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Exit times of Symmetric α-Stable Processes from unbounded convex domains

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Abstract

Let X_t be a d-dimensional symmetric stable process with parameter $\alpha \in (0, 2)$. Consider τ_D the first exit time of X_t from the domain $D = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R}^{d-1} : 0 < x, |y| < \phi(x) \right\}$, where ϕ is concave and $\lim_{x\to\infty} \phi(x) = \infty$. We obtain upper and lower bounds for $P^z \{\tau_D > t\}$ and for the harmonic measure of X_t killed upon leaving $D \cap B(0, r)$. These estimates are, under some mild assumptions on ϕ, asymptotically sharp as $t \to \infty$. In particular, we determine the critical exponents of integrability of τ_D for domains given by $\phi(x) = x^\beta \left[\ln(x + 1)\right]^\gamma$, where $0 \le \beta < 1$, and $\gamma \in \mathbb{R}$. These results extend the work of R. Bañuelos and R. Bogdan (2).

Key words: stable process, exit times, unbounded domains.

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1 Introduction

Let X_t be a d-dimensional symmetric α-stable process of order $\alpha \in (0,2]$. The process X_t has stationary independent increments and its transition density $p^\alpha(t,z,w) = f_t^\alpha(z-w)$ is determined by its Fourier transform

$$\exp(-t|z|^\alpha) = \int_{\mathbb{R}^d} e^{iz\cdot w} f_t^\alpha(w)dw.$$

These processes have right continuous sample paths and their transition densities satisfy the scaling property

$$p^\alpha(t,x,y) = t^{-d/\alpha} p^\alpha(1, t^{-1/\alpha}x, t^{-1/\alpha}y) .$$

When $\alpha = 2$, the process X_t is a d-dimensional Brownian motion running at twice the usual speed.

Let D be a domain in $\mathbb{R}^d$, and let X_t^D be the symmetric α-stable process killed upon leaving D. If $\alpha \in (0,2)$, H_α the self-adjoint positive operator associated to X_t^D is non-local. Analytically this operator is obtained by imposing Dirichlet boundary conditions on D to the pseudo-differential operator $(-\Delta)^{\alpha/2}$, where Δ is the Laplace operator in $\mathbb{R}^d$. The transition density of X_t^D is denoted by $p_D^\alpha(t,x,y)$ and

$$\tau_D = \inf\{t > 0 : X_t \notin D\},$$

is the first exit time of X_t from D.

It is well known, see (7), that if D has finite Lebesgue measure then the spectrum of H_α is discrete and

$$\lim_{t\to\infty} \frac{P^x[\tau_D > t]}{\exp[-t\lambda_1^\alpha]\,\varphi_1^\alpha(x)\int_D \varphi_1^\alpha(y)dy} = 1, \tag{1.1}$$

where λ_1^α is the smallest eigenvalue of H_α and $\varphi_1^\alpha(x)$ is its associated eigenfunction.

On the other hand, if the domain is the cone given by

$$C = \{x \in \mathbb{R}^d : x \neq 0, \pi - \theta < \varphi(x) \leq \pi\},$$

where $0 < \theta < \pi$ and $\varphi(x)$ is the angle between x, and the point $(0,\ldots,0,1)$. Then there exists $0 < a$ such that

$$E^x\left[\tau_C^p\right] < \infty, \text{ if and only if } p < a, \tag{1.2}$$

see (2), (14), (18), and (20). T. Kulczycki (18) also proved, for $\alpha \in (0,2)$, that $a < 1$ and a converges to one as θ approaches zero. The behavior of the critical exponent of integrability a is significantly different for $\alpha = 2$. D. Burkholder (9) proved that a goes to infinity as θ goes to zero. These results were extended, for $\alpha \in (0,2]$, to cones generated by a domain Ω of $\mathbb{S}^n$ with vertex at the origin in (1),(6), (13), and (19).

In the Brownian motion case, it is known there are domains such that the distribution of the exit time has sub exponential behavior. As a matter of fact, consider the domain $D = D_p$ given by

$$D_p = \left\{(x_1,x) \in \mathbb{R}\times\mathbb{R}^{d-1} : x_1 > 0, x_1^p > |x|\right\}, \tag{1.3}$$

where $p < 1$ and $|x|$ is the euclidean norm in $\mathbb{R}^{d-1}$. R. Bañuelos et al. (4), W. Li (23), and Z. Shi et al. (21) prove

$$\lim_{t\to\infty} \frac{-\ln\left(P^x\left[\tau_{D_p} > t\right]\right)}{t^{\frac{1-p}{1+p}}} = c, \tag{1.4}$$

for some $c > 0$. Similar results were obtained by M. van den Berg (22) for the asymptotic behavior of $p_D^2(t,x,y)$.

Notice D_p is obtained by moving $B(0,x_1^p)$, the ball centered at the origin $0 \in \mathbb{R}^{d-1}$ of radius x_1^p, along the straight line $l_{x_1} = (x_1, 0, \ldots, 0)$. R. D. DeBlassie and R. Smits (15) extend (1.4) to domains generated in a similar way by a curve γ. We should also mentioned the work of Collet et al. ((10),(11)) where the authors study domains of the form $D = \mathbb{R}^d \setminus K$, for K a compact subset of $\mathbb{R}^d$. In this case, there exits $c > 0$ such that

$$\lim_{t\to\infty} -\ln\left(t\, P^x\left[\tau_D > t\right]\right) = c. \tag{1.5}$$

It is then natural to ask if, for $\alpha \in (0,2)$, there are domains in $\mathbb{R}^d$ such that

$$P^x\left(\tau_D > t\right), \tag{1.6}$$

has subexponential behavior as $t \to \infty$.

In this paper we will study the behavior of (1.6) and the behavior of the harmonic measure for unbounded domains of the form

$$D = \left\{ (x,y) \in \mathbb{R} \times \mathbb{R}^{d-1} : 0 < x, |y| < \phi(x) \right\}, \tag{1.7}$$

where ϕ is an increasing concave function, such that

$$\lim_{r\to\infty} \frac{\phi(r)}{r} = 0\ , \text{ and } \int_1^\infty \left(\frac{\phi(\rho)}{\rho}\right)^{d-1} \frac{1}{\rho}\, d\rho \ < \ \infty. \tag{1.8}$$

As shown in §5, for $\phi(x) = [\,\ln(x+1)\,]^\mu$, $\mu \in \mathbb{R}$, (1.6) has sub exponential behavior.

We will denote the ball of radius r centered at the origin, 0 of $\mathbb{R}^n$, by $B(0,r)$. The following result, which we believe is of independent interest, will be fundamental in the study of (1.6).

Theorem 1.1. *Let $0 < x$, and $D_r = D \cap B(0,r)$ where D is given by (1.7). Then there exists $M > 0$ and $c_d^\alpha > 0$ such that for all $r \geq M$*

$$\begin{aligned} & \frac{1}{c_d^\alpha}\, [\,\phi(x)\,]^\alpha \int_{2r}^\infty \left(\frac{\phi(\rho)}{\rho}\right)^{d-1} \frac{1}{\rho^{1+\alpha}} d\rho \\ \leq\ & P^z\left[\, X_{\tau_{D_r}} \in D \,\right] \\ \leq\ & c_d^\alpha\, |z|^\alpha \int_r^\infty \left(\frac{\phi(\rho)}{\rho}\right)^{d-1} \frac{1}{(\rho - r)^{\alpha/2} \rho^{1+\alpha/2}} d\rho, \end{aligned} \tag{1.9}$$

where $z = (x, 0, \ldots, 0)$.

