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# Integral representations of periodic and cyclic fractional stable motions*

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**Abstract**

Stable non-Gaussian self-similar mixed moving averages can be decomposed into several components. Two of these are the periodic and cyclic fractional stable motions which are the subject of this study. We focus on the structure of their integral representations and show that the periodic fractional stable motions have, in fact, a canonical representation. We study several examples and discuss questions of uniqueness, namely how to determine whether two given integral representations of periodic or cyclic fractional stable motions give rise to the same process.

**Key words:** stable, self-similar processes with stationary increments, mixed moving averages, periodic and cyclic flows, cocycles, semi-additive functionals.

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# 1 Introduction

*Periodic and cyclic fractional stable motions* (*PFSMs* and *CFSMs*, in short) were introduced by Pipiras and Taqqu (2004*b*) in the context of a decomposition of symmetric $\alpha$-stable ($S\alpha S$, in short), $\alpha \in (0,2)$, self-similar processes $X_\alpha(t)$, $t \in \mathbb{R}$, with stationary increments having a mixed moving average representation

$$\{X_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_X \int_{\mathbb{R}} \Big(G(x,t+u) - G(x,u)\Big) M_\alpha(dx,du)\right\}_{t\in\mathbb{R}}, \tag{1.1}$$

where $\stackrel{d}{=}$ stands for the equality of the finite-dimensional distributions. Here, $M_\alpha$ is a symmetric $\alpha$-stable random measure with control measure $\mu(dx)du$ (see Samorodnitsky and Taqqu (1994)), $(X,\mathcal{X},\mu)$ is a standard Lebesgue space (defined, for example in Appendix A of Pipiras and Taqqu (2004*d*)) and

$$\{G_t\}_{t\in\mathbb{R}} \in L^\alpha(X\times\mathbb{R},\mu(dx)du), \tag{1.2}$$

where

$$G_t(x,u) = G(x,t+u) - G(x,u), \quad x\in X, u \in \mathbb{R}. \tag{1.3}$$

A process $X_\alpha$ is said to be self-similar with index of self-similarity $H > 0$ if for any $a > 0$, $X_\alpha(at)$ has the same finite-dimensional distributions as $a^H X_\alpha(t)$. We will be interested in self-similar processes which have stationary increments as well ($H$-sssi processes, in short). Note that a process $X_\alpha(t)$ of the form (1.1) always has stationary increments. In order for the process (1.1) to be self-similar, it is necessary to impose additional conditions on the function $G$. There are various ways of doing this and different choices may give rise to different types of processes. In order to understand the nature of the resulting processes, one can associate the process $X_\alpha$ in (1.1) (or its kernel $G$) to flows. Flows are deterministic maps $\psi_c$, $c > 0$, satisfying $\psi_{c_1c_2} = \psi_{c_1}\circ\psi_{c_2}$, $c_1, c_2 > 0$. One can then use the characteristics of the flows to classify the corresponding $H$-sssi processes as well as to decompose a given $H$-sssi process into sub-processes that belong to disjoint classes. Periodic flows are such that each point of the space comes back to its initial position in a finite period of time. Cyclic flows are periodic flows such that the shortest return time is positive (nonzero).

In this work, we examine PFSMs and CFSMs in greater depth. We show that PFSMs can be defined as those self-similar mixed moving averages having the representation (1.1) with

$$X = Z \times [0,q(\cdot)), \quad \mu(dx) = \sigma(dz)dv \tag{1.4}$$

where $x = (z,v)$, and

$$G(z,v,u) = b_1(z)^{[v+\ln|u|]_{q(z)}}\Big(F_1(z,\{v+\ln|u|\}_{q(z)})\,u_+^\kappa + F_2(z,\{v+\ln|u|\}_{q(z)})\,u_-^\kappa\Big)$$
$$+\ 1_{\{b_1(z)=1\}}1_{\{\kappa=0\}}F_3(z)\ln|u|, \tag{1.5}$$

where $(Z,\mathcal{Z},\sigma)$ is a standard Lebesgue space, $b_1(z) \in \{-1,1\}$, $q(z) > 0$ a.e. $\sigma(dz)$, $F_1, F_2 : Z\times[0,q(\cdot)) \mapsto \mathbb{R}$, $F_3 : Z \mapsto \mathbb{R}$ are some functions. We used here the convenient notation

$$\kappa = H - \frac{1}{\alpha}, \tag{1.6}$$

where $H > 0$ is the self-similarity parameter of the process $X_\alpha$. Thus, in particular, $\kappa > -1/\alpha$. We also let

$$[x]_a = \max\{n \in \mathbb{Z} : na \le x\}, \quad \{x\}_a = x - a[x]_a, \quad x \in \mathbb{R}, a > 0, \tag{1.7}$$

and suppose by convention that

$$u_+^\kappa = 1_{(0,\infty)}(u), \quad u_-^\kappa = 1_{(-\infty,0]}(u), \tag{1.8}$$

when $\kappa = 0$. Since the representation (1.1) with $X$ defined as in (1.4) and $G$ defined as in (1.5) characterizes a PFSM, it is called *a canonical representation* for PFSMs. We will provide other canonical representations for PFSMs as well. We are not aware of canonical representations for CFSMs. CFSMs, however, do admit the representation (1.1) with (1.4) and (1.5) because they are PFSMs.

By using the representation (1.1) with (1.4) and (1.5), we will generate and study various PFSMs and CFSMs. For these processes to be well-defined, one must choose functions $F_1, F_2, F_3$ in (1.5) so that the functions $\{G_t\}_{t\in\mathbb{R}}$ in (1.3) belong to the space $L^\alpha(Z\times[0,q(\cdot))\times\mathbb{R}, \sigma(dz)dvdu)$. This is in general quite difficult (see Section 6). We will also address the uniqueness problem of PFSMs and CFSMs, namely, how to determine whether two given PFSMs or CFSMs are different, that is, when their finite dimensional distributions are not the same up to a constant.

The paper is organized as follows. In Section 2, we briefly recall the definitions of PFSMs and CFSMs, and related notions. In Section 3, we establish several canonical representations for PFSMs. In Section 4, we discuss the representation problem for CFSMs. Examples of PFSMs and CFSMs are given in Section 6. Uniqueness questions are addressed in Section 7. In Section 8, we study some functionals related to cyclic flows. Section 9 contains the proofs of some results of Section 3.

## 2 Periodic and cyclic fractional stable motions

Periodic and cyclic fractional stable motions (PFSMs and CFSMs, in short) can be defined in two equivalent ways (see Pipiras and Taqqu (2004*b*)). The first definition uses the kernel function $G$ in the representation (1.1) of stable self-similar mixed moving average. Consider the sets

$$\begin{aligned} C_P \ = \ \Big\{ & x \in X : \exists\ c = c(x) \neq 1 : G(x, cu) = b\, G(x, u+a) + d \ \text{ a.e. } du \\ & \text{for some } a = a(c,x), b = b(c,x) \neq 0, d = d(c,x) \in \mathbb{R} \Big\}, \end{aligned} \tag{2.1}$$

and

$$C_L = C_P \setminus C_F, \tag{2.2}$$

where

$$\begin{aligned} C_F \ = \ \Big\{ & x \in X : \exists\ c_n = c_n(x) \to 1\ (c_n \neq 1) : G(x, c_n u) = b_n\, G(x, u+a_n) + d_n \text{ a.e. } du \\ & \text{for some } a_n = a_n(c_n,x), b_n = b_n(c_n,x) \neq 0, d_n = d_n(c_n,x) \in \mathbb{R} \Big\}. \end{aligned} \tag{2.3}$$

The sets $C_P$, $C_L$ and $C_F$ are called, respectively, the PFSM set, the CFSM set and the mixed LFSM set.

**Definition 2.1.** A $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average $X_\alpha$ having a representation (1.1) is called

$$\begin{aligned} \text{PFSM} \quad &\text{if} \quad X = C_P, \\ \text{CFSM} \quad &\text{if} \quad X = C_L, \\ \text{mixed LFSM} \quad &\text{if} \quad X = C_F. \end{aligned}$$

**Remark.** A given $S\alpha S$ self-similar mixed moving average can be characterized by different kernels $G$, that is, different integral representations (1.1). Definition 2.1 and other results of the paper can be extended to include all $\alpha \in (0,2)$. Their validity depends on the existence of the so-called "minimal representations". If $\alpha \in (1,2)$, there is always at least one minimal representation (Theorem 4.2 in Pipiras and Taqqu (2002*a*)). If $\alpha \in (0,1]$, a minimal representation exists if some additional conditions are satisfied (see the Remark on page 436 in Pipiras and Taqqu (2002*a*)). For the sake of simplicity, we chose here not to involve these additional conditions and hence, we suppose that $\alpha \in (1,2)$, unless otherwise specified. We will not need here to use explicitly the definition of minimal representations. Minimal representations are studied, for example, in Rosiński (2006) and also used in Pipiras and Taqqu (2002*a*, 2004*c*).

Mixed LFSMs were studied in Pipiras and Taqqu (2002*b*), Section 7, where it is shown that they have the following canonical representation.

**Proposition 2.1.** *A $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average $X_\alpha$ is a mixed LFSM if and only if*

$$\begin{cases} \int_X \int_{\mathbb{R}} \Big( F_1(x)((t+u)_+^\kappa - u_+^\kappa) + F_2(x)((t+u)_-^\kappa - u_-^\kappa) \Big) M_\alpha(dx,du), & \kappa \neq 0, \\ \int_X \int_{\mathbb{R}} \Big( F_1(x) \ln \frac{|t+u|}{|u|} + F_2(x) 1_{(-t,0)}(u) \Big) M_\alpha(dx,du), & \kappa = 0, \end{cases} \tag{2.4}$$

*where $F_1, F_2 : X \mapsto \mathbb{R}$ are some functions and $M_\alpha$ has the control measure $\nu(dx)du$.*

If the space $X$ reduces to a single point, then $X_\alpha$ becomes the usual linear fractional stable motion (LFSM) process if $\kappa \neq 0$, and it becomes a linear combination of a log-fractional stable motion and a Lévy stable motion if $\kappa = 0$ (see Samorodnitsky and Taqqu (1994), Section 7, for an introduction to these processes). The process $X_\alpha$ in (2.4) is called a mixed LFSM because when $\kappa \neq 0$, it differs from a LFSM by the additional variable $x$.

Our goal is to study integral representations of PFSMs and CFSMs. These processes are defined in Definition 2.1 in terms of the kernel $G$ in (1.1) via the sets $C_P$ and $C_L$. An alternative definition of PFSMs and CFSMs is related to the notion of a flow. A (multiplicative) *flow* $\{\psi_c\}_{c>0}$ is a collection of deterministic maps satisfying

$$\psi_{c_1 c_2}(x) = \psi_{c_1}(\psi_{c_2}(x)), \quad \text{for all } c_1, c_2 > 0, \ x \in X, \tag{2.5}$$

and $\psi_1(x) = x$, for all $x \in X$. In addition to flows, we shall also use a number of related real-valued functionals as *cocycles*, *1-semi-additive functionals* and *2-semi-additive functionals*. The definitions of these functionals and related results are given in Section 8. We say henceforth that a $S\alpha S$, $\alpha \in (0,2)$, self-similar process $X_\alpha$ having a mixed moving average representation

(1.1) is *generated by a nonsingular measurable flow* $\{\psi_c\}_{c>0}$ on $(X, \mathcal{X}, \mu)$ (through the kernel function $G$) if, for all $c > 0$,

$$c^{-\kappa} G(x, cu) = b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha} G\Big(\psi_c(x), u + g_c(x)\Big) + j_c(x) \quad \text{a.e. } \mu(dx)du, \tag{2.6}$$

where $\{b_c\}_{c>0}$ is a cocycle for the flow $\{\psi_c\}_{c>0}$ taking values in $\{-1, 1\}$, $\{g_c\}_{c>0}$ is a *1*-semi-additive functional for the flow $\{\psi_c\}_{c>0}$ and $\{j_c\}_{c>0}$ is a *2*-semi-additive functional for the flow $\{\psi_c\}_{c>0}$ and the cocycle $\{b_c\}_{c>0}$, and if

$$\operatorname{supp} \{G(x, t + u) - G(x, u), t \in \mathbb{R}\} = X \times \mathbb{R} \quad \text{a.e. } \mu(dx)du. \tag{2.7}$$

This definition can be found in Pipiras and Taqqu (2004*c*). It differs from that used in Pipiras and Taqqu (2002*a*, 2002*b*, 2004*b*) in the statement that $\{j_c\}_{c>0}$ is a *2*-semi-additive functional, but this, it turns out, is no restriction.

The following is an alternative definition of PFSMs and CFSMs.

**Definition 2.2.** A $S\alpha S$, $\alpha \in (1, 2)$, self-similar mixed moving average $X_\alpha$ is a PFSM (a CFSM, resp.) if its minimal representation is generated by a periodic (cyclic, resp.) flow.

