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A Compensator Characterization of Point Processes on Topological Lattices*

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Abstract

We resolve the longstanding question of how to define the compensator of a point process on a general partially ordered set in such a way that the compensator exists, is unique, and characterizes the law of the process. We define a family of one-parameter compensators and prove that this family is unique in some sense and characterizes the finite dimensional distributions of a totally ordered point process. This result can then be applied to a general point process since we prove that such a process can be embedded into a totally ordered point process on a larger space. We present some examples, including the partial sum

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multiparameter process, single line point processes, multiparameter renewal processes, and obtain a new characterization of the two-parameter Poisson process.

Key words: point process, compensator, partial order, single jump process, partial sum process, adapted random set, renewal process, Poisson process, multiparameter martingale.

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1 Introduction

When working on the dynamical properties of a point process defined on a partially ordered space, one of the main tools is the *compensator*: the unique predictable increasing process which compensates the point process, i.e. the difference between the point process and the compensator is a martingale. In the classical case, a remarkable result of J. Jacod in (9) established that the law (i.e. the finite dimensional distributions) of a simple point process on the real line is characterized by the knowledge of its compensator. The power of martingale methods in the statistical analysis of point processes on $\mathbf{R}_+$ is well-established, and some of these techniques have recently been extended to point processes on more general spaces in (7).

In this paper, we provide a positive answer to the question of whether such a characterization exists for general point processes on partially ordered sets. It is clear that for such processes, this problem becomes much more complicated and has remained open except in special cases (cf. (5), (6)). First, compensators can be defined in several ways for any simple point process, depending on how one defines the "history" or "past" at a point. As well, only under stringent conditions can one prove existence and uniqueness of the compensator. Moreover, even if we have a compensator that is unique in some sense, it will not generally characterize the law of the point process. To illustrate this, we recall an example studied in detail in (6).

Suppose our parameter set T is the positive quadrant of the plane $\mathbf{R}_+^2$, let $\tau = (\tau_1, \tau_2)$ be a random point in this space and for any point $t = (t_1, t_2)$ in T denote the following sets: $A_t = \{(s_1, s_2) \in T : s_1 \le t_1 \text{ and } s_2 \le t_2\}$, $A_t^i = \{(s_1, s_2) \in T : s_i \le t_i\}$ for $i = 1, 2$ and $D_t = A_t^1 \cup A_t^2$. Let G denote both the distribution function of τ and the associated measure; i.e. for any Borel set B in T, $G(B) = P(\tau \in B)$, $G(t) := G(A_t)$. The simplest point process on $\mathbf{R}_+^2$ is the *single jump process*; that is a process of the form $N_t = I(t \ge \tau) = I(\tau \in A_t)$. This process possesses several compensators depending on the information available at t about the random point τ (i.e. the "history" at t). Three of them are the following:

$$\Lambda_t^w = \int_{A_t} I_{\{\tau \notin A_u\}} \frac{G(du)}{G(A_u^c)}$$

$$\Lambda_t^i = \int_{A_t} I_{\{\tau \notin A_u^i\}} \frac{G(du)}{G((A_u^i)^c)}$$

$$\Lambda_t^* = \int_{A_t} I_{\{\tau \notin D_u\}} \frac{G(du)}{G(D_u^c)}.$$

In other words, each of the processes $N - \Lambda^w$, $N - \Lambda^i$, $N - \Lambda^*$ possesses a type of martingale property with respect to the relevant history, and in each case, $\Lambda^\cdot$ is predictable in some sense. The history at t consists of the values of N_s for all s in A_t for Λ^w, to the left of t for Λ^1, below t for Λ^2, and in D_t for Λ^*. Since T is in this example a product space, we observe that N_t can be viewed as a marked point process on $\mathbf{R}_+$ (τ_1 is interpreted as the time of the jump and τ_2 as the mark), so according to Jacod's (9) characterization result, the law of N is determined by Λ^1 (or Λ^2, by symmetry). In other words, we can reconstruct the distribution G. However, as is well known from multivariate survival analysis, neither the ratio $G(dt)/G(A_t^c)$ nor the ratio $G(dt)/G(D_t^c)$ (the *hazard function*) determines G, so neither Λ^w nor Λ^* can characterize the law of the point process N (cf. (2), pg. 690).

In this example, the compensator Λ^1 does the job. However, it works only because T has a product structure (as required for Jacod's marked point process approach). Moreover, at $t = (t_1, t_2)$ this compensator requires complete information about N to the left of t_1 (i.e. in A_t^1) or by symmetry, below t_2 (A_t^2). In survival analysis, this means that one needs the entire sample in one direction to estimate the value of the distribution function G at t. However, in practice it may happen that the only information available at time $t = (t_1, t_2) \in T$ consists of the locations of jumps in $A_t = \{s \in T : s \leq t\}$.

Another example is the Poisson process $M = \{M_t, t \in \mathbf{R}_+^2\}$. As is explained in (5), since the increments are independent each of the compensators of M is a deterministic function and in fact at each point t, it is the measure of the rectangle A_t (up to a constant). In this special case, the compensator Λ^* has been shown to characterize the Poisson process (cf. (6), Theorem 5.3.1) whereas Λ^w does not. The compensator Λ^* works well if it is deterministic (for example, it also characterizes the set-indexed Brownian motion among the class of square integrable continuous multiparameter strong martingales ((6), Theorem 5.2.1)), but as observed above, it fails for other processes such as the single jump process.

Here, we resolve the open question by defining the compensator of a general point process N on an arbitrary topological lattice T in such a way that it always exists, is unique, and completely characterizes the law of N. T need not have the product structure required for Jacod's marked point process approach, and in the case of the single jump process or more generally, point processes whose jumps are strictly increasing, the only information required at time $t \in T$ is the location of jumps in $A_t = \{s \in T : s \leq t\}$. For more general point processes, the information required at t is the location of any jump point τ for which $N(A_\tau) \leq N(A_t)$. In fact, the compensator will be defined to be a class of (one-dimensional) compensators of point processes on $\mathbf{R}_+$ generated by projections of N along *flows* (increasing maps from $\mathbf{R}_+$ to a class of subsets of T). We show that the projection of a point process along a flow is a one-parameter point process, and therefore the compensator of the projection is well defined. As will be explained later in full detail, this compensator will be called "*$\mathcal{U}-$ flow compensator*" . This approach was first introduced by Plante in (11) in the special case $T = \mathbf{R}_+^2$. Notice that our new flow compensator is not identifiable with any of the compensators Λ^w, Λ^1 or Λ^*.

In addition to the theoretical interest of this result, there are important areas of application such as multiparameter renewal theory and survival analysis (see (7) and (8)). In addition, we give a new characterization of the Poisson process on the plane.

The paper is organized as follows. In Section 2, we develop all the prerequisites needed for point processes on a partially ordered parameter set. We define strictly simple point processes in a very general framework using the concept of flow and we discuss how this allows us to exploit results from classical one-parameter theory. In Section 3, we study point processes whose jump points are totally ordered. The *flow compensator* is defined for flows on the original space T and we prove that the flow compensator characterizes the law of the (totally ordered) point process. As examples, we re-analyse the single jump process and then consider a partial sum process. Section 4 is devoted to general point processes. We introduce the notion of embedding a point process into a larger space in such a way that the embedded point process is totally ordered. We show that the law of the original process determines and is determined by the law of the embedded process. This idea, introduced in (10) in the special case $T = \mathbf{R}_+^2$, allows us to apply the results of Section 3 to the totally ordered embedded process: we define the *$\mathcal{U}-$ flow compensator* of the general point process via an identification with the flow compensator of the embedded process. Therefore, we get a complete compensator characterization for any strictly simple point process. In Section 5, we present more examples. In particular, we investigate the *single line process* and apply it to two special cases: the renewal point process and the Poisson point process on the plane. We obtain a new characterization of the Poisson process and compare it to a similar result given by Aletti and Capasso in (1).

2 Prerequisites

The point processes are defined on a complete separable metric space (T, d). We will require that T be partially ordered.

Assumption 2.1. *T is endowed with a partial order "$\leq$" for which T is a complete distributive lattice (i.e. every subset of T has a sup and an inf, and for $u, s, t \in T$,*

$$u \vee (s \wedge t) = (u \vee s) \wedge (u \vee t)) \text{ and } u \wedge (s \vee t) = (u \wedge s) \vee (u \wedge t).)$$

In particular, T contains a minimum element denoted by $\mathbf{0}$ and a maximum element denoted by $\mathbf{1}$. This setup includes compact rectangles in $\mathbf{R}_+^p$ with the usual partial order and we shall use $[0,1]^2$ as the fundamental example to illustrate our approach.

For $t \in T$, the partial order generates the following sets:

$$\begin{aligned} A_t &:= \downarrow t = \{s \in T : s \leq t\} \\ E_t &:= \uparrow t = \{s \in T : s \geq t\} \end{aligned}$$

We observe that $A_{s \wedge t} = A_s \wedge A_t$ and $E_{s \vee t} = E_s \wedge E_t$, $\forall s, t, \in T$.

Assumption 2.2. *For every $t \in T$, both A_t and E_t are d-closed.*

Now we introduce the "dyadics" in T. The notation "$s < t$" should be interpreted as "$s \leq t$ and $s \neq t$".

Assumption 2.3. *The partial order on T is sufficiently rich that there exists an increasing sequence of finite sublattices (T_n) of T each containing $\mathbf{0}$ and $\mathbf{1}$, such that for any $t \in T, t \neq \mathbf{0}, \mathbf{1}$, each open neighbourhood of t contains points $t', t'' \in T_n$ distinct from t such that $t' < t < t''$. Each open neighbourhood of $\mathbf{0}$ (respectively, $\mathbf{1}$) contains a point $t' \in T_n$ distinct from $\mathbf{0}$ (respectively, $\mathbf{1}$). The elements of T_n can be thought of as the dyadics of order n.*

In particular, defining

$$t_n^+ := \wedge_{t' \in T_n; t \leq t'} t' \text{ and } t_n^- := \vee_{t' \in T_n; t' \leq t} t',$$

it follows that $t = \wedge_n t_n^+ = \lim_n t_n^+ = \lim_n t_n^- = \vee_n t_n^-$, and any point t can therefore be approximated from above (respectively, below) by elements in T_n either as an infimum (respectively, supremum) or as a limit in the metric d.

Assumptions 2.1-2.3 suffice for the results in Section 3; for Section 4, a further mild assumption will be required.

Assumption 2.4.

1. *For $u, s, t \in T, d(s \wedge u, t \wedge u) \leq d(s,t)$ and $d(s \vee u, t \vee u) \leq d(s,t)$.*

2. *The sublattices T_n of Assumption 2.3 can be chosen so that*

$$\lim_n \sup_{t \in T} d(t, t_n^+) = 0 \text{ and } \lim_n \sup_{t \in T} d(t, t_n^-) = 0.$$

Comment 2.5. If T is bounded, then it is reasonable to suppose that T has a maximum element and that T_n is finite. This is not a restriction if there exists an increasing sequence of bounded subsets $B_1 \subseteq B_2 \subseteq ... \subseteq T$, such that $T = \cup_{n=1}^{\infty} B_n$ and each subset B_n satisfies the preceding assumptions. The law of any point process on T is expressed in terms of its finite dimensional distributions on bounded Borel subsets of T (cf. (3), pg.166), and so it suffices to characterize the law of the point process on each B_n. This allows us to extend our results to point processes on $\mathbf{R}_+^d$, for example.

We now define the *left neighbourhoods* $\mathcal{C}_n$ associated with T_n:

$$C \in \mathcal{C}_n \text{ if and only if } C = C_t^n := A_t \setminus \cup_{t' \in T_n, t' \not\geq t} A_{t'} \text{ for some } t \in T_n.$$

The sequence $(\mathcal{C}_n)$ is a nested sequence of finite partitions of T that ultimately separates the points of T, and so forms a *dissecting system*. Consequently, any closed set K can be expressed as

$$K = \cap_n \left(\cup_{t \in K} C_t^n \right).$$

Each of the unions in the expression above is finite since $\mathcal{C}_n$ is finite. As a result, the Borel sets of T are generated by the sets $\{A_t : t \in \cup_n T_n\}$.