This theorem can be combined with the results of (19) to obtain upper and lower bounds on (1.6). For instance one can show that there exist $c_d^\alpha > 0$ and $M > 0$

$$
\begin{aligned}
& \frac{1}{c_d^\alpha} \exp\left(\frac{-\lambda_1 t}{[\,\phi(r)\,]^\alpha}\right) \int_{2r}^\infty \left(\frac{\phi(\rho)}{\rho}\right)^{d-1} \frac{1}{\rho^{1+\alpha}} d\rho \\
\leq \;& P^z\left(\tau_D > t\right) \\
\leq \;& c_d^\alpha \left[\exp\left(\frac{-\lambda_1 t}{[\,\phi(r)\,]^\alpha}\right) + \int_r^\infty \left(\frac{\phi(\rho)}{\rho}\right)^{d-1} \frac{1}{(\rho - r)^{\alpha/2} \rho^{1+\alpha/2}} d\rho\right],
\end{aligned}
$$

for all $z \in D$ and all $t, r > M$. Our bounds on (1.6) will imply the following result.

Theorem 1.2. *Let D be the domain given by (1.7) with*

$$\phi(x) = x^\beta \,[\,\ln(x+1)\,]^\mu\,. \tag{1.10}$$

(i) *If $0 < \beta < 1$ and $\mu \in \mathbb{R}$, or $\beta = 1$ and $\mu < -1$. Then there exist $M > 0$ and $c > 0$, depending only on d, α, β and μ, such that for all $t \geq M$ and all $z \in D$*

$$
\begin{aligned}
\frac{1}{c} \frac{[\,\ln t\,]^q}{t^p} \;&\leq\; P^z\left(\,\tau_D > t\,\right) \\
&\leq\; c \, \frac{[\,\ln t\,]^q}{t^p} \;\left[\,\ln\,\left(\,\ln t\,\right)^p\,\right]^p ,
\end{aligned} \tag{1.11}
$$

where $p = \frac{(1-\beta)(d-1)+\alpha}{\alpha\beta}$, and $q = p\alpha\mu + (d-1)\mu$. In particular

$$E^z\left[\,\tau_D^r\,\left[\,\ln\left(1+\tau_D\right)\,\right]^s\,\right] < \infty, \tag{1.12}$$

if and only if either $r < p$, or $r = p$ and $s < -1 - q$.

(ii) *If $\beta = 0$ and $\mu \in \mathbb{R}$. Then there exist $M > 0$ and $c > 0$, depending only on d, α and μ, such that for all $t \geq M$ and all $z \in D$*

$$
\begin{aligned}
\frac{1}{c}\, t^{\frac{\mu(d-1)}{1+\mu\alpha}} \exp\left[\,-2\eta\, t^{\frac{1}{\mu\alpha+1}}\,\right] \;&\leq\; P^z\left(\,\tau_D > t\,\right) \\
&\leq\; c\, t^{\frac{\mu(d-1)}{1+\mu\alpha}} \exp\left[\,-\eta\, t^{\frac{1}{\mu\alpha+1}}\,\right],
\end{aligned} \tag{1.13}
$$

where

$$\eta \;=\; (d-1+\alpha)\left(\frac{\lambda_1}{d-1+\alpha}\right)^{\frac{1}{d-1+\alpha}}.$$

In particular, if we take $\mu = 0$ and $0 < \beta < 1$ in (1.10), then

$$E^z\left[\tau_D^p\right] < \infty, \tag{1.14}$$

if and only if $p < \frac{(1-\beta)(d-1)+\alpha}{\alpha\beta}$. This result was first obtained by R. Bañuelos and K. Bogdan in (2).

The paper is organized as follows. In §2 we setup more notation and give some preliminary lemmas. Theorem 1.1 is proved in §3. We obtained bounds on the asymptotic behavior of (1.6) in §4, and finish by proving Theorem 1.2 in §5.

Throughout the paper, the letters c,C, will be used to denote constants which may change from line to line but which do not depend on the variables x, y, z, etc. To indicate the dependence of c on α, or any other parameter, we will write $c = c(\alpha)$, c_α or c^α.

2 Preliminary results.

Throughout this paper the norm in the Euclidean space, regardless of dimension, will be denote by $|\cdot|$, and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ will be an increasing concave function such that

$$\phi(0) = 0 \text{ and } \lim_{x\to\infty} \frac{\phi(x)}{x} = 0. \tag{2.1}$$

Notice that the concavity of ϕ implies

$$\phi'(x) \leq \frac{\phi(x)}{x}. \tag{2.2}$$

Thus

$$\lim_{x\to\infty} \phi'(x) = 0, \text{ and } \frac{\phi(x)}{x} \text{ is decreasing.} \tag{2.3}$$

For any domain $D \subset \mathbb{R}^d$, we denote by $d_D(z)$ to the distance from z to the boundary ∂D.

Lemma 2.1. *Let D be the domain given by (1.7). If $u > 0$ and $z = (u, 0, \ldots, 0)$. Then*

$$\lim_{u\to\infty} \frac{\phi(u)}{d_D(u)} = 1.$$

Proof. Let $u > 0$. A simple computation shows that there exists $x_0 > 0$ such that

$$u = x_0 + \phi(x_0)\phi'(x_0),$$

and

$$d_D(z) = \sqrt{(u - x_0)^2 + [\phi(x_0)]^2} = \phi(x_0)\sqrt{1 + [\phi'(x_0)]^2}.$$

Then the monotonicity of ϕ and (2.3) imply

$$d_D(z) = \phi(x_0)\,[1 + o(1)] \leq \phi(u)\,[1 + o(1)]. \tag{2.4}$$

On the other hand, thanks to (2.3)

$$u = x_0\,(\,1 + o(1)\,),$$

and

$$\phi(u) \leq \frac{u}{x_0}\,\phi(x_0) \;=\; [\,1 + o(1)\,]\,\phi(x_0).$$

Thus

$$d_D(z) = \phi(x_0)\,[\,1 + o(1)\,] \;\geq\; \phi(u)\,[\,1 + o(1)\,]\,, \tag{2.5}$$

the desired result immediately follows. □

In the next section we will approximate certain integrals over D using spherical coordinates. For this we will need to study the behavior of the cross section angles.

For $r > 0$, let x_r be the solution of

$$[\,x_r\,]^2 + [\,\phi(x_r)\,]^2 = r^2, \tag{2.6}$$

and

$$\theta(r) = \arctan\left(\frac{\phi(x_r)}{x_r}\right) \;\geq\; \arctan\left(\frac{\phi(r)}{r}\right), \tag{2.7}$$

the angle between the x-axis and $(x_r, 0, \ldots, 0, \phi(x_r))$.

