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Parabolic SPDEs degenerating on the boundary of non-smooth domain

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Abstract

Degenerate stochastic partial differential equations of divergence and non-divergence forms are considered in non-smooth domains. Existence and uniqueness results are given in weighted Sobolev spaces, and Hölder estimates of the solutions are presented

Key words: SPDEs degenerating on the boundary, Weighted Sobolev spaces

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1 Introduction

We are dealing with an L_p-theory of parabolic stochastic partial differential equations (SPDEs) of the types

$$du = (a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f)\,dt + (\sigma^{ik}u_{x^i} + \nu^ku + g^k)\,dw_t^k \tag{1.1}$$

$$du = (D_i(a^{ij}u_{x^j} + \bar{b}^iu + \bar{f}^i) + b^iu_{x^i} + cu + f)\,dt + (\sigma^{ik}u_{x^i} + \nu^ku + g^k)\,dw_t^k \tag{1.2}$$

considered for $t > 0$ and $x \in G$. Here w_t^k are independent one-dimensional Wiener processes and G is a bounded domain in $\mathbb{R}^d$.

In this article we assume that the equations have the "degeneracy α" near ∂G : $\exists \delta_0, K > 0$ such that for any $\lambda \in \mathbb{R}^d$,

$$\delta_0\rho^{2\alpha}(x)|\lambda|^2 \le (a^{ij}(t,x) - \alpha^{ij}(t,x))\lambda^i\lambda^j \le K\rho^{2\alpha}(x)|\lambda|^2 \tag{1.3}$$

where $\rho(x) := \mathrm{dist}(x, \partial G)$ and $\alpha^{ij} := \frac{1}{2}\sum_k \sigma^{ik}\sigma^{jk}$. Note that if $\alpha = 0$ then the equations are uniformly nondegenerate. In this case, unique solvability of the equations in appropriate Banach spaces has been widely studied in many articles. See, for instance, [3], [4], [5], [8], [10], [14], [15] and [17].

Our motivation of considering SPDEs with such degeneracy comes from several articles related to PDEs with different types of degeneracies. We refer to [16], [19] and [20] for degenerate elliptic equations. For parabolic PDEs we refer to [1], [18] (and references therein), where interior Schauder estimates for equations with the degeneracy $\alpha < 1/2$ were established.

An L_p-theory of equation (1.1) with the degeneracy $\alpha = 1$ can be found in [12]. In this article, we extend the results in [12]. We prove the unique solvability of equations (1.1) and (1.2) with arbitrary degeneracy $\alpha \in [1, \infty)$ in appropriate Sobolev spaces. Also we give some Hölder estimates of the solutions.

One of main applications of the theory of SPDEs is a nonlinear filtering problem. Consider a pair of diffusion processes $(X_t, Y_t) \in \mathbb{R}^d \times \mathbb{R}^{d_1-d}$,

$$dX_t = \rho^\alpha(X_t)b(t, X_t, Y_t)dt + \rho^\alpha(X_t)r(t, X_t, Y_t)dW_t, \quad X(0) = X_0$$

$$dY_t = B(t, X_t, Y_t)dt + R(t, Y_t)dW_t, \quad Y(0) = Y_0,$$

where W_t is d_1-dimensional Wiener process and b, r, B, R are Lipschitz continuous matrices. The nonlinear filtering problem is computing the conditional density π_t of X_t given by the observations $\{Y_s : s \le t\}$. It was shown in [8] that when $\alpha = 0$, there exists a conditional density π_t and π_t satisfies a SPDE of type (1.1). Based on our L_p-theory, one can easily construct the corresponding results when $\alpha \ge 1$. The motivations of considering the case $\alpha > 0$ were discussed at length in [12]. We only mention that usually the process X_t evolves in a bounded region due to, for instance, mechanical restrictions, and therefore the above model is suitable when the process X_t stays in the bounded domain. Note that since ρ^α ($\alpha \ge 1$) is Lipschitz continuous in $\mathbb{R}^d$ ($\rho(x) := 0$ if $x \notin G$), by the unique solvability of the above SDE, if X_0 is in G then the process X_t never cross the boundary of G.

Here are notations used in the article. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, ..., x^d)$ and $B_r(x) := \{y \in \mathbb{R}^d : |x-y| < r\}$. For $i = 1, ..., d$, multi-indices $\beta = (\beta_1, ..., \beta_d)$, $\beta_i \in \{0, 1, 2, ...\}$, and functions $u(x)$ we set

$$u_{x^i} = \partial u/\partial x^i = D_iu, \quad D^\beta u = D_1^{\beta_1} \cdot ... \cdot D_d^{\beta_d}u, \quad |\beta| = \beta_1 + ... + \beta_d.$$

We also use the notation D^m for a partial derivative of order m with respect to x.