We say that two sets $A, B \subset T$ are *incomparable* if a and b are incomparable for every $a \in A$ and $b \in B$.

Proposition 2.6. *If $t, t' \in T_n$ are incomparable, then C_t^n and $C_{t'}^n$ are incomparable sets.*

Proof. This is an easy consequence of the observation that by definition, $C_t^n \cap A_{t'} = C_{t'}^n \cap A_t = \emptyset$ when t, t' are incomparable. □

We now turn to point processes on T. Let $\mathcal{N}$ denote the set of finite integer-valued measures μ on T with $\mu(\{x\}) = 0$ or 1 for all $x \in T$ and $\mu(\{\mathbf{0}\}) = 0$. Let $\mathcal{F}(\mathcal{N})$ be the smallest σ-algebra on $\mathcal{N}$ with respect to which the mappings $\mu \to \mu(A_t)$ are measurable, for every $t \in T$. For $C \in \mathcal{C}_n$, $\mu(C)$ can be calculated as a finite linear combination of random variables $\mu(A_t), t \in T_n$. Since $\cup_n \mathcal{C}_n$ is a semiring which generates the Borel sets in T, by Proposition A.2.5.IV of (3), $\mathcal{F}(\mathcal{N})$ coincides with the σ-algebra generated by the topology of weak convergence on $\mathcal{N}$.

Definition 2.7. *A point process N is a measurable mapping from a probability space $(\Omega, \mathcal{F}, P)$ into $(\mathcal{N}, \mathcal{F}(\mathcal{N}))$.*

Assumption 2.3 ensures that the law of a point process is uniquely determined by its finite dimensional distributions of the form $P(N(A_{t_1}) = k_1, ..., N(A_{t_j}) = k_j)$ for $t_1, ..., t_j \in T_n$, some n. Hence, equivalently the law is determined by the finite dimensional distributions of N on the left-neighbourhoods $\mathcal{C}_n$, for every n.

Notice that our definition automatically ensures that there is at most one jump point at any $t \in T$; this is frequently called a simple point process in the literature. We will require a stronger type of simplicity; for this we need to introduce *flows*.

Definition 2.8. *A flow f on T is a continuous strictly increasing map from $[0,1]$ to T (i.e., if $s < t$, then $f(s) < f(t)$).*

Given any finite totally ordered subset $t_0 < t_1 < t_2 < ... < t_n$ of T, Assumptions 2.3 and 2.4 ensure that there exists a flow f *connecting* $t_0, ..., t_n$: that is, there exist points $0 = u_0 < u_1 < ... < u_n = 1$ such that $f(u_i) = t_i, i = 0, ..., n$. This can be proven in a manner similar to that of Theorem 5.1.6 in (6).

Given a flow f, we can define the *projection* N^f of a point process N along the flow: N^f is an integer-valued process on $[0,1]$ defined by

$$N^f(u) := N(A_{f(u)}), u \in [0,1].$$

Although N^f can be regarded as an (integer-valued) point measure on $[0,1]$, it is not necessarily the case that N^f is a simple point process.

Definition 2.9. *A point process N on T is strictly simple if for any finite totally ordered subset $0 = t_0 < t_1 < t_2 < ... < t_m$ of $\cup_n T_n$, there exists a flow f connecting $t_0, ..., t_m$ such that N^f is a simple point process on $[0,1]$. Denote by F_N any countable class of flows containing at least one flow f connecting each finite totally ordered subset of $\cup_n T_n$ and such that N^f is simple.*

As an example, consider $T = [0,1]^2$ endowed with the usual partial order. A point process N on T is strictly simple if and only if there is at most one jump point of N on any horizontal or vertical line. F_N includes all continuous increasing maps $f : [0,1] \to [0,1]$ that are increasing in only one component at a time: i.e. the trajectory of f is continuous, consisting of horizontal and vertical line segments. Formally, if $u < 1$ and $f(u) = (f_1(u), f_2(u))$ then $\exists \epsilon > 0$ such that either $f_1(u) = f_1(u+\delta)$ $\forall \delta \in (0,\epsilon)$ or $f_2(u) = f_2(u+\delta)$ $\forall \delta \in (0,\epsilon)$. We note that not all flows yield simple projections if N has more than one jump point. For example, consider the point process N with jumps at $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{4})$. N is strictly simple since N^f is simple for every $f \in F_N$. However, consider the flow f defined by $f(u) = (u,u), 0 \le u \le 1$. In this case N^f is not simple, since $N^f(\{\frac{1}{2}\}) = 2$.

We now consider a simple point process N defined on the unit interval. Let $\mathcal{F}^N = (\mathcal{F}^N(u) : u \in [0,1]) = (\sigma\{N(v) : 0 \le v \le u\}; u \in [0,1])$ be the minimal filtration generated by N and denote by Λ_N the compensator of N with respect to $\mathcal{F}^N$ (i.e. the unique predictable increasing process in the Doob-Meyer decomposition of N). According to the now classic result of Jacod (9), the law of N is uniquely characterized by Λ_N.

As discussed in Section 1, there is no obvious definition for the compensator of a point process on a general poset since there is no unique way of defining the Doob-Meyer decomposition, and none of the known compensators characterize the law (for a more detailed discussion see (6)). We will instead see that flows enable us to exploit Jacod's one dimensional characterization. The first step is to consider point processes whose jumps are totally ordered; we do this in the next section (§3). We shall see in §4 that in fact this is enough to permit us to handle general point processes.

3 Totally ordered point processes

In this section, we analyze *totally ordered* point processes N on T: by this we mean that with probability 1, N takes its values in $\mathcal{N}^{to}$, which consists of all $\mu \in \mathcal{N}$ whose set of jump points

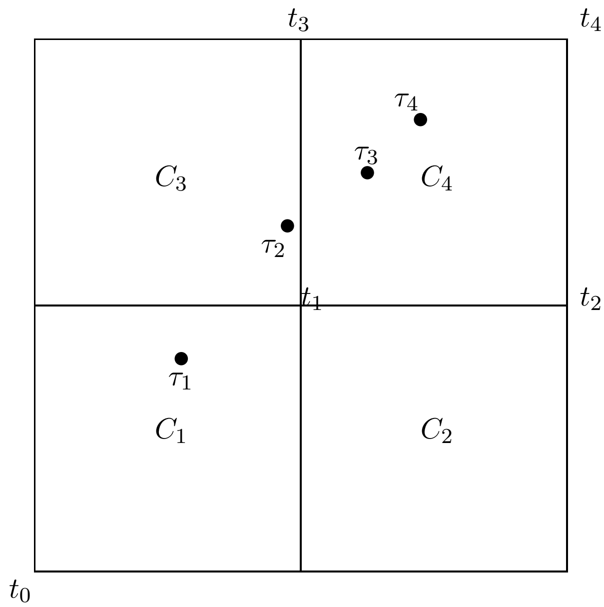

Figure 1: A totally ordered point process

$\{x : \mu(\{x\}) = 1\}$ is totally ordered (see Figure 1, for example). Such processes have a special structure that allow us to identify the law of N given the laws of $N^f, f \in F_N$; the following theorem is the key result. Here we give only a sketch of the proof; the complete proof may be found in Appendix A.

Theorem 3.1. *Let (T, d) be a complete separable metric space satisfying Assumptions 2.1, 2.2 and 2.3. Let N be a totally ordered point process on T and let F be any class of flows containing at least one flow connecting each finite totally ordered subset of $\cup_n T_n$. The law of N determines and is determined by the laws of the family of the projected point processes $N^f, f \in F$.*

Proof. (Sketch) It is trivial that the law of N^f is determined by the law of N. For the converse, we assume that the laws of N^f are known for every $f \in F$ and it is enough to show that we can reconstruct the finite dimensional distributions of N on the sets in $\mathcal{C}_n$; i.e. if $T_n = (t_0 = \mathbf{0}, t_1, ..., t_{j_n} = \mathbf{1})$ and $C_i = C^n_{t_i}$, we must be able to find

$$P(N(C_1) = k_1, ..., N(C_{j_n}) = k_{j_n})$$

from the laws of $N^f, f \in F$. The crucial point behind the proof is that if t_i and t_j are incomparable, then so are C_i and C_j. Consequently, when N is totally ordered, at most one of any collection of incomparable sets can contain any jump points. Below we give an illustration of how this simple fact is used.

Consider the unit square $T = [0, 1]^2$ and the sublattice consisting of the points $t_0 = (0, 0), t_1 = (\frac{1}{2}, \frac{1}{2}), t_2 = (1, \frac{1}{2}), t_3 = (\frac{1}{2}, 1), t_4 = (1, 1)$. The corresponding left neighbourhoods $C_i = C_{t_i}$ are illustrated in Figure 1, along with a realization of an increasing sequence $(\tau_i; i = 1, ..., 4)$ of jump points of N. We see that C_2 and C_3 are incomparable, so if $N(C_2) > 0$, then necessarily $N(C_3) = 0$ (likewise, if $N(C_3) > 0$, then necessarily $N(C_2) = 0$). We now need to construct

$$P(N(C_1) = k_1, N(C_2) = k_2, N(C_3) = k_3, N(C_4) = k_4)$$

for $k_i \geq 0, i = 1,2,3,4$. The law of N^f for any flow f in F connecting t_0, t_1, t_4 with $f(0) = 0, f(u_1) = t_1, f(1) = t_4$ will give

$$\begin{aligned} & P(N(C_1) = k_1, N(C_2 \cup C_3 \cup C_4) = k_2 + k_3 + k_4) \\ = \; & P(N^f(u_1) = k_1, N^f(1) - N^f(u_1) = k_2 + k_3 + k_4). \end{aligned} \tag{1}$$

The law of N^f for any flow f in F connecting t_0, t_1, t_2, t_4 with $f(0) = 0, f(u_1) = t_1, f(u_2) = t_2, f(1) = t_4$ will give

$$\begin{aligned} & P(N(C_1) = k_1, N(C_2) = k_2, N(C_3 \cup C_4) = k_3 + k_4) \\ = \; & P(N^f(u_1) = k_1, N^f(u_2) - N^f(u_1) = k_2, N^f(1) - N^f(u_2) = k_3 + k_4). \end{aligned} \tag{2}$$

If $k_2 > 0$, then necessarily $k_3 = 0$ and (2) becomes

$$P(N(C_1) = k_1, N(C_2) = k_2, N(C_3) = 0, N(C_4) = k_4). \tag{3}$$

Similarly, by using a flow connecting t_0, t_1, t_3, t_4 we can obtain probabilities of the form

$$P(N(C_1) = k_1, N(C_3) = k_3, N(C_2 \cup C_4) = k_3 + k_4) \tag{4}$$

which becomes

$$P(N(C_1) = k_1, N(C_2) = 0, N(C_3) = k_3, N(C_4) = k_4) \tag{5}$$

if $k_3 > 0$. Therefore, we need only find a formula for

$$P(N(C_1) = k_1, N(C_2) = 0, N(C_3) = 0, N(C_4) = k_4)$$

using (1), (3) and (5). But this is straightforward, since

$$\begin{aligned} & P(N(C_1) = k_1, N(C_2) = 0, N(C_3) = 0, N(C_4) = k_4) \\ = \; & P(N(C_1) = k_1, N(C_2 \cup C_3 \cup C_4) = k_4) \\ & -P(N(C_1) = k_1, N(C_2) > 0, N(C_3) = 0, N(C_4) = k_4 - N(C_2)) \qquad (6) \\ & -P(N(C_1) = k_1, N(C_2) = 0, N(C_3) > 0, N(C_4) = k_4 - N(C_3)) \qquad (7) \end{aligned}$$

and the probabilities in (6) and (7) are easily reconstructed from (3) and (5), respectively. For the complete proof, see Appendix A. □

We observe that in Theorem 3.1, there is no requirement that N be strictly simple. However, in order to have a compensator characterization of the law of the projection N^f along a flow f, it is necessary that N^f be simple. Thus, we now restrict our attention to strictly simple point processes and a class of flows F_N as defined in Definition 2.9. We recall the following notation: for $f \in F_N$, Λ_{N^f} is the (predictable) compensator of N^f with respect to the minimal filtration $\mathcal{F}^{N^f} = (\mathcal{F}^{N^f}(u) : 0 \leq u \leq 1)$, where $\mathcal{F}^{N^f}(u) := \sigma(N^f(v) : 0 \leq v \leq u)$. This leads us to the following definition of the flow compensator of N and the characterization theorem for totally ordered point processes.