We now introduce a number of concepts related to Definition 2.2. For more details, see Pipiras and Taqqu (2004*b*). A measurable flow $\{\psi_c\}_{c>0}$ on $(X, \mathcal{X}, \mu)$ is called *periodic* if $X = P$ $\mu$-a.e. where $P$ is the set of periodic points of the flow defined as

$$P = \{x : \exists\ c = c(x) \neq 1 : \psi_c(x) = x\}. \tag{2.8}$$

It is called *cyclic* if $X = L$ $\mu$-a.e. where $L$ is the set of cyclic points of the flow defined by

$$L = P \setminus F, \tag{2.9}$$

where

$$F = \{x : \psi_c(x) = x \text{ for all } c > 0\} \tag{2.10}$$

is the set of the fixed points.

Note that the sets $C_L$, $C_P$ and $C_F$ in (2.2), (2.1) and (2.3) are defined in terms of the kernel $G$, whereas the sets $L$, $P$ and $F$ are defined in terms of the flow. If the representation is minimal, one has $C_L = L$, $C_P = P$ and $C_F = F$ $\mu$-a.e. (Theorem 3.2, Pipiras and Taqqu (2004*b*)).

A cyclic flow can also be characterized as a flow which is null-isomorphic (mod 0) to the flow

$$\widetilde{\psi}_c(z, v) = (z, \{v + \ln c\}_{q(z)}) \tag{2.11}$$

on $(Z \times [0, q(\cdot)), \mathcal{Z} \times \mathcal{B}([0, q(\cdot))), \sigma(dz)dv)$, where $q(z) > 0$ a.e. is a measurable function. Null-isomorphic (mod 0) means that there are two null sets $N \subset X$ and $\widetilde{N} \subset Z \times [0, q(\cdot))$, and a Borel measurable, one-to-one, onto and nonsingular map with a measurable nonsingular inverse (a so-called "null-isomorphism") $\Phi : Z \times [0, q(\cdot)) \setminus \widetilde{N} \mapsto X \setminus N$ such that

$$\psi_c(\Phi(z, v)) = \Phi(\widetilde{\psi}_c(z, v)) \tag{2.12}$$

for all $c > 0$ and $(z, v) \in Z \times [0, q(\cdot)) \setminus \widetilde{N}$. The null sets $N$ and $\widetilde{N}$ are required to be invariant under the flows $\psi_c$ and $\widetilde{\psi}_c$, respectively. This result, established in Theorem 2.1 of Pipiras and Taqqu (2004*d*), will be used in the sequel to establish the canonical representation (1.5) for PFSMs.

## 3 Canonical representations for PFSM

We show in this section that a PFSM can be characterized as a self-similar mixed moving average represented in one of the explicit ways specified below. We say that these representations are *canonical* for a PFSM. Canonical representations for a PFSM allow the construction of specific examples of PFSMs and also help to identify them. The first canonical representation is given in the next result.

**Theorem 3.1.** *A $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average $X_\alpha$ is a PFSM if and only if $X_\alpha$ can be represented by the sum of two independent processes:*

*(i) The first process has the representation*

$$\int_Z \int_{[0,q(z))} \int_{\mathbb{R}} \Big\{ \Big( b_1(z)^{[v+\ln|t+u|]_{q(z)}} F_1(z, \{v+\ln|t+u|\}_{q(z)})\,(t+u)_+^{\kappa} - b_1(z)^{[v+\ln|u|]_{q(z)}} F_1(z, \{v+\ln|u|\}_{q(z)})\,u_+^{\kappa} \Big)$$
$$+ \Big( b_1(z)^{[v+\ln|t+u|]_{q(z)}} F_2(z, \{v+\ln|t+u|\}_{q(z)})\,(t+u)_-^{\kappa} - b_1(z)^{[v+\ln|u|]_{q(z)}} F_2(z, \{v+\ln|u|\}_{q(z)})\,u_-^{\kappa} \Big)$$
$$+ 1_{\{b_1(z)=1\}} 1_{\{\kappa=0\}} F_3(z) \ln \frac{|t+u|}{|u|} \Big\} M_\alpha(dz,dv,du), \qquad (3.1)$$

*where $(Z,\mathcal{Z},\sigma)$ is a standard Lebesgue space, $b_1(z) \in \{-1,1\}$, $q(z)>0$ a.e. $\sigma(dz)$ and $F_1, F_2 : Z \times [0,q(\cdot)) \mapsto \mathbb{R}$, $F_3 : Z \mapsto \mathbb{R}$ are measurable functions, and $M_\alpha$ has the control measure $\sigma(dz)dvdu$.*

*(ii) The second process has the representation*

$$\begin{cases} \int_Y \int_{\mathbb{R}} \Big( F_1(y)((t+u)_+^{\kappa} - u_+^{\kappa}) + F_2(y)((t+u)_-^{\kappa} - u_-^{\kappa}) \Big) M_\alpha(dy,du), & \kappa \neq 0, \\ \int_Y \int_{\mathbb{R}} \Big( F_1(y) \ln \frac{|t+u|}{|u|} + F_2(y) 1_{(-t,0)}(u) \Big) M_\alpha(dy,du), & \kappa = 0, \end{cases} \qquad (3.2)$$

*where $(Y,\mathcal{Y},\nu)$ is a standard Lebesgue space, $F_1, F_2 : Y \mapsto \mathbb{R}$ are some functions and $M_\alpha$ has the control measure $\nu(dy)du$.*

Observe that the processes (3.2) are the mixed LFSM (2.4) introduced in Section 2. As we will see below (Corollary 3.1), Theorem 3.1 is also true without part *(ii)*. It is convenient, however, to state it with Part *(ii)* because this facilitates the identification of PFSMs and helps in understanding the distinction between PFSMs and CFSMs. The proof of Theorem 3.1 can be found in Section 9. It is based on results on flow functionals established in Section 4 and on the following proposition.

**Proposition 3.1.** *If $X_\alpha$ is a $S\alpha S$, $\alpha \in (0,2)$, self-similar mixed moving average generated by a cyclic flow, then $X_\alpha$ can be represented by (3.1).*

The proof of this proposition is given in Section 9. It is used in the proof of Theorem 3.1 in the following way. If $X_\alpha$ is a PFSM, it has a minimal representation generated by a periodic flow (see Definition 2.2). Since the periodic points of a flow consist of cyclic and fixed points, the process $X_\alpha$ can be expressed as the sum of two processes: one generated by a cyclic flow and the other generated by an identity flow. Proposition 3.1 is used to show that the process generated by a cyclic flow has the representation (3.1). The process generated by an identity flow has the representation (3.2) according to Theorem 5.1 in Pipiras and Taqqu (2002*b*).

The representation (3.1) is not specific to processes generated by cyclic flows. The next result shows that mixed LFSMs (they are generated by identity flows) can also be represented by (3.1).

**Proposition 3.2.** *A mixed LFSM having the representation (2.4) can be represented by (3.1).*

PROOF: Consider first the case $\kappa \neq 0$. Taking $b_1(z) \equiv 1$, $q(z) \equiv 1$, $F_1(z,v) \equiv F_1(z)$ and $F_2(z,v) \equiv F_2(z)$ in (3.1), we obtain the process

$$\int_Z \int_0^1 \int_{\mathbb{R}} 1_{[0,1)}(v) \Big( F_1(z)((t+u)_+^\kappa - u_+^\kappa) + F_2(z)((t+u)_-^\kappa - u_-^\kappa) \Big) M_\alpha(dz,dv,du).$$

Since the kernel above involves the variable $v$ only through the indicator function $1_{[0,1)}(v)$ and since the control measure of $M_\alpha(dz,dv,du)$ in variable $v$ is $dv$, the latter process has the same finite-dimensional distributions as

$$\int_Z \int_{\mathbb{R}} \Big( F_1(z)((t+u)_+^\kappa - u_+^\kappa) + F_2(z)(t+u)_-^\kappa - u_-^\kappa) \Big) M_\alpha(dz,du),$$

which is the representation (2.4) of a mixed LFSM when $\kappa \neq 0$. In the case $\kappa = 0$, one can arrive at the same conclusion by taking $b_1(z) \equiv 1$, $q(z) \equiv 1$, $F_1(z,v) = F_2(z)$, $F_2(z,v) = 0$ and $F_3(z) = F_1(z)$. $\square$

The next corollary which is an immediate consequence of Theorem 3.1 and Proposition 3.2 states, as indicated earlier, that it is not necessary to include Part (*ii*) in Theorem 3.1.

**Corollary 3.1.** *A $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average $X_\alpha$ is a PFSM if and only if it can be represented by (3.1).*

The following result provides another canonical representation of a PFSM which is often useful in practice. The difference between this result and Theorem 3.1 is that the function $s(z)$ appearing in the expressions $v + s(z)\ln|u|$ below is not necessarily equal to 1. One can interpret $|s(z)|$ as the "speed" with which the point $(z,v)$ moves under the multiplicative flow $\psi_c(z,v) = (z, \{v + s(z)\ln c\}_{q(z)})$. The greater $|s(z)|$, the faster does the fractional part $\{v+s(z)\ln c\}_{q(z)}$ regenerates itself.

**Theorem 3.2.** *A $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average $X_\alpha$ is a PFSM if and only if $X_\alpha$ can be represented by*

$$\int_Z \int_{[0,q(z))} \int_{\mathbb{R}} \Big\{ \Big( b_1(z)^{[v+s(z)\ln|t+u|]_{q(z)}} F_1(z, \{v+s(z)\ln|t+u|\}_{q(z)})\, (t+u)_+^\kappa$$

$$
\begin{aligned}
&\qquad\qquad - b_1(z)^{[v+s(z)\ln|u|]_{q(z)}} F_1(z, \{v+s(z)\ln|u|\}_{q(z)})\, u_+^{\kappa}\Big) \\
&+ \Big(b_1(z)^{[v+s(z)\ln|t+u|]_{q(z)}} F_2(z, \{v+s(z)\ln|t+u|\}_{q(z)})\, (t+u)_-^{\kappa} \\
&\qquad\qquad - b_1(z)^{[v+s(z)\ln|u|]_{q(z)}} F_2(z, \{v+s(z)\ln|u|\}_{q(z)})\, u_-^{\kappa}\Big) \\
&\quad + 1_{\{b_1(z)=1\}} 1_{\{\kappa=0\}} F_3(z) \ln \frac{|t+u|}{|u|}\Big)\Big\} M_\alpha(dz, dv, du), \qquad (3.3)
\end{aligned}
$$

*where* $(Z, \mathcal{Z}, \sigma)$ *is a standard Lebesgue space,* $b_1(z) \in \{-1, 1\}$*,* $s(z) \neq 0$*,* $q(z) > 0$ *a.e.* $\sigma(dz)$ *and* $F_1, F_2 : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$*,* $F_3 : Z \mapsto \mathbb{R}$ *are measurable functions, and* $M_\alpha$ *has the control measure* $\sigma(dz)dvdu$*.*

PROOF: By Corollary 3.1 above, it is enough to show that the process (3.3) can be represented by (3.1). As in Example 2.2 of Pipiras and Taqqu (2004*d*), the flow

$$\psi_c(z, v) = (z, \{v + s(z)\ln c\}_{q(z)}), \quad c > 0,$$

is cyclic because each point of the space $Z \times [0, q(\cdot))$ comes back to its initial position in a finite (nonzero) time. Moreover,

$$b_c(z, v) = b_1(z)^{[v+s(z)\ln c]}, \quad c > 0,$$

is a cocycle for the flow $\{\psi_c\}_{c>0}$. Indeed, by using the second relation in (1.7) and the fact that $\{\psi_c\}_{c>0}$ is a multiplicative flow, we have

$$
\begin{aligned}
q[v + s(\ln c_1 + \ln c_2)]_q &= v + s(\ln c_1 + \ln c_2) - \{v + s(\ln c_1 + \ln c_2)\}_q \\
&= v + s\ln c_1 - \{v + s\ln c_1\}_q \\
&\qquad + \{v + s\ln c_1\}_q + v\ln c_2 - \{\{v + s\ln c_1\}_q + s\ln c_2\}_q \\
&= q[v + s\ln c_1]_q + q[\{v + s\ln c_1\}_q + s\ln c_2]_q.
\end{aligned}
$$

Hence,

$$b_1(z)^{q(z)[v+s(z)(\ln c_1+\ln c_2)]_{q(z)}} = b_1(z)^{[v+s(z)\ln c_1]_{q(z)}} b_1(z)^{[\{v+s(z)\ln c_1\}_{q(z)}+s(z)\ln c_2]_{q(z)}}$$

which shows that $b_c(z, v)$ is a cocycle. Since $\{b_c\}_{c>0}$ is a cocycle, relation (2.6) holds and the process (3.3) is generated by the flow $\{\psi_c\}_{c>0}$. Since the flow is cyclic, the process (3.3) has a representation (3.1) by Proposition 3.1. $\square$

**Remark.** Observe that the kernel function $G$ corresponding to the process (3.3) is defined as

$$
\begin{aligned}
G(z, v, u) = b_1(z)^{[v+s(z)\ln|u|]_{q(z)}} &\Big(F_1(z, \{v+s(z)\ln|u|\}_{q(z)})\, u_+^{\kappa} + F_2(z, \{v+s(z)\ln|u|\}_{q(z)})\, u_-^{\kappa}\Big) \\
&+ 1_{\{b_1(z)=1\}} 1_{\{\kappa=0\}} F_3(z) \ln|u|, \qquad (3.4)
\end{aligned}
$$

an expression which will be used a number of times in the sequel.

