left(\pi^{(n)}_{t_n} u_n^e\right)\right)^2\right] \\ &= \lim_{n\to+\infty} \mathbb{E}\left[\left(J^{\pi^{(n)}}_{X_n}\left(\pi^{(n)}_{t_n} u_n^e\right)\right)^2\right]. \end{aligned} \tag{45}$$

Without loss of generality, we can always suppose that u_n^e has the form

$$u_n^e = \sum_{i=1}^{N_n}\left[\sum_{j=1}^{M_n(i)} \Phi_j^{(n)}\left(t_{i-1}^{(n)}\right)\left(\pi^{(n)}_{t_i^{(n)}} - \pi^{(n)}_{t_{i-1}^{(n)}}\right) f_j^{(n)}\right]$$

where $0 = t_0^{(n)} < \ldots < t_{N_n}^{(n)} = 1$, $f_j^{(n)} \in \mathfrak{H}_n$, $N_n, M_n(i) \geq 1$, $\Phi_j^{(n)}\left(t_{i-1}^{(n)}\right)$ is square integrable and measurable with respect to $\mathcal{F}^{\pi^{(n)}}_{t_{i-1}^{(n)}}(X_n)$ where one of the $t_0^{(n)}, \ldots, t_{N_n}^{(n)}$ equals t_n. Moreover, we have

$$\begin{aligned} J^{\pi^{(n)}}_{X_n}(u_n^e) &= \sum_{i=1}^{N_n}\left[\sum_{j=1}^{M_n(i)} \Phi_j^{(n)}\left(t_{i-1}^{(n)}\right) X_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_j^{(n)}\right)\right] \\ J^{\pi^{(n)}}_{\widetilde{X}_n}(u_n^e) &= \sum_{i=1}^{N_n}\left[\sum_{j=1}^{M_n(i)} \Phi_j^{(n)}\left(t_{i-1}^{(n)}\right) \widetilde{X}_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_j^{(n)}\right)\right]. \end{aligned}$$

Now define for $n \geq 1$ and $i = 1, \ldots, N_n$

$$\begin{aligned} X^{(1)}_{n,i} &= \sum_{j=1}^{M_n(i)} \Phi_j^{(n)}\left(t_{i-1}^{(n)}\right) X_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_j^{(n)}\right) \\ X^{(2)}_{n,i} &= \sum_{j=1}^{M_n(i)} \Phi_j^{(n)}\left(t_{i-1}^{(n)}\right) \widetilde{X}_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_j^{(n)}\right) \end{aligned}$$

as well as $X^{(\ell)}_{n,0} = 0$, $\ell = 1, 2$; introduce moreover the filtration

$$\widehat{\mathcal{F}}_t^{\left(\pi^{(n)},\mathfrak{H}_n\right)} = \mathcal{F}^{\pi^{(n)}}_t(X_n) \vee \sigma\left\{\widetilde{X}\left(\pi^{(n)}_t f\right) : f \in \mathfrak{H}_n\right\}, \;\; t \in [0,1], \tag{46}$$

and let $\mathcal{G}_n = \sigma(X_n)$, $n \geq 1$. We shall verify that the array $X^{(2)} = \left\{X^{(2)}_{n,i} : 0 \leq i \leq N_n,\; n \geq 1\right\}$ is decoupled and tangent to $X^{(1)} = \left\{X^{(1)}_{n,i} : 0 \leq i \leq N_n,\; n \geq 1\right\}$, in the sense of Definition C of Section 2. Indeed, for $\ell = 1, 2$, the sequence $\left\{ X_{n,i}^{(\ell)} : 0 \leq i \leq N_n \right\}$ is adapted to the discrete filtration

$$\mathcal{F}_{n,i} \triangleq \widehat{\mathcal{F}}_{t_i^{(n)}}^{\left(\pi^{(n)}, \mathfrak{H}_n\right)}, \quad i = 1, ..., N_n; \tag{47}$$

also (5) is satisfied, since, for every j and every $i = 1, ..., N_n$,

$$\Phi_j^{(n)}\left(t_{i-1}^{(n)}\right) \in \mathcal{F}_{t_{i-1}^{(n)}}^{\pi^{(n)}}(X_n) \subset \mathcal{F}_{n,i-1},$$

and

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\lambda X_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}})f_j^{(n)}\right)\right) \mid \mathcal{F}_{n,i-1}\right] &= \mathbb{E}\left[\exp\left(i\lambda X_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}})f_j^{(n)}\right)\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\lambda \widetilde{X}_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}})f_j^{(n)}\right)\right) \mid \mathcal{F}_{n,i-1}\right].
\end{aligned}$$

Since $\mathcal{G}_n = \sigma(X_n)$, we obtain immediately (6), because $\widetilde{X}_n$ is an independent copy of X_n. We now want to apply Theorem 1 with

$$\begin{aligned}
J_{X_n}^{\pi}\left(\pi_{t_n}^{(n)} u_n^e\right) &= \sum_{i=1}^{r_n}\left[\sum_{l=1}^{M_n(i)} \Phi_l^{(n)}\left(t_{i-1}^{(n)}\right) X_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}})f_l^{(n)}\right)\right] = \sum_{i=1}^{r_n} X_{n,i}^{(1)} = S_{n,r_n}^{(1)} \qquad (48) \\
J_{\widetilde{X}_n}^{\pi}\left(\pi_{t_n} u_n^e\right) &= \sum_{i=1}^{r_n}\left[\sum_{l=1}^{M_n(i)} \Phi_l^{(n)}\left(t_{i-1}^{(n)}\right) \widetilde{X}_n\left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}})f_l^{(n)}\right)\right] = \sum_{i=1}^{r_n} X_{n,i}^{(1)} = S_{n,r_n}^{(2)},
\end{aligned}$$

where r_n is the element of $\{1, ..., N_n\}$ such that $t_{r_n}^{(n)} = t_n$. To do so, we need to verify the remaining conditions of that theorem. To prove (8), use (46), (47) and (38), to obtain

$$\mathcal{F}_{n,r_n} = \widehat{\mathcal{F}}_{t_{r_n}^{(n)}}^{\left(\pi^{(n)}, \mathfrak{H}_n\right)} \supset \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n) \supseteq \mathcal{U}_n,$$

and hence (8) holds with $\mathcal{V}_n = \mathcal{U}_n$. To prove (9), observe that the asymptotic relation in (45) can be rewritten as

$$\lim_{n \to +\infty} \mathbb{E}\left[\left(S_{n,r_n}^{(\ell)}\right)^2\right] = 0, \quad \ell = 1, 2, \tag{49}$$

which immediately yields, as $n \to +\infty$,

$$\mathbb{E}\left[\exp\left(i\lambda S_{n,r_n}^{(2)}\right) \mid \mathcal{G}_n\right] \xrightarrow{\mathbb{P}} 1$$

for every $\lambda \in \mathbb{R}$. To justify the last relation, just observe that (49) implies that

$$\mathbb{E}\left[\left(S_{n,r_n}^{(2)}\right)^2\right] = \mathbb{E}\left[\mathbb{E}\left[\left(S_{n,r_n}^{(2)}\right)^2 \mid \mathcal{G}_n\right]\right] \to 0,$$

and therefore $\mathbb{E}\left[\left(S_{n,r_n}^{(2)}\right)^2 \mid \mathcal{G}_n\right] \to 0$ in $L^1(\mathbb{P})$. Thus, for every diverging sequence n_k, there exists a subsequence n'_k such that, a.s.- $\mathbb{P}$,

$$\mathbb{E}\left[\left(S_{n'_k, r_{n'_k}}^{(2)}\right)^2 \mid \mathcal{G}_{n'_k}\right] \underset{k \to +\infty}{\to} 0,$$

which in turn yields that, a.s.-$\mathbb{P}$,

$$\mathbb{E}\left[\exp\left(i\lambda S^{(2)}_{n'_k, r_{n'_k}}\right) \mid \mathcal{G}_{n'_k}\right] \underset{k\to+\infty}{\to} 1.$$