One easily sees that (2.1) and (2.6) imply

$$\lim_{r\to\infty} \frac{x_r}{r} = 1. \tag{2.8}$$

Thus

$$\lim_{r\to\infty} \frac{\phi(x_r)}{x_r} = 0,$$

and there exists $M > 0$ such that

$$\begin{array}{lll} \frac{1}{2}\frac{\phi(r)}{r} & \leq \theta(r) & \leq 2\frac{\phi(r)}{r} \\ \frac{1}{2}\varphi & \leq \sin(\varphi) & \leq \varphi, \end{array} \tag{2.9}$$

for all $r \geq M$ and $0 \leq \varphi \leq \theta(r)$.

3 Harmonic measure estimates

In this section we study the harmonic measure of the domain

$$D_r = D \cap B(0, r),$$

for D given by (1.7). Our arguments follow the ideas of T. Kulczycki in (18). As a matter of fact, we are interested in the behavior of

$$P^z\left[X_{\tau_{D_r}} \in B \right],$$

as $r \to \infty$, where $z \in D$, and B is a borelian subset of D. For $m \in \mathbb{Z}$, we define

$$D^m = D_{r2^m}, A_m = D^m \setminus D^{m-1}, \tag{3.1}$$

and $\mathcal{B}(A_m)$ to be the Borel subsets of A_m.
To simplify the notation we set

$$\tau_m = \tau_{D^m}.$$

If $x \in A_m$ the probability that X jumps directly to B, $B \setminus D^{m+1} \neq \emptyset$, when leaving the subdomain D^{m+1} is

$$q_m(x, B) = P^x\left(X_{\tau_{m+1}} \in B\right) = \int_B p_m(x, y)dy, \tag{3.2}$$

where p_m is the Poisson kernel of X_t killed upon leaving the domain D^{m+1}.
However the process X_t, starting at $x \in A_m$, could also jump out of D^{m+1} and reach $B \subset A_n$ after precisely k successive jumps to $A_{i_1}, \ldots, A_{i_k}$, $m < i_1 < i_2 < \ldots < i_k = n$. Thus we are interested in the behavior of

$$\begin{aligned} & q_{i_1,\ldots,i_k}(x, B) \\ = \quad & P^z\left\{X_{\tau_{i_0+1}} \in A_{i_1}, \ldots, X_{\tau_{i_{k-2}+1}} \in A_{i_{k-1}}, X_{\tau_{i_{k-1}+1}} \in B\right\}, \end{aligned} \tag{3.3}$$

where $i_0 = m$. The Markov property implies that

$$q_{i_1,\ldots,i_k}(x, B) = \int_{A_{i_1}} \cdots \int_{A_{i_{k-1}}} \int_B \prod_{i=0}^{k-1} p_{i_k}(y_i, y_{i+1})\, dy_1 \ldots dy_k, \tag{3.4}$$

where $i_0 = m$. Notice that the event

$$\left\{ X_{\tau_{k+1}} \in A_l \right\},$$

is not empty if and only if $k \leq l - 2$. Thus

$$q_{i_1,\dots,i_k}(x, B_n) = 0$$

for all borelian sets $B_n \subset A_n$, unless $(i_1, \dots, i_k) \in J_k(m, n)$ where

$$J_k(m,n) = \left\{(i_1,\dots,i_k) \in \mathbb{Z}^k : i_1 \geq m+2, i_k = n, i_{j+1} - i_j \geq 2\right\}, \tag{3.5}$$

for $m, n \in \mathbb{Z}$ and $k \in \mathbb{N}$ with $m < n$.

Therefore the probability that X starts at $x \in D$ and goes to $B \cap D$ after k jumps, of the type (3.3), is

$$\begin{aligned} P_k(x,B) &= \sum_{(i_1,\dots,i_k)\in\mathbb{Z}^k} q_{i_1,\dots,i_k}(x,B) \qquad (3.6)\\ &= \sum_{(i_1,\dots,i_k)\in J_k(m,n)} q_{i_1,\dots,i_k}(x,B). \end{aligned}$$

Let

$$\sigma(x,B) = \sum_{k=1}^{\frac{n-m}{2}} P_k(x,B). \tag{3.7}$$

T. Kulzcycki prove that if $x \in D_{-1}$ and $B_1 \subset A_1$, then

$$P^x\left(X_{\tau_D} \in B_1\right) = \sigma(x, B_1) + \int_{A_0} P^y\left(X_{\tau_D} \in B_1\right) d\sigma(x,y). \tag{3.8}$$

Thus to estimate the harmonic measure it is enough to have good estimates of $\sigma(x,\cdot)$. We will start by estimating the function $q_m(x,\cdot)$.

Lemma 3.1. *Let $m, n \in \mathbb{Z}$ with $n \geq m+2$. If $x \in A_m$ and $B_n \in \mathcal{B}(A_n)$. Then there exists $c_d^\alpha > 0$ such that*

$$q_m(x, B_n) \leq \frac{c_d^\alpha}{2^{(n-m)\alpha}} \int_{B_n} \frac{\psi_m(y)}{|y|^d} dy, \tag{3.9}$$

where

$$\psi_m(|y|) = \begin{cases} 1 & \text{if } |y| \geq 2^{m+2} \\ \left(\frac{2^{m+1}r}{|y|-2^{m+1}r}\right)^{\alpha/2} & \text{if } 2^{m+1} \leq |y| < 2^{m+2} \end{cases}.$$

In particular

$$q_m(x, A_n) \leq \frac{c_d^\alpha}{2^{(n-m)\alpha}} \int_{2^{n-1}}^{2^n} \left(\frac{\phi(\rho)}{\rho}\right)^{d-1} \frac{\psi_m(\rho)}{\rho}\, d\rho. \tag{3.10}$$

Proof. Since $x \in A_m \subset D^{m+1}$

$$
\begin{aligned}
q_m(x, B_n) &\le P^x\left(X_{\tau_{B(0,r2^{m+1})}} \in B_n\right) \\
&= c_d^\alpha \int_{B_n} \frac{\left(2^{2m+2}r^2 - |x|^2\right)^{\alpha/2}}{\left(|y|^2 - 2^{2m+2}r^2\right)^{\alpha/2} |x-y|^d} dy. \qquad (3.11)
\end{aligned}
$$

If $n - m \ge 2$, and $y \in B_n$ we have

$$|x - y| \ge 2^{n-1}r - 2^m r \ge 2^{n-2}r \ge |y|/4.$$

Besides, if $n > m + 2$, then

$$|y|^2 - 2^{2m+2}r^2 \ge 2^{2n}r^2 - 2^{2m+2}r^2 \ge 2^{2n-3}r^2.$$

Thus

$$P(x, B_n) \le c_d^\alpha \int_{B_n} \frac{2^{(m+1)\alpha}}{|y|^d 2^{(n-1)\alpha}} dy = \frac{c_d^\alpha}{2^{(n-m)\alpha}} \int_{B_n} \frac{1}{|y|^d} dy.$$

On the other hand, if $n = m + 2$, we have

$$\frac{2^{m+1}r + |x|}{|y| + 2^{m+1}r} \le 1.$$

Then

$$P(x, B_n) \le \frac{c_d^\alpha}{2^{(n-m)\alpha}} \int_{B_n} \frac{2^{(m+1)\alpha/2}}{|y|^d \left(y - 2^{m+1}r\right)^{\alpha/2}} dy.$$