# 4 Representations for CFSM

As shown in Pipiras and Taqqu (2003), CFSMs do not have a (nontrivial) mixed LFSM component, that is, they cannot be expressed as a sum of two independent processes one of which is a mixed LFSM. Since a mixed LFSM can be represented by (3.1) (Proposition 3.2 above), a CFSM cannot be characterized as a process having the representation (3.1). The following result can be used instead.

**Proposition 4.1.** *Let $\alpha \in (1,2)$.*

*$(i)$ If $X_\alpha$ is a CFSM, then it can be represented as (3.1) and also as (3.3).*

*$(ii)$ If the process $X_\alpha$ has the representation (3.1) or (3.3) and $C_F = \emptyset$ a.e. where the set $C_F$ is defined by (2.3) using the representation (3.1) or (3.3), then $X_\alpha$ is a CFSM. Moreover, to show that $C_F = \emptyset$ a.e., it is enough to prove that $(z,0) \notin C_F$ a.e., that is, $(z,v) \notin C_F$ a.e. when $v = 0$.*

PROOF: $(i)$ If $X_\alpha$ is a CFSM, then $X_\alpha$ given by its minimal representation, is generated by a cyclic flow by Definition 2.2. Hence, $X_\alpha$ has the representation (3.1) by Proposition 3.1.

$(ii)$ If $X_\alpha$ is represented by (3.1), then by Corollary 3.1, $X_\alpha$ is a PFSM. By Definition 2.1, the PFSM set $C_P$ associated with the representation (3.1) is the whole space a.e. Since $C_L = C_P \backslash C_F$, the assumption $C_F = \emptyset$ a.e. implies that $C_L$ is the whole space a.e. as well. Therefore, $X_\alpha$ is a CFSM by Definition 2.1.

The last statement of Part $(ii)$ follows if we show that $(z,v) \in C_F$ if and only if $(z,0) \in C_F$. Suppose for simplicity that $X_\alpha$ has the representation (3.1) defined through the kernel $G$ in (1.5). Observe that

$$G(z,v,u) = e^{-\kappa v} G(z,0,e^v u) - 1_{\{b_1(z)=1\}} 1_{\{\kappa=0\}} F_3(z) v \tag{4.1}$$

for all $z, v, u$, and that, by making the change of variables $e^v u = w$, $v = v$ and $z = z$ in (4.1),

$$G(z,0,w) = e^{\kappa v} G(z,v,e^{-v} w) + 1_{\{b_1(z)=1\}} 1_{\{\kappa=0\}} F_3(z) v e^{\kappa v}$$

for all $z, v, w$. By using these relations and the definition (2.3) of $C_F$, we obtain that there is $c_n(z,v) \to 1$ ($c_n(z,v) \neq 1$) such that

$$G(z,v,c_n(z,v)u) = b_n(z,v) G(z,v,u + a_n(z,v)) + d_n(z,v), \quad \text{a.e. } du,$$

for some $a_n(z,v), b_n(z,v) \neq 0, d_n(z,v)$, if and only if there is $\widetilde{c}_n(z) \to 1$ ($\widetilde{c}_n(z) \neq 1$) such that

$$G(z,0,\widetilde{c}_n(z)u) = \widetilde{b}_n(z) G(z,0,u + \widetilde{a}_n(z)) + \widetilde{d}_n(z), \quad \text{a.e. } du,$$

for some $\widetilde{a}_n(z), \widetilde{b}_n(z) \neq 0, \widetilde{d}_n(z)$. This shows that $(z,v) \in C_F$ if and only if $(z,0) \in C_F$. $\square$

We do not know of any canonical representation for CFSMs, and feel that such representations may not exist at all.

## 5 Equivalent representations and space of integrands

The following result shows that there are other representations for PFSMs and CFSMs which are equivalent to (3.1). As we will see below, this result is useful for comparing PFSMs and CFSMs to other self-similar stable mixed moving averages, and for characterizing the space of integrands.

**Proposition 5.1.** *A process $X_\alpha$ has a representation (3.1) if and only if*

$$\begin{aligned} X_\alpha(t) &\stackrel{d}{=} \int_Z \int_0^{q(z)} \int_{\mathbb{R}} e^{-\kappa v}(K(z, e^v(t+u)) - K(z, e^v u)) M_\alpha(dz, dv, du) && (5.1) \\ &\stackrel{d}{=} \int_Z \int_1^{e^{q(z)}} \int_{\mathbb{R}} w^{-H}(K(z, w(t+u)) - K(z, wu)) M_\alpha(dz, dw, du) && (5.2) \\ &\stackrel{d}{=} \int_Z \int_1^{e^{-q(z)}} \int_{\mathbb{R}} w^{H-\frac{2}{\alpha}}(K(z, w^{-1}(t+u)) - K(z, w^{-1}u)) M_\alpha(dz, dw, du) && (5.3) \\ &\stackrel{d}{=} \int_Z \int_1^{e^{q(z)}} \int_{\mathbb{R}} w^{-H-\frac{1}{\alpha}}(K(z, wt+u)) - K(z, u)) M_\alpha(dz, dw, du) && (5.4) \\ &\stackrel{d}{=} \int_Z \int_1^{e^{q(z)}} \int_{\mathbb{R}} w^{H-\frac{1}{\alpha}}(K(z, w^{-1}t+u)) - K(z, u)) M_\alpha(dz, dw, du), && (5.5) \end{aligned}$$

*where $M_\alpha(dz, dv, du)$ has the control measure $\sigma(dz)dvdu$, and*

$$\begin{aligned} K(z, u) = b_1(z)^{[\ln|u|]_{q(z)}} \Big( F_1(z, \{\ln|u|\}_{q(z)})\, u_+^\kappa + F_2(z, \{\ln|u|\}_{q(z)})\, u_-^\kappa \Big) \\ + 1_{\{b_1(z)=1\}} 1_{\{\kappa=0\}} F_3(z) \ln|u|, \qquad (5.6) \end{aligned}$$

*for some functions $b_1(z) \in \{-1, 1\}$, $q(z) > 0$ a.e. and $F_1, F_2 : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$, $F_3 : Z \mapsto \mathbb{R}$. Moreover, PFSMs and CFSMs can be represented by either of the representation (5.1)–(5.5) with $K$ defined by (5.6), and each of these representations (5.1)–(5.5) is canonical for PFSMs.*

PROOF: To see that the representations (3.1) and (5.1) are equivalent, observe that $[v + \ln|u|]_{q(z)} = [\ln|e^v u|]_{q(z)}$, $\{v + \ln|u|\}_{q(z)} = \{\ln|e^v u|\}_{q(z)}$, and hence

$$G(z, v, u) = e^{-\kappa v} K(z, e^v u), \qquad (5.7)$$

where $G$ is the kernel function of (3.1) defined by (1.5) and $K$ is defined by (5.6). The relations (5.2)–(5.5) follow by making suitable changes of variables. For example, to obtain (5.2), make the change of variables $e^v = w$. The last statement of the proposition follows by using Corollary 3.1 and Proposition 4.1. □

**Remark.** When the integral over $v$ is $\int_{-\infty}^{\infty}$ in (5.1) instead of $\int_0^{q(z)}$, the integrals over $w$ are $\int_0^\infty$ in (5.2)–(5.5) and $K : Z \times \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary function (not necessarily of the form (5.6)), the resulting processes (5.1)–(5.5) are self-similar stable mixed moving averages as well. These processes, called *dilated fractional stable motions* (*DFSMs*, in short), are generated by the so-called dissipative flows, and are studied in detail by Pipiras and Taqqu (2004$a$). DFSMs and PFSMs (in particular, CFSMs) have different finite-dimensional distributions because

DFSMs are generated by dissipative flows and PFSMs are generated by conservative flows (see Theorem 5.3 in Pipiras and Taqqu (2002$a$)). Despite this fact, Proposition 5.1 shows that the representations of DFSMs and PFSMs have a common structure.

The condition for a PFSM or a CFSM given by (3.1) to be well-defined is that its kernel function satisfies condition (1.2). By using Proposition 5.1, we can replace this condition by one which is often easier to verify in practice. Let $\alpha \in (0,2)$, $H > 0$, $(Z, \mathcal{Z}, \sigma)$ be a measure space, and $g : Z \mapsto \mathbb{R}$ be a function such that $q(z) > 0$ a.e. $\sigma(dz)$. Consider the space of functions

$$\mathcal{C}^q_{\sigma,\alpha,H} = \Big\{ K : Z \times \mathbb{R} \mapsto \mathbb{R} \text{ such that } \|K\|_{\mathcal{C}^q_{\sigma,\alpha,H}} < \infty \Big\}, \tag{5.8}$$

where

$$\begin{aligned} \|K\|^\alpha_{\mathcal{C}^q_{\sigma,\alpha,H}} &= \int_Z \int_{\mathbb{R}} \int_1^{e^{q(z)}} h^{-\alpha H-1} |K(z,u+h) - K(u)|^\alpha \sigma(dz) du dh & (5.9) \\ &= \int_Z \left( \int_1^{e^{q(z)}} h^{-\alpha H-1} \|\triangle_h K(z,\cdot)\|^\alpha dh \right) \sigma(dz) & (5.10) \end{aligned}$$

with $\triangle_h g(\cdot) = g(\cdot + h) - g(\cdot)$.

**Proposition 5.2.** *A PFSM or CFSM represented by (3.1) is well-defined, that is, the condition (1.2) holds where $G$ is defined by (1.5), if and only if $K \in \mathcal{C}^q_{\sigma,\alpha,H}$, where $\mathcal{C}^q_{\sigma,\alpha,H}$ is given by (5.8) and $K$ is defined in (5.6).*

Proof: A process $X_\alpha$ represented by (3.1) is well-defined if and only if it is well-defined when represented by (5.4). Since $X_\alpha$ is self-similar, it is well-defined if and only if the integral

$$\int_Z \int_1^{e^{q(z)}} \int_{\mathbb{R}} w^{-H-\frac{1}{\alpha}} (K(z, w+u)) - K(z,u)) M_\alpha(dz, dw, du)$$

corresponding to $X_\alpha(1)$, is well-defined. The latter condition is equivalent to $K \in \mathcal{C}^q_{\sigma,\alpha,H}$, where $\mathcal{C}^q_{\sigma,\alpha,H}$ is given by (5.8). $\square$

Though PFSMs and CFSMs were defined for $\alpha \in (1,2)$, Proposition 5.2 continues to hold for processes represented by (3.1) when $\alpha \in (0,2)$. For this reason, we defined the space $\mathcal{C}^q_{\sigma,\alpha,H}$ in (5.8) above for $\alpha \in (0,2)$. When $Z = \{1\}$ and $\sigma(dz) = \delta_{\{1\}}(dz)$ is the point mass at $z = 1$, we shall use the notation

$$\mathcal{C}^q_{\alpha,H} := \mathcal{C}^q_{\sigma,\alpha,H}, \tag{5.11}$$

where $q > 0$. Thus, $K : \mathbb{R} \mapsto \mathbb{R}$ is in $\mathcal{C}^q_{\alpha,H}$ if and only if

$$\int_1^{e^q} h^{-\alpha H-1} \|\triangle_h K(\cdot)\|^\alpha dh < \infty. \tag{5.12}$$

The following result provides sufficient conditions for a function to belong to the space $\mathcal{C}^q_{\alpha,H}$.

**Lemma 5.1.** *Let $\alpha \in (0,2)$, $H \in (0,1)$ and $\kappa = H - 1/\alpha$, and let $K$ be defined by (5.6) with $Z = \{1\}$ and $\sigma(dz) = \delta_{\{1\}}(dz)$.*

*(i) Suppose that $\kappa < 0$. If $F_1, F_2 : [0,q) \mapsto \mathbb{R}$ are such that $F_1, F_2$ are absolutely continuous with derivatives $F_1', F_2'$, and*

$$\sup_{u\in[0,q)} |F_i(u)| \le C, \quad i = 1,2, \tag{5.13}$$

$$\operatorname{ess}\sup_{u\in[0,q)} |F_i'(u)| \le C, \quad i = 1,2, \tag{5.14}$$

*then $K \in \mathcal{C}^q_{\alpha,H}$.*

*(ii) Suppose that $\kappa \ge 0$. If, in addition to (i),*

$$F_i(0) = b_1 F_i(q-), \quad i = 1,2, \tag{5.15}$$

*then $K \in \mathcal{C}^q_{\alpha,H}$.*

PROOF: By using (5.12) and $-\alpha H - 1 = -\alpha\kappa - 2$, it is enough to show that

$$\int_1^{e^q} dh\, h^{-\alpha\kappa-2} \int_{\mathbb{R}} du |K(u+h) - K(u)|^\alpha < \infty. \tag{5.16}$$

Since $H \in (0,1)$, we have $\kappa = 0$ only when $\alpha \in (1,2)$. Since, for $\alpha \in (1,2)$,

$$\int_{\mathbb{R}} \Big| \ln|u+h| - \ln|u| \Big|^\alpha du = h \int_{\mathbb{R}} \Big| \ln|w+1| - \ln|w| \Big|^\alpha dw = Ch,$$

where $0 < C < \infty$, the function $K(u) = \ln|u|$ satisfies (5.16) when $\kappa = 0$. We may therefore suppose that the function $K$ is defined by (5.6) without the last term.