To prove (10), observe that

$$\mathbb{E}\left|\exp\left(i\lambda J^{\pi^{(n)}}_{\widetilde{X}_n}(u^e_n)\right) - \exp\left(i\lambda J^{\pi^{(n)}}_{\widetilde{X}_n}(u_n)\right)\right| \leq |\lambda| \, \mathbb{E}\left|J^{\pi^{(n)}}_{\widetilde{X}_n}(u^e_n) - J^{\pi^{(n)}}_{\widetilde{X}_n}(u_n)\right| \underset{n\to+\infty}{\to} 0,$$

by (44). Hence, since (39) holds for u_n, it also holds when u_n is replaced by the elementary sequence u^e_n. Since $J^{\pi^{(n)}}_{\widetilde{X}_n}(u^e_n) = J^{\pi^{(n)}}_{\widetilde{X}_n}\left(\pi^{(n)}_1 u^e_n\right) = S^{(2)}_{n,N_n}$ and $\mathcal{G}_n = \sigma(X_n)$, relation (10) holds. It follows that the assumptions of Theorem 1 are satisfied, and we deduce that necessarily, as $n \to +\infty$,

$$\begin{aligned} &\mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u^e_n)\right) \mid \mathcal{F}^{\pi^{(n)}}_{t_n}(X_n)\right] \\ = \;& \mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u^e_n)\right) \mid \widehat{\mathcal{F}}^{\left(\pi^{(n)}, \mathfrak{H}_n\right)}_{t_n}\right] \overset{\mathbb{P}}{\to} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \end{aligned}$$

(the equality follows from the fact that X_n and $\widetilde{X}_n$ are independent). Theorem 1 also yields

$$J^{\pi^{(n)}}_{X_n}(u^e_n) \to_{(s,\mathcal{U}^*)} \mu(\cdot). \tag{50}$$

To go back from u^e_n to u_n, we use

$$\mathbb{E}\left|\exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u^e_n)\right) - \exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u_n)\right)\right| \underset{n\to+\infty}{\to} 0, \tag{51}$$

which follows again from (44), and we deduce that

$$\mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u^e_n)\right) - \exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u_n)\right) \mid \mathcal{F}^{\pi^{(n)}}_{t_n}(X_n)\right] \underset{n\to+\infty}{\overset{L^1}{\to}} 0,$$

and therefore

$$\mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u_n)\right) \mid \mathcal{F}^{\pi^{(n)}}_{t_n}(X_n)\right] \underset{n\to+\infty}{\overset{\mathbb{P}}{\to}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

Finally, by combining (50) and (51), we obtain

$$J^{\pi^{(n)}}_{X_n}(u_n) \to_{(s,\mathcal{U}^*)} \mu(\cdot).$$

■

By using the same approximation procedure as in the preceding proof, we may use Proposition 2 to prove the following refinement of Theorem 7.

Proposition 10. *With the notation of Theorem 7, suppose that the sequence $J^{\pi^{(n)}}_{X_n}(u_n)$ verifies (40), and that there exists a finite random variable $C(\omega) > 0$ such that, for some $\eta > 0$,*

$$\mathbb{E}\left[\left|J^{\pi^{(n)}}_{X_n}(u_n)\right|^{\eta} \mid \mathcal{F}^{\pi^{(n)}}_{t_n}\right] < C(\omega), \;\; \forall n \geq 1, \;\; a.s.\text{-}\mathbb{P}.$$

Then, there is a subsequence $\{n(k) : k \geq 1\}$ such that, a.s. - $\mathbb{P}$,

$$\mathbb{E}\left[\exp\left(i\lambda J^{\pi^{(n)}}_{X_n}(u_n)\right) \mid \mathcal{F}^{\pi^{(n(k))}}_{t_{n(k)}}\right] \underset{k\to+\infty}{\to} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

Theorem 7 can also be extended to a slightly more general framework. To this end, we introduce some further notation. Fix a closed subspace $\mathfrak{H}^* \subseteq \mathfrak{H}$. For every $t \in [0,1]$, we denote by $\pi_{s \le t}\mathfrak{H}^*$ the closed linear subspace of $\mathfrak{H}$, generated by the set $\{\pi_s f : f \in \mathfrak{H}^*,\, s \le t\}$. Of course, $\pi_{\le t}\mathfrak{H}^* \subseteq \pi_t \mathfrak{H} = \pi_{\le t}\mathfrak{H}$. For a fixed $\pi \in \mathcal{R}_X(\mathfrak{H})$, we set $\mathcal{E}_\pi(\mathfrak{H}, \mathfrak{H}^*, X)$ to be the subset of $\mathcal{E}_\pi(\mathfrak{H}, X)$ composed of $\mathfrak{H}$-valued random variables of the kind

$$h = \Psi^*(t_1)(\pi_{t_2} - \pi_{t_1})\, g, \tag{52}$$

where $t_2 > t_1$, $g \in \mathfrak{H}^*$ and $\Psi^*(t_1)$ is a square integrable random variable verifying the measurability condition

$$\Psi^*(t_1) \in \sigma\{X(f) : f \in \pi_{\le t_1}\mathfrak{H}^*\},$$

whereas $L^2_\pi(\mathfrak{H}, \mathfrak{H}^*, X)$ is defined as the closure of $\mathcal{E}_\pi(\mathfrak{H}, \mathfrak{H}^*, X)$ in $L^2_\pi(\mathfrak{H}, X)$. Note that, plainly, $\mathcal{E}_\pi(\mathfrak{H}, X) = \mathcal{E}_\pi(\mathfrak{H}, \mathfrak{H}, X)$ and $L^2_\pi(\mathfrak{H}, X) = L^2_\pi(\mathfrak{H}, \mathfrak{H}, X)$. Moreover, for every $Y \in L^2_\pi(\mathfrak{H}, \mathfrak{H}^*, X)$ and every $t \in [0,1]$, the following two poperties are verified: (i) the random element $\pi_t Y$ takes values in $\pi_{\le t}\mathfrak{H}^*$, a.s.-$\mathbb{P}$, and (ii) the random variable $J^\pi_X(\pi_t h)$ is measurable with respect to the σ-field $\sigma\{X(f) : f \in \pi_{\le t}\mathfrak{H}^*\}$ (such claims are easily verified for h as in (52), and the general results follow once again by standard density arguments).