Finally if $B_n = A_n$. Using spherical coordinates we obtain from (2.9)

$$
\begin{aligned}
\int_{B_n} \frac{\psi_m(y)}{|y|^d} dy &\le c_d^\alpha \int_{2^{n-1}r}^{2^n r} \int_0^{\theta(\rho)} \frac{\psi_m(\rho)}{\rho} \sin^{d-2}(\varphi)\, d\varphi d\rho \qquad (3.12) \\
&\le c_d^\alpha \int_{2^{n-1}r}^{2^n r} \int_0^{\theta(\rho)} \frac{\psi_m(\rho)}{\rho} \varphi^{d-2}\, d\varphi d\rho \\
&\le c_d^\alpha \int_{2^{n-1}r}^{2^n r} \int_0^{2\frac{\phi(\rho)}{\rho}} \frac{\psi_m(\rho)}{\rho} \varphi^{d-2}\, d\varphi d\rho \\
&\le c_d^\alpha \int_{2^{n-1}r}^{2^n r} \frac{\psi_m(\rho)}{\rho} \left(\frac{\phi(\rho)}{\rho}\right)^{d-1} d\rho,
\end{aligned}
$$

and (3.10) follows. □

The following corollary is an immediate consequence of the definition of $q_{i_1,\dots,i_k}(x,\cdot)$.

Corollary 3.2. *Let $m, n \in \mathbb{Z}$ be such that $n \ge m + 2$. If $x \in A_m$ and $B_n \in \mathcal{B}(A_n)$. Then there exists $c_d^\alpha > 0$ such that*

$$q_{i_1,\dots,i_k}(x, B_n) \le \frac{[\,c_d^\alpha\,]^k}{2^{(n-m)\alpha}} I(i_{k-1}, B_n) \prod_{j=0}^{k-1} I(i_{j-1}, A_{i_j}), \qquad (3.13)$$

where

$$I(l,B_k)=\int_{B_k}\frac{\psi_l(y)}{|y|^d}\,dy, \tag{3.14}$$

for all $l,k\in\mathbb{Z}$ and all borelian sets B_k contained in A_k.

In order to estimate $\sigma(\cdot,\cdot)$, we will need the following monotonicity result.

Lemma 3.3. *Let $k,m\in\mathbb{Z}$ be such that $k\geq m$. Then*

$$I(k,A_{k+2})\leq 2I(m,A_{m+2}). \tag{3.15}$$

Proof. Recall that the function $\phi(x)/x$ is decreasing. Following the arguments of Lemma 3.1, we obtained a constant c_α^d such that

$$\begin{aligned}
& I(k,A_{k+2}) \qquad (3.16)\\
\leq\ & c_\alpha^d\int_{2^{k+1}r}^{2^{k+2}r}\left[\frac{\phi(\rho)}{\rho}\right]^{d-1}\left[\frac{2^{k+1}r}{(\rho-2^{k+1}r)}\right]^{\alpha/2}\frac{1}{\rho}\,d\rho\\
\leq\ & c_\alpha^d\left[\frac{\phi(2^{k+1}r)}{2^{k+1}r}\right]^{d-1}\frac{1}{2^{k+1}r}\int_{2^{k+1}r}^{2^{k+2}r}\left[\frac{2^{k+1}r}{\rho-2^{k+1}r}\right]^{\alpha/2}d\rho\\
=\ & c_\alpha^d\left[\frac{\phi(2^{k+1}r)}{2^{k+1}r}\right]^{d-1}\frac{1}{1-\alpha/2}.
\end{aligned}$$

On the other hand, using spherical coordinates and (2.9)

$$\begin{aligned}
& I(m,B_{m+2}) \qquad (3.17)\\
\geq\ & c_\alpha^d\int_{2^{m+1}r}^{2^{m+2}r}\int_0^{\theta(\rho)}\frac{\psi_m(\rho)}{\rho}\sin^{d-2}(\varphi_1)\,d\varphi_1 d\rho\\
\geq\ & c_\alpha^d\int_{2^{m+1}r}^{2^{m+2}r}\left[\frac{\phi(\rho)}{\rho}\right]^{d-1}\left[\frac{2^{m+1}r}{(\rho-2^{m+1}r)}\right]^{\alpha/2}\frac{1}{\rho}\,d\rho\\
\geq\ & c_\alpha^d\left[\frac{\phi(2^{m+2}r)}{2^{m+2}r}\right]^{d-1}\frac{1}{2^{m+2}r}\int_{2^{m+1}r}^{2^{m+2}r}\left[\frac{2^{m+1}r}{\rho-2^{m+1}r}\right]^{\alpha/2}d\rho\\
=\ & c_\alpha^d\left[\frac{\phi(2^{m+2}r)}{2^{m+2}r}\right]^{d-1}\frac{1}{2-\alpha},
\end{aligned}$$

and the result follows. □

Let $(i_1,\ldots,i_k)\in J(m,n)$, and $1\leq s<k-1$. By the definition of $J(m,n)$ we have

$$m+2s\leq i_s.$$

Now if $i_s+2=i_{s+1}$, Lemma 3.3 implies that

$$\begin{aligned}
\int_{r2^{i_s+1}}^{r2^{i_s+2}}\frac{\psi_{i_s}(\rho)}{|\rho|^d}\,d\rho &= I(i_s,A_{i_{s+1}})\\
&\leq 2\,I(m+2s,A_{m+2s+1}) \qquad (3.18)\\
&= 2\int_{r2^{m+2s+1}}^{r2^{m+2s+2}}\frac{\psi_{m+2s}(\rho)}{|\rho|^d}\,d\rho.
\end{aligned}$$

In addition, if $i_{k-1} < n-2$, then for all $\rho \le 2^n r$

$$\frac{2^{n-1} r}{\rho - 2^{n-1} r} \ge 1.$$

Thus

$$\begin{aligned} I(i_{k-1}, B_n) &= \int_{B_n} \frac{1}{|y|^d} dy \\ &\le c_d^\alpha \int_{B_n} \left[\frac{2^{(n-1)} r}{|y| - 2^{n-1} r} \right]^{\alpha/2} \frac{1}{|y|^d} \, dy. \\ &= \mu_n(B_n). \end{aligned}$$

Since this inequality also holds when $i_{k-1} = n-2$, we conclude that

$$I(i_{k-1}, B_n) \le \mu_n(B_n). \tag{3.19}$$

In order to obtain and upper bound on $\sigma(x, B)$, we need to estimate

$$P_k(x, B_n) = \sum_{(i_1,\dots,i_k) \in J_k(m,n)} q_{i_1,\dots,i_k}(x, B_n).$$

Lemma 3.4. *Let $m, n \in \mathbb{Z}$ be such that $n \ge m+2$. If $x \in A_m$ and $B_n \in \mathcal{B}(A_n)$. Then there exists $c_d^\alpha > 0$ such that for $k \ge 2$*

$$P_k(x, B_n) \le \frac{[c_d^\alpha]^k}{2^{(n-m)\alpha}} \mu_n(B_n) \prod_{i=0}^{k-2} \int_{r 2^{m+2i+1}}^{r 2^{n-2(k-i)}} \frac{\psi_{m+2i}(\rho)}{|\rho|^d} \, d\rho. \tag{3.20}$$