For simplicity, we will prove (5.16) in the case $F_1 = F_2 = F$, that is,

$$K(u) = b_1^{[\ln|u|]_q} F(\{\ln|u|\}_q)|u|^\kappa, \tag{5.17}$$

where the function $F$ satisfies (5.13)–(5.14) in Part $(i)$ and, in addition, (5.15) in Part $(ii)$. The general case for arbitrary $F_1$ and $F_2$ can be proved in a similar way. Since $F$ satisfies (5.13), we have $\int_1^{e^q} dh\, h^{-\alpha\kappa-2} \int_{-N}^{N} du|K(u+h) - K(u)|^\alpha < \infty$ for any constant $N$. Since $K(u) = K(-u)$, we have $\int_1^{e^q} dh\, h^{-\alpha\kappa-2} \int_N^\infty du|K(u+h) - K(u)|^\alpha < \infty$ if and only if $\int_1^{e^q} dh\, h^{-\alpha\kappa-2} \int_{-\infty}^{N} du|K(u+h) - K(u)|^\alpha < \infty$. It is therefore enough to show that

$$\int_N^\infty |K(u+h) - K(u)|^\alpha du \le C, \quad h \in (1, e^q), \tag{5.18}$$

where $C$ and $N$ are some constants. Observe that $0 < e^{qk} < e^{q(k+1)} - h < e^{q(k+1)}$ for large enough $k$ and all $h \in [0,q)$. Then, by taking $N = e^{qk_0}$ with a fixed large $k_0$ in (5.18), we have

$$\int_N^\infty |K(u+h) - K(u)|^\alpha du = \sum_{k=k_0}^{\infty} \int_{e^{qk}}^{e^{q(k+1)}-h} |K(u+h) - K(u)|^\alpha du$$

$$+\sum_{k=k_0}^{\infty}\int_{e^{q(k+1)}-h}^{e^{q(k+1)}}|K(u+h)-K(u)|^{\alpha}du =: I_1+I_2.$$

We will show that $I_1<\infty$ and $I_2<\infty$.

To show $I_1<\infty$, observe that $qk\le \ln|u+h|<q(k+1)$ for large $u\in[e^{qk}-h,e^{q(k+1)}-h)$, and $qk\le\ln|u|<q(k+1)$ for large $u\in[e^{qk},e^{q(k+1)})$. Hence, for large $u\in[e^{qk},e^{q(k+1)}-h)\subset[e^{qk}-h,e^{q(k+1)}-h)\cap[e^{qk},e^{q(k+1)})$, we have

$$[\ln|u+h|]_q=[\ln|u|]_q=k,\quad \{\ln|u+h|\}_q=\ln|u+h|-qk,\quad \{\ln|u|\}_q=\ln|u|-qk.$$

By using these relations and since $b_1\in\{-1,1\}$, we obtain that

$$I_1=\sum_{k=k_0}^{\infty}\int_{e^{qk}}^{e^{q(k+1)}-h}\Big|b_1^{[\ln|u+h|]_q}F(\{\ln|u+h|\}_q)|u+h|^{\kappa}-b_1^{[\ln|u|]_q}F(\{\ln|u|\}_q)|u|^{\kappa}\Big|^{\alpha}du$$

$$=\sum_{k=k_0}^{\infty}\int_{e^{qk}}^{e^{q(k+1)}-h}\Big|F(\ln|u+h|-qk)|u+h|^{\kappa}-F(\ln|u|-qk)|u|^{\kappa}\Big|^{\alpha}du$$

$$\le C\sum_{k=k_0}^{\infty}\int_{e^{qk}}^{e^{q(k+1)}-h}\Big|F(\ln|u+h|-qk)-F(\ln|u|-qk)\Big|^{\alpha}|u+h|^{\kappa\alpha}du$$

$$+C\sum_{k=k_0}^{\infty}\int_{e^{qk}}^{e^{q(k+1)}-h}|F(\ln|u|-qk)|^{\alpha}\Big||u+h|^{\kappa}-|u|^{\kappa}\Big|^{\alpha}du =: C(I_{1,1}+I_{1,2}).$$

By using (5.14) and the mean value theorem and by making the change of variables $u=hw$ below, we obtain that

$$I_{1,1}\le C\sum_{k=k_0}^{\infty}\int_{e^{qk}}^{e^{q(k+1)}-h}\Big|\ln\frac{|u+h|}{|u|}\Big|^{\alpha}|u+h|^{\kappa\alpha}du\le C\int_h^{\infty}\Big|\ln\frac{|u+h|}{|u|}\Big|^{\alpha}|u+h|^{\kappa\alpha}du$$

$$=Ch^{\kappa\alpha+1}\int_1^{\infty}\Big|\ln\frac{|w+1|}{|w|}\Big|^{\alpha}|w+1|^{\kappa\alpha}dw\le C'h^{\kappa\alpha+1}\int_1^{\infty}|w|^{\kappa\alpha-\alpha}dw\le C,\quad \text{for } h\in(1,e^q),$$

since $(\kappa\alpha-\alpha)+1=(H-1)\alpha<0$. By using (5.13), we can similarly show that

$$I_{1,2}\le C\int_{\mathbb{R}}\Big||u+h|^{\kappa}-|u|^{\kappa}\Big|^{\alpha}du=Ch^{\kappa\alpha+1}\int_{\mathbb{R}}\Big||w+1|^{\kappa}-|w|^{\kappa}\Big|^{\alpha}dw\le C',\quad \text{for } h\in(1,e^q).$$

Hence, $I_1\le C$ for $h\in(1,e^q)$.

We will now show that $I_2<\infty$. Consider first the case $(i)$ where $\kappa<0$. Then, by using (5.13) and for $h\in(1,e^q)$,

$$I_2\le C\sum_{k=k_0}^{\infty}\int_{e^{q(k+1)}-h}^{e^{q(k+1)}}\Big(|u+h|^{\kappa\alpha}+|u|^{\kappa\alpha}\Big)du\le C'\sum_{k=k_0}^{\infty}e^{q(k+1)\kappa\alpha}<\infty,$$

since $\kappa<0$. Consider now the case $(ii)$ where $\kappa\ge0$. Observe as above that, for large $u\in[e^{q(k+1)}-h,e^{q(k+1)})$,

$$[\ln|u+h|]_q=k+1,\quad [\ln|u|]_q=k,\quad \{\ln|u+h|\}_q=\ln|u+h|-q(k+1),\quad \{\ln|u|\}_q=\ln|u|-qk.$$

By using these relations and since $b_1 \in \{-1,1\}$, we obtain that

$$
\begin{aligned}
I_2 &= \sum_{k=k_0}^{\infty} \int_{e^{q(k+1)}-h}^{e^{q(k+1)}} \Big| b_1^{[\ln|u+h|]_q} F(\{\ln|u+h|\}_q)|u+h|^{\kappa} - b_1^{[\ln|u|]_q} F(\{\ln|u|\}_q)|u|^{\kappa} \Big|^{\alpha} du \\
&= \sum_{k=k_0}^{\infty} \int_{e^{q(k+1)}-h}^{e^{q(k+1)}} \Big| b_1 F(\ln|u+h| - q(k+1))|u+h|^{\kappa} - F(\ln|u| - qk)|u|^{\kappa} \Big|^{\alpha} du \\
&\le C \sum_{k=k_0}^{\infty} \int_{e^{q(k+1)}-h}^{e^{q(k+1)}} \Big| b_1 F(\ln|u+h| - q(k+1)) - F(\ln|u| - qk) \Big|^{\alpha} |u+h|^{\kappa\alpha} du \\
&\quad + C \sum_{k=k_0}^{\infty} \int_{e^{q(k+1)}-h}^{e^{q(k+1)}} |F(\ln|u| - qk)|^{\alpha} \Big| |u+h|^{\kappa} - |u|^{\kappa} \Big|^{\alpha} du =: C(I_{2,1} + I_{2,2}).
\end{aligned}
$$

We can show that $I_{2,2} \le C$ for $h \in (1, e^q)$ as in the case $I_{1,2}$ above. Consider now the term $I_{2,1}$. By using (5.15), we can extend the function $F$ to $[q, 2q)$ so that $F(u) = b_1 F(u+q)$ for $u \in [0,q)$ or $b_1 F(u) = F(u+q)$ for $u \in [0,q)$ since $b_1 \in \{-1,1\}$, and so that $F$ is absolutely continuous on $[0,2q)$ with the derivative $F'$ satisfying (5.14). Then,

$$
I_{2,1} = \sum_{k=k_0}^{\infty} \int_{e^{q(k+1)}-h}^{e^{q(k+1)}} \Big| F(\ln|u+h| - qk) - F(\ln|u| - qk) \Big|^{\alpha} |u+h|^{\kappa\alpha} du.
$$

We get $I_{2,1} \le C$ for $h \in (1, e^q)$ as in the case $I_{1,1}$ above. Hence, $I_2 \le C$ for $h \in (1, e^q)$ and, since $I_1 \le C$ for $h \in (1, e^q)$ as shown above, we have $I \le C$ for $h \in (1, e^q)$ . $\square$

# 6 Examples of PFSMs and CFSMs

In order for a process given by (3.3) to be well-defined, its kernel must belong to the space $L^{\alpha}(Z \times [0, q(\cdot)) \times \mathbb{R}, \sigma(dz)dvdu)$. In this section, we provide examples of such kernels and hence examples of well-defined PFSMs. We show that the processes in these examples are also CFSMs.

**Example 6.1.** Let $\alpha \in (0,2)$ and $H \in (0,1)$ and hence $\kappa = H - 1/\alpha \in (-1/\alpha, 1-1/\alpha)$. The process

$$
X_{\alpha}(t) = \int_0^1 \int_{\mathbb{R}} \Big( F(\{v + \ln|t+u|\}_1)(t+u)_+^{\kappa} - F(\{v + \ln|u|\}_1)u_+^{\kappa} \Big) M_{\alpha}(dv, du), \tag{6.1}
$$

has a representation (3.1) with $Z = \{1\}$, $\sigma(dz) = \delta_{\{1\}}(dz)$, $b_1(1) = 1$, $q(1) = 1$, $F_1(1,u) = F(u)$, $F_2(1,u) = 0$ and $F_3(1) = 0$. It is well-defined by Lemma 5.1 if the function $F : [0,1) \mapsto \mathbb{R}$ satisfies the conditions (5.13) and (5.14) when $\kappa < 0$ and, in addition, the condition (5.15) when $\kappa \ge 0$. We can take, for example,

$$
F(u) = u, \quad u \in [0,1), \tag{6.2}
$$

when $\kappa < 0$, and

$$
F(u) = u 1_{[0,1/2)}(u) + (1-u) 1_{[1/2,1)}(u), \quad u \in [0,1), \tag{6.3}
$$

when no additional conditions on $\kappa$ are imposed. (The function $F$ satisfies (5.15) because $F(0) = 0 = F(1-)$.) The sufficient conditions of Lemma 5.1 are not necessary. For example, Lemma 8.1 in Pipiras and Taqqu (2002$a$) shows that the process $X_\alpha$ is also well-defined with the function

$$F(u) = 1_{[0,1/2)}(u), \quad u \in [0,1), \tag{6.4}$$

when $\kappa < 0$, a function which does not satisfy the (sufficient) conditions of Lemma 5.1. When $\alpha \in (1,2)$, the process (6.1) with the function $F$ in (6.2), (6.3) or (6.4) is therefore a well-defined PFSM.

Note also that, depending on the parameter values $H$ and $\alpha$, the processes (6.1) may have different sample behavior (even for the same function $F$). Sample behavior of stable self-similar processes is discussed in Section 12.4 of Samorodnitsky and Taqqu (1994), and that of general stable processes is studied in Chapter 10 of that book. When $\kappa > 0$, for example, as with the kernel $F$ in (6.3), the process (6.1) is always sample continuous. When $\kappa < 0$, we expect the sample paths to be unbounded (and hence not continuous) on every interval of positive length for most functions $F$. For example, when $F$ is càdlàg (that is, right-continuous and with left limits) on the interval $[0,1]$, then

$$\sup_{t\in\mathbb{Q}} \Big| F(\{v+\ln|t+u|\}_1)(t+u)_+^\kappa - F(\{v+\ln|u|\}_1)u_+^\kappa \Big| = \infty,$$

for any $v \in [0,1)$ and $u \in \mathbb{R}$ (taking $\mathbb{Q} \ni t \to -u$). The unboundedness of the sample paths follows from Corollary 10.2.4 in Samorodnitsky and Taqqu (1994). The case $\kappa = 0$ is more difficult to analyze.