Remark – Note that, in general, even when $rank(\pi) = 1$ as in (16), and $\mathfrak{H}^*$ is non-trivial, for $0 < t \le 1$ the set $\pi_{\le t}\mathfrak{H}^*$ may be strictly contained in $\pi_t\mathfrak{H}$. It follows that the σ-field $\sigma\{X(f) : f \in \pi_{\le t}\mathfrak{H}^*\}$ can be strictly contained in $\mathcal{F}^\pi_t(X)$, as defined in (20). To see this, just consider the case $\mathfrak{H} = L^2([0,1], dx)$, $\mathfrak{H}^* = \left\{f \in L^2([0,1], dx) : f = f\mathbf{1}_{[0,1/2]}\right\}$, $\pi_s f = f\mathbf{1}_{[0,s]}$ ($s \in [0,1]$), and take $t \in (1/2, 1]$. Indeed, in this case $X\left(\mathbf{1}_{[0,t]}\right)$ is $\mathcal{F}^\pi_t(X)$-measurable but is not $\sigma\{X(f) : f \in \pi_{\le t_1}\mathfrak{H}^*\}$-measurable.

The following result can be proved along the lines of Lemma 3.

Lemma 11. *For every closed subspace $\mathfrak{H}^*$ of $\mathfrak{H}$, a random element Y is in $L^2_\pi(\mathfrak{H}, \mathfrak{H}^*, X)$ if, and only if, $Y \in L^2(\mathfrak{H}, X)$ and, for every $t \in [0,1]$,*

$$(Y, \pi_t h)_{\mathfrak{H}} \in \sigma\{X(f) : f \in \pi_{\le t}\mathfrak{H}^*\}.$$

The next theorem can be proved by using arguments analogous to the ones in the proof of Theorem 7. Here, $\mathfrak{H}_n = \mathfrak{H}$ and $X_n(\mathfrak{H}_n) = X(\mathfrak{H})$ for every n.

Theorem 12. *Under the above notation and assumptions, for every $n \ge 1$ let $\mathfrak{H}^{(n)}$ be a closed subspace of $\mathfrak{H}$, $\pi^{(n)} \in \mathcal{R}_X(\mathfrak{H})$, and $u_n \in L^2_{\pi^{(n)}}\left(\mathfrak{H}, \mathfrak{H}^{(n)}, X\right)$. Suppose also that there exists a sequence $\{t_n : n \ge 1\} \subset [0,1]$ and a collection of closed subspaces of $\mathfrak{H}$, noted $\{\mathfrak{U}_n : n \ge 1\}$, such that*

$$\lim_{n\to+\infty} \mathbb{E}\left[\left\|\pi^{(n)}_{t_n} u_n\right\|^2_{\mathfrak{H}}\right] = 0$$

and

$$\mathfrak{U}_n \subseteq \mathfrak{U}_{n+1} \cap \pi^{(n)}_{\le t_n}\mathfrak{H}^{(n)}.$$

If

$$\exp\left[\psi_{\mathfrak{H}}(u_n; \lambda)\right] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda, \omega), \quad \forall \lambda \in \mathbb{R},$$

where $\phi \in \widehat{\mathbf{M}}_0$ *and,* $\forall \lambda \in \mathbb{R}$,

$$\phi(\lambda) \in \vee_n \sigma \{X(f) : f \in \mathfrak{U}_n\} \triangleq \mathcal{U}^*,$$

then, as $n \to +\infty$,

$$\mathbb{E}\left[\exp\left(i\lambda J_X^{\pi^{(n)}}(u_n)\right) \mid X(f) : f \in \pi_{\leq t_n}^{(n)} \mathfrak{H}^{(n)}\right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R},$$

and

$$J_X^{\pi^{(n)}}(u_n) \to_{(s,\mathcal{U}^*)} \mu(\cdot),$$

where $\mu \in \mathbf{M}$ *verifies (2).*

5 Stable convergence of functionals of Gaussian processes

As an example, we shall now use Theorem 7 to prove general sufficient conditions, ensuring the stable convergence of functionals of Gaussian processes towards mixtures of normal distributions. This extends part of the results contained in (19) and (27), and leads to quite general criteria for the stable convergence of Skorohod integrals and multiple Wiener-Itô integrals. As explained in the Introduction, we have deferred the discussion about multiple Wiener-Itô integrals, as well as some relations with Brownian martingales to a separate paper, see (24). We also recall that the stable convergence of multiple Wiener-Itô integrals, with respect to independently scattered and not necessarily Gaussian random measures, is studied in detail in (23).

5.1 Preliminaries

Consider a real separable Hilbert space $\mathfrak{H}$, as well as a continuous resolution of the identity $\pi = \{\pi_t : t \in [0,1]\} \in \mathcal{R}(\mathfrak{H})$ (see Definition D). Throughout this paragraph, $X = X(\mathfrak{H}) = \{X(f) : f \in \mathfrak{H}\}$ stands for a centered Gaussian family, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by the elements of $\mathfrak{H}$ and satisfying the isomorphic condition (15). Note, that due to the Gaussian nature of X, every vector as in (19) is composed of independent random variables, and therefore, in this case, $\mathcal{R}(\mathfrak{H}) = \mathcal{R}_X(\mathfrak{H})$. When (15) is satisfied and $X(\mathfrak{H})$ is a Gaussian family, one usually says that $X(\mathfrak{H})$ is an *isonormal Gaussian process*, or a *Gaussian measure*, over $\mathfrak{H}$ (see e.g. (17, Section 1) or (18)). As before, we write $L^2(\mathfrak{H}, X)$ to indicate the (Hilbert) space of $\mathfrak{H}$-valued and $\sigma(X)$-measurable random variables. The filtration $\mathcal{F}^\pi(X) = \{\mathcal{F}_t^\pi(X) : t \in [0,1]\}$ (which is complete by definition) is given by formula (20).

In what follows, we shall apply to the Gaussian measure X some standard notions and results from Malliavin calculus (the reader is again referred to (17) and (18) for any unexplained notation or definition). For instance, $D = D_X$ and $\delta = \delta_X$ stand, respectively, for the usual Malliavin derivative and Skorohod integral with respect to the Gaussian measure X (the dependence on X will be dropped, when there is no risk of confusion); for $k \geq 1$, $\mathbb{D}_X^{1,2}$ is the space of differentiable functionals of X, endowed with the norm $\|\cdot\|_{1,2}$ (see (17, Chapter 1) for a definition of this norm); *dom* (δ_X) is the domain of the operator δ_X. Note that D_X is an operator from $\mathbb{D}_X^{1,2}$ to $L^2(\mathfrak{H}, X)$, and also that *dom* $(\delta_X) \subset L^2(\mathfrak{H}, X)$. For every $d \geq 1$, we define $\mathfrak{H}^{\otimes d}$ and $\mathfrak{H}^{\odot d}$ to be, respectively, the dth tensor product and the dth *symmetric* tensor product of $\mathfrak{H}$. For $d \geq 1$ we will denote by I_d^X the isometry between $\mathfrak{H}^{\odot d}$ equipped with the norm $\sqrt{d!}\,\|\cdot\|_{\mathfrak{H}^{\otimes d}}$ and the dth Wiener chaos of X.