Proof. Thanks to Corollary 3.2 it is enough to prove that

$$\begin{aligned} &\sum_{(i_1,\dots,i_k) \in J_k(m,n)} I(i_{k-1}, B_n) \prod_{j=1}^{k-1} I(i_{j-1}, A_{i_j}) \\ &\le \mu_n(B_n) \prod_{i=0}^{k-2} \int_{r 2^{m+2i+1}}^{r 2^{n-2(k-i)}} \frac{\psi_{m+2i}(\rho)}{|\rho|^d} \, d\rho. \end{aligned} \tag{3.21}$$

We will prove (3.21) by induction in k. Notice that

$$J_2(m, n) = \{(i, n) : m+2 \le i \le n-2\}.$$

Then (3.14) and (3.19) imply that

$$\begin{aligned} &\sum_{i=m+2}^{n-2} I(m, A_i) \, I(i, B_n) \\ &\le \frac{[c_d^\alpha]^2}{2^{(n-m)\alpha}} \mu_n(B_n) \sum_{i=m+2}^{n-2} \int_{r 2^{i-1}}^{r 2^i} \frac{\psi_m(\rho)}{|\rho|^d} \, d\rho, \end{aligned}$$

and the result follows for $k = 2$.

On the other hand, Lemma 3.3, and (3.18) imply

$$\begin{aligned}
&\sum_{(i_1,\ldots,i_k)\in J_k(m,n)} I(i_{k-1}, B_n)\prod_{j=1}^{k-1} I(i_{j-1}, A_{i_j}) \\
= &\sum_{i_1=m+2}^{n-2(k-1)} I(m, A_{i_1})\left\{\sum_{(i_2,\ldots,i_k)\in J_{k-1}(i_1,n)} I(i_{k-1}, B_n)\prod_{j=2}^{k-1} I(i_{j-1}, A_{i_j})\right\} \\
\leq &\sum_{i_1=m+2}^{n-2(k-1)} I(m, A_{i_1})\left\{\mu_n(B_n)\prod_{j=0}^{k-3}\int_{r2^{i_1+2j+1}}^{r2^{n-2(k-1-j)}} \frac{\psi_{i_1+2j}(\rho)}{|\rho|^d}\,d\rho\right\} \\
\leq &\sum_{i_1=m+2}^{n-2(k-1)} \int_{r2^{i_1-1}}^{r2^{i_1}} \frac{\psi_m(\rho)}{|\rho|^d}d\rho\left\{\mu_n(B_n)\prod_{j=0}^{k-3}\int_{r2^{m+2+2j+1}}^{r2^{n-2(k-(j+1))}} \frac{\psi_{m+2+2j+1}(\rho)}{|\rho|^d}\,d\rho\right\} \\
\leq &\sum_{i_1=m+2}^{n-2(k-1)} \int_{r2^{i-1}}^{r2^{i}} \frac{\psi_m(\rho)}{|\rho|^d}d\rho\left\{\mu_n(B_n)\prod_{j=0}^{k-3}\int_{r2^{m+2+2j+1}}^{r2^{n-2(k-(j+1))}} \frac{\psi_{m+2+2j}(\rho)}{|\rho|^d}\,d\rho\right\} \\
\leq &\int_{r2^{m+1}}^{r2^{n-2(k-1)}} \frac{\psi_m(\rho)}{|\rho|^d}\,d\rho\left\{\mu_n(B_n)\prod_{j=0}^{k-3}\int_{r2^{m+2+2j+1}}^{r2^{n-2(k-(j+1))}} \frac{\psi_{m+2+2j}(\rho)}{|\rho|^d}\,d\rho\right\},
\end{aligned}$$

and the result follows. □

We finally obtain an upper bound on $\sigma(x, B)$.

Lemma 3.5. *Let $m, n \in \mathbb{Z}$ be such that $n \geq m + 2$. If $x \in A_m$, $B_n \in \mathcal{B}(A_n)$, and*

$$\int_1^\infty \left(\frac{\phi(\rho)}{\rho}\right)^{d-1} \frac{1}{\rho}\,d\rho < \infty. \tag{3.22}$$

Then there exists a constant c_d^α such that

$$\sigma(x, B_n) \;\leq\; c_d^\alpha\,|x|^\alpha \int_{B_n}\left[\frac{1}{|y| - 2^{n-1}r}\right]^{\alpha/2} \frac{1}{|y|^{d+\alpha/2}}\,dy. \tag{3.23}$$

Proof. The previous result implies that

$$\begin{aligned}
\sigma(x, B_n) &= \sum_{k=1}^{\frac{n-m}{2}} P_k(x, B_n) \\
&\leq \frac{c_d^\alpha\,\mu_n(B_n)}{2^{(n-m)\alpha}}\left[1 + \sum_{k=2}^{\frac{n-m}{2}}\prod_{i=0}^{k-2}\left\{c_d^\alpha\int_{r2^{m+2i+1}}^{r2^{n-2(k-i)}} \frac{\psi_{m+2i}(\rho)}{|\rho|^d}\,d\rho\right\}\right].
\end{aligned} \tag{3.24}$$

Let $y \in B_n$, since $x \in A_m$ we have

$$2^{m\alpha}r^\alpha \leq 2|x|^\alpha, \text{ and } |y|^{\alpha/2} \leq 2^{n\alpha/2}r^{\alpha/2}.$$

Then

$$
\begin{aligned}
\frac{1}{2^{(n-m)\alpha}}\,\mu_n(B_n) &= \frac{2^{m\alpha}r^\alpha}{2^{n\alpha}r^\alpha}\int_{B_n}\left(\frac{2^{n-1}r}{|y|-2^{n-1}r}\right)^{\alpha/2}\frac{1}{|y|^d}\,dy\\
&\le |x|^\alpha\int_{B_n}\frac{1}{(|y|-2^{n-1}r)^{\alpha/2}}\frac{1}{|y|^{d+\alpha/2}}\,dy.
\end{aligned}
$$

On the other hand

$$
\sum_{k=2}^{\frac{n-m}{2}}\prod_{i=0}^{k-2}\left\{c_d^\alpha\int_{r2^{m+2i+1}}^{r2^{n-2(k-i)}}\frac{\psi_{m+2i+1}(\rho)}{|\rho|}d\rho\right\} \le \sum_{k=2}^{\infty}\prod_{i=0}^{k-2}\left\{c_d^\alpha\int_{r2^{m+2i+1}}^{\infty}\frac{\psi_{m+2i+1}(\rho)}{|\rho|^d}\,d\rho\right\},
$$

one easily sees that (3.22) implies the converges of this series.

□

The proof of Lemma 3.8, Lemma 3.9 and Lemma 3.10 of (18) can be followed step by step to obtain the following result, which is the upper bound on Theorem 1.1.

Proposition 3.6. *Let $x \in D_{r/2}$ and B a Borelian subset of $D \setminus D_r$. Then there exists $c_d^\alpha > 0$ such that*

$$
P^z\left[\,X_{\tau_{D_r}} \in B\,\right] \le c_d^\alpha\,|x|^\alpha\int_B\left(\frac{1}{|y|-r}\right)^{\alpha/2}\frac{1}{|y|^{d+\alpha/2}}dy, \tag{3.25}
$$

In particular

$$
P^z\left[\,X_{\tau_{D_r}} \in D\,\right] \le c_d^\alpha\,|x|^\alpha\,\Lambda(r). \tag{3.26}
$$

We shall now obtain the lower bound in (1.10) of Theorem 1.1.