**Example 6.2.** Let $\alpha \in (0,2)$ and $H \in (0,1)$. Consider the process

$$X_\alpha(t) = \int_{\mathbb{R}} \int_0^{2\pi} \int_{\mathbb{R}} \Big( \cos(v + z\ln|t+u|)(t+u)_+^\kappa - \cos(v+z\ln|u|)u_+^\kappa \Big) M_\alpha(dz,dv,du), \tag{6.5}$$

where $M_\alpha(dz,dv,du)$ has the control measure $\lambda(dz)dvdu$. Suppose that the measure $\lambda(dz)$ is such that

$$\int_{\mathbb{R}} \lambda(dz) < \infty, \quad \int_{\mathbb{R}} |z|^\alpha \lambda(dz) < \infty.$$

The process (6.5) is well-defined since, by the mean value theorem, one has $|\cos(x) - \cos(z)| \le |x-z|$ and $|\ln|t+u| - \ln|u|| \le C|u|^{-1}$ for fixed $t$ and large $|u|$,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^{2\pi} \int_{\mathbb{R}} \Big| \cos(v+z\ln|t+u|)(t+u)_+^\kappa - \cos(v+z\ln|u|)u_+^\kappa \Big|^\alpha \lambda(dz)dvdu \\
&\le 2^\alpha \int_{\mathbb{R}} \int_0^{2\pi} \int_{\mathbb{R}} \Big| (t+u)_+^\kappa - u_+^\kappa \Big|^\alpha \lambda(dz)dvdu \\
&+ 2^\alpha \int_{\mathbb{R}} \int_0^{2\pi} \int_{\mathbb{R}} |u_+|^{\kappa\alpha} \Big| \cos(v+z\ln|t+u|) - \cos(v+z\ln|u|)\Big|^\alpha \lambda(dz)dvdu \\
&\le 2^\alpha \int_{\mathbb{R}} \lambda(dz) \int_{\mathbb{R}} \Big| (t+u)_+^\kappa - u_+^\kappa \Big|^\alpha du \\
&+ 2^\alpha \int_{\mathbb{R}} |z|^\alpha \lambda(dz) \int_{\mathbb{R}} |u_+|^{\kappa\alpha} \Big| \ln|t+u| - \ln|u| \Big|^\alpha du \ < \infty.
\end{aligned}$$

Thus, when $\alpha \in (1,2)$, (6.5) is a well-defined PFSM represented by (3.3) with $Z = \mathbb{R}$, $\sigma(dz) = \lambda(dz)$, $q(z) = 1$, $b_1(z) = 1$, $s(z) = z$, $F_1(z, \{u\}_{2\pi}) = \cos(\{u\}_{2\pi}) = \cos(u)$, $F_2(z,u) = 0$ and $F_3(z) = 0$.

**Remark.** The PFSM (6.5) is related to the well-known Lamperti transformation and harmonizable processes. Let $\widetilde{M}(dz,du)$ be a rotationally invariant (isotropic) $S\alpha S$ random measure on $Z \times \mathbb{R}$ with the control measure $\lambda(dz)du$. (Recall that a rotationally invariant $S\alpha S$ random measure $\widetilde{M}(ds)$ on a space $S$ with the control measure $\widetilde{\nu}(ds)$ is a complex-valued random measure satisfying

$$E\exp\Big\{i\Re\Big(\overline{\theta}\int_S f(s)\widetilde{M}_\alpha(ds)\Big)\Big\} = \exp\Big\{-|\theta|^\alpha \int_S |f(s)|^\alpha \widetilde{\nu}(ds)\Big)\Big\}$$

for all $\theta \in \mathbb{C}$ and $f = f_1 + i f_2 \in L^\alpha(S, \widetilde{\nu})$. See Samorodnitsky and Taqqu (1994) for more information.) Arguing as in Example 2.5 of Rosiński (2000), we can show that, up to a multiplicative constant, the PFSM (6.5) has the same finite-dimensional distributions as the real part of the complex-valued process

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(e^{iz\ln|t+u|}(t+u)_+^\kappa - e^{iz\ln|u|}u_+^\kappa\Big)\widetilde{M}_\alpha(dz,du). \tag{6.6}$$

The process (6.6), can also be constructed by the following procedure. Consider the so-called "harmonizable" *stationary* process

$$Y_\alpha^1(t) = \int_{\mathbb{R}} e^{izt}\widetilde{M}(dz),$$

where $\widetilde{M}(dz)$ is rotationally invariant and has the control measure $\mu(dz)$. By applying the Lamperti transformation to the stationary process $Y_\alpha^1$ (see Samorodnitsky and Taqqu (1994)), one obtains a self-similar process

$$Y_\alpha^2(t) = \int_{\mathbb{R}} e^{iz\ln t} t^H \widetilde{M}(dz),\ t > 0.$$

The self-similar process $Y_\alpha^2$ can be made also stationary (in the sense of generalized processes) by introducing an additional variable in its integral representation, namely,

$$Y_\alpha^3(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{iz\ln|t+u|}(t+u)_+^\kappa \widetilde{M}(dz,du),$$

where the measure $\widetilde{M}(dz,du)$ is the same as in (6.6). The stationary increments PFSM process (6.6) can be obtained from the stationary process $Y_\alpha^3$ in the usual way by considering $Y_\alpha^3(t) - Y_\alpha^3(0)$.

**Proposition 6.1.** *The PFSMs of Example 6.1 (with $F$ given by either of (6.2), (6.3) or (6.4)) and Example 6.2 are CFSMs.*

PROOF: Consider first the process (6.1) defined through the kernel function

$$G(v,u) = F(\{v + \ln|u|\}_1)u_+^\kappa,$$

where $F$ is given by either of (6.2), (6.3) or (6.4). By Proposition 4.1, $(ii)$, it is enough to show that $0 \notin C_F$ where the set $C_F$ is defined by (2.3). If $0 \in C_F$, then there is $c_n \to 1$ ($c_n \neq 1$) such that

$$F(\{\ln|c_n u|\}_1)u_+^\kappa = b_n F(\{\ln|u + a_n|\}_1)(u + a_n)_+^\kappa + d_n \quad \text{a.e. } du, \tag{6.7}$$

for some $a_n, b_n \neq 0, d_n$. By taking large enough negative $u$ such that $u_+^\kappa = 0$ and $(u + a_n)_+^\kappa = 0$, we get that $d_n = 0$ and hence (6.7) becomes

$$F(\{\ln|c_n u|\}_1)u_+^\kappa = b_n F(\{\ln|u + a_n|\}_1)(u + a_n)_+^\kappa, \quad \text{a.e. } du. \tag{6.8}$$

We shall distinguish between the cases $\kappa < 0$ and $\kappa \geq 0$. Observe that we need to consider the functions (6.2), (6.3) and (6.4) when $\kappa < 0$, and only the function (6.3) when $\kappa \geq 0$.

If $\kappa < 0$, since $F$ is bounded and not identically zero, by letting $u \to 0$ in (6.8), we obtain that $a_n = 0$. Hence, when $\kappa < 0$, $F(\{\ln|c_n u|\}_1)1_{(0,\infty)}(u) = b_n F(\{\ln|u|\}_1)1_{(0,\infty)}(u)$ a.e. $du$ or, by setting $e_n = \ln c_n$ and $v = \ln u$,

$$F(\{e_n + v\}_1) = b_n F(\{v\}_1), \quad \text{a.e. } dv, \tag{6.9}$$

for $e_n \to 0$ ($e_n \neq 0$) and $b_n \neq 0$. Neither of the functions $F$ in (6.2), (6.3) or (6.4) satisfies the relation (6.9). For example, if the function (6.2) satisfies (6.9), then $e_n + v = b_n v$ for $v \in [0, 1 - e_n)$ and some $0 < e_n < 1$ which is a contradiction (for example, if $v = 0$, we get $e_n = 0$). Indeed, if $\kappa \geq 0$, we can also get $a_n = 0$ in (6.8). If, for example, $a_n < 0$, then since $(u + a_n)_+^\kappa = 0$ for $u \in [0, -a_n)$, we have $F(\{\ln|c_n u|\}_1) = 0$ for $u \in [0, -a_n)$ or that the function $F(v) = 0$ on an interval of $[0, 1)$. The function (6.3) does not have this property.

Consider now the process (6.5) defined through the kernel function

$$G(z,v,u) = \cos(v + z\ln|u|)u_+^\kappa.$$

By Proposition 4.1, $(ii)$, it is enough to show that $(z, 0) \notin C_F$ a.e. $dz$. If $(z, 0) \in C_F$, then there is $c_n = c_n(z) \to 1$ ($c_n \neq 1$) such that

$$\cos(z\ln|c_n u|)u_+^\kappa = b_n \cos(z\ln|u + a_n|)(u + a_n)_+^\kappa + d_n, \quad \text{a.e. } du,$$

for some $a_n = a_n(z)$, $b_n = b_n(z) \neq 0$, $d_n = d_n(z)$. If $z \neq 0$, we may argue as above to obtain $d_n = 0$ and $a_n = 0$. Then,

$$\cos(z e_n + v) = b_n \cos(v), \quad \text{a.e. } dv,$$

for $e_n \to 0$ ($e_n \neq 0$) and $b_n \neq 0$. This relation cannot hold when $z \neq 0$, since $e_n \neq 0$, $e_n \to 0$ and because of the shape of the function $\cos(v)$. □

**Remark.** Observe that the $S\alpha S$ $H$–self-similar processes of Examples 6.1 and 6.2 are well-defined when $\alpha \in (0, 2)$ and $H \in (0, 1)$. By Corollary 7.1.1 in Samorodnitsky and Taqqu (1994), self-similar stable processes can also be defined when $\alpha \in (0, 1)$ and $1 \leq H \leq 1/\alpha$, and when $\alpha \in [1, 2)$ and $H = 1$. We do not know of examples of processes having a representation (3.1) for these ranges of $\alpha$ and $H$.

## 7 Uniqueness results for PFSMs and CFSMs

We are interested in determining whether two given PFSMs or CFSMs are in fact the same process. Since we don't want to distinguish between processes which differ by a multiplicative constant, we will say that two processes $X(t)$ and $Y(t)$ are *essentially identical* if $X(t)$ and $cY(t)$ have the same finite-dimensional distributions for some constant $c$. If the processes are not essentially identical, we will say that they are *essentially different*.

The next result can often be used to conclude that two PFSMs and CFSMs are essentially different.

**Theorem 7.1.** *Suppose that $X_\alpha$ and $\widetilde{X}_\alpha$ are two PFSMs or CFSMs having representations (3.3): the process $X_\alpha$ on the space $Z \times [0, q(\cdot)) \times \mathbb{R}$ with the kernel function $G$ defined through the functions $b_1, s, q, F_1, F_2, F_3$, and the process $\widetilde{X}_\alpha$ on the space $\widetilde{Z} \times [0, \widetilde{q}(\cdot)) \times \mathbb{R}$ with the kernel $\widetilde{G}$ define through the functions $\widetilde{b}_1, \widetilde{s}, \widetilde{q}, \widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3$. If $X_\alpha$ and $\widetilde{X}_\alpha$ are essentially identical, then there are maps $h : Z \mapsto \mathbb{R} \setminus \{0\}$, $\psi : Z \mapsto \widetilde{Z}$, $k : Z \mapsto (0, \infty)$, $g, j : Z \mapsto \mathbb{R}$ such that*

$$G(z, 0, u) = h(z)\widetilde{G}(\psi(z), 0, k(z)u + g(z)) + j(z), \quad \textit{a.e. } \sigma(dz)du. \tag{7.1}$$

**Remark.** The use of $v = 0$ in (7.1) should not be surprising because the function $G(z, v, u)$ in (3.4) can be expressed through $G(z, 0, w)$. Indeed, as in (4.1), it follows from (3.4) that

$$G(z, v, u) = e^{-\kappa v/s(z)} G(z, 0, e^{v/s(z)}u) - 1_{\{b_1(z)=1\}} 1_{\{\kappa=0\}} F_3(z)v/s(z), \quad \text{for all } z, v, u. \tag{7.2}$$

Proof: Let $G$ and $\widetilde{G}$ be the kernel functions for the processes $X_\alpha$ and $\widetilde{X}_\alpha$, respectively, as defined in the theorem. By Theorem 5.2 in Pipiras and Taqqu (2002), there are maps $\psi = (\psi_1, \psi_2) : Z \times [0, q(\cdot)) \mapsto \widetilde{Z} \times [0, \widetilde{q}(\cdot))$, $h : Z \times [0, q(\cdot)) \mapsto \mathbb{R} \setminus \{0\}$ and $g, j : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$ such that

$$G(z, v, u) = h(z, v)\widetilde{G}(\psi_1(z, v), \psi_2(z, v), u + g(z, v)) + j(z, v) \tag{7.3}$$

a.e. $\sigma(dz)dvdu$. By applying (7.2) to both sides of (7.3) and replacing $u$ by $e^{-v/s(z)}u$, we get

$$\begin{aligned} G(z, 0, u) = {} & e^{\kappa v/s(z)} e^{-\kappa \psi_2(z,v)/\widetilde{s}(\psi_1(z,v))} h(z, v)\widetilde{G}(\psi_1(z, v), 0, e^{\psi_2(z,v)/\widetilde{s}(\psi_1(z,v))}(e^{-v/s(z)}u + g(z, v))) \\ & - e^{\kappa v/s(z)} h(z, v) 1_{\{\widetilde{b}_1(\psi_1(z,v))=1\}} 1_{\{\kappa=0\}} \widetilde{F}_3(\psi_1(z, v))\psi_2(z, v)/\widetilde{s}(\psi_1(z, v)) \\ & + e^{\kappa v/s(z)} 1_{\{b_1(z)=1\}} 1_{\{\kappa=0\}} F_3(z)v/s(z) + e^{\kappa v/s(z)} j(z, v) \end{aligned}$$

a.e. $\sigma(dz)dvdu$. By making the change of variables $e^v u = w$, $z = z$, $v = v$, we obtain that the last relation holds a.e. $\sigma(dz)dvdw$. By fixing $v$, for which this relation holds a.e. $\sigma(dz)dw$, we obtain (7.1). $\square$

The following result shows that we can obtain an "if and only if" condition in Theorem 7.1 in the case when $Z = \{1\}$, $\widetilde{Z} = \{1\}$, $s = \widetilde{s} = 1$ and $q = \widetilde{q}$. More general cases, for example, even when $Z = \{1\}$, $\widetilde{Z} = \{1\}$, $s = \widetilde{s} = 1$ but $q \neq \widetilde{q}$, are much more difficult to analyze. The measure $\delta_{\{1\}}$ below denotes the point mass at the point $\{1\}$.