Definition 3.2. *Let N be a strictly simple point process on T and F_N a class of flows as defined in Definition 2.9. The flow compensator Λ of N is the family of processes*

$$\Lambda := (\Lambda_{N^f}, f \in F_N).$$

In particular, the value of the compensator at $t \in T$ may be regarded as the family of random variables $\Lambda(t) := (\Lambda_{N^f}(u) : f \in F_N, f(u) = t)$.

Theorem 3.3. *Let (T, d) be a complete separable metric space satisfying Assumptions 2.1, 2.2 and 2.3. Let N be a strictly simple totally ordered point process on T amd F_N a class of flows as defined in Definition 2.9. The flow compensator of N exists and is unique (i.e. if Λ and Λ' are flow compensators of N, then for P-almost all $\omega \in \Omega$, the paths of Λ_{N^f} and Λ'_{N^f} coincide for every $f \in F_N$). The law of N determines and is determined by its flow compensator Λ.*

Proof. Existence and uniqueness of the flow compensator follow from existence and uniqueness of the predictable increasing process Λ_{N^f} in the Doob-Meyer decomposition of N^f, $f \in F_N$, and the fact that F_N is countable. Since N^f is simple, Jacod's result (cf. (9)) ensures that the law of N^f determines and is determined by Λ_{N^f}. The general result follows immediately from Theorem 3.1. □

Comment 3.4. We may define the minimal T-indexed filtration generated by N as follows:

$$\mathcal{F}^N = \{\mathcal{F}^N(t) : t \in T\} = (\sigma\{N(A_s) : s \in A_t\} : t \in T).$$

$\mathcal{F}^N(t)$ may be regarded as the *strict* past of N at $t \in T$. For any $f \in F_N$ and $u \in [0,1]$, it is trivial that $\mathcal{F}^{N^f}(u) \subseteq \mathcal{F}^N(f(u))$, and so the random variables defining $\Lambda(t)$ are $\mathcal{F}^N(t)$-measurable. In this sense, Λ is $\mathcal{F}^N$-adapted.

The preceding comment may appear to contradict the observation in Section 1 that even in the single jump case, the compensator based on the strict past $\mathcal{F}^N$ will not determine the law of N (i.e. the distribution function G of the jump point τ). For $T = [0,1]^2$, the compensator $\Lambda^w(dt)$ is defined in terms of the conditional hazard of the jump point τ, given that $\tau \not\leq t$. Even more surprising is the fact that $\Lambda^*(dt)$ does not determine the survival function $G(D_t^c)$, where now the hazard is conditioned on the event $\{t < \tau\}$ which lies in the *strong* past of N:

$$\mathcal{F}^{N^*} := \{\mathcal{F}^{N^*}(t) : t \in T\} = (\sigma\{N(A_s) : s \in D_t\} : t \in T).$$

Since neither hazard determines $G(A_t)$ or $G(D_t^c)$, finding good hazard-based estimators of G has remained a problem in multivariate survival analysis. The apparent contradiction mentioned above is resolved by the fact that the flow compensator is not based on a two-dimensional hazard, as seen in the following example. This observation may have useful applications in survival analysis.

Example 3.5. The single jump process:
If τ is a T-valued random variable with continuous distribution G (where $G(t) := G(A_t) = P(\tau \leq t)$), and $N(A_t) = I(t \geq \tau)$, then F_N can include all flows $f : [0,1] \to T$. For $f \in F_N$ and $u \in [0,1]$, let $G^f(u) := G(f(u))$. Then N^f is a single jump point process on $[0,1]$ whose jump

time $\tau^f = \inf\{v : N^f(v) = N(A_{f(v)}) = 1\}$ has distribution G^f. The compensator of N^f is the integrated hazard function

$$\Lambda_{N^f}(u) = \int_0^{u\wedge\tau^f} \frac{dG^f(v)}{1 - G^f(v)}.$$

As is well known from one-dimensional theory,

$$\Lambda_{N^f}(u) = -\ln(1 - G^f(u \wedge \tau^f)) = -\ln(1 - G(f(u \wedge \tau^f))),$$

and so $\Lambda_{N^f}(u)$ depends only on the value $f(u)$ if $f(u) \leq \tau$ (and not on the path of f between $\mathbf{0}$ and $f(u)$), and clearly the family $\Lambda = (\Lambda_{N^f}, f \in F_N)$ determines G. However, $d\Lambda_{N^f}(u)$ does depend on the parametrization f, and so cannot be interpreted as a hazard on T.

A more general example is the partial sum process on $\mathbf{R}_+^2$:

Example 3.6. The partial sum process:
Let $T = [0, n]^2$ and let $Y_1, Y_2, ..., Y_n$ be i.i.d. $(0, 1]^2$-valued random variables with continuous distribution G and density g. Let N be the strictly simple totally ordered point process with jump points $\tau_1 < \tau_2 < ... < \tau_n$, where $\tau_i = \sum_{j=1}^i Y_j$. As before, F_N can include all flows $f : [0, 1] \to [0, 1]^2$, where $f(u) = (f_1(u), f_2(u))$. Fix a flow f and denote the jump times of N^f by $\tau_1^f < ... < \tau_n^f$. For $1 \leq i < n$, given $\tau_1^f, ..., \tau_i^f$, the conditional distribution of τ_{i+1}^f depends only on τ_i^f, and must take into account where τ_i is on the boundary of $A_{f(\tau_i^f)} = [0, f_1(\tau_i^f)]\times[0, f_2(\tau_i^f)]$. Letting G^{*i} denote the i-fold convolution of G and g^{*i} the corresponding density, the conditional distribution of τ_{i+1}^f given τ_i^f is:

$$\begin{aligned} G_{i+1}^f(u \mid \tau_i^f) &= \int_0^{f_2(\tau_i^f)} G(f_1(u) - f_1(\tau_i^f), f_2(u) - y)\frac{g^{*i}(f_1(\tau_i^f), y)}{g^{*i}(\partial A_{f(\tau_i^f)})}dy \\ &+ \int_0^{f_1(\tau_i^f)} G(f_1(u) - x, f_2(u) - f_2(\tau_i^f))\frac{g^{*i}(x, f_2(\tau_i^f))}{g^{*i}(\partial A_{f(\tau_i^f)})}dx \end{aligned}$$

where $g^{*i}(\partial(A_{(t_1,t_2)}) := \int_0^{t_2} g^{*i}(t_1, y)dy + \int_0^{t_1} g^{*i}(x, t_2)dx$. The usual (one-dimensional) integrated hazard H_i^f can be calculated from G_i^f (see, for example, (3)) to yield

$$\Lambda_{N^f}(u) = \sum_{i=1}^n \Lambda_{N^f}^{(i)}(u)$$

where $\Lambda_{N^f}^{(1)}(u)$ is as defined in Example 3.5 and for $i > 1$

$$\Lambda_{N^f}^{(i)}(u) = \begin{cases} 0 & u < \tau_{i-1}^f \\ H_i^f(u - \tau_{i-1}^f) & \tau_{i-1}^f \leq u < \tau_i^f \\ H_i^f(\tau_i^f - \tau_{i-1}^f) & \tau_i^f \leq u \end{cases}.$$

Comment 3.7. A type of flow compensator can be defined for any point process N (cf. Definition 3.2) since if f is any flow, N^f will be an increasing process on $[0, 1]$ with a unique Doob-Meyer decomposition with respect to its minimal filtration. However, the proof of the characterization in Theorem 3.3 makes use of both strict simplicity and totally ordered jump points. We conjecture that at least in some special cases the assumption of strict simplicity may be weakened through a more detailed analysis of the projections N^f, but this will likely require stronger assumptions on the structure of T. It will be seen that the assumption of totally ordered jump points in Theorem 3.3 is not always necessary. On the contrary, if the jump points of N are all incomparable, under certain conditions the flow compensator characterizes the law of N in this situation as well (cf. Corollary 5.4). However, for arbitrary strictly simple point processes, a more general compensator will be needed to characterize the law of the process; this is the topic of the next section.

4 General point processes

We now turn to the structure of general point processes, and in particular to an embedding of a point process on T into a totally ordered point process on a larger space, $\mathcal{U}$. A closely related approach is described in (10) and developed in detail in (8) for Euclidean space.

To motivate what follows, we observe that a point process N on $[0,1]$ is characterized by its successive jump times $\tau_i = \inf\{s : N_s \geq i\}, i = 1, 2, \ldots$. In particular, we may identify τ_i with the random set $\xi_i := [\tau_i, 1] = E_{\tau_i}$, and we note that $\{t \in \xi_i\} \in \mathcal{F}^N(t), \forall t \in \mathbf{R}_+$. Also, $N(A_t) = N([0,t]) \geq i \Leftrightarrow t \geq \tau_i \Leftrightarrow E_t \subseteq \xi_i$.

As defined, τ_i is a stopping time with respect to $\mathcal{F}^N$. However, the jump points of point processes on partially ordered sets ($\mathbf{R}_+^p$ for example) are not in general stopping times, and it has long been recognized (cf. (6)) that the natural analogue of the stopping time is an *adapted random set*, a concept that will be made precise shortly.

We generalize the approach above to an arbitrary point process on a lattice T. If $\Delta N = \{x \in T : N(\{x\}) = 1\}$ denotes the (unordered) set of jump points of N, we have that $N(A_t) \geq k \Leftrightarrow t \in E_{\tau_1 \vee \ldots \vee \tau_k}$ for some distinct $\tau_1, \ldots, \tau_k \in \Delta N$ and we define

$$\xi_k := \{t \in T : N(A_t) \geq k\} = \cup_{\tau_1, \ldots \tau_k \in \Delta N} E_{\tau_1 \vee \ldots \vee \tau_k}. \tag{8}$$

for $1 \leq k \leq |\Delta N|$ (the cardinality of ΔN), and consider ξ_k undefined for $k > |\Delta N|$. The union in (8) is taken over all collections of k distinct points in ΔN. Clearly, the random sets are *decreasing*: $\xi_1 \supseteq \xi_2 \supseteq \ldots$ and the following lemma is obvious:

Lemma 4.1. *For any $t \in T$, $N(A_t) \geq k \Leftrightarrow \xi_k \supseteq E_t$. Hence, the sets ξ_k determine and are determined by N.*

Comment 4.2. The random sets ξ_k are $\mathcal{F}^N$-adapted random sets: i.e. $\{t \in \xi_k\} \in \mathcal{F}^N(A_t)$ for each t. Since they are totally ordered and determine N, this property supports the conclusion that adapted random sets are the natural analogue of stopping times.

In order to exploit the sets ξ_i, we need Assumption 2.4 which implies some useful topological properties of T, including Assumption 2.2. First, it is easily seen that Assumption 2.4.1 implies that T is bounded. Recall that the Hausdorff metric d_H is defined on the class of closed subsets of T as follows: if $F, G \subseteq T$ are closed, then

$$d_H(F,G) := \inf\{\epsilon > 0 : F \subseteq G^\epsilon \text{ and } G \subseteq F^\epsilon\},$$

where $F^\epsilon = \{t \in T : d(t,F) \leq \epsilon\}$.