Notice that D_r is a bounded domain that satisfies the exterior cone condition. It is well know that, see (17),

$$
\begin{aligned}
P^x\left[X_{\tau_{D_r}} \in D\setminus D_r\,\right] &= \int_{D_r}G_{D_r}(x,y)\int_{D\setminus D_r}\frac{c_d^\alpha}{|y-z|^{d+\alpha}}\,dz\,dy\\
&\ge \int_{D_r}G_{D_r}(x,y)\int_{D\setminus D_{2r}}\frac{c_d^\alpha}{|y-z|^{d+\alpha}}\,dz\,dy.
\end{aligned}
$$

Notice that for all $y \in D_r$ and all $z \in D_{2r}$,

$$
\frac{|z|}{2} \le |z|-|y| \le |z-y| \le 2|z|.
$$

Then

$$
P^x\left[X_{\tau_{D_r}} \in D\setminus D_r\,\right] \ge \int_{D_r}G_{D_r}(x,y)\int_{D\setminus D_{2r}}\frac{c_d^\alpha}{|z|^{d+\alpha}}\,dz\,dy. \tag{3.27}
$$

We will estimate the integral on z using polar coordinates. Thanks to (2.9) there exists $M \in \mathbb{R}$ such that for all $r \geq M$,

$$\begin{aligned}
\int_{D\setminus D_{2r}} \frac{c_d^\alpha}{|z|}\, dz &= \int_{2r}^{\infty}\int_0^{\theta(\rho)} \sin^{d-2}(\varphi)\, \frac{1}{\rho^{1+\alpha}}\, d\varphi\, d\rho \\
&\geq c_d^\alpha \int_{2r}^{\infty} [\,\theta(\rho)\,]^{d-1}\, \frac{1}{\rho^{1+\alpha}}\, d\rho \\
&\geq c_d^\alpha \int_{2r}^{\infty} \left[\frac{\phi(\rho)}{\rho}\right]^{d-1} \frac{1}{\rho^{1+\alpha}}\, d\rho.
\end{aligned} \tag{3.28}$$

Finally

$$\begin{aligned}
\int_{D_r} G_{D_r}(x,y)dy &= E^x\left[\,\tau_{D_r}\,\right] \\
&\geq E^0\left[\,\tau_{B(\,0,d_{D_r}(x)\,)}\,\right] \\
&= c_d^\alpha\, [\,d_{D_r}(x)\,]^\alpha\,.
\end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29) we obtain the desired inequality.

4 Exit time estimates

T. Kulczycki proved the semigroup associated to the killed symmetric α-stable process on any bounded domain is intrinsic ultracontractive. Thus there exists $c_d^\alpha > 0$ such that

$$\frac{1}{c_d^\alpha}\, \exp\left[-\frac{t\lambda_d}{r^\alpha}\right] \,\leq\, P^0\left[\,\tau_{B(0,r)} > t\,\right] \,\leq\, c_d^\alpha\, \exp\left[-\frac{t\lambda_d}{r^\alpha}\right], \tag{4.1}$$

for all $t > 1$, where λ_d is the principal eigenvalue of X_t killed upon leaving $B(0,1) \subset \mathbb{R}^d$.

We now use the results of §4 to obtained estimates for the distribution of the exit time.

Lemma 4.1. *Let $r > 0$ and $D_r = D \cap B(0,r)$. If λ_1 is the principal eigenvalue of the one dimensional symmetric α-stable process killed upon leaving $(-1,1)$. Then there exists $c = c(d,\alpha)$ such that*

$$P^z\left[\,\tau_{D_r} > t\right] \,\leq\, c\, \exp\left[-\frac{[\,\lambda_1 + o(1)\,]\, t}{[\,\phi(r)\,]^\alpha}\right], \tag{4.2}$$

for all $z \in D$ and all $t > 1$.

Proof. Notice D_r is a convex domain in $\mathbb{R}^d$. Let $r(D_r)$ be the inradius of D_r and

$$I_r = (\,-r(D_r), r(D_r)\,).$$

Then Theorem 5.1 in (19) and (4.1) imply

$$P^z\left[\,\tau_{D_r} > t\,\right] \,\leq\, P^0\left[\,\tau_{I_r} > t\right] \,\leq\, c_1^\alpha \exp\left[\frac{-\lambda_1\, t}{r^\alpha(D_r)}\right]. \tag{4.3}$$

One easily proves that for all $z \in D_r$

$$d_{D_r}(z) = \min\{d_D(z), r - |z|\},$$

and that there exists $u = (x, 0, \ldots, 0)$ such that

$$r(D_r) = d_{D_r}(u) = d_D(u) = r - |x| \leq \phi(r).$$

Since $d_D(u) \leq \phi(x)$, then

$$\lim_{r\to\infty} \left(1 - \frac{|x|}{r}\right) = \lim_{r\to\infty} \frac{d_D(u)}{r} \leq \lim_{r\to\infty} \frac{\phi(x)}{r} \leq \lim_{r\to\infty} \frac{\phi(r)}{r} = 0.$$

On the other hand, Lemma 2.1 implies that

$$\phi(r) \leq \frac{r}{x}\phi(x) = [\,1 + o(1)\,]\, d_D(u) = [\,1 + o(1)\,]\, r(D_r).$$

Hence

$$\lim_{r\to\infty} \frac{r(D_r)}{\phi(r)} = 1,$$

and (4.2) follows from (4.3). □

We now obtained our lower bound on the asymptotic behavior of $P^z\,(\,\tau_D > t\,)$.

Proposition 4.2. *Let $z = (x, 0, \ldots, 0) \in D$ and $D_r = D \cap B(0, r)$. Then there exist $M > 0$ and $c > 0$, depending only on d and α, such that*

$$P^z\,[\,\tau_D > t\,] \geq c\, \exp\left[-\frac{\lambda_1\, t}{[\,\phi(r)\,]^\alpha}\right] [\,d_{D_r}(z)\,]^\alpha \int_{2r}^{\infty} \frac{[\,\phi(\rho)\,]^{d-1}}{\rho^{d+\alpha}}\, d\rho,$$

for all $r \geq M$ and all $t > 1$.