**Corollary 7.1.** *Suppose the conditions of Theorem 7.1 above hold with* $Z = \{1\}$, $\sigma = \delta_{\{1\}}$, $\widetilde{Z} = \{1\}$, $\widetilde{\sigma} = \delta_{\{1\}}$, $s = \widetilde{s} = 1$ *and* $q = \widetilde{q}$. *Then, the processes* $X_\alpha$ *and* $\widetilde{X}_\alpha$ *are essentially identical if and only if there are constants* $h \neq 0$, $k > 0$, $g, j \in \mathbb{R}$ *such that*

$$G(0,u) = h\widetilde{G}(0, k\,u + g) + j, \quad a.e.\ du, \tag{7.4}$$

*or equivalently*

$$\begin{aligned} &b_1^{[\ln|u|]_q}\Big(F_1(\{\ln|u|\}_q)\,u_+^\kappa + F_2(\{\ln|u|\}_q)\,u_-^\kappa\Big) + 1_{\{b_1=1\}}1_{\{\kappa=0\}}F_3\ln|u| \\ &= h\widetilde{b}_1^{[\ln|ku+g|]_{\widetilde{q}}}\Big(\widetilde{F}_1(\{\ln|ku+g|\}_q)\,(ku+g)_+^\kappa + \widetilde{F}_2(\{\ln|ku+g|\}_q)\,(ku+g)_-^\kappa\Big) \\ &\quad + 1_{\{\widetilde{b}_1=1\}}1_{\{\kappa=0\}}h\widetilde{F}_3\ln|ku+g| + j, \quad a.e.\ du. \end{aligned} \tag{7.5}$$

*Moreover, the processes* $X_\alpha$ *and* $\widetilde{X}_\alpha$ *have identical finite-dimensional distributions if and only if* $|h| = 1$.

Proof: The "only if" part follows from Theorem 7.1. We now show the "if" part. By using (7.2), we have, for $v \in [0,q)$,

$$\begin{aligned} G(v,u) &= e^{-\kappa v}G(0,e^v u) - 1_{\{b_1=1\}}1_{\{\kappa=0\}}F_3 v \\ &= he^{-\kappa v}\widetilde{G}(0,ke^v u+g) + e^{-\kappa v}j - 1_{\{b_1=1\}}1_{\{\kappa=0\}}F_3 v \\ &= h\Big(e^{-\kappa v}\widetilde{G}(0,ke^v u+g) - 1_{\{\widetilde{b}_1=1\}}1_{\{\kappa=0\}}\widetilde{F}_3 v\Big) \\ &\quad + h1_{\{\widetilde{b}_1=1\}}1_{\{\kappa=0\}}\widetilde{F}_3 v - 1_{\{b_1=1\}}1_{\{\kappa=0\}}F_3 v \\ &= h\widetilde{G}(v, ku + e^{-v}g) + \widetilde{F}(v) = h\widetilde{G}(\{v+\ln k\}_q, u + e^{-v}k^{-1}g) + \widetilde{F}(v) \end{aligned}$$

a.e. $du$, for some function $\widetilde{F}(v)$. Hence, by making the change of variables $u + e^{-v}k^{-1}g \to u$ and $\{v+\ln k\}_q \to v$ below,

$$\begin{aligned} X_\alpha(t) &\stackrel{d}{=} \int_0^q\int_{\mathbb{R}}\Big(G(v,t+u) - G(v,u)\Big)M_\alpha(dv,du) \\ &= h\int_0^q\int_{\mathbb{R}}\Big(\widetilde{G}(\{v+\ln k\}_q, t+u+e^{-v}k^{-1}g) - \widetilde{G}(\{v+\ln k\}_q, u+e^{-v}k^{-1}g)\Big)M_\alpha(dv,du) \\ &\stackrel{d}{=} h\int_0^q\int_{\mathbb{R}}\Big(\widetilde{G}(v,t+u) - \widetilde{G}(v,u)\Big)M_\alpha(dv,du) \stackrel{d}{=} h\widetilde{X}_\alpha(t). \end{aligned}$$

This relation also implies the last statement of the result. □

We now apply Theorem 7.1 and Corollary 7.1 to examples of PFSMs in Section 6.

**Proposition 7.1.** *The four PFSMs considered in Examples 6.1 and 6.2 are essentially different.*

PROOF: Let $X_\alpha$ and $Y_\alpha$ be two PFSMs of Example 6.1 defined through two different functions $F_1$ and $F_2$ in (6.2), (6.3) or (6.4). To show that $X_\alpha$ and $Y_\alpha$ are essentially different, we can suppose that $\kappa < 0$ because only the function (6.3) involved $\kappa \geq 0$. If $X_\alpha$ and $Y_\alpha$ are essentially identical, it follows from Corollary 7.1 that

$$F_1(\{\ln|u|\}_q)\, u_+^\kappa = hF_2(\{\ln|ku+g|\}_q)\,(ku+g)_+^\kappa + j$$

a.e. $du$, for $h_1 \neq 0$, $k > 0$, $g, j \in \mathbb{R}$. By arguing as in the proof of Proposition 6.1, we can obtain that $j = 0$ and $g = 0$. Then, after the change of variables $\ln u = v$, we have

$$F_1(\{v\}_q) = h_1 F_2(\{k_1 + v\}_q)$$

a.e. $dv$, for some $h_1 \neq 0$, $k_1 \neq 0$. This relation does not hold for any two different functions $F_1$ and $F_2$ in (6.2), (6.3) or (6.4).

Suppose now that $X_\alpha$ and $Y_\alpha$ are the PFSMs of Examples 6.2 and 6.1, respectively, defined through the kernel functions $G_1(z,v,u) = \cos(v+\ln|u|)u_+^\kappa$ and $G_2(z,v) = F(\{v+\ln|u|\})u_+^\kappa$, where $F$ is defined by (6.2), (6.3) or (6.4). If $X_\alpha$ and $Y_\alpha$ are essentially identical, it follows from Theorem 7.1 that

$$\cos(z\ln|u|)u_+^\kappa = k(z)F(\{\ln|k(z)u+g(z)|\}_1)(k(z)u+g(z))_+^\kappa + j(z)$$

a.e. $du$, for some $h(z) \neq 0$, $k(z) > 0$ and $g(z), j(z) \in \mathbb{R}$. When $z \neq 0$, by arguing as in the proof of Proposition 6.1, we get that $j(z) = 0$ and $g(z) = 0$. Then, by making the change of variables $\ln u = v$, we have

$$\cos(zv) = h_1(z)F(\{k_1(z)+v\}_1)$$

a.e. $dv$, for some $h_1(z) \neq 0$, $k_1(z) \neq 0$. The function $F$ in (6.2), (6.3) or (6.4) does not satisfy this relation. Hence, $X_\alpha$ and $Y_\alpha$ are essentially different. $\square$

# 8 Functionals of cyclic flows

We shall characterize here the functionals appearing in (2.6) which are related to cyclic flows. These results are used in Section 9 below to establish a canonical representation for PFSMs in Theorem 3.1. We start by providing a precise definition of flows and related functionals. See Pipiras and Taqqu (2004c) for motivation. A flow $\{\psi_c\}_{c>0}$ on a standard Lebesque space $(X, \mathcal{X}, \mu)$ is a collection of measurable maps $\psi_c : X \to X$ satisfying (2.5). The flow is *nonsingular* if each map $\psi_c$, $c > 0$, is nonsingular, that is, $\mu(A) = 0$ implies $\mu(\psi_c^{-1}(A)) = 0$. It is *measurable* if a map $\psi_c(x) : (0,\infty) \times X \to X$ is measurable. A *cocycle* $\{b_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ taking values in $\{-1, 1\}$ is a measurable map $b_c(x) : (0,\infty) \times X \to \{-1, 1\}$ such that

$$b_{c_1c_2}(x) = b_{c_1}(x)b_{c_2}(\psi_{c_1}(x)), \quad \text{for all } c_1, c_2 > 0,\ x \in X. \tag{8.1}$$

A *1-semi-additive functional* $\{g_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ is a measurable map $g_c(x) : (0,\infty) \times X \to \mathbb{R}$ such that

$$g_{c_1c_2}(x) = c_2^{-1}g_{c_1}(x) + g_{c_2}(\psi_{c_1}(x)), \quad \text{for all } c_1, c_2 > 0,\ x \in X. \tag{8.2}$$

A *2-semi-additive functional* $\{j_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ and a related cocycle $\{b_c\}_{c>0}$ is a measurable map $j_c(x) : (0, \infty) \times X \to \mathbb{R}$ such that

$$j_{c_1c_2}(x) = c_2^{-\kappa} j_{c_1}(x) + b_{c_1}(x) \left\{ \frac{d(\mu \circ \psi_{c_1})}{d\mu}(x) \right\}^{1/\alpha} j_{c_2}(\psi_{c_1}(x)), \quad \text{for all } c_1, c_2 > 0,\ x \in X. \tag{8.3}$$

The Radon-Nikodym derivatives $\widetilde{b}_c(x) = (d(\mu \circ \psi_c)/d\mu)(x)$ in (8.3) can and will be viewed as a cocycle taking values in $\mathbb{R}\backslash\{0\}$, that is, a measurable map $\widetilde{b}_c(x) : (0, \infty) \times X \to \mathbb{R}\backslash\{0\}$ satisfying $\widetilde{b}_{c_1c_2}(x) = \widetilde{b}_{c_1}(x)\widetilde{b}_{c_2}(\psi_{c_1}(x))$, for all $c_1, c_2 > 0$ and $x \in X$ (see Pipiras and Taqqu (2004*c*)).

In the following three lemmas, we characterize cocycles and *1*- and *2*-semi-additive functionals associated with cyclic flows.