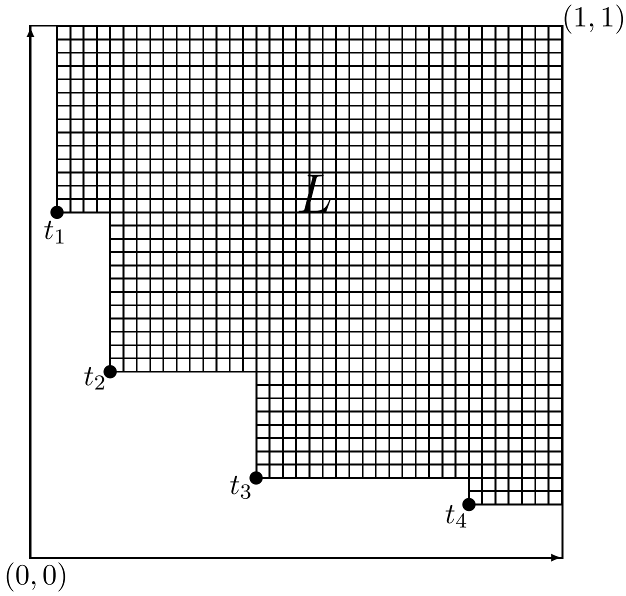

Figure 2: An upper layer $L = \cup_1^4 E_{t_i}$

Lemma 4.3. *If T satisfies Assumption 2.4 and $s, t \in T$, then $d_H(A_s, A_t)$, $d_H(E_s, E_t) \leq d(s, t)$.*

Proof. Let $\epsilon = d(s,t)$. Since both $A_s, A_t \subseteq A_{s \vee t}$, it is enough to show that $A_{s\vee t} \subseteq A_s^\epsilon$ (and so by symmetry, $A_{s\vee t} \subseteq A_t^\epsilon$) . If $u \in A_{s\vee t}$, then by Assumption 2.4

$$\begin{aligned} d(u, u\wedge s) &= d(u \wedge (s \vee t), (u \wedge s) \wedge (s \vee t)) \\ &= d(((u\wedge s) \vee (u \wedge t)), (u \wedge s)) \\ &\leq d(u \wedge s, u \wedge t) \\ &\leq d(s,t) = \epsilon, \end{aligned}$$

and if follows that $u \in A_s^\epsilon$, as required.

The proof for $d_H(E_s, E_t)$ is similar. □

We now introduce the space $\mathcal{U}$ of *upper layers:*

Definition 4.4. *A nonempty closed set $B \subseteq T$ is called an* upper layer *if $t \in B \Leftrightarrow E_t \subseteq B$. The collection of upper layers is denoted by $\mathcal{U}$.*

The shaded region in Figure 2 illustrates an upper layer L. Note that $L = \cup_{t\in L} E_t = \cup_{i=1}^4 E_{t_i}$.

Lemma 4.5. *If (T, d) is a complete separable metric space satisfying Assumptions 2.1-2.4, then $(\mathcal{U}, d_H)$ is a complete separable metric space satisfying Assumptions 2.1, 2.2 and 2.3 under the partial order of reverse set inclusion: i.e. $L_1 \leq L_2 \Leftrightarrow L_1 \supseteq L_2,\ \forall L_1, L_2, \in \mathcal{U}$.*

Proof. The space $\mathcal{U}$ is closed under arbitrary intersections (sups) and finite unions (infs). Therefore, since $E_{\mathbf{1}} = \{\mathbf{1}\}$ and $E_{\mathbf{0}} = T$, $\mathcal{U}$ has both a maximum and minimum, and is therefore a complete distributive lattice (cf. (4), Proposition 0.2.2). Thus, Assumption 2.1 is satisfied.

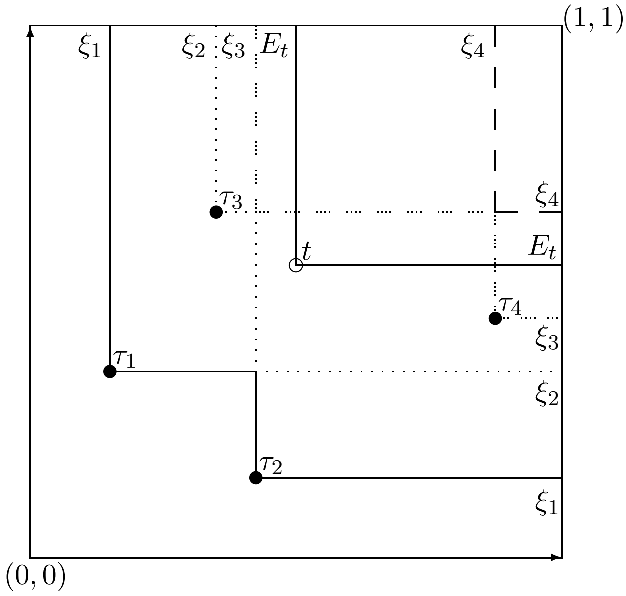

Figure 3: N and the upper layers ξ_i

It is an easy exercise to show that $(\mathcal{U}, d_H)$ is complete since T is, and that $(\mathcal{U}, d_H)$ satisfies Assumption 2.2.

To show that $(\mathcal{U}, d_H)$ satisfies Assumption 2.3 (and is therefore separable), we define $\mathcal{U}_n$ to be the (finite) lattice of upper layers generated by the sets $E_t, t \in T_n$: i.e. $L \in \mathcal{U}_n$ if and only if L is a (finite) union of sets $E_t, t \in T_n$. Since $t' < t < t'' \Rightarrow E_{t'} \supset E_t \supset E_{t''}$, Assumption 2.3 is an immediate consequence of Assumption 2.4.3 and Lemma 4.3. For $L \in \mathcal{U}$, we have

$$
\begin{aligned}
L_n^+ &= \cup_{L' \in \mathcal{U}_n, L \supseteq L'} L' = \cup_{t \in L} E_{t_n^+}, \text{ and} \\
L_n^- &= \cap_{L' \in \mathcal{U}_n, L \subseteq L'} L' = \cup_{t \in L} E_{t_n^-},
\end{aligned}
$$

(where the unions above are finite since T_n is finite) and $L = \overline{\cup_n L_n^+} = \wedge_n L_n^+ = \cap_n L_n^- = \vee L_n^-$, where "$\overline{(\cdot)}$" denotes the closure of a set. $\square$

We now may use the fact that ξ_k is an upper layer to define the embedding $N \to \tilde{N}$, where $\tilde{N}$ is a random point measure on $\mathcal{U}$ induced by N. For $L \in \mathcal{U}$, recall the notation $A_L := \downarrow L = \{L' \in \mathcal{U} : L' \supseteq L\}$.

Definition 4.6. *Given a point process N on T, the induced random point measure $\tilde{N}$ on $\mathcal{U}$ has (ordered) jumps at $(\xi_k, k \geq 1)$ for all k such that ξ_k is defined. In particular, for $L \in \mathcal{U}$, $\tilde{N}(A_L) \geq k$ if and only if $\xi_k \supseteq L$.*

Figure 3 illustrates the embedding $N \to \tilde{N}$ for a realization of a point process N with jump points $\tau_1, ..., \tau_4$. The lower boundaries of the adapted random sets $\xi_i, i = 1, ...,$ are illustrated, as is the lower boundary of the upper layer E_t. We see that $\tilde{N}(A_{E_t}) = 2$, since $E_t \subset \xi_2$ (i.e. $\xi_2 \in A_{E_t}$), but $E_t \not\subseteq \xi_3$ (i.e. $\xi_3 \notin A_{E_t}$). This corresponds to the fact that $N(A_t) = 2$.

The following theorem gives us the essential properties of $\tilde{N}$. The proof is given in Appendix A.

Theorem 4.7. *Let N be a strictly simple point process on a complete separable metric space T that satisfies Assumptions 2.1-2.4. If $\tilde{N}$ is defined as in Definition 4.6,*

1. *$\tilde{N}$ is a totally ordered strictly simple point process on $(\mathcal{U}, \mathcal{F}_{\mathcal{U}})$.*
2. *The law of N determines and is determined by the law of $\tilde{N}$.*

According to the above theorem, we are able to construct a flow $\tilde{f}$ connecting any finite increasing (in the partial order of reverse set inclusion) sequence $(L_j) \subseteq \cup_n \mathcal{U}_n$ so that $\tilde{N}^{\tilde{f}}$ is a simple point process on $[0,1]$. Denote any countable class of such flows by $F_{\tilde{N}}$. For $f \in F_{\tilde{N}}$, the law of $\tilde{N}^{\tilde{f}}$ is uniquely determined by $\Lambda_{\tilde{N}\tilde{f}}$, the compensator of $\tilde{N}^{\tilde{f}}$ with respect to its minimal filtration $\mathcal{F}^{\tilde{N}\tilde{f}}$. This leads us to the following definition:

Definition 4.8. *Let N be a strictly simple point process on T and $\tilde{N}$ its embedding in $\mathcal{U}$. The $\mathcal{U}$-flow compensator Λ of N is the family of processes*

$$\tilde{\Lambda} := \{\Lambda_{\tilde{N}\tilde{f}}; \tilde{f} \in F_{\tilde{N}}\}$$

where $F_{\tilde{N}}$ is a family of flows as defined above.

Our main result is now straightforward:

Theorem 4.9. *Let N be a strictly simple point process on a complete separable metric space T that satisfies Assumptions 2.1-2.4. The $\mathcal{U}$-flow compensator of N exists and is unique. The law of N determines and is determined by its $\mathcal{U}$-flow compensator $\tilde{\Lambda}$. In fact, the flows $\tilde{f}$ used to determine the $\mathcal{U}$-flow compensator may be restricted to the family of* basic *flows: $\tilde{f}$ is a basic flow if for each $u \in [0,1]$, $f(u)$ can be expressed as a finite union of the form $f(u) = \cup_{i=1}^{k} E_{t_i}, t_1, ..., t_k \in T$.*

Proof. Lemma 4.5 and statement 1 of Theorem 4.7 permit us to apply Theorem 3.3 to $\tilde{N}$: i.e. the law of $\tilde{N}$ determines and is determined by $\tilde{\Lambda}$. In the proof of statement 1 of Theorem 4.7, it is seen that the basic flows in $F_{\tilde{N}}$ suffice (cf. Lemma A.2). The result follows by statement 2 of Theorem 4.7. □

Comment 4.10. It is important to distinguish between flow and $\mathcal{U}$-flow compensators. The flow compensator is defined by flows whose range is on the *same* space as the point process, while the $\mathcal{U}$-flow compensator is defined by flows whose range is on a larger space. Therefore, the $\mathcal{U}$-flow compensator of N can be interpreted as the flow compensator of $\tilde{N}$.

Comment 4.11. To conclude this section, we compare Theorems 3.3 and 4.9 in the case when N is totally ordered. If $f \in F_N$, then we may identify f with the basic flow $\tilde{f}$ on $\mathcal{U}$ such that $\tilde{f}(u) = E_{f(u)}$ for all $u \in [0,1]$. Denote this class of flows by $F_{\tilde{N}}^{-}$. From Lemma A.3 we have that $N^f(u) = \tilde{N}^{\tilde{f}}(u)\ \forall u \in [0,1]$, and so $\Lambda_{Nf} = \Lambda_{\tilde{N}\tilde{f}}$. Therefore, when N is totally ordered, the family of flows in Theorem 4.9 may be restricted to $F_{\tilde{N}}^{-}$, and either the flow or the $\mathcal{U}$-flow compensator may be used to characterize the law of N. In one dimension, the flow and $\mathcal{U}$-flow compensator are equivalent to the usual compensator.

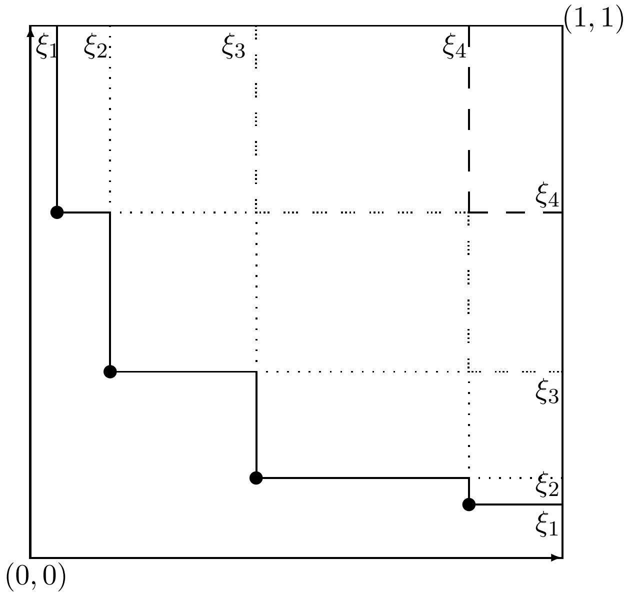

Figure 4: A single line process on $[0,1]^2$

5 More examples

5.1 The single line process:

A *single line process* is a process whose jump points are all *incomparable* (see (8) for a detailed discussion). Such processes are strictly simple and the $\mathcal{U}$-flow compensator can be reduced to the case of a single jump process. If N is a single line point process, it is completely characterized by the adapted random set $\xi \equiv \xi_1 = \cup_{\tau\in\Delta N} E_\tau$, since the set of jump points, ΔN, is in fact the set of the *minimal* points of ξ_1 (see Figure 4):

$$\Delta N = \{t \in \xi_1 : \nexists s \in \xi_1 \setminus \{t\} \text{ such that } s \leq t\}.$$

The process $M(t) = I\{t \in \xi\}$ is not a point process on T, but the embedding $\tilde{M}$ in $\mathcal{U}$ defined by $\tilde{M}(L) = I\{L \subseteq \xi\}$ is a single jump point process on $\mathcal{U}$ with jump point ξ. $\tilde{M}$ will be referred to as the single jump process on $\mathcal{U}$ associated with the single line process N on T. Although $\tilde{M} \neq \tilde{N}$, we do have the following:

Proposition 5.1. *: The law of $\tilde{M}$ determines and is determined by the law of N.*

Proof. Since $\tilde{M}$ is the first jump point of $\tilde{N}$, its law is determined by N.