The vector spaces $L^2_\pi(\mathfrak{H}, X)$ and $\mathcal{E}_\pi(\mathfrak{H}, X)$, composed respectively of adapted and elementary adapted elements of $L^2(\mathfrak{H}, X)$, are once again defined as in Section 3.2. We now want to link the above defined operators δ_X and D_X to the theory developed in the previous sections. In particular, we shall use the facts that (i) for any $\pi \in \mathcal{R}_X(\mathfrak{H})$, $L^2_\pi(\mathfrak{H}, X) \subseteq dom(\delta_X)$, and (ii) for any $u \in L^2_\pi(\mathfrak{H}, X)$ the random variable $J_X^\pi(u)$ can be regarded as a Skorohod integral. They are based on the following (simple) result, proved for instance in (39, Lemme 1).

Proposition 13. *Let the assumptions of this section prevail. Then, $L^2_\pi(\mathfrak{H}, X) \subseteq dom(\delta_X)$, and for every $h_1, h_2 \in L^2_\pi(\mathfrak{H}, X)$*

$$\mathbb{E}\left(\delta_X(h_1)\,\delta_X(h_2)\right) = (h_1, h_2)_{L^2_\pi(\mathfrak{H},X)}. \tag{53}$$

Moreover, if $h \in \mathcal{E}_\pi(\mathfrak{H}, X)$ has the form $h = \sum_{i=1}^n h_i$, where $n \geq 1$, and $h_i \in \mathcal{E}_\pi(\mathfrak{H}, X)$ is such that

$$h_i = \Phi_i \times \left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}}\right) f_i, \quad f_i \in \mathfrak{H}, \quad i = 1, ..., n,$$

with $t_2^{(i)} > t_1^{(i)}$ and Φ_i square integrable and $\mathcal{F}^\pi_{t_1^{(i)}}(X)$-measurable, then

$$\delta_X(h) = \sum_{i=1}^n \Phi_i \times \left[X\left(\pi_{t_2^{(i)}} f_i\right) - X\left(\pi_{t_1^{(i)}} f_i\right)\right]. \tag{54}$$

Relation (53) implies, in the terminology of (39), that $L^2_\pi(\mathfrak{H}, X)$ is a closed subspace of the *isometric subset of* $dom(\delta_X)$, defined as the class of those $h \in dom(\delta_X)$ s.t. $\mathbb{E}\left(\delta_X(h)^2\right) = \|h\|^2_{L^2(\mathfrak{H},X)}$ (note that, in general, such an isometric subset is not even a vector space; see (39, p. 170)). Relation (54) applies to simple integrands h, but by combining (53), (54) and Proposition 4, we deduce immediately that, for every $h \in L^2_\pi(\mathfrak{H}, X)$,

$$\delta_X(h) = J_X^\pi(h), \quad \text{a.s.-}\mathbb{P}. \tag{55}$$

where the random variable $J_X^\pi(h)$ is defined according to Proposition 4 and formula (28). Observe that the definition of J_X^π involves the resolution of the identity π, whereas the definition of δ does not involve any notion of resolution.

The next crucial result, which is partly a consequence of the continuity of π, is an abstract version of the *Clark-Ocone formula* (see (17)): it is a direct corollary of (39, Théorème 1, formula (2.4) and Théorème 3), to which the reader is referred for a detailed proof.

Proposition 14 (Abstract Clark-Ocone formula; Wu, 1990)**.** *Under the above notation and assumptions (in particular, π is a* continuous *resolution of the identity as in Definition D), every $F \in \mathbb{D}_X^{1,2}$ can be represented as*

$$F = \mathbb{E}(F) + \delta\left(proj\left\{D_X F \mid L^2_\pi(\mathfrak{H}, X)\right\}\right), \tag{56}$$

where $D_X F$ is the Malliavin derivative of F, and $proj\left\{\cdot \mid L^2_\pi(\mathfrak{H}, X)\right\}$ is the orthogonal projection operator on $L^2_\pi(\mathfrak{H}, X)$.

Remarks – (a) Note that the right-hand side of (56) is well defined since $D_X F \in L^2(\mathfrak{H}, X)$ by definition, and therefore

$$proj\left\{D_X F \mid L^2_\pi(\mathfrak{H}, X)\right\} \in L^2_\pi(\mathfrak{H}, X) \subseteq dom\,(\delta_X)\,,$$

where the last inclusion is stated in Proposition 13.

(b) Formula (56) has been proved in (39) in the context of abstract Wiener spaces, but in the proof of (56) the role of the underlying probability space is immaterial. The extension to the framework of isonormal Gaussian processes (which is defined, as above, on an arbitrary probability space) is therefore standard. See e.g. (18, Section 1.1).

(c) Since $\mathbb{D}^{1,2}_X$ is dense in $L^2(\mathbb{P})$ and $\delta_X\left(L^2_\pi(\mathfrak{H}, X)\right)$ is an isometry (due to relation (53)), the Clark-Ocone formula (56) implies that every $F \in L^2(\mathbb{P}, \sigma(X))$ admits a unique "predictable" representation of the form

$$F = \mathbb{E}(F) + \delta_X(u)\,, \quad u \in L^2_\pi(\mathfrak{H}, X)\,; \tag{57}$$

see also (39, Remarque 2, p. 172).

(d) Since (55) holds, formula (56) can be rewritten as

$$F = \mathbb{E}(F) + J^\pi_X\left(proj\left\{D_X F \mid L^2_\pi(\mathfrak{H}, X)\right\}\right). \tag{58}$$

Now consider, as before, an independent copy of X, noted $\widetilde{X} = \left\{\widetilde{X}(f) : f \in \mathfrak{H}\right\}$, and, for $h \in L^2_\pi(\mathfrak{H}, X)$, define the random variable $J^\pi_{\widetilde{X}}(h)$ according to Proposition 4 and (29). The following result is an immediate consequence of Proposition 6, and characterizes $J^\pi_{\widetilde{X}}(h)$, $h \in L^2_\pi(\mathfrak{H}, X)$, as a conditionally Gaussian random variable.

Proposition 15. *For every $h \in L^2_\pi(\mathfrak{H}, X)$ and for every $\lambda \in \mathbb{R}$,*

$$\mathbb{E}\left[\exp\left(i\lambda J^\pi_{\widetilde{X}}(h)\right) \mid \sigma(X)\right] = \exp\left(-\frac{\lambda^2}{2}\|h\|^2_{\mathfrak{H}}\right).$$

5.2 Stable convergence of Skorohod integrals to a mixture of Gaussian distributions

The following result, based on Theorem 7, gives general sufficient conditions for the stable convergence of Skorohod integrals to a conditionally Gaussian distributions. In what follows, $\mathfrak{H}_n$, $n \geq 1$, is a sequence of real separable Hilbert spaces, and, for each $n \geq 1$, $X_n = X_n(\mathfrak{H}_n) = \{X_n(g) : g \in \mathfrak{H}_n\}$, is an isonormal Gaussian process over $\mathfrak{H}_n$; for $n \geq 1$, $\widetilde{X}_n$ is an independent copy of X_n (note that $\widetilde{X}_n$ appears in the proof of the next result, but not in the statement). Recall that $\mathcal{R}(\mathfrak{H}_n)$ is a class of resolutions of the identity π (see Definition D), and that the Hilbert space $L^2_\pi(\mathfrak{H}_n, X_n)$ is defined after Relation (25).