**Lemma 8.1.** *Let $\{b_c\}_{c>0}$ be a cocycle taking values in $\{-1, 1\}$ for a cyclic flow $\{\psi_c\}_{c>0}$. Set $\widetilde{b}_c(z, v) = b_c(\Phi(z, v))$ if $(z, v) \in Z \times [0, q(\cdot)) \setminus \widetilde{N}$, and $\widetilde{b}_c(z, v) = 1$ if $(z, v) \in \widetilde{N}$, where $\Phi : Z \times [0, q(\cdot)) \setminus \widetilde{N} \mapsto X \setminus N$ is the null-isomorphism appearing in (2.12). Then, $\{\widetilde{b}_c\}_{c>0}$ is a cocycle for the cyclic flow $\{\widetilde{\psi}_c\}_{c>0}$ defined by (2.11), and it can be expressed as*

$$\widetilde{b}_c(z, v) = \frac{\widetilde{b}(z, \{v + \ln c\}_{q(z)})}{\widetilde{b}(z, v)}\ \widetilde{b}_1(z)^{[v+\ln c]_{q(z)}}, \tag{8.4}$$

*for some functions $\widetilde{b} : Z \times [0, q(\cdot)) \mapsto \{-1, 1\}$ and $\widetilde{b}_1 : Z \mapsto \{-1, 1\}$.*

Proof: The result can be deduced from Lemma 2.2 in Pipiras and Taqqu (2004*d*) by using the following relation between multiplicative and additive flows: $\{\psi_c\}_{c>0}$ is a multiplicative flow and $\{b_c\}_{c>0}$ is a related cocycle if and only if $\phi_t := \psi_{e^t}, t \in \mathbb{R}$, is an additive flow and $a_t := b_{e^t}, t \in \mathbb{R}$, is a related cocycle. (Additive flows $\{\phi_t\}_{t\in\mathbb{R}}$ are such that $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}, t_1, t_2 \in \mathbb{R}$.) □

**Lemma 8.2.** *Let $\{g_c\}_{c>0}$ be a 1-semi-additive functional for a cyclic flow $\{\psi_c\}_{c>0}$. Set $\widetilde{g}_c(z, v) = g_c(\Phi(z, v))$ if $(z, v) \in Z \times [0, q(\cdot)) \backslash \widetilde{N}$, and $\widetilde{g}_c(z, v) = 0$ if $(z, v) \in \widetilde{N}$, where $\Phi : Z \times [0, q(\cdot)) \backslash \widetilde{N} \mapsto X \backslash N$ is the null-isomorphism appearing in (2.12). Then, $\{\widetilde{g}_c\}_{c>0}$ is a 1-semi-additive functional for the cyclic flow $\{\widetilde{\psi}_c\}_{c>0}$ defined by (2.11), and it can be expressed as*

$$\widetilde{g}_c(z, v) = \widetilde{g}(z, \{v + \ln c\}_{q(z)}) - c^{-1}\widetilde{g}(z, v), \tag{8.5}$$

*for some function $\widetilde{g} : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$.*

Proof: We may suppose without loss of generality $N = \widetilde{N} = \emptyset$ because $\widetilde{g}_c(z, v) = 0$ obviously satisfies the *1*-semi-additive functional relation (8.2) on the set $\widetilde{N}$ (which is invariant for the flow $\widetilde{\psi}_c$). By substituting $x = \Phi(z, v)$ in the equation (8.2), we obtain that

$$g_{c_1c_2}(\Phi(z, v)) = c_2^{-1} g_{c_1}(\Phi(z, v)) + g_{c_2}(\psi_{c_1}(\Phi(z, v)))$$

and, since $\psi_c \circ \Phi = \Phi \circ \widetilde{\psi}_c$ by (2.12), we get

$$\widetilde{g}_{c_1c_2}(z, v) = c_2^{-1}\widetilde{g}_{c_1}(z, v) + \widetilde{g}_{c_2}(\widetilde{\psi}_{c_1}(z, v)). \tag{8.6}$$

Relation (8.6) shows that $\{\widetilde{g}_c\}_{c>0}$ is a *1*-semi-additive functional for the flow $\{\widetilde{\psi}_c\}_{c>0}$. The expression (8.5) follows from Proposition 5.1 in Pipiras and Taqqu (2004*c*). □

Let $T$ be an arbitrary index set, e.g. $T = (0, \infty)$, and $\{f_t^1\}_{t\in T}$, $\{f_t^1\}_{t\in T}$ be two collections of measurable functions on $(X, \mathcal{X}, \mu)$. We will say that $\{f_t^1\}_{t\in T}$ is a version of $\{f_t^2\}_{t\in T}$ if $\mu\{x : f_t^1(x) \neq f_t^2(x)\} = 0$ for all $t \in T$. We now characterize *2*-semi-additive functionals related to cyclic flows.

**Lemma 8.3.** *Let $\{\widetilde{j}_c\}_{c>0}$ be a 2-semi-additive functional for a cyclic flow $\{\psi_c\}_{c>0}$. Set*

$$\widetilde{j}_c(z,v) = \left\{ \frac{d(\mu \circ \Phi)}{d(\sigma \otimes \mathbb{L})}(z,v) \right\}^{1/\alpha} j_c(\Phi(z,v)) \tag{8.7}$$

*if $(z,v) \in Z \times [0, q(\cdot)) \setminus \widetilde{N}$, and let $\widetilde{j}_c(z,v)$ be defined arbitrarily for $(z,v) \in \widetilde{N}$, where $\Phi : Z \times [0, q(\cdot)) \setminus \widetilde{N} \mapsto X \setminus N$ is the isomorphism appearing in (2.12). Then, $\{\widetilde{j}_c\}_{c>0}$ has a version which is a 2-semi-additive functional for the flow $\{\widetilde{\psi}_c\}_{c>0}$ defined by (2.11) and the cocycle $\{\widetilde{b}_c\}_{c>0}$ defined in Lemma 8.1. Moreover, for any $c > 0$,*

$$\begin{aligned} \widetilde{j}_c(z,v) &= \widetilde{b}_c(z,v)\widetilde{j}(z, \{v + \ln c\}_{q(z)}) - c^{-\kappa}\widetilde{j}(z,v) \\ &+ \widetilde{j}_1(z)(\widetilde{b}(z,v))^{-1}[v + \ln c]_{q(z)} 1_{\{\widetilde{b}_1(z)=1\}} 1_{\{\kappa=0\}}, \quad \textit{a.e. } \sigma(dz)dv, \end{aligned} \tag{8.8}$$

*where $\widetilde{j} : Z \times [0, q(\cdot)) \mapsto \mathbb{R}$, $\widetilde{j}_1 : Z \mapsto \mathbb{R}$ are some measurable functions, and the functions $\widetilde{b}_1(z)$ and $\widetilde{b}(z,v)$ appear in (8.4) of Lemma 8.1.*

Proof: We suppose without loss of generality that $N = \widetilde{N} = \emptyset$. Substituting $x = \Phi(z,v)$ in the *2*-semi-additive functional equation (8.3), we obtain that

$$j_{c_1 c_2}(\Phi(z,v)) = c_2^{-\kappa} j_{c_1}(\Phi(z,v)) + b_{c_1}(\Phi(z,v)) \left\{ \frac{d(\mu \circ \psi_{c_1})}{d\mu}(\Phi(z,v)) \right\}^{1/\alpha} j_{c_2}(\psi_{c_1}(\Phi(z,v))). \tag{8.9}$$

We first show that $\{\widetilde{j}_c\}_{c>0}$ is an almost *2*-semi-additive functional, that is, it satisfies relation (8.3) for all $c_1, c_2 > 0$ a.e. $\sigma(dz)dv$. Observe that, for any $c > 0$ and $\mathbb{L}$ denoting the Lebesgue measure,

$$\begin{aligned} \frac{d(\mu \circ \psi_c)}{d\mu} \circ \Phi &= \frac{d((\mu \circ \Phi) \circ \widetilde{\psi}_c)}{d(\mu \circ \Phi)} \\ &= \frac{d((\sigma \otimes \mathbb{L}) \circ \widetilde{\psi}_c)}{d(\sigma \otimes \mathbb{L})} \frac{d(\sigma \otimes \mathbb{L})}{d(\mu \circ \Phi)} \left( \frac{d(\mu \circ \Phi)}{d(\sigma \otimes \mathbb{L})} \circ \widetilde{\psi}_c \right) \\ &= \left( \frac{d(\mu \circ \Phi)}{d(\sigma \otimes \mathbb{L})} \right)^{-1} \left( \frac{d(\mu \circ \Phi)}{d(\sigma \otimes \mathbb{L})} \circ \widetilde{\psi}_c \right), \quad (\sigma \otimes \mathbb{L})\text{-a.e.}, \end{aligned} \tag{8.10}$$

where in the last equality above we used the identity $d((\sigma \otimes \mathbb{L}) \circ \widetilde{\psi}_c)/d(\sigma \otimes \mathbb{L}) = 1$ $(\sigma \otimes \mathbb{L})$-a.e., which follows from (2.11) because $d\{v + \ln c\}/dv = 1$ a.e. By using the relation (8.10), we can write (8.9) as

$$\begin{aligned} \left\{ \frac{d(\mu \circ \Phi)}{d(\sigma \otimes \mathbb{L})}(z,v) \right\}^{1/\alpha} j_{c_1 c_2}(\Phi(z,v)) &= c_2^{-\kappa} \left\{ \frac{d(\mu \circ \Phi)}{d(\sigma \otimes \mathbb{L})}(z,v) \right\}^{1/\alpha} j_{c_1}(\Phi(z,v)) \\ + \; b_{c_1}(\Phi(z,v)) \left\{ \frac{d(\mu \circ \Phi)}{d(\sigma \otimes \mathbb{L})}(\widetilde{\psi}_c(z,v)) \right\}^{1/\alpha} & j_{c_2}(\psi_{c_1}(\Phi(z,v))), \quad \text{a.e. } \sigma(dz)dv. \end{aligned} \tag{8.11}$$

Since $\psi_c \circ \Phi = \Phi \circ \widetilde{\psi}_c$ by (2.12), $b_c \circ \Phi = \widetilde{b}_c$ by using the notation of Lemma 8.1 and $\widetilde{j}_c = \{d(\mu \circ \Phi)/d(\sigma \otimes \mathbb{L})\}^{1/\alpha} j_c$ by (8.7), we deduce from (8.11) that, for any $c_1, c_2 > 0$,

$$\widetilde{j}_{c_1 c_2}(z,v) = c_2^{-\kappa} \widetilde{j}_{c_1}(z,v) + \widetilde{b}_{c_1}(z,v) \widetilde{j}_{c_2}(\widetilde{\psi}_{c_1}(z,v)), \quad \text{a.e. } \sigma(dz)dv, \tag{8.12}$$

that is, $\{\widetilde{j}_c\}_{c>0}$ is an almost *2*-semi-additive functional. By Theorem 2.1 in Pipiras and Taqqu (2004*c*), $\{\widetilde{j}_c\}_{c>0}$ has a version which is a *2*-semi-additive functional.

Since $\{\widetilde{j}_c\}_{c>0}$ has a version which is a *2*-semi-additive functional, we may suppose without loss of generality that $\{\widetilde{j}_c\}_{c>0}$ is a *2*-semi-additive functional. The expression (8.8) for $\{\widetilde{j}_c\}_{c>0}$ then follows from Proposition 5.2 in Pipiras and Taqqu (2004*c*). $\Box$

# 9 The proofs of Proposition 3.1 and Theorem 3.1

PROOF OF PROPOSITION 3.1: Suppose that the process $X_\alpha$ is generated by a cyclic flow $\{\psi_c\}_{c>0}$ on $(X, \mathcal{X}, \mu)$. Then, by a discussion following Definition 2.2, there are a standard Lebesgue space $(Z, \mathcal{Z}, \sigma)$, function $q(z) > 0$ and a null-isomorphism $\Phi : Z \times [0, q(\cdot)) \mapsto X$ such that

$$\psi_c(\Phi(z,v)) = \Phi(z, \{v + \ln c\}_{q(z)}) \tag{9.1}$$

for all $c > 0$ and $(z,v) \in Z \times [0, q(\cdot))$. In other words, the flow $\{\psi_c\}_{c>0}$ on $(X, \mu)$ is null-isomorphic to the flow $\{\widetilde{\psi}_c\}_{c>0}$ on $(Z \times [0, q(\cdot)), \sigma(dz)dv)$ defined by $\widetilde{\psi}_c(z,v) = (z, \{v + \ln c\}_{q(z)})$. (We may suppose that the null sets in (2.12) are empty because, otherwise, we can replace $X$ by $X \setminus N$ in the definition of $X_\alpha$ without changing its distribution.) By replacing $x$ by $\Phi(z,v)$ in (2.6) and using (9.1), we get that, for all $c > 0$,

$$c^{-\kappa} G(\Phi(z,v), cu) = b_c(\Phi(z,v)) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(\Phi(z,v)) \right\}^{1/\alpha} \times$$
$$\times\, G\Big(\Phi(\widetilde{\psi}_c(z,v)), u + g_c(\Phi(z,v))\Big) + j_c(\Phi(z,v)) \tag{9.2}$$

a.e. $\sigma(dz)dvdu$. By using the relation

$$\begin{aligned}
\frac{d(\mu \circ \psi_c)}{d\mu} \circ \Phi &= \frac{d(\mu \circ \Phi \circ \widetilde{\psi}_c)}{d(\mu \circ \Phi)} \\
&= \left( \frac{d\mu}{d((\sigma \otimes \mathbb{L}) \circ \Phi^{-1})} \circ \Phi \circ \widetilde{\psi}_c \right) \frac{d((\sigma \otimes \mathbb{L}) \circ \widetilde{\psi}_c)}{d(\sigma \otimes \mathbb{L})} \frac{d(\sigma \otimes \mathbb{L})}{d(\mu \circ \Phi)} \\
&= \left( \frac{d\mu}{d((\sigma \otimes \mathbb{L}) \circ \Phi^{-1})} \circ \Phi \circ \widetilde{\psi}_c \right) \frac{d((\sigma \otimes \mathbb{L}) \circ \Phi^{-1})}{d\mu} \circ \Phi \\
&= \left( \frac{d\mu}{d((\sigma \otimes \mathbb{L}) \circ \Phi^{-1})} \circ \Phi \circ \widetilde{\psi}_c \right) \left( \frac{d\mu}{d((\sigma \otimes \mathbb{L}) \circ \Phi^{-1})} \circ \Phi \right)^{-1},
\end{aligned}$$

where $\mathbb{L}$ is the Lebesgue measure, setting

$$\widetilde{G}(z,v,u) = \left\{ \frac{d(\mu \circ \Phi)}{d(\sigma \otimes \mathbb{L})}(z,v) \right\}^{1/\alpha} G(\Phi(z,v), u) \tag{9.3}$$

and using the notation of Lemmas 8.1, 8.2 and 8.3, we obtain that, for all $c > 0$,

$$c^{-\kappa}\widetilde{G}(z,v,cu) = \widetilde{b}_c(z,v)\widetilde{G}(\widetilde{\psi}_c(z,v), u + \widetilde{g}_c(z,v)) + \widetilde{j}_c(z,v) \tag{9.4}$$

a.e. $\sigma(dz)dvdu$. We next consider the cases $\kappa \neq 0$ and $\kappa = 0$ separately.