To prove the converse, we first show that the law of M (i.e. the finite dimensional distributions of M) is determined by that of $\tilde{M}$, and then that the law of M determines that of N.

Since

$\{M_{t_1}=0,...,M_{t_k}=0\}=\cap_{i=1}^k\{t_i\in\xi\}^c=(\cup_{i=1}^k\{E_{t_i}\subseteq\xi\})^c$, then

$$\begin{aligned}
&P(M_{t_1}=0,...,M_{t_k}=0) \\
&= 1-P(\cup_{i=1}^k\{E_{t_i}\subseteq\xi\}) \\
&= 1-\sum_{i=1}^k P(E_{t_i}\subseteq\xi)+\sum_{i<j}P(E_{t_i}\cup E_{t_j}\subseteq\xi)+...+(-1)^kP(\cup_{i=1}^k E_{t_i}\subseteq\xi),
\end{aligned} \tag{9}$$

which is equal to:

$$\begin{aligned}
1-\sum_{i=1}^k P(\tilde{M}(E_{t_i})=1)+\sum_{i<j}P(\tilde{M}(E_{t_i}\cup E_{t_j})=1)+... \\
+(-1)^kP(\tilde{M}(\cup_{i=1}^k E_{t_i})=1).
\end{aligned}$$

Observing that for any k

$$\begin{aligned}
&P(M_{t_1}=...=M_{t_{k-1}}=0,M_{t_k}=1) \\
&= P(M_{t_1}=...=M_{t_{k-1}}=0)-P(M_{t_1}=...=M_{t_k}=0)
\end{aligned}$$

and for any $j<k$

$$\begin{aligned}
&P(M_{t_1}=...=M_{t_j}=0,M_{t_{j+1}}=...=M_{t_k}=1) \\
&= P(M_{t_1}=...=M_{t_j}=0,M_{t_{j+2}}=...=M_{t_k}=1) \\
&\quad -P(M_{t_1}=...=M_{t_j}=M_{t_{j+1}}=0,M_{t_{j+2}}=...=M_{t_k}=1),
\end{aligned}$$

it follows by induction that the finite dimensional distributions of M can be reconstructed from probabilities of the form (9), and so are determined by the law of $\tilde{M}$.

Now, it remains to show that the law of N is determined by the law of M. Clearly, in order to find terms of the form $P(N(A_{t_1})=k_1,...,N(A_{t_j})=k_j)$, we can suppose that the t_i's are dyadic. For any such event, we observe that all realizations of the point process N, for $n(=n(\omega))$ sufficiently large, the sets $C_t,t\in T_n$ will separate all the jump points and that the left-neighbourhoods containing jump points will be incomparable (since the jump points are incomparable). Then

$$\begin{aligned}
\{N(A_t)=k\} &= \cup_m\cap_{n\geq m}\{\exists \text{ exactly } k \text{ incomparable left-neighbourhoods} \\
&\qquad C_{t'}\in\mathcal{C}_n \text{ with } N(C_{t'})=1 \text{ and } t'\leq t\} \\
&= \cup_m\cap_{n\geq m}\{\exists \text{ exactly } k \text{ dyadics } t'\leq t \text{ with } M(t')=1 \\
&\qquad \text{and } M(s)=0,\forall s<t',s\in T_n\}
\end{aligned}$$

Notice that the fact that the left-neighbourhoods of the jump points are incomparable is very important. Otherwise, there could exist $s<t'$ with $N(C_s)=1$ and $N(C_{t'})=1$, in which case $M_s=1$ and $M_{t'}=1$. This cannot happen and so the left-neighbourhoods $C_{t'}$ with $M_{t'}=1,M_s=0,\forall s<t'$ are the only left-neighbourhoods with jump points. Then, we can obtain $\{N(A_{t_1})=k_1,...,N(A_{t_j})=k_j\}$ as countable unions and intersections of events involving the values of M on finite sets of points, and so the law of M determines that of N. □

We have therefore reduced the problem of characterizing the law of a single line process N on T to characterizing the law of a single jump process $\tilde{M}$ on $\mathcal{U}$. This can be done by applying Example 3.5. We need to calculate $G(L) = P(\xi \supseteq L)$ for any L in the range of a basic flow on $\mathcal{U}$ (cf. Definition 4.8). Such a set will be of the form $L = \cup_{i=1}^{k} E_{t_i}$; then apply Lemma 4.1 and an inclusion-exclusion argument to obtain

$$\begin{aligned}
G(L) &= P(\xi \supseteq \cup_{i=1}^{k} E_{t_i}) = P(\cap_{i=1}^{k}(\xi \supseteq E_{t_i})) \\
&= P(\cap_{i=1}^{k}(N(A_{t_i}) \geq 1)) \\
&= 1 - P(\cup_{i=1}^{k}(N(A_{t_i}) = 0)) \\
&= 1 - (\sum_{i=1}^{k} P(N(A_{t_i}) = 0) - \sum_{i<j} P(N(A_{t_i}) = N(A_{t_j}) = 0) \\
&\quad +... + (-1)^{k+1} P(N(A_{t_1}) = ... = N(A_{t_k}) = 0)) \\
&= 1 - (\sum_{i=1}^{k} P(N(A_{t_i}) = 0) - \sum_{i<j} P(N(A_{t_i} \cup A_{t_j}) = 0) \\
&\quad +... + (-1)^{k+1} P(N(A_{t_1} \cup ... \cup A_{t_k}) = 0)) \qquad (10)
\end{aligned}$$

Let $\tilde{f}$ be a basic flow on $\mathcal{U}$, and consider the projection $\tilde{M}^{\tilde{f}}(v) = \tilde{M}(A_{\tilde{f}(v)})$. Let $\tau^{\tilde{f}} = \inf\{v : \tilde{M}^{\tilde{f}}(v) = 1\} = \inf\{v : \tilde{f}(v) \subseteq \xi\}$. The following Proposition is an immediate consequence of Example 3.5 and Proposition 5.1:

Proposition 5.2. *Let N be a single line process on T and $\tilde{M}$ its associated single jump process on $\mathcal{U}$. The flow compensator of the single jump process $\tilde{M}$ on $\mathcal{U}$ is defined as follows: for a basic flow $\tilde{f}$ on $\mathcal{U}$,*

$$\Lambda_{\tilde{M}^{\tilde{f}}}(u) = -\ln(1 - G(\tilde{f}(u \wedge \tau^{\tilde{f}}))$$

where G is defined in (10). Conversely, the law of the point process N can be recovered from the flow compensator of $\tilde{M}$.

We remark here that the flow compensator of $\tilde{M}$ is defined on $\mathcal{U}$, but is not the $\mathcal{U}$-flow compensator of N since $\tilde{M} \neq \tilde{N}$.

It should be noted that we have a generalization of Theorem 7.3.II in (3) which states that a simple point process N on a complete separable metric space is characterized by the values of its *avoidance function* $P_0(D) \equiv P(N(D) = 0)$ for all D in a dissecting ring for the Borel sets. Proposition 5.2 and (10) imply that if N is a single line process, then its law is determined by the values of P_0 on sets of the form

$$D = (A_{t_1} \cup ... \cup A_{t_k}), t_1, ..., t_k \text{ dyadic and incomparable.} \qquad (11)$$

There is a further simplification that can be made in the case $T = [0,1]^2$ when N is a single line process satisfying a condition known in the literature as $(F4)$:

$(F4)$: Given $t = (t_1, t_2) \in [0,1]^2$, the σ-fields $\mathcal{F}^1(t)$ and $\mathcal{F}^2(t)$ are conditionally independent given $\mathcal{F}^N(t)$, where

$$\begin{aligned}
\mathcal{F}^1(t) &= \sigma\{N(u) : 0 \leq u_1 \leq t_1, 0 \leq u_2 \leq 1\} \text{ and} \\
\mathcal{F}^2(t) &= \sigma\{N(u) : 0 \leq u_1 \leq 1, 0 \leq u_2 \leq t_2\}.
\end{aligned}$$

Lemma 5.3. *Let N be a single line point process on $[0,1]^2$ satisfying $(F4)$. Then the values of the avoidance function P_0 on sets of the form (11) are determined by the values $P_0(A_t) = P(N(A_t) = 0), t \in T$. (These values will be referred to as* one-dimensional avoidance probabilities.)

Proof. The proof is by induction on k. The case $k = 1$ is trivial and we assume that probabilities of the form

$$P(N(A_{t_1} \cup ... \cup A_{t_k}) = 0), t_1, ..., t_k \text{ dyadic and incomparable}$$

are determined by one-dimensional avoidance probabilities if $k \leq n-1$. If points $t_i = (t_{i,1}, t_{i,2}), i = 1, ..., n$ are incomparable, we may assume that $t_{1,1} < ... < t_{n,1}$ and $t_{1,2} > ... > t_{n,2}$. Let $B_j := \cup_{i=1}^{j} A_{t_i}, j = 1, ..., n$ and $A := A_{(t_{n-1,1}, t_{n,2})}$. From $(F4)$, the random variables $N(B_{n-1} \setminus A)$ and $N(A_{t_n} \setminus A)$ are conditionally independent given the σ-algebra $\mathcal{F}^N(A)$, and so

$$\begin{aligned}
&P(N(B_n) = 0) \\
&= P(N(B_{n-1} \setminus A) = 0, N(A_{t_n} \setminus A) = 0, N(A) = 0) \\
&= P(N(B_{n-1} \setminus A) = 0, N(A_{t_n} \setminus A) = 0 \mid N(A) = 0) \times P(N(A) = 0) \\
&= P(N(B_{n-1} \setminus A) = 0 \mid N(A) = 0) \times P(N(A_{t_n} \setminus A) = 0 \mid N(A) = 0) \\
&\qquad \times P(N(A) = 0) \\
&= P(N(B_{n-1} \setminus A) = 0, N(A) = 0) \times P(N(A_{t_n} \setminus A) = 0, N(A) = 0) \\
&\qquad \div P(N(A) = 0) \\
&= \frac{P(N(B_{n-1}) = 0) P(N(A_{t_n}) = 0)}{P(N(A) = 0)}.
\end{aligned}$$

By the induction hypothesis, this is a function of one-dimensional avoidance probabilities of the form $P_0(A_t) = P(N(A_t) = 0), t \in T$. □

The preceding Lemma leads to an interesting corollary, which is an analogue of Theorems 3.1 and 3.3.

Corollary 5.4. *Let N be a single line point process on $[0,1]^2$ satisfying $(F4)$. The law of N determines and is determined by the laws of the family of the projected (single jump) point processes N^f, $f \in F_N$. Consequently, the law of N determines and is determined by its flow compensator $\Lambda = (\Lambda_{N^f} : f \in F_N)$.*

Proof. The law of N^f for any flow f passing through t will yield $P(N(A_t) = 0)$. By Lemma 5.3, these probabilities determine the law of N. □

We now see that the flow compensator Λ of N (on $[0,1]^2$) characterizes the law of the point process in two extreme cases: the totally ordered point process (all jump points are comparable) and if $(F4)$ is satisfied, the single line point process (all jump points are incomparable).