Theorem 16. *Suppose that the isonormal Gaussian processes $X_n(\mathfrak{H}_n)$, $n \geq 1$, are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let, for $n \geq 1$, $\pi^{(n)} \in \mathcal{R}(\mathfrak{H}_n)$ and $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$. Suppose also that there exists a sequence $\{t_n : n \geq 1\} \subset [0,1]$ and σ-fields $\{\mathcal{U}_n : n \geq 1\}$, such that*

$$\left\|\pi^{(n)}_{t_n} u_n\right\|^2_{\mathfrak{H}_n} \xrightarrow{\mathbb{P}} 0 \tag{59}$$

and

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n). \tag{60}$$

If

$$\|u_n\|_{\mathfrak{H}}^2 \xrightarrow{\mathbb{P}} Y, \tag{61}$$

for some $Y \in L^2(\mathbb{P})$ *such that* $Y \neq 0$, $Y \geq 0$ *and* $Y \in \mathcal{U}^* \triangleq \vee_n \mathcal{U}_n$, *then, as* $n \to +\infty$,

$$\mathbb{E}\left[\exp\left(i\lambda\delta_{X_n}(u_n)\right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n)\right] \xrightarrow{\mathbb{P}} \exp\left(-\frac{\lambda^2}{2}Y\right), \quad \forall\lambda \in \mathbb{R},$$

and

$$\delta_{X_n}(u_n) \to_{(s,\mathcal{U}^*)} \mu(\cdot),$$

where $\mu \in \mathbf{M}$ *verifies* $\widehat{\mu}(\lambda) = \exp\left(-\frac{\lambda^2}{2}Y\right)$ *(see (3) for the definition of* $\widehat{\mu}$*).*

Proof. Since $\delta_{X_n}(u_n) = J_{X_n}^{\pi^{(n)}}(u_n)$ for every n, the result follows immediately from Theorem 7 by observing that, due to Proposition 15,

$$\mathbb{E}\left[\exp\left(i\lambda J_{\widetilde{X}_n}^{\pi^{(n)}}(u_n)\right) \mid \sigma(X_n)\right] = \exp\left(-\frac{\lambda^2}{2}\|u_n\|_{\mathfrak{H}}^2\right),$$

and therefore (61) that $\mathbb{E}\left[\exp\left(i\lambda J_{\widetilde{X}_n}^{\pi^{(n)}}(u_n)\right) \mid \sigma(X_n)\right] \to \exp\left(-\lambda^2 Y/2\right)$ if, and only if, (61) is verified. □

By using the Clark-Ocone formula stated in Proposition 14, we deduce immediately, from Theorem 16, a useful criterion for the stable convergence of (Malliavin) differentiable functionals.

Corollary 17. *Let* $\mathfrak{H}_n$, $X_n(\mathfrak{H}_n)$, $\pi^{(n)}$, t_n *and* $\mathcal{U}_n$, $n \geq 1$, *satisfy the assumptions of Theorem 16 (in particular, (38) holds), and consider a sequence of random variables* $\{F_n : n \geq 1\}$, *such that* $\mathbb{E}(F_n) = 0$ *and* $F_n \in \mathbb{D}_{X_n}^{1,2}$ *for every* n. *Then, a sufficient condition to have that*

$$F_n \to_{(s,\mathcal{U}^*)} \mu(\cdot)$$

and

$$\mathbb{E}\left[\exp\left(i\lambda F_n\right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n)\right] \xrightarrow{\mathbb{P}} \exp\left(-\frac{\lambda^2}{2}Y\right), \quad \forall\lambda \in \mathbb{R},$$

where $\mathcal{U}^* \triangleq \vee_n \mathcal{U}_n$, $Y \geq 0$ *is s.t.* $Y \in \mathcal{U}^*$ *and* $\widehat{\mu}(\lambda) = \exp\left(-\frac{\lambda^2}{2}Y\right)$, $\forall\lambda \in \mathbb{R}$, *is*

$$\left\|\pi_{t_n}^{(n)} proj\left\{D_{X_n}F_n \mid L_{\pi^{(n)}}^2(\mathfrak{H}_n, X_n)\right\}\right\|_{\mathfrak{H}_n}^2 \xrightarrow{\mathbb{P}} 0 \quad and \quad \left\|proj\left\{D_{X_n}F_n \mid L_{\pi^{(n)}}^2(\mathfrak{H}_n, X_n)\right\}\right\|_{\mathfrak{H}_n}^2 \underset{n\to+\infty}{\xrightarrow{\mathbb{P}}} Y. \tag{62}$$

Proof. Since, for every n, F_n is a centered random variable in $\mathbb{D}_{X_n}^{1,2}$, the abstract Clark-Ocone formula ensures that $F_n = \delta_{X_n}\left(proj\left\{D_{X_n}F_n \mid L_{\pi^{(n)}}^2(\mathfrak{H}_n, X_n)\right\}\right)$, the result follows from Theorem 16, by putting

$$u_n = proj\left\{D_{X_n}F_n \mid L_{\pi^{(n)}}^2(\mathfrak{H}_n, X_n)\right\}.$$

□

6 Sequences of quadratic Brownian functionals

As an application, we provide a generalization, as well as a new proof, of a result contained in (28, Proposition 2.1) concerning the stable convergence of quadratic Brownian functionals. To this end, we define $\mathfrak{H}_1 \triangleq L^2([0,1], ds)$ and consider, for every $n \geq 1$, a unitary transformation $T_n : \mathfrak{H}_1 \mapsto \mathfrak{H}_1$, from $\mathfrak{H}_1$ onto itself. We will work under the following assumption.

Assumption I – (**I-i**) For every $n \geq 1$, $T_n \mathbf{1}_{[0,1]} = \mathbf{1}_{[0,1]}$. (**I-ii**) For every increasing sequence $t_n \uparrow 1$, there exists an integer n_* and a sequence of subspaces $\mathfrak{H}_{n,1}$, $n \geq n_*$, such that $\mathfrak{H}_{n,1} \subset T_n \mathbf{1}_{[0,t_n]} \mathfrak{H}_1$, $\forall n \geq n_*$, and $\mathfrak{H}_{n,1} \uparrow \mathfrak{H}_1$, that is, every function in $\mathfrak{H}_{n,1}$ can be viewed as the transform of a function of $\mathfrak{H}_1$ restricted to $[0, t_n]$.