*The case $\kappa \neq 0$:* By using Lemmas 8.1, 8.2 and 8.3, and setting

$$\widehat{G}(z,v,u) = \widetilde{b}(z,v)\Big(\widetilde{G}(z,v,u+\widetilde{g}(z,v)) - \widetilde{j}(z,v)\Big), \tag{9.5}$$

we obtain from (9.4) that, for all $c > 0$,

$$\widehat{G}(z,v,cu) = c^{\kappa}\widetilde{b}_1(z)^{[v+\ln c]_{q(z)}}\widehat{G}(z,\{v+\ln c\}_{q(z)},u)$$

a.e. $\sigma(dz)dvdu$. By making a change of variables $cu = w$, we get that, for all $c > 0$,

$$\widehat{G}(z,v,w) = c^{\kappa}\widetilde{b}_1(z)^{[v+\ln c]_{q(z)}}\widehat{G}(z,\{v+\ln c\}_{q(z)},c^{-1}w)$$

$$= |w^{-1}c|^{\kappa}|w|^{\kappa}\widetilde{b}_1(z)^{[v+\ln|w||w^{-1}c|]_{q(z)}}\widehat{G}(z,\{v+\ln|w||w^{-1}c|\}_{q(z)},c^{-1}w)$$

a.e. $\sigma(dz)dvdw$. By the Fubini's theorem, this relation holds a.e. $\sigma(dz)dvdwdc$ as well. Then, for $w > 0$, by making the change of variables $c = yw$ and then fixing $y = y_0$, we get

$$\widehat{G}(z,v,w) = w^{\kappa}\widetilde{b}_1(z)^{[v+\ln a_1|w|]_{q(z)}}\widehat{F}_1(z,\{v+\ln a_1|w|\}_{q(z)})$$

a.e. $\sigma(dz)dvdw$, for some $a_1 > 0$ and function $\widehat{F}_1$. By using identities $\{v+\ln a_1|w|\}_{q(z)} = \{\{v+\ln|w|\}_{q(z)}+\ln a_1\}_{q(z)}$ and $[v+\ln a_1|w|]_{q(z)} = [v+\ln|w|]_{q(z)} + [\{v+\ln|w|\}_{q(z)}+\ln a_1]_{q(z)}$, we can simplify the last relation as

$$\widehat{G}(z,v,w) = w^{\kappa}\widetilde{b}_1(z)^{[v+\ln|w|]_{q(z)}}F_1(z,\{v+\ln|w|\}_{q(z)}) \tag{9.6}$$

a.e. $\sigma(dz)dvdw$, for some function $F_1$. Similarly, for $w < 0$, we may get that

$$\widehat{G}(z,v,w) = w_-^{\kappa}\widetilde{b}_1(z)^{[v+\ln|w|]_{q(z)}}F_2(z,\{v+\ln|w|\}_{q(z)}) \tag{9.7}$$

a.e. $\sigma(dz)dvdw$, for some function $F_2$. Observe now that, by writing characteristic functions and using (9.3) and (9.5),

$$\{X_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_X\int_{\mathbb{R}}(G(x,t+u)-G(x,u))M_\alpha(dx,du)\right\}_{t\in\mathbb{R}}$$

$$\stackrel{d}{=} \left\{\int_Z\int_{[0,q(\cdot))}\int_{\mathbb{R}}(\widetilde{G}(z,v,t+u)-\widetilde{G}(z,v,u))\widetilde{M}_\alpha(dz,dv,du)\right\}_{t\in\mathbb{R}}$$

$$\stackrel{d}{=} \left\{\int_Z\int_{[0,q(\cdot))}\int_{\mathbb{R}}(\widehat{G}(z,v,t+u)-\widehat{G}(z,v,u))\widetilde{M}_\alpha(dz,dv,du)\right\}_{t\in\mathbb{R}}, \tag{9.8}$$

where $\widetilde{M}_\alpha(dz,dv,du)$ is a $S\alpha S$ random measure on $(Z\times[0,q(\cdot)))\times\mathbb{R}$ with the control measure $\sigma(dz)dvdu$. The result of the theorem when $\kappa \neq 0$ then follows by using (9.6) and (9.7).

*The case $\kappa = 0$:* In this case, by using Lemmas 8.1, 8.2 and 8.3, and the notation (9.5), we get that, for all $c > 0$,

$$\widehat{G}(z, v, cu) = \widetilde{b}_1(z)^{[v+\ln c]_{q(z)}} \widehat{G}(z, \{v + \ln c\}_{q(z)}, u) + \widetilde{j}_1(z)[v + \ln c]_{q(z)} 1_{\{\widetilde{b}_1(z)=1\}}$$

a.e. $\sigma(dz)dvdu$. Arguing as in the case $\kappa \neq 0$, we may show that, for $w > 0$,

$$\widehat{G}(z, v, w) = \widetilde{b}_1(z)^{[v+\ln a_1|w|]_{q(z)}} \widehat{F}_1(z, \{v + \ln a_1|w|\}_{q(z)}) 1_{(0,\infty)}(w)$$

$$+\widetilde{j}_1(z)[v + \ln a_1|w|]_{q(z)} 1_{\{\widetilde{b}_1(z)=1\}} 1_{(0,\infty)}(w),$$

a.e. $\sigma(dz)dvdu$. By using the identities preceding (9.6), we conclude that, for $w > 0$,

$$\widehat{G}(z, v, w) = \widetilde{b}_1(z)^{[v+\ln |w|]_{q(z)}} F_1(z, \{v + \ln |w|\}_{q(z)}) 1_{(0,\infty)}(w)$$

$$+F_3(z)[v + \ln |w|]_{q(z)} 1_{\{\widetilde{b}_1(z)=1\}} 1_{(0,\infty)}(w),$$

a.e. $\sigma(dz)dvdu$, for some functions $F_1$ and $F_3$. Similarly, for $w < 0$,

$$\widehat{G}(z, v, w) = \widetilde{b}_1(z)^{[v+\ln |w|]_{q(z)}} F_2(z, \{v + \ln |w|\}_{q(z)}) 1_{(-\infty,0)}(w)$$

$$+F_3(z)[v + \ln |w|]_{q(z)} 1_{\{\widetilde{b}_1(z)=1\}} 1_{(-\infty,0)}(w),$$

a.e. $\sigma(dz)dvdu$, for some functions $F_2$ and $F_3$. As in the case $\kappa \neq 0$ above, we can conclude that $X_\alpha$ can be represented by (3.1) where $\ln |t+u|/|u|$ in the last integrand term of (3.1) is replaced by

$$[v + \ln |t + u|]_{q(z)} - [v + \ln |u|]_{q(z)}.$$

Observe that, by using (1.7), this difference can be expressed as

$$\frac{1}{q(z)}\Big(\{v + \ln |t + u|\}_{q(z)} - \{v + \ln |u|\}_{q(z)} - \ln \frac{|t + u|}{|u|}\Big). \tag{9.9}$$

By including the first two terms of (9.9) into the first four terms of (3.1), we can deduce that $X_\alpha$ can indeed be represented by (3.1). $\square$

Proof of Theorem 3.1: Suppose that $X_\alpha$ is represented by the sum of two independent processes (3.1) and (3.2). To show that $X_\alpha$ is a PFSM, it is enough to prove that (3.1) and (3.2) are PFSMs. The process (3.1) has the representation (1.1) with (1.4)–(1.5). It is easy to verify that

$$G(z, v, c(z)u) = c(z)^\kappa b_1(z)G(z, v, u) + F_3(z)q(z)1_{\{b_1(z)=1\}}1_{\{\kappa=0\}},$$

where $c(z) = e^{q(z)}$. Hence, $C_P = Z \times [0, q(\cdot))$ where $C_P$ is the PFSM set defined by (2.1). Definition 2.1 yields that (3.1) is a PFSM. One can show that the process (3.2) is a PFSM in a similar way.

Suppose now that $X_\alpha$ is a PFSM. By Definition 2.2, a minimal representation of the process $X_\alpha$ is generated by a periodic flow. Suppose that the process $X_\alpha$ has the minimal representation

$$X_\alpha(t) \stackrel{d}{=} \int_{\widetilde{X}} \int_{\mathbb{R}} \widetilde{G}_t(\widetilde{x}, u)\widetilde{M}_\alpha(d\widetilde{x}, du),$$

where $(\widetilde{X}, \widetilde{\mathcal{X}}, \widetilde{\mu})$ is a standard Lebesgue space, $\widetilde{G}_t(\widetilde{x}, u) = \widetilde{G}(\widetilde{x}, t+u) - \widetilde{G}(\widetilde{x}, u)$, $\widetilde{x} \in \widetilde{X}, u \in \mathbb{R}$, $\{\widetilde{G}_t\}_{t\in\mathbb{R}} \subset L^\alpha(\widetilde{X} \times \mathbb{R}, \widetilde{\mu}(dx)du)$, and $\widetilde{M}_\alpha(d\widetilde{x}, du)$ has the control measure $\widetilde{\mu}(d\widetilde{x})du$, and that it is generated by a periodic flow $\{\widetilde{\psi}_c\}_{c>0}$ on $\widetilde{X}$. Since the flow is periodic, we have $\widetilde{X} = \widetilde{P}$, where $\widetilde{P}$ is the set (2.8) of periodic points of the flow $\{\widetilde{\psi}_c\}_{c>0}$. Partitioning $\widetilde{P}$ into the set $\widetilde{L}$ of the cyclic points of the flow in (2.9) and the set $\widetilde{F}$ of the fixed points of the flow in (2.10), we get

$$X_\alpha(t) \stackrel{d}{=} \int_{\widetilde{L}} \int_{\mathbb{R}} \widetilde{G}_t(\widetilde{x}, u) \widetilde{M}_\alpha(d\widetilde{x}, du) + \int_{\widetilde{F}} \int_{\mathbb{R}} \widetilde{G}_t(\widetilde{x}, u) \widetilde{M}_\alpha(d\widetilde{x}, du) =: X_\alpha^L(t) + X_\alpha^F(t).$$

The processes $X_\alpha^L$ and $X_\alpha^F$ are independent since the sets $\widetilde{L}$ and $\widetilde{F}$ are disjoint. The process $X_\alpha^L$ is generated by a cyclic flow, the flow $\{\widetilde{\psi}_c\}_{c>0}$ restricted to the set $\widetilde{L}$, and has a representation (3.1) by Proposition 3.1 below. The process $X_\alpha^F$ is generated by an identity flow satisfying $\widetilde{\psi}_c(\widetilde{x}) = \widetilde{x}$ for all $\widetilde{x} \in \widetilde{F}$, and has a representation (2.4) by Theorem 5.1 in Pipiras and Taqqu (2002*b*). $\square$

## References


Kubo, I. (1969), 'Quasi-flows', *Nagoya Mathematical Journal* **35**, 1–30. MR0247032

Pipiras, V. & Taqqu, M. S. (2002*a*), 'Decomposition of self-similar stable mixed moving averages', *Probability Theory and Related Fields* **123**(3), 412–452.

Pipiras, V. & Taqqu, M. S. (2002*b*), 'The structure of self-similar stable mixed moving averages', *The Annals of Probability* **30**(2), 898–932.

Pipiras, V. & Taqqu, M. S. (2004*a*), 'Dilated fractional stable motions', *The Journal of Theoretical Probability* **17**(1), 51–84.

Pipiras, V. & Taqqu, M. S. (2004*b*), Identification of periodic and cyclic fractional stable motions, Preprint.

Pipiras, V. & Taqqu, M. S. (2004*c*), Semi-additive functionals and cocycles in the context of self-similarity, Preprint. To appear in *Discussiones Mathematicae - Probability and Statistics*.

Pipiras, V. & Taqqu, M. S. (2004*d*), 'Stable stationary processes related to cyclic flows', *The Annals of Probability* **32**(3A), 2222–2260.

Rosiński, J. (1995), 'On the structure of stationary stable processes', *The Annals of Probability* **23**(3), 1163–1187.MR1349166

Rosiński, J. (2006), 'Minimal integral representations of stable processes', *Probability and Mathematical Statistics* **26**(1), 121–142.

Samorodnitsky, G. & Taqqu, M. S. (1994), *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance*, Chapman and Hall, New York, London.