5.2 Renewal Processes:

Renewal processes on $\mathbf{R}_+^d$ are defined and studied in (8). As the definition is very technical and requires the introduction of new notation, it will be given in Appendix B. The only facts required here are that a renewal process is a strictly simple point process and its law is characterized by the law of N_1, the single line process associated with ξ_1. Consequently, we can apply the characterization given in the preceding example to renewal processes.

Corollary 5.5. *Let N be a renewal process on $[0,1]^2$ satisfying $(F4)$. The law of N determines and is determined by the laws of the family of the projected point processes N^f, $f \in F_N$. Consequently, the law of N determines and is determined by its flow compensator $\Lambda = (\Lambda_{N^f} : f \in F_N)$.*

Proof. Λ_{N^f} characterizes the law of the first jump of N^f, which in turn is the only jump of N_1^f. The result now follows from Corollary 5.4 and the fact that if N is renewal, the law of N_1 determines the law of N. □

Comment 5.6. Since $N(A_t) = 0 \Leftrightarrow N_1(A_t) = 0$, Lemma 5.3 implies that the law of a renewal process N satisfying $(F4)$ is determined by its one-dimensional avoidance probabilities $P_0(A_t) = P(N(A_t) = 0), t \in T$.

5.3 The Homogeneous Poisson Process:

Corollary 5.5 leads to a new characterization of the two-parameter homogeneous Poisson process. Indeed, the minimal filtration of any Poisson process satisfies $(F4)$ and it is proven in (8) that the two-parameter homogeneous Poisson process is renewal. Therefore, we have the following:

Theorem 5.7. *A strictly simple point process on $[0,1]^2$ is a homogeneous Poisson process with intensity c (c is a positive constant) if and only if it is renewal, $(F4)$ is satisfied, and its flow compensator Λ is deterministic with $\Lambda_{N^f}(u) = c\lambda(A_{f(u)})\ \forall u \in T$ (λ denotes Lebesgue measure).*

Proof. If N is a homogeneous Poisson process with intensity c, then N^f is Poisson with mean measure Λ_{N^f}. This and the discussion preceding the theorem prove the "only if" statement.

Conversely, since N is renewal and $(F4)$ is satisfied, as in Corollary 5.5 the law of N_1 is determined by Λ^f, and this in turn determines the law of N. For any homogeneous Poisson process N^p with intensity c, the law of N_1^p must equal that of N_1 (this follows from "only if"). Since both N^p and N are renewal, their distributions are equal. This completes the proof of "if".□

Since Λ_{N^f} is deterministic, this means that the projection N^f is a Poisson process along each flow.

It is interesting to compare this characterization to the characterization given by G. Aletti and V. Capasso in (1): A strictly simple point process on $[0,1]^2$ is a homogeneous Poisson process if and only if $(F4)$ is satisfied and for each flow f, the projection N^f has the deterministic compensator Λ_{N^f} (as defined above) with respect to the filtration $\mathcal{G}^f = (\mathcal{G}^f(u), 0 \leq u \leq 1)$ where $\mathcal{G}^f(u) = \mathcal{F}^N(f(u))$. As observed in Section 3, the filtration $\mathcal{G}^f$ is strictly larger than $\mathcal{F}^{N^f}$ and the fact that N^f is Poisson does not immediately imply that its compensator will be deterministic with respect to a filtration larger than its minimal filtration $\mathcal{F}^{N^f}$. Therefore, these characterizations are different and the following question remains an open problem: Can the two-parameter homogeneous Poisson process can be characterized by the $(F4)$ property and the fact its projection along any flow f is Poisson with mean measure Λ_{N^f}?

5.4 A comparison of histories:

As a final general example, if N is a strictly simple point process on $T = [0,1]^2$ we would like to compare the "history" associated with the family $\tilde{\Lambda}$ of the $\mathcal{U}$-flow compensator with the histories associated with Λ^w and Λ^* (see (11) for more details). First we recall that Λ^w is the compensator associated with the strict past

$$\mathcal{F}^N = \{\mathcal{F}^N(t) : t \in T\} = (\sigma\{N(A_s) : s \in A_t\} : t \in T),$$

while Λ^* is associated with the strong past

$$\mathcal{F}^{N^*} = \{\mathcal{F}^{N^*}(t) : t \in T\} = (\sigma\{N(A_s) : s \in D_t = \overline{E_t^c}\} : t \in T).$$

For any flow $\tilde{f} \in F_{\tilde{N}}$, we have

$$\mathcal{F}^{\tilde{N}^{\tilde{f}}} = \{\mathcal{F}^{\tilde{N}^{\tilde{f}}}(u) : 0 \le u \le 1\} = (\sigma\{\tilde{N}^{\tilde{f}}(v) : 0 \le v \le u\} : 0 \le u \le 1),$$

and for $t \in T$, let

$$\mathcal{F}^{F_{\tilde{N}}}(t) = \sigma(\mathcal{F}^{\tilde{N}^{\tilde{f}}}(u) : \tilde{f} \in F_{\tilde{N}}, u = \tilde{f}^{-1}(E_t)).$$

$\mathcal{F}^{F_{\tilde{N}}}(t)$ may be regarded as the information associated with all the projections $\tilde{N}^{\tilde{f}}$ up to time $\tilde{f}^{-1}(E_t)$ for flows $\tilde{f} \in F_{\tilde{N}}$ passing through E_t. It may be seen that $\mathcal{F}^N(t) \subset \mathcal{F}^{F_{\tilde{N}}}(t) \subset \mathcal{F}^{N^*}(t)$. As an illustration, refer to Figure 3: t is an arbitrary point of the plane, τ_1, τ_2, τ_3 and τ_4 are the jump points of a given realization of the point process N and $\xi_1, ..., \xi_4$ are the associated adapted random sets. Note that $\xi_1, \xi_2 \supseteq E_t$, and so the family of flows through E_t will allow us to trace back the locations of τ_1, τ_2 and τ_3. However, since $\xi_3 \not\subseteq E_t$, $\mathcal{F}^{F_{\tilde{N}}}(t)$ will contain no information about ξ_3 or ξ_4, and so τ_4 will not be captured. Consequently, the jump points τ_1 and τ_2 are captured by all three σ-algebras. Point τ_4 is only captured by $\mathcal{F}^{N^*}(t)$; the "strict past" $\mathcal{F}^N(t)$ of t does not capture τ_4 because τ_4 is not smaller than t and, as just noted, the location of τ_4 cannot be reconstructed from $\mathcal{F}^{F_{\tilde{N}}}(t)$ either. Finally, point τ_3 is captured by both $\mathcal{F}^{N^*}(t)$ and $\mathcal{F}^{F_{\tilde{N}}}(t)$, but obviously not by $\mathcal{F}^N(t)$.

More generally, $\mathcal{F}^{F_{\tilde{N}}}(t)$ allows us to reconstruct all the sets ξ_k such that $\xi_k \supseteq E_t$ but contains no information about ξ_k if $\xi_k \not\supseteq E_t$. Therefore, $\mathcal{F}^{F_{\tilde{N}}}(t)$ identifies all jump points τ such that $N(A_\tau) \le N(A_t)$ (but no others). All of these jump points are contained in D_t and so are captured by $\mathcal{F}^{N^*}(t)$ (which may contain information about other jump points), but are not necessarily contained in A_t and so may not be captured by $\mathcal{F}^N(t)$. Therefore,

$$\mathcal{F}^N(t) \subset \mathcal{F}^{F_{\tilde{N}}}(t) \subset \mathcal{F}^{N^*}(t).$$

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A Appendix: Proofs of Theorems 3.1 and 4.7

Proof of Theorem 3.1

In what follows, we prove that the finite dimensional distributions of N on the sets in $\mathcal{C}_n$ can be reconstructed from the laws of $N^f, f \in F_N$.

To avoid multiple subscripts and superscripts, we fix n and let $T_n = (t_0 = \mathbf{0}, t_1, ..., t_m = \mathbf{1})$ and denote $A_j = A_{t_j}$ and $C_j = C^n_{t_j}$. Recall that C_i and C_j are incomparable if and only if t_i and t_j are incomparable and that $N(C_0) = N(\{\mathbf{0}\}) = 0$. Using the laws of $N^f, f \in F$ we must be able to construct

$$P(N(C_1) = k_1, ..., N(C_m) = k_m). \tag{12}$$

If $k_i > 0$ and $k_h > 0$ for C_i, C_h incomparable, the probability in (12) is 0. Thus, we can assume without loss of generality that $k_i \neq 0$ if and only if $i \in \{i_1, ...i_h\} =: H$ where $t_{i_1} < t_{i_2} < ... < t_{i_h}$. We can make a few simplifications. First, we observe that

$$\{N(C_i) = 0 \;\forall C_i \subseteq T \setminus A_{i_h}\} = \{N(T \setminus A_{i_h}) = 0\}. \tag{13}$$

Next, note that any flow f connecting $(t_{i_1}, t_{i_2}, ..., t_{i_h})$ can be reparameterized and extended to connect $(t_{i_1}, t_{i_2}, ..., t_{i_h}, \mathbf{1})$. If $f(u) = t_{i_h}$, $f(1) = \mathbf{1}$ then $N^f(1) - N^f(u) = N(T \setminus A_{i_h})$. It follows that if the finite dimensional distributions of $(N(C_i) : C_i \subseteq A_{i_h})$ can be determined by N^f for flows f connecting $(t_{i_1}, t_{i_2}, ...t_{i_h})$, then by (13), the probabilities in (12) can be determined by extending these flows to $\mathbf{1}$. Therefore, we need only consider the finite dimensional distributions of $(N(C_i) : C_i \subseteq A_{i_h})$ and so to avoid changing notation, we will now assume without loss of generality that in (12) the sets $C_1, ..., C_m$ are the left neighbourhoods of $\mathcal{C}_n$ contained in A_{i_h}.

Let H' denote the set of all $i \notin H$ such that $t_i \in A_{i_h}$ and t_i is comparable with t_ℓ for every $\ell \in H$:

$$\begin{aligned} H' &= \{i : t_i < t_{i_h} \text{ and } \forall \ell \in H, \text{ either } t_i < t_\ell \text{ or } t_i > t_\ell\} \\ &= \{i : t_i < t_{i_1} \text{ or } \exists \ell, 2 \leq \ell \leq h \text{ such that } t_{i_{\ell-1}} < t_i < t_{i_\ell}\}. \end{aligned}$$

Next, let

$$H'' = \{i : t_i < t_{i_h}, i \notin H' \cup H\}.$$

We note here that if $\ell \in H''$, then there exists $i \in H$ such that C_ℓ and C_i are incomparable. Therefore, if $N(C_i) > 0 \;\forall i \in H$, it automatically follows that $N(C_\ell) = 0 \;\forall \ell \in H''$. Now we see that if $k_i > 0$ if and only if $i \in H$,

$$\begin{aligned} &P(N(C_1) = k_1, ...N(C_m) = k_m) \\ &= P(\cap_{i \in H}\{N(C_i) = k_i\} \cap_{j \in H'} \{N(C_j) = 0\} \cap_{\ell \in H''} \{N(C_\ell) = 0\}) \qquad (14) \\ &= P(\cap_{i \in H}\{N(C_i) = k_i\} \cap_{j \in H'} \{N(C_j) = 0\}). \qquad (15) \end{aligned}$$

The equality of (14) and (15) follows from the fact that if $\ell \in H''$, then necessarily $N(C_\ell) = 0$.

We proceed by induction on $|H'|$:=the cardinality of H'. In what follows, keep in mind that $|H'|$ is the number of left neighbourhoods $C \in \mathcal{C}_n$ such that $C \subseteq A_{i_h}$, $N(C) = 0$ and C is comparable with *all* of the left neighbourhoods $\mathcal{C}' \subseteq A_{i_h}$, $C' \in \mathcal{C}_n$ with $N(C') > 0$.