Proof. Let $\eta < 1$. The strong Markov property implies

$$\begin{aligned} P^z\,[\,\tau_D > t\,] &\geq P^z\left[\,\tau_D > t,\, X_{\tau_{D_r}} \in D\,\right] \\ &\geq P^z\left[\,X_{\tau_{D_r}} \in D\,, P^{X_{\tau_{D_r}}}\,(\,\tau_D > t\,)\,\right] \\ &\geq P^z\left[\,X_{\tau_{D_r}} \in \hat{D} \setminus D_r\,, P^{X_{\tau_{D_r}}}\,(\,\tau_D > t\,)\,\right], \end{aligned} \tag{4.4}$$

where

$$\hat{D} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1} : \, 0 < x, |y| < \,\eta\,\phi(x)\right\}.$$

Let $w \in \hat{D} \setminus D_r$. Then Lemma 2.1 implies that there exists $M > 0$ such that for all $r \geq M$

$$B = B\,(\,w, \phi(r)\,[1-2\eta]\,)\ \subset\ D.$$

Thus, thanks to (4.1), we have

$$P^w\,(\,\tau_D > t\,) \geq P^w\,(\,\tau_B > t\,)\ \geq\ c_d^\alpha\ \exp\left[-\frac{\lambda_d\ t}{[\,\phi(r)\ (1-2\eta)\,]^\alpha}\right],$$

for some $c > 0$. Now equation (58) in (19) implies

$$\lambda_d < \lambda_1.$$

Take $0 < \eta = \eta(\alpha, d) < 1$ such that

$$\frac{\lambda_d}{(1-2\eta)^\alpha} = \lambda_1.$$

Hence for all $w \in \hat{D} \setminus D_r$

$$P^w\,(\,\tau_D > t\,)\ \geq\ c_d^\alpha\ \exp\left[-\frac{\lambda_1\ t}{[\,\phi(r)\,]^\alpha}\right].$$

We conclude

$$P^z\,[\,\tau_D > t,\, \tau_{D_r} < \tau_D\,]\ \geq\ c\ \exp\left[\frac{-\lambda_1\ t}{[\,\phi(r)\,]^\alpha}\right] P^z\left[\,X_{\tau_{D_r}} \in \hat{D} \setminus D_r\,\right],$$

where c depends only on d and α.

On the other hand, following the arguments used to prove Proposition 3.7 one easily shows

$$P^z\left[\,X_{\tau_{D_r}} \in \hat{D} \setminus D_r\,\right] \geq c\,[\,d_{D_r}(z)\,]^\alpha\ \int_{2r}^{\infty} \frac{[\,\phi(\rho)\,]^{d-1}}{\rho^{d+\alpha}}\,d\rho,$$

for some c depending only on d and α.

□

We end this section with an upper bound for the distribution of the exit time.

Proposition 4.3. *Let $z = (x, 0, \ldots, 0) \in D$. Then there exist $M > 0$ and $c > 0$, depending only on d and α, such that*

$$\begin{aligned} P^z\,[\,\tau_D > t\,]\ &\leq\ c\ \exp\left[-\frac{[\lambda_1 + o(1)]\ t}{2[\,\phi(r_1)\,]^\alpha}\right] + c\,|x|^\alpha\,\Lambda(r_2) \\ &+\ c\,|x|^\alpha\,\Lambda(r_1)\ \exp\left[-\frac{[\lambda_1 + o(1)]\ t}{2[\,\phi(r_2)\,]^\alpha}\right], \end{aligned} \tag{4.5}$$

for all $r_2 > r_1 \geq M$ and all $t > 1$.

Proof. Let $0 < r_1 < r_2$, then

$$\begin{aligned} P^z [\, \tau_D > t \,] &= P^z \left[\, \tau_D > t,\, \tau_{D_{r_2}} < \tau_D \,\right] + P^z \left[\, \tau_D > t,\, \tau_{D_{r_1}} = \tau_D \,\right] \\ &+ P^z \left[\, \tau_D > t,\, \tau_{D_{r_1}} < \tau_D \le \tau_{D_{r_2}} \,\right] \\ &\le P^z \left[\, X_{\tau_{D_{r_2}}} \in D \setminus D_{r_2} \,\right] + P^z \left[\, \tau_{D_{r_1}} > t \,\right] \qquad (4.6) \\ &+ P^z \left[\, \tau_{D_{r_2}} > t,\, \tau_{D_{r_1}} < \tau_D \le \tau_{D_{r_2}} \,\right]. \end{aligned}$$

Besides

$$\begin{aligned} P^z \left[\, \tau_{D_{r_2}} > t,\, \tau_{D_{r_1}} < \tau_D \le \tau_{D_{r_2}} \,\right] &= P^z \left[\, \tau_{D_{r_1}} > \frac{t}{2},\, \tau_{D_{r_2}} > t,\, \tau_{D_{r_1}} < \tau_D \le \tau_{D_{r_2}} \,\right] \\ &+ P^z \left[\, \tau_{D_{r_1}} \le \frac{t}{2},\, \tau_{D_{r_2}} > t,\, \tau_{D_{r_1}} < \tau_D \le \tau_{D_{r_2}} \,\right] \\ &\le P^z \left[\, \tau_{D_{r_1}} > \frac{t}{2} \,\right] + P^z \left[\, \tau_{D_{r_2}} - \tau_{D_{r_1}} > \frac{t}{2},\, \tau_{D_{r_1}} < \tau_D \,\right]. \end{aligned}$$

The strong Markov property and Theorem 5.1 in (19) imply

$$\begin{aligned} P^z \left[\, \tau_{D_{r_2}} - \tau_{D_{r_1}} > \frac{t}{2},\, \tau_{D_{r_1}} < \tau_D \,\right] &= E^z \left[\, P^{X_{\tau_{r_1}}} \left[\tau_{D_{r_2}} > \frac{t}{2}\right] ,\, \tau_{D_{r_1}} < \tau_D \,\right] \\ &\le P^z [\, \tau_{D_{r_1}} < \tau_D \,]\; P^0 \left[\tau_{D_{r_2}} > \frac{t}{2}\right]. \end{aligned}$$

The result follows from Lemma 4.1 and Proposition 3.6. □

5 Applications and examples

In this section we will apply the results of the previous section to the function

$$\phi(x) = x^\beta \, [\, \ln(x+1)\,]^\mu \,.$$

A straight forward computation shows that ϕ satisfies the assumptions of Theorem 1.1 and §4, if either

$$0 \le \beta < 1, \text{ and } \mu \in \mathbb{R},$$

or

$$\beta = 1, \text{ and } \mu < -1.$$

Case I: Let us first assume that $0 < \beta < 1$, and $\mu \in \mathbb{R}$, or $\beta = 1$ and $\mu < -1$. First we obtain a lower bound for $P^z\,(\, \tau_D > t \,)$. Let

$$r + 1 = \frac{t^{\frac{1}{\beta\alpha}}}{[\ln t\,]^{\frac{\mu}{\beta}}}.$$

Then

$$\begin{aligned}
\lim_{t\to\infty}\ \exp\left[-\frac{\lambda_1\ t}{[\,\phi(r)\,]^{\alpha\beta}}\right] &= \lim_{t\to\infty}\exp\left[-\frac{\lambda_1\ [\,\ln t\,]^{\mu\alpha}}{\left[\frac{1}{\beta\alpha}\ln t-\frac{\mu}{\alpha}\ln(\ln t)\,\right]^{\mu\alpha}}\right]\\
&= \exp\left[-\lambda_1\,(\,\beta\alpha\,)^{\mu\alpha}\,\right].
\end{aligned}$$