Now let W_t, $t \in [0,1]$, be a standard Brownian motion initialized at zero. For every $n \geq 1$, we define the Brownian motion $W^{(n)}$ by

$$W_t^{(n)} = \int_0^1 T_n \mathbf{1}_{[0,t]}(s) \, dW_s, \quad t \in [0,1]. \tag{63}$$

Observe that, by construction and for every $n \geq 1$,

$$W_0^{(n)} = 0 \quad \text{and} \quad W_1^{(n)} = \int_0^1 T_n \mathbf{1}_{[0,1]}(s) \, dW_s = W_1. \tag{64}$$

Examples – (i) The sequence $T_n = Id.$, $n \geq 1$, trivially satisifies Assumption I. In this case, $W_t^{(n)} = W_t$ for every n, and for every increasing sequence $t_n \uparrow 1$ one can choose $\mathfrak{H}_{n,1}$ as the closed subspace generated by functions with support in $[0, t_n]$.

(ii) We can also choose $T_n = Id.$ for n odd, and $T_n f(x) = f(1-x)$ for n even. In this case, for $k \geq 1$, $W_t^{(2k)} = W_t$ and $W_t^{(2k-1)} = W_1 - W_{1-t}$. Moreover, for every increasing sequence $t_n \uparrow 1$, one can define $n_* = \inf\{m : t_m > 1/2\}$, and it is easily verified that the increasing sequence of subspaces

$$\mathfrak{H}_{n,1} = \{f \in \mathfrak{H}_1 : f \text{ has support in } [1 - t_n, t_n]\}, \quad n \geq n_*,$$

satisfies the requirements of Assumption I-ii (for instance, if n is odd any function on $[1 - t_n, t_n]$ can be viewed as a function on $[1 - t_n, 1]$). We refer the reader to (26, Section 4) for further stochastic analysis results involving the two Brownian motions W_t and $W_1 - W_{1-t}$.

(iii) Consider points $y_n \in \left(\frac{1}{3}, \frac{2}{3}\right)$, as well as a sequence $\eta_n \in \left(0, \frac{1}{12}\right)$. Observe that

$$\frac{1}{4} < \frac{1}{3} < y_n - \eta_n < y_n < y_n + \eta_n < \frac{2}{3} < \frac{3}{4} < 1.$$

Divide the interval $[0,1]$ into the subintervals $[0, y_n - \eta_n]$, $(y_n - \eta_n, y_n + \eta_n)$, $[y_n + \eta_n, 1]$ and define the transformation T_n to preserve the values of a function $f(x)$ unless $x \in (y_n - \eta_n, y_n + \eta_n)$ in which case it flips the value of $f(x)$, $x \in (y_n - \eta_n, y_n)$, into the value $f(x')$, where $x' \in (y_n, y_n + \eta_n)$ is the symmetric of x around the center point y_n, and viceversa. Formally, for every $n \geq 1$ define the unitary transformation T_n as follows: for every $f \in \mathfrak{H}_1$

$$T_n f(x) = \begin{cases} f(x), & x \in [0, y_n - \eta_n] \cup [y_n + \eta_n, 1] \\ f(2y_n - x), & x \in (y_n - \eta_n, y_n + \eta_n). \end{cases}$$

For every $n \geq 1$, one has therefore $T_n \mathbf{1}_{[0,1]} = \mathbf{1}_{[0,1]}$. Moreover,

$$W_t^{(n)} = \begin{cases} W_t, & t \in [0, y_n - \eta_n] \cup [y_n + \eta_n, 1] \\ W_{y_n - \eta_n} + W_{y_n + \eta_n} - W_{2y_n - t}, & t \in (y_n - \eta_n, y_n + \eta_n). \end{cases}$$

Thus, if for instance $t \in (y_n - \eta_n, y_n)$, $W_t^{(n)}$ cumulates the increments of W up to $y_n - \eta_n$, to which instead of adding the increments of W over $(y_n - \eta_n, t)$, one adds the increments of W over $(2y_n - t, y_n + \eta_n)$, by peaking into the future. Also, for every $t_n \uparrow 1$ we may set $n_* = \inf \left\{ m : t_m > \frac{3}{4} \right\}$, so that the sequence of subspaces $\mathfrak{H}_{n,1} = \{ f : f \text{ has support in } [0, t_n] \}$, $n \geq n_*$, satisfies Assumption I-ii, since the transform of the function with support in $[0, t_n]$, $t_n > \frac{3}{4}$ is a function which has similar support.

We are interested in the asymptotic behavior, for $n \to +\infty$, of the sequence

$$A_n = \int_0^1 t^{2n} \left[\left(W_1^{(n)} \right)^2 - \left(W_t^{(n)} \right)^2 \right] dt, \quad n \geq 1,$$

where the Brownian motions $W^{(n)}$, $n \geq 1$, are defined according to (63).

In particular, we would like to determine the speed at which A_n converges to zero as $n \to +\infty$, by establishing a stable convergence result. We start by observing that the asymptotic study of A_n can be reduced to that of a sequence of double stochastic integrals, because

$$\left(W_t^{(n)} \right)^2 = t + 2 \int_0^t W_s^{(n)} dW_s^{(n)}. \tag{65}$$

Thus,

$$A_n = \int_0^1 t^{2n} \left[2 \int_0^1 W_s^{(n)} \mathbf{1}_{(t \leq s)} dW_s^{(n)} + 1 - t \right] dt,$$

and it is easily deduced that

$$\begin{aligned} \sqrt{n} (2n+1) A_n &= 2\sqrt{n} \int_0^1 dW_s^{(n)} W_s^{(n)} s^{2n+1} + \sqrt{n} (2n+1) \int_0^1 (1-t) t^{2n} dt \\ &= 2\sqrt{n} \int_0^1 dW_s^{(n)} W_s^{(n)} s^{2n+1} + o(1). \end{aligned}$$

Now define $\sigma(W)$ to be the σ-field generated by W (or, equivalently, by any of the $W^{(n)}$'s): we have the following

Theorem 18. *Under Assumption I, as $n \to +\infty$,*

$$\sqrt{n} (2n+1) A_n \to_{(s, \sigma(W))} \mathbb{E}\mu_1(\cdot), \tag{66}$$

where $\mu_1(\cdot)$ verifies, for $\lambda \in \mathbb{R}$,

$$\widehat{\mu}_1(\lambda) = \exp\left(-\frac{\lambda^2}{2} W_1^2 \right),$$

or, equivalently, for every $Z \in \sigma(W)$

$$\left(Z, \sqrt{n}\,(2n+1)\,A_n\right) \overset{law}{\to} \left(Z, W_1 \times N'\right), \tag{67}$$

where N' is a standard Gaussian random variable independent of W. In particular, $\sqrt{n}\,(2n+1)\,A_n \overset{law}{\to} N \times N'$, where N, N' are standard normal random variables.

Example – One can take $Z = \sup_{s\in[0,1]} W_s$, and deduce from (67) that

$$\left(\sup_{s\in[0,1]} W_s, \sqrt{n}\,(2n+1)\,A_n\right) \overset{\text{law}}{\to} \left(\sup_{s\in[0,1]} W_s, W_1 \times N'\right).$$

Remark – In particular, if $W^{(n)} = W$ for every n, one gets the same convergence in law (67). This last result was proved in (28, Proposition 2.1) by completely different methods.