If $|H'| = 0$, then for $i_j \in H$ (and defining $A_{i_0} = \emptyset$), $A_{i_j} \setminus A_{i_{j-1}} = C_{i_j} \cup \cup_{\ell \in H'', t_\ell \in A_{i_j} \setminus A_{i_{j-1}}} C_\ell$ and since $N(C_\ell) = 0$ necessarily for $\ell \in H''$,

$$(12) = (15) = P(\cap_{i \in H}\{N(C_i) = k_i\}) = P(\cap_{j=1}^h \{N(A_{i_j} \setminus A_{i_{j-1}}) = k_{i_j}\}). \tag{16}$$

The probability in (16) is determined by the finite dimensional distributions of N^f for any flow $f \in F$ connecting $(t_{i_1}, ...t_{i_h})$: if $f(u_j) = t_{i_j}, j = 1, ..., h$,

$$P(\cap_{j=1}^h \{N(A_{i_j} \setminus A_{i_{j-1}}) = k_{i_j}\}) = P(\cap_{j=1}^h \{N^f(u_j) - N^f(u_{j-1}) = k_{i_j}\}).$$

Proceeding inductively, we assume that any probabilities of the form in (15) can be determined by the laws of N^f for $f \in F_n$ whenever $|H'| \leq \ell - 1$. (This assumption is made for *arbitrary* A_{i_h}, and corresponding H'.) Assume that $|H'| = \ell$. There exists $1 \leq j \leq h$ such that $\{t_i : i \in H'\} \subseteq A_{i_j}$ but $\{t_i : i \in H'\} \not\subseteq A_{i_{j-1}}$.

Since $\{t_i : i \in H'\} \subseteq A_{i_j}$, it follows that for $j < k \leq h$, if $N(C_{i_k}) > 0$ then $N(A_{i_k} \setminus A_{i_{k-1}}) = N(C_{i_k})$ and (as in (16)), probabilities of the form

$$P(\cap_{r=j+1}^h \{N(C_{i_r}) = k_{i_r}\} \cap_{q \in H'', t_q < t_{i_h}, t_q \not\leq t_{i_j}} \{N(C_q) = 0\})$$

are determined by the laws of N^f for flows f connecting $(t_{i_j}, ..., t_{i_h})$. If it can be shown that probabilities of the form (assuming $k_{i_r} > 0$ for $r = 1, ..., j$)

$$\begin{aligned} &P(\cap_{1 \leq r \leq j}\{N(C_{i_r}) = k_{i_r}\} \cap_{j \in H'} \{N(C_j) = 0\} \cap_{q \in H'', t_q < t_{i_j}} \{N(C_q) = 0\}) \\ &\qquad = P(\cap_{1 \leq r \leq j}\{N(C_{i_r}) = k_{i_r}\} \cap_{j \in H'} \{N(C_j) = 0\}) \end{aligned} \tag{17}$$

can be determined by the laws of N^f for flows f connecting $(t_0, t_1, ..., t_{i_{j-1}}, t_{i_j})$, then flows from the two families can be joined at t_{i_j} to yield a single family of flows $F' \subseteq F$ such that (15) is determined by the laws of $\{N^f : f \in F'\}$.

Therefore, the final step in the proof is to show that probabilities of the form (17) can be determined by the laws of N^f for flows $f \in F$ connecting $(t_0, t_1, ..., t_{i_{j-1}}, t_{i_j})$. By the induction hypothesis, such flows will determine the following probabilities:

$$\begin{aligned} P(\cap_{1 \leq r \leq j-1}\{N(C_{i_r}) = k_{i_r}\} &\cap_{q \in H', t_q < t_{i_{j-1}}} \{N(C_q) = 0\} \\ &\cap \{N(A_{i_j} \setminus A_{i_{j-1}}) = k_{i_j}\}) \end{aligned} \tag{18}$$

where we must have $k_{i_r} > 0, i = 1, ..., j-1$, but k_{i_j} can take on any value. We note that since $k_{i_{j-1}} > 0$, $N(C_v) = 0$ if $t_v \in A_{i_j} \setminus A_{i_{j-1}}$ and $t_v \not> t_{i_{j-1}}$. Therefore,

$$N(A_{i_j} \setminus A_{i_{j-1}}) = N(C_{i_j}) + \sum_{v \in H' : t_{i_{j-1}} < t_v < t_{i_j}} N(C_v). \tag{19}$$

Consider probabilities of the form

$$\begin{aligned} P(\cap_{r=1}^j \{N(C_{i_r}) = k_{i_r}\} \cap \cap_{v \in H', t_{i_{j-1}} < t_v < t_{i_j}} \{N(C_v) = k_v\} \\ \cap_{q \in H', t_q < t_{i_{j-1}}} \{N(C_q) = 0\}). \end{aligned} \tag{20}$$

where $k_{i_r} > 0, i = 1, ..., j-1$ and $k_v > 0$ for at least one $v \in H^{'}$ such that $t_{i_{j-1}} < t_v < t_{i_j}$. Different scenarios must be considered. The first scenario is as follows:

- $k_{i_j} > 0$ and there exist $r \geq 1$ and $v_1, ...v_r \in H'$ such that $t_{i_{j-1}} < t_{v_1} < ... < t_{v_r} < t_{i_j}$, $k_{v_h} > 0 \ \forall 1 \leq h \leq r$, and $k_v = 0$ for all other $v \in H'$ with $t_{i_{j-1}} < t_v < t_{i_j}$.

The induction hypothesis may be applied to $P(N(C_q) = k_q : t_q \leq t_{i_j})$, and the probability in (20) is determined by the laws of N^f for $f \in F$ in the family of flows connecting $(t_0, t_1, ..., t_{i_{j-1}}, t_{v_1}, ..., t_{v_r}, t_{i_j})$. The other scenario is:

- $k_{i_j} = 0$ and there exist $r \geq 1$ and $v_1, ...v_r \in H'$ such that $t_{i_{j-1}} < t_{v_1} < ... < t_{v_r} < t_{i_j}$, $k_{v_h} > 0 \ \forall 1 \leq h \leq r$, and $k_v = 0$ for all other $v \in H'$ with $t_{i_{j-1}} < t_v \leq t_{i_j}$.

In this case we may write

$$\begin{aligned}(20) \quad = \quad & P(\cap_{r=1}^{j-1}\{N(C_{i_r}) = k_{i_r}\} \cap \cap_{h=1}^{r}\{N(C_{v_h}) = k_{v_h}\} \cap \\ & \cap_{q \in H', t_q < t_v, q \neq v_1, ..., v_r} \{N(C_q) = 0\} \cap \{N(A_{i_j} \setminus A_{v_r}) = 0\}). \qquad (21)\end{aligned}$$

Now the induction hypothesis may be applied to $P(N(C_q) = k_q : t_q \leq t_{v_r})$, and by extending the flows from t_{v_r} to t_{i_j}, the probability in (21) is determined by the laws of N^f for $f \in F$ in the family of flows connecting $(t_0, t_1, ..., t_{i_{j-1}}, t_{v_1}, ..., t_{v_r}, t_{i_j})$.

To summarize, we have shown that the probability in (20) is completely determined by the laws of N^f for $f \in F$ in the family of flows connecting $(t_0, t_1, ..., t_{i_{j-1}}, t_{i_j})$ provided that $k_{i_r} > 0, r = 1, ..., j-1$ and $k_v > 0$ for at least one $v \in H^{'}, t_{i_{j-1}} < t_v < t_{i_j}$. In particular, if $k_{i_r} > 0, r = 1, ..., j-1$ and $k > 0$, the laws of N^f for $f \in F$ in the family of flows connecting $(t_0, t_1, ..., t_{i_{j-1}}, t_{i_j})$ determine

$$P(\cap_{r=1}^{j}\{N(C_{i_r}) = k_{i_r}\} \cap \{N(S_j) = k\} \cap_{q \in H', t_q < t_{i_{j-1}}} \{N(C_q) = 0\}), \qquad (22)$$

where $S_j := \cup_{v \in H', t_{i_{j-1}} < t_v < t_{i_j}} C_v$.

Returning to (17), if $k_{i_1}, ..., k_{i_j} > 0$ we may apply (22) to obtain

$$\begin{aligned} & P(\cap_{1 \leq r \leq j}\{N(C_{i_r}) = k_{i_r}\} \cap_{j \in H'} \{N(C_j) = 0\}) \\ & = P(\cap_{r=1}^{j}\{N(C_{i_r}) = k_{i_r}\} \cap \{N(S_j) = 0\} \cap_{q \in H', t_q < t_{i_{j-1}}} \{N(C_q) = 0\}) \\ & = P(\cap_{1 \leq r \leq j-1}\{N(C_{i_r}) = k_{i_r}\} \cap \{N(A_{i_j} \setminus A_{i_{j-1}}) = k_{i_j}\}) \\ & \quad - \sum_{k=1}^{k_{i_j}} P(\cap_{r=1}^{j-1}\{N(C_{i_r}) = k_{i_r}\} \cap \{N(C_{i_j}) = k_{i_j} - k\} \cap \{N(S_j) = k\} \\ & \qquad \cap_{q \in H', t_q < t_{i_{j-1}}} \{N(C_q) = 0\}). \qquad (23)\end{aligned}$$

>From (22) we see that the probabilities in (23) are determined by the laws of N^f for $f \in F$ in the family of flows connecting $(t_0, t_1, ..., t_{i_{j-1}}, t_{i_j})$. This completes the proof. □

Proof of Theorem 4.7:
The proof will proceed in a series of Lemmas. First, we must verify that in fact $\tilde{N}$ is a totally ordered point process on $\mathcal{U}$; i.e. a measurable mapping from $(\Omega, \mathcal{F})$ to the space of counting measures on $(\mathcal{U}, \mathcal{F}_{\mathcal{U}})$, where $\mathcal{F}_{\mathcal{U}}$ is the Borel σ-field generated by d_H, that $N(\{L\}) = 0$ or 1, and that the jumps are totally ordered.

Lemma A.1. *If N is a strictly simple point process on T, then $\tilde{N}$ is a totally ordered point process on $(\mathcal{U}, \mathcal{F}_{\mathcal{U}})$.*

Proof. By Proposition 7.1.VIII of (3), $\tilde{N}$ is a point measure if it can be shown that $\tilde{N}(U)$ is a random variable for all U in a semiring of bounded Borel sets in $\mathcal{U}$ generating $\mathcal{F}_{\mathcal{U}}$. (Keep in mind that U is a set of sets.) By the proof of the preceding Lemma, it is easily seen that the class of left-neighbourhoods associated with $\mathcal{U}_n$ for $n \geq 1$ form such a semiring.

Next, for any left-neighbourhood U associated with $\mathcal{U}_n$, $\tilde{N}(U)$ can be expressed as a linear combination of random elements of the form $\tilde{N}(A_L)$ for some $L \in \mathcal{U}_n$, so it suffices to show that $\tilde{N}(A_L)$ is a random variable. Since $\cup_n T_n$ is dense in T, measurability follows from the representation in (24) below:

$$\begin{aligned} \{\tilde{N}(A_L) \geq k\} &= \{\xi_k \supseteq L\} \\ &= \{L \subseteq \cup_{\tau_1,\dots\tau_k \in \Delta N} E_{\tau_1 \vee \dots \vee \tau_k}\} \\ &= \cap_n \cap_{t \in T_n, t \in L} \{N(A_t) \geq k\}. \end{aligned} \tag{24}$$

$\tilde{N}$ is an integer-valued measure putting mass 1 on $\xi_1, \xi_2, \dots$, and by definition, we have $\xi_1 \leq \xi_2 \leq \dots$ (in the partial order of reverse set-inclusion). That $\tilde{N}$ is simple follows from that fact that N is strictly simple. First, no jump point in ΔN can be the sup of other jump points, since if $\tau_k = \vee_1^{k-1} \tau_i$ where $\tau_{k-1} \neq \vee_1^{k-2} \tau_i$, then any flow f connecting $(\vee_1^{k-2} \tau_i)_n^+$ with $(\tau_k)_n^+$ would have a projection N^f with a double jump at u, where $u = \inf\{v : f(v) \geq \tau_k\}$. Since $s \neq t \Leftrightarrow E_s \neq E_t$, it follows that $\xi_{k-1} \neq \xi_k$. $\diamond$

To complete the proof of the first statement of Theorem 4.7, we must prove that the embedded process $\tilde{N}$ is strictly simple.