On the other hand, if $p=\frac{(1-\beta)(d-1)+\alpha}{\alpha\beta}$, then

$$\begin{aligned}
\int_{2r}^{\infty}\frac{[\,\phi(\rho)\,]^{d-1}}{\rho^{d+\alpha}}\,d\rho &= \int_{2r}^{\infty}\frac{[\,\ln(\rho+1)\,]^{\mu(d-1)}}{\rho^{p\beta\alpha+1}}\,d\rho\\
&= \frac{[\,\ln(r+1)\,]^{\mu(d-1)}}{r^{p\beta\alpha}}\int_{2}^{\infty}\frac{\left(1+\frac{\ln(t+1)}{\ln(r+1)}\right)^{\mu(d-1)}}{t^{p\beta\alpha+1}}\,dt.
\end{aligned}$$

One easily proves that the function

$$\int_{2}^{\infty}\left(1+\frac{\ln(t+1)}{\ln(r+1)}\right)^{\mu(d-1)}\frac{1}{t^{p\beta\alpha+1}}\,dt,$$

is bounded in r. Then there exists $c=c(d,\alpha)>0$ such that

$$\begin{aligned}
P^{z}\,(\,\tau_D>t) &\geq c\left[\frac{1}{\beta\alpha}\ln t-\frac{\mu}{\beta}\ln(\ln t)\right]^{\mu(d-1)}\ \frac{[\,\ln t\,]^{\mu p\alpha}}{t^{p}}\\
&= c\left[\frac{1}{\beta\alpha}-\frac{\mu}{\beta}\frac{\ln(\ln t)}{\ln t}\right]^{\mu(d-1)}\ \frac{[\,\ln t\,]^{q}}{t^{p}}\\
&\geq c\,\frac{[\,\ln t\,]^{q}}{t^{p}},
\end{aligned}$$

where $q=p\alpha\mu+(d-1)\mu$.

We now obtain the upper bound. A simple computation shows that there exists $c=c(d,\alpha)$ such that

$$\begin{aligned}
\Lambda(r) &= \int_{r}^{\infty}\left[\frac{[\,\ln(\rho+1)\,]^{\mu}}{\rho^{\beta+1}}\right]^{d-1}\frac{1}{(\rho-r)^{\alpha/2}\ \rho^{1+\alpha/2}}d\rho.\\
&\leq c\,\frac{[\,\ln(r+1)\,]^{\mu(d-1)}}{r^{p\beta\alpha}}.
\end{aligned}$$

Consider r_1 and r_2 given by

$$r_1+1=\left[\frac{t\lambda_1}{2\,(\,\ln t^{p}-\ln(\,\ln t)^{q}\,)}\right]^{\frac{1}{\beta\alpha}}\ \frac{1}{\left[\,\frac{1}{\beta\alpha}\ \ln t-\frac{1}{\beta\alpha}\ \ln(\,\ln t^{p}-\ln[\,\ln t\,]^{q}\,)\,\right]^{\frac{\mu}{\beta}}},$$

$$r_2 + 1 = \left[\frac{\lambda_1 t}{2}\right]^{\frac{1}{\alpha\beta}} \frac{1}{[\ln t]^{\frac{\mu}{\beta}} \, [\ln(\ln t)^p]^{\frac{1}{\alpha\beta}}}.$$

Then there exists $c = c(d, \alpha, \beta, \mu)$ such that

$$\exp\left[-\frac{\lambda_1 \, t}{[\phi(r_1)]^\alpha}\right] \le c \, \frac{[\ln t]^q}{t^p},$$

$$\Lambda(r_1) \le c \, \frac{1}{t^p} \, [\ln t]^{q+p},$$

$$\exp\left[-\frac{\lambda_1 \, t}{[\phi(r_2)]^\alpha}\right] \le c \, \frac{1}{[\ln t]^p},$$

and

$$\Lambda(r_2) \le c \, \frac{[\ln t]^q}{t^p} \, [\ln(\ln t)^p]^p.$$

Proposition 4.3 immediately implies that

$$P^z(\tau_D > t) \le c \, \frac{[\ln t]^q}{t^p} \, [\ln(\ln t)^p]^p,$$

which is the desired result.

The case $\mu = 0$ was first study by R. Bañuelos and K. Bogdan in (1), where the authors obtain (1.12).

As mention in (23) the asymptotic behavior of $P^z(\tau_D > t)$ is not known for $\alpha = 2$, $\beta = 1$ and $\mu < -1$. However using the well known behavior of the exit times from cones and a suitable approximation of the domain D we can prove that there exists $M > 0$, $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} \frac{1}{c_1} \, \exp\left[-\frac{1}{c_2}(\ln t)^{1-\mu}\right] &\le P^z(\tau_D > t) \\ &\le c_1 \, \exp\left[-c_2(\ln t)^{1-\mu}\right], \end{aligned}$$

for all $t \ge M$.

Case II: We now study the case $\beta = 0$ and $\mu \in \mathbb{R}$. That is we now consider

$$\phi(x) = (\ln[x+1])^\mu,$$

where $\mu \in \mathbb{R}$. In this case we will have subexponential behavior of (1.6).

Let

$$r+1=\exp\left[\eta_1\, t^{\frac{1}{(1+\mu\alpha)}}\right],\ \text{where}\ \eta_1=\left(\frac{\lambda_1}{d-1+\alpha}\right)^{\frac{1}{d-1+\alpha}}.$$

Consider $z=(x,0,\ldots,0)$ with $0<x\le r/2$, and $\eta=(d-1+\alpha)\eta_1$, then

$$\exp\left[-\frac{\lambda_1\ t}{[\ln(r+1)\,]^{\alpha\mu}}\right]=\exp\left[-\eta\, t^{\frac{1}{\mu\alpha+1}}\right],$$

and

$$\begin{aligned}\int_{2r}^{\infty}\frac{[\,\phi(\rho)\,]^{d-1}}{\rho^{d+\alpha}}\,d\rho &= \int_{2r}^{\infty}\frac{[\,\ln(\rho+1)\,]^{\mu(d-1)}}{\rho^{d+\alpha}}\,d\rho\\ &\ge c\,\frac{[\,\ln(r+1)\,]^{\mu(d-1)}}{r^{d-1+\alpha}},\end{aligned}$$

for some $c>0$. Proposition 4.2 implies that there exists $M>0$ and $c>0$ such that

$$c\,t^{\frac{\mu(d-1)}{1+\mu\alpha}}\ \exp\left[-2\eta\, t^{\frac{1}{\mu\alpha+1}}\right]\ \le\ P^z\,(\,\tau_D>t\,),$$

for all $t>M$.

An argument similar to the one used to prove Proposition 4.3 shows that there exists $M>0$ and $c>0$ such that

$$P^z\,(\,\tau_D>t\,)\le c\left[\Lambda(r)+\exp\left(-\frac{\lambda_1 t}{[\,\phi(r)\,]^{\alpha}}\right)\right],$$

and

$$\Lambda(r)\ \le\ c\,\frac{[\,\ln(r+1)\,]^{\mu(d-1)}}{r^{d-1+\alpha}}\ \le\ c\,t^{\frac{\mu(d-1)}{1+\mu\alpha}}\ \exp\left[-\eta\, t^{\frac{1}{\mu\alpha+1}}\right],$$

for all $r>M$ and all $t>M$. Then

$$P^z\,(\,\tau_D>t\,)\ \le\ c\,t^{\frac{\mu(d-1)}{1+\mu\alpha}}\ \exp\left[-\eta\, t^{\frac{1}{\mu\alpha+1}}\right],$$

which is the upper bound of (1.13).

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