Proof of Theorem 18. The proof of (66) is based on Theorem 16. First observe that the Gaussian family

$$X_W(h) = \int_0^1 h(s)\,dW_s, \quad h \in L^2([0,1], ds), \tag{68}$$

defines an isonormal Gaussian process over the Hilbert space $\mathfrak{H}_1 \triangleq L^2([0,1], ds)$; we shall write X_W to indicate the isonormal Gaussian process given by (68). Now define the following sequence of continuous resolutions of the identity on $\mathfrak{H}_1$: for every $n \geq 1$, every $t \in [0,1]$ and every $h \in \mathfrak{H}_1$,

$$\pi_t^{(n)} h = T_n \mathbf{1}_{[0,t]}\left(T_n^{-1} h\right). \tag{69}$$

Observe that $\pi_t^{(n)} \pi_t^{(n)} h = \pi_t^{(n)} h$, that is, $\pi_t^{(n)}$ is indeed a projection. To show that it is an orthogonal projection one needs to show that for any $g \in \mathfrak{H}_1$, $\pi_t^{(n)} g = 0$, if, and only if, $\left(\pi_t^{(n)} h, g\right)_{\mathfrak{H}_1} = 0$ for every $h \in \mathfrak{H}_1$. But,

$$\left(\pi_t^{(n)} h, g\right)_{\mathfrak{H}_1} = \left(\mathbf{1}_{[0,t]} T_n^{-1} h, T_n^{-1} g\right)_{\mathfrak{H}_1} = \left(T_n^{-1} h, \mathbf{1}_{[0,t]} T_n^{-1} g\right)_{\mathfrak{H}_1} = \left(h, \pi_t^{(n)} g\right)_{\mathfrak{H}_1} = 0,$$

and therefore $\pi_t^{(n)}$ is orthogonal. For $t \in [0,1]$

$$\mathcal{F}_t^{\pi^{(n)}}(X_W) = \sigma\left\{W_u^{(n)} : u \leq t\right\}. \tag{70}$$

In this case, the class of adapted processes $L^2_{\pi^{(n)}}(\mathfrak{H}_1, X_W)$, $n \geq 1$, is given by those elements of $L^2(\mathfrak{H}_1, X_W)$ that are adapted to the filtration $\mathcal{F}_\cdot^{\pi^{(n)}}(X_W)$, as defined in (70). Define, for $n \geq 1$,

$$u_n(t) = 2\sqrt{n} W_t^{(n)} t^{2n+1}, \quad t \in [0,1].$$

Since $\mathbb{E}\left[\int_0^1 u_n(s)^2\,ds\right] < +\infty$, $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_1, X_W)$ for every n, and hence

$$2\sqrt{n} \int_0^1 W_s^{(n)} s^{2n+1} dW_s^{(n)} = \int_0^1 u_n(s)\,dW_s^{(n)} = \delta_W(u_n), \tag{71}$$

where δ_W stands for a Skorohod integral with respect to X_W. Indeed, for every n, $\int_0^1 u_n(s)\,dW_s^{(n)}$ can be approximated in $L^2(\mathbb{P})$ by a sequence of random variables of the type

$$\begin{aligned}\sum_{j=1}^{N} u_n(t_{j-1})\left(W_{t_j}^{(n)}-W_{t_{j-1}}^{(n)}\right) &= \sum_{j=1}^{N} u_n(t_{j-1})\,W\left(T_n\mathbf{1}_{(t_{j-1},t_j]}\right)\\ &= \delta_W\left(\sum_{j=1}^{N} u_n(t_{j-1})\times T_n\mathbf{1}_{(t_{j-1},t_j]}\right), \qquad (72)\end{aligned}$$

where $0=t_0<t_1<\cdots<t_N=1$ and (72) derives from standard properties of Skorohod integrals (see e.g. (17, Ch. 1)), so that (71) is obtained by using the fact that δ_W is a closed operator. Now fix $\varepsilon\in(0,1)$, and set $t_n=\varepsilon^{1/\sqrt{n}}$, $t_n\uparrow 1$. Then,

$$\mathbb{E}\left[\left\|\pi_{t_n}^{(n)}u_n\right\|_{\mathfrak{H}_1}^2\right]=4n\int_0^{t_n}s^{4n+3}ds=\frac{4n}{4n+4}\varepsilon^{\frac{4n+4}{\sqrt{n}}}\underset{n\to+\infty}{\longrightarrow}0,$$

and, by (65),

$$\begin{aligned}\|u_n\|_{\mathfrak{H}_1}^2 &= 4n\int_0^1 ds\left(W_s^{(n)}\right)^2 s^{4n+2}\\ &= 4n\int_0^1 s^{4n+3}ds+8n\int_0^1 s^{4n+2}\int_0^s dW_u^{(n)}W_u^{(n)}\\ &= \frac{4n}{4n+4}+\frac{8n}{4n+3}\int_0^1 dW_u^{(n)}W_u^{(n)}\left(1-u^{4n+3}\right)\\ &= o_{\mathbb{P}}(1)+1+\frac{8n}{4n+3}\int_0^1 dW_u^{(n)}W_u^{(n)}\overset{\mathbb{P}}{\to}W_1^2, \qquad (73)\end{aligned}$$

by (64), where $o_{\mathbb{P}}(1)$ stands for a sequence converging to zero in probability (as $n\to+\infty$). We thus have shown that relations (59) and (61) of Theorem 16 are satisfied. It remains to verify relation (38), namely to show that there exists an integer $n\geq n_*$ as well as a sequence of σ-fields $\{\mathcal{U}_n:n\geq n_*\}$ verifying $\mathcal{U}_n\subseteq\mathcal{U}_{n+1}\cap\mathcal{F}_{t_n}^{\pi^{(n)}}(X_W)$ and $\vee_n\mathcal{U}_n=\sigma(W)$. The sequence

$$\mathcal{U}_n\triangleq\sigma\left\{\int_0^1 h(s)\,dW_s:h\in\mathfrak{H}_{n,1}\right\},\quad n\geq n_*,$$

where the spaces $\mathfrak{H}_{n,1}$ are defined in Assumption I-ii, is increasing and such that $\mathcal{U}_n\subseteq\mathcal{F}_{t_n}^{\pi^{(n)}}(X_W)$ (see (70)), and therefore verifies the required properties. As a consequence, Theorem 16 applies, and we obtain the stable convergence result (66). ■

Remark – The sequence $A_n'\triangleq\sqrt{n}\int_0^1 W_s^{(n)}s^{2n+1}dW_s^{(n)}$, although stably convergent and such that (73) is verified, *does not* admit a limit in probability. Indeed,

$$\begin{aligned}\underline{\lim}_{n,m\to+\infty}\mathbb{E}\left[\left(A_m'-A_n'\right)^2\right] &= \underline{\lim}_{n,m\to+\infty}\int_0^1\left(\sqrt{n}s^{2n+1}-\sqrt{m}s^{2m+1}\right)^2 sds\\ &= \underline{\lim}_{n,m\to+\infty}\left[\frac{n}{4n+4}+\frac{m}{4m+4}-\frac{\sqrt{nm}}{n+m+2}\right]\\ &> 0\end{aligned}$$

(for instance, take $m = 2n$). It follows that A'_n is not a Cauchy sequence in $L^2(\mathbb{P})$ and therefore, since the L^2 and L^0 topologies coincide on any finite sum of Wiener chaoses (see e.g. (35)), A'_n cannot converge in probability.