Lemma A.2. *If N is strictly simple, then so is $\tilde{N}$. $F^{\tilde{N}}$ can be restricted to the class of basic flows on $\mathcal{U}$.*

Proof. It is enough to show that for $L_0 = T \supset L_1 \supset \dots \supset L_m = \{\mathbf{1}\} \in \mathcal{U}_n$ there exists a flow $\tilde{f} : [0,1] \to \mathcal{U}$ with $\tilde{f}(\frac{k}{m}) = L_k, k = 0, \dots, m$ such that $\tilde{N}^{\tilde{f}}$ is simple. Without loss of generality, we may assume that for $0 \leq j \leq m$, $L_j = \cup_{i=j}^m E_{t_i}$ where $T_n = (t_0 = \mathbf{0}, t_1, \dots, t_m = \mathbf{1})$ is a *consistent* ordering of the elements of T_n: i.e. $t_i > t_j \Rightarrow i > j$.

To construct the flow $\tilde{f}$, for each $0 \leq j \leq m-1$, choose $t_{j+} \in \{t_{j+1}, \dots, t_m\}$ such that $t_{j+} > t_j$ and there exist no other points $t_h \in \{t_{j+1}, \dots, t_m\}$ with $t_j < t_h < t_{j+}$. Note that $t_{(m-1)+} = \mathbf{1}$ since the ordering is consistent. Since N is strictly simple, for $0 \leq j \leq m-1$ we may choose a flow f_j connecting t_j and t_{j+} such that (with an appropriate reparametrization) $f_j(\frac{j}{m}) = t_j$, $f_j(\frac{j+1}{m}) = t_{j+}$ and $N^{f_j}(u)$ is simple for $u \in (\frac{j}{m}, \frac{j+1}{m}]$. Now define $\tilde{f}(0) := E_{\mathbf{0}} = T$ and for $u \in (\frac{j}{m}, \frac{j+1}{m}]$,

$$\tilde{f}(u) := E_{f_j(u)} \cup \cup_{h=j+1}^m E_{t_h}. \tag{25}$$

To see that $\tilde{f}$ is a flow on $\mathcal{U}$, we first observe that $\tilde{f}$ is strictly increasing since f_j is and so $\frac{j}{m} < u < v \leq \frac{j+1}{m} \Rightarrow f_j(u) < f_j(v) \Rightarrow E_{f_j(u)} \supset E_{f_j(v)}$; by the choice of t_{j+}, $E_{f_j(u)} \not\subseteq \cup_{h=j+1}^m E_{t_h}$ for $u \in (\frac{j}{m}, \frac{j+1}{m})$. By Lemma 4.3, if $t_r \to_d t$ then $E_{t_r} \to_{d_H} E_t$, and since the f_j's are continuous on T, continuity of $\tilde{f}$ needs to be checked only at points of the form $\frac{j}{m}$. If $u_r \uparrow \frac{j}{m}$ then by definition of t_{j+} and continuity of f_j, $\tilde{f}(u_r) \to_{d_H} \cup_{h=j}^m E_{t_h} = f(\frac{j}{m})$. Conversely, if $u_r \downarrow \frac{j}{m}$, then using continuity of f_j again we see that $\tilde{f}(u_r) \to_{d_H} E_{t_j} \cup \cup_{h=j+1}^m E_{t_h} = \cup_{h=j}^m E_{t_h} = f(\frac{j}{m})$. We see from (25) that $\tilde{f}$ is basic.

We now prove that $\tilde{N}^{\tilde{f}}$ is simple where $\tilde{N}^{\tilde{f}}(u) = \tilde{N}(A_{\tilde{f}(u)})$. If $\tilde{N}^{\tilde{f}}(\{u\}) > 1$ for $u \in (\frac{j}{m}, \frac{j+1}{m}]$ then since

$$\tilde{N}^{\tilde{f}}(\{u\}) \quad = \quad \tilde{N}(\{E_{f_j(u)} \cup \cup_{h=j+1}^m E_{t_h}\})$$

there exists k such that $\xi_k \supset \xi_{k+1} \supseteq E_{f_j(u)} \cup \cup_{h=j+1}^m E_{t_h}$, but $\xi_k \not\supseteq E_{f_j(u-)} \cup \cup_{h=j+1}^m E_{t_h}$ and $\xi_{k+1} \not\supseteq E_{f_j(u-)} \cup \cup_{h=j+1}^m E_{t_h}$. This implies the simpler statement that $\xi_k \supset \xi_{k+1} \supseteq E_{f_j(u)}$, but $\xi_k \not\supseteq E_{f_j(u-)}$ and $\xi_{k+1} \not\supseteq E_{f_j(u-)}$. From this it follows that $N_{A_{f_j(u)}} \geq k+1$ while $N_{A_{f_j(u-)}} \leq k-1$ - in other words, $N^{f_j}(\{u\}) > 1$. This is a contradiction, since f_j was chosen so that N^{f_j} is simple. Therefore, we are able to construct a basic flow $\tilde{f}$ connecting any increasing (in the partial order of reverse set inclusion) sequence $(L_j) \subseteq \cup_n \mathcal{U}_n$ so that $\tilde{N}^{\tilde{f}}$ is simple. By definition, $\tilde{N}$ is strictly simple and we can restrict $F^{\tilde{N}}$ to the class of basic flows. $\diamond$

To prove the second statement in Theorem 4.7, we show that N and $\tilde{N}$ are dual: each determines the law of the other. First, we need the following:

Lemma A.3. *For $t \in T$, $N(A_t) = k$ if and only if $\tilde{N}(A_{E_t}) = k$.*

Proof. Lemma 4.1 implies that

$$\begin{aligned}\{N_{A_t} = k\} &= \{N_{A_t} \geq k\} \setminus \{N_{A_t} \geq k+1\} \\ &= \{\xi_k \supseteq E_t\} \setminus \{\xi_{k+1} \supseteq E_t\} \\ &= \{\tilde{N}_{A_{E_t}} \geq k\} \setminus \{\tilde{N}_{A_{E_t}} \geq k+1\} \\ &= \{\tilde{N}_{A_{E_t}} = k\}. \qquad \diamond\end{aligned}$$

Lemma A.4. *The law of N determines and is determined by the law of $\tilde{N}$.*

Proof. Since the laws of N and $\tilde{N}$ are determined by the finite dimensional distributions on the left-neighbourhoods generated by T_n and $\mathcal{U}_n$ respectively, for $n \geq 1$ (cf. (3). Proposition 6.2.III), by additivity it is enough to consider the finite dimensional distributions of the form

$$P(N(A_{t_1}) = k_1, ..., N(A_{t_j}) = k_j) \text{ and } P(\tilde{N}(A_{L_1}) \geq k_1, ..., \tilde{N}(A_{L_j}) \geq k_j).$$

>From Lemma A.3, we have that for $t_1, ..., t_j \in T$

$$P(N(A_{t_1}) = k_1, ..., N(A_{t_j}) = k_j) = P(\tilde{N}(A_{E_{t_1}}) = k_1, ..., \tilde{N}(A_{E_{t_j}}) = k_j),$$

and so the law of $\tilde{N}$ determines that of N. Conversely, since T_n is finite, (24) implies that the law of N determines that of $\tilde{N}$:

$$\begin{aligned}&P(\tilde{N}(A_{L_1}) \geq k_1, ..., \tilde{N}(A_{L_j}) \geq k_j) \\ &\quad = \lim_n P\left(\cap_{t \in T_n, t \in L_1}\{N(A_t) \geq k_1\}, ..., \cap_{t \in T_n, t \in L_j}\{N(A_t) \geq k_j\}\right).\end{aligned}$$

$\diamond$

This completes the proof of Theorem 4.7. $\square$

B Appendix: Renewal Processes

Here we define renewal processes on $[0,1]^2$, adapting the definition given in (8) to the notation used in this paper. We refer the reader to (8) for details.

Definition B.1. *For an arbitrary Borel set $B \subseteq [0,1]^2$, the set of minimal points of B is*

$$\min(B) := \{t \in B : s \not\leq t,\ \forall s \in B \text{ such that } s \neq t\}.$$

Definition B.2. *Let N be a strictly simple point process on $[0,1]^2$. With ξ_n defined as in (8), denote:*

- $\Delta_N := \{\tau : N(\{\tau\}) = 1\}$. *This is the set of jump points of the process N.*
- $\varepsilon(\xi_n)$ *is the finite set of minimal points of ξ_n, and let $\varepsilon_N := \cup_{n=1}^{\infty}\varepsilon(\xi_n)$. The points in $\varepsilon(\xi_n)$ will be denoted by $\{\tau_j^{(n)}, j = 1, ...,\}$; the numbering $\tau_1^{(n)}, \tau_2^{(n)}...$ may be defined arbitrarily.*
- $\xi_n^+ := \cup_{\ k\neq j}(E_{\tau_k^{(n)}} \cap E_{\tau_j^{(n)}})$. *If ξ_n has only one minimal point, then $\xi_n^+ := \emptyset$.*

Comments B.3.

1. In general, $\Delta_N \subseteq \varepsilon_N$ and ε_N is the closure of Δ_N under suprema; both sets are finite. If $\tau \in \Delta_N$ and $N(A_\tau) = i$, then $\tau = \tau_j^{(i)}$ for some j. Conversely, while all of the minimal points of ξ_1 are in Δ_N (in fact, $\varepsilon(\xi_1) = \min(\Delta_N)$), if $i > 1$, $\tau_j^{(i)}$ is not necessarily a jump point of N but will always be the supremum of i jump points.
2. The random sets ξ_n are determined by $\varepsilon(\xi_n)$ and vice versa. Likewise, ξ_n^+ is determined by $\varepsilon(\xi_n)$; each of its minimal points is the sup of a pair of minimal points of ξ_n, and the set $\xi_n \setminus \xi_n^+$ is the disjoint union of the mutually incomparable sets $E_{\tau_j^{(n)}} \setminus \xi_n^+,\ j = 1, 2, ...$
3. By definition, $N(A_t) \geq i+1$ if $t \in \xi_i^+$, and so $\xi_i \subseteq \xi_{i+1} \subseteq \xi_i^+,\ \forall i$. The set of minimal points of ξ_{i+1} consists of the minimal points of ξ_i^+ as well as the minimal jump points of N contained in the set $\xi_i \setminus \xi_i^+$: i.e.

$$\begin{aligned}\varepsilon(\xi_{i+1}) &= \varepsilon(\xi_i^+) \cup \min(\Delta_N \cap (\xi_i \setminus \xi_i^+)) \\ &= \varepsilon(\xi_i^+) \cup \cup_j \min(\Delta_N \cap (E_{\tau_j^{(i)}} \setminus \xi_i^+)).\end{aligned} \tag{26}$$

The representation in (26) motivates the following definition of the renewal property.

Definition B.4. *Let N be a (strictly simple) point process on $[0,1]^2$ with associated adapted random sets $\xi_i, i \geq 1$, and let $\tau_1^{(i)}, \tau_2^{(i)}, ...$ denote the minimal points of ξ_i. Let N_1 be the single line process with jump points $\varepsilon(\xi_1)$. N is a renewal point process if for every $i \geq 1$,*

- *Given ξ_i, the process N behaves independently on each of the disjoint incomparable sets $E_{\tau_j^{(i)}} \setminus \xi_i^+, j = 1, 2, ...$.*
- *Given ξ_i, the law of $\min(\Delta_N \cap (E_{\tau_j^{(i)}} \setminus \xi_i^+))$ is the same as the law of $((\Delta_M \oplus \tau_j^{(i)}) \cap (E_{\tau_j^{(i)}} \setminus \xi_i^+))$, where M is an independent copy of N_1, and $(\Delta_M \oplus \tau_j^{(i)})$ is the set of jump points of M each translated by $\tau_j^{(i)}$.*

We see that the law of ξ_{i+1} given ξ_i does not depend on i, and is determined by the law of $\min(\Delta_N \cap (E_{\tau_j^{(i)}} \setminus \xi_i^+))$; this in turn is determined by the law of N_1. Therefore, the law of the renewal process N is completely characterized by the law of N_1. See (8) for a rigorous development.

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