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References

[1] Billingsley P. (1969). *Convergence of probability measures.* Birkhäuser. MR0233396

[2] Brodskii M.S. (1971), *Triangular and Jordan Representations of Linear Operators.* Transl. Math. Monographs **32**, AMS, Providence. MR0322542

[3] Cohen S. and Taqqu M. (2004). Small and large scale behavior of the Poissonized Telecom process. *Methodology and Computing in Applied Probability* **6**, 363-379 MR2108557

[4] Engel D.D. (1982). The multiple stochastic integral. *Mem. Am. Math. Society* **38**, 1-82. MR0660396

[5] Feigin P. D. (1985). Stable convergence of semimartingales. *Stochastic Processes and their Applications* **19**, 125-134. MR0780725

[6] Giné E. and de la Peña V.H. (1999). *Decoupling.* Springer Verlag. MR1666908

[7] Jacod, J. (1984). Une généralisation des semimartingales : les processus admettant un processus à accroissements indépendants tangent. In: *Séminaire de probabilités XVIII*, 91-118. LNM **1059**, Springer Verlag. MR0770952

[8] Jacod, J. (2003). On processes with conditional independent increments and stable convergence in law. In: *Séminaire de probabilités XXXVI*, 383-401. LNM **1801**, Springer Verlag. MR1971599

[9] Jacod J., Klopotowski A. and Mémin J. (1982). Théorème de la limite centrale et convergence fonctionnelle vers un processus à accroissements indépendants : la méthode des martingales. *Annales de l'I.H.P. section B*, **1**, 1-45

[10] Jacod J. and Sadi H. (1987). Processus admettant un processus à accroissements indépendants tangent : cas général. In: *Séminaire de Probabilités XXI*, 479-514. LNM **1247**, Springer Verlag. MR0942000

[11] Jacod J. and Shiryaev A.N. (1987). *Limit Theorems for Stochastic Processes.* Springer, Berlin. MR0959133

[12] Jakubowski A. (1986). Principle of conditioning in limit theorems for sums of random variables. *The Annals of Probability* **11**(3), 902-915 MR0841592

[13] Y. Kabanov (1975). On extended stochastic integrals. *Theory of Probability and its applications* **20**, 710-722. MR0397877

[14] Kwapien S. and Woyczyński W.A. (1991). Semimartingale integrals via decoupling inequalities and tangent processes. *Probability and Mathematical Statisitics* **12**(2), 165-200 MR1199772

[15] Kwapien S. and Woyczyński W.A. (1992). *Random Series and Stochastic Integrals: Single and Multiple.* Birkhäuser. MR1167198

[16] Lipster R.Sh. and Shiryaev A.N. (1980). A functional central limit theorem for semimartingales. *Theory of Probability and Applications* **XXV**, 667-688.

[17] Nualart D. (1995). *The Malliavin Calculus and related topics.* Springer Verlag. MR1344217

[18] Nualart D. (1998). Analysis on Wiener space and anticipating stochastic calculus. In: *Lectures on Probability Theory and Statistics. École de probabilités de St. Flour XXV (1995)*, LNM **1690**, Springer, 123-227. MR1668111

[19] Nualart D. and Peccati G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *The Annals of Probability* **33**(1), 177-193. MR2118863

[20] Nualart D. and Schoutens W. (2000). Chaotic and predictable representation for Lévy processes. *Stochastic Processes and their Applications* **90**, 109-122. MR1787127

[21] Nualart D. and J. Vives J. (1990). Anticipative calculus for the Poisson space based on the Fock space. *Séminaire de Probabilités XXIV*, LNM **1426**, Springer, 154-165. MR1071538

[22] Peccati G. and Prünster I. (2006). Linear and quadratic functionals of random hazard rates: an asymptotic analysis. Preprint.

[23] Peccati G. and Taqqu M.S. (2006a). Central limit theorems for double Poisson integrals. Preprint.

[24] Peccati G. and Taqqu M.S. (2006b). Stable convergence of multiple Wiener-Itô integrals. Preprint.

[25] Peccati G. and Taqqu M.S. (2007). Limit theorems for multiple stochastic integrals. Preprint.

[26] Peccati G., Thieullen M. and Tudor C.A. (2006). Martingale structure of Skorohod integral processes. *The Annals of Probability*, **34**(3), 1217-1239. MR2243882

[27] Peccati G. and Tudor C.A. (2004). Gaussian limits for vector-valued multiple stochastic integrals. In: *Séminaire de Probabilités XXXVIII*, 247-262, Springer Verlag MR2126978

[28] Peccati G. and Yor M. (2004). Four limit theorems for quadratic functionals of Brownian motion and Brownian bridge. In: *Asymptotic Methods in Stochastics*, AMS, Fields Institute Communication Series, 75-87 MR2106849

[29] Protter P. (1992). *Stochastic Integration and Differential Equation.* Springer-Verlag, Berlin-New York

[30] Rajput B.S. and Rosinski J. (1989). Spectral representation of infinitely divisble processes. *Probability Theory and Related Fields* **82**, 451-487 MR1001524

[31] Revuz D. and Yor M. (1999). *Continuous martingales and Brownian motion.* Springer-Verlag MR1725357

[32] Rota G.-C. and Wallstrom C. (1997). Stochastic integrals: a combinatorial approach. *The Annals of Probability*, **25**(3), 1257-1283. MR1457619

[33] Samorodnitsky G. and Taqqu M.S. (1994). *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance*, Chapman and Hall. New York, London. MR1280932

[34] Sato K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Studies in Advanced Mathematics **68**. Cambridge University Press. MR1739520

[35] Schreiber M. (1969). Fermeture en probabilité de certains sous-espaces d'un espace L^2. *Zeitschrift W. v. G.* **14**, 36-48 MR0413290

[36] Surgailis D. (2003). CLTs for Polynomials of Linear Sequences: Diagram Formulae with Applications. In: *Long Range Dependence*, 111-128, Birkhäuser. MR1956046

[37] Surgailis D. (2003). Non-CLT's: U-Statistics, Multinomial Formula and Approximations of Multiple Wiener-Itô integrals. In: *Long Range Dependence*, 129-142, Birkhäuser. MR1956047

[38] Üstünel A.S. and Zakai M. (1997). The Construction of Filtrations on Abstract Wiener Space. *Journal of Functional Analysis*, **143**, 10-32. MR1428114

[39] Wu L.M. (1990). Un traitement unifié de la représentation des fonctionnelles de Wiener. In: *Séminaire de Probabilités XXIV*, 166-187. LNM **1426**, Springer Verlag. MR1071539

[40] Xue, X.-H. (1991). On the principle of conditioning and convergence to mixtures of distributions for sums of dependent random variables. *Stochastic Processes and their Applications* **37**(2), 175-186. MR1102868

[41] Yosida K. (1980). *Functional analysis.* Springer-Verlag, Berlin-New York. MR0617913