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2 Main results

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all $(\mathcal{F}, P)$-null sets. By $\mathcal{P}$ we denote the predictable σ-field generated by $\{\mathcal{F}_t, t \geq 0\}$ and we assume that on Ω we are given independent one-dimensional Wiener processes $w_t^1, w_t^2, \ldots$, each of which is a Wiener process relative to $\{\mathcal{F}_t, t \geq 0\}$.

Choose and fix a smooth function ψ such that $\psi(x) \sim \rho(x)$ (see (2.9)). We rewrite equations (1.1) and (1.2) in the following forms.

$$\begin{aligned} du = &(\psi^{2\alpha} a^{ij} u_{x^i x^j} + \psi^\alpha b^i u_{x^i} + cu + f)\, dt \\ &+ (\psi^\alpha \sigma^{ik} u_{x^i} + \nu^k u + g^k)\, dw_t^k, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} du = &(D_i(\psi^{2\alpha} a^{ij} u_{x^j} + \psi^\alpha \bar{b}^i u + \bar{f}^i) + \psi^\alpha b^i u_{x^i} + cu + f)\, dt \\ &+ (\psi^\alpha \sigma^{ik} u_{x^i} + \nu^k u + g^k)\, dw_t^k, \end{aligned} \tag{2.5}$$

Here, i and j go from 1 to d, and k runs through $\{1, 2, \ldots\}$. The coefficients $a^{ij}, \bar{b}^i, b^i, c, \sigma^{ik}, \nu^k$ and the free terms $\bar{f}^i, f, g^k$ are random functions depending on t and x. Throughout the article, for functions defined on $\Omega \times [0, T] \times G$, the argument $\omega \in \Omega$ will be omitted.

To describe the assumptions of $\bar{f}^i, f$ and g we use Sobolev spaces introduced in [8], [9] and [13]. If $\theta \in \mathbb{R}$ and n is a nonnegative integer, then

$$H_p^n = H_p^n(\mathbb{R}^d) = \{u : u, Du, \ldots, D^n u \in L_p\},$$

$$\begin{aligned} L_{p,\theta}(G) &:= H_{p,\theta}^0(G) = L_p(G, \rho^{\theta - d} dx), \\ H_{p,\theta}^n(G) &:= \{u : u, \rho u_x, \ldots, \rho^n D^n u \in L_{p,\theta}(G)\}. \end{aligned} \tag{2.6}$$

In general, by $H_p^\gamma = H_p^\gamma(\mathbb{R}^d) = (1 - \Delta)^{-\gamma/2} L_p$ we denote the space of Bessel potential. We define

$$\|u\|_{H_p^\gamma} = \|(1 - \Delta)^{\gamma/2} u\|_{L_p}.$$

The space $H_{p,\theta}^\gamma(G)$ is defined as the set of all distributions u on G such that

$$\|u\|_{H_{p,\theta}^\gamma(G)}^p := \sum_{n=-\infty}^{\infty} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|_{H_p^\gamma}^p < \infty, \tag{2.7}$$

where $\{\zeta_n : n \in \mathbb{Z}\}$ is a sequence of smooth functions such that

$$|D^m \zeta_n(x)| \leq N(m) e^{mn}, \quad \sum_n \zeta_n \geq \text{const} > 0, \tag{2.8}$$

$$\zeta_n \in C_0^\infty(G_n), \quad G_n := \{x \in G : e^{-n-1} < \rho(x) < e^{-n+1}\}.$$

If G_n is empty set, then we put $\zeta_n = 0$. One can construct the function ζ_n, for instance, by mollifying the indicator function of G_n. It is known that up to equivalent norms the space $H^{\gamma}_{p,\theta}(G)$ and its norm are independent of $\{\zeta_n\}$ (see Lemma 2.1(iv)).
We also use the above notations for ℓ_2-valued functions $g = (g_1, g_2, ...)$. We define

$$\|g\|_{H^{\gamma}_p} = \|g\|_{H^{\gamma}_p(\ell_2)} = \||(1-\Delta)^{\gamma/2} g|_{\ell_2}\|_{L_p}.$$

$$\|g\|_{H^{\gamma}_{p,\theta}(G)} = \sum_{n=-\infty}^{\infty} e^{n\theta} \|\zeta_{-n}(e^n\cdot) g(e^n\cdot)\|^p_{H^{\gamma}_p}.$$

Fix a smooth function ψ in G such that

$$\sup_x |\rho(x)^m D^{m+1}\psi(x)| < \infty,$$

$$\rho(x) \le N\psi(x) \le N\rho(x), \quad \forall x \in G. \tag{2.9}$$

For instance one can take $\psi(x) = \sum_n e^{-n}\zeta_n(x)$.
In the following lemma we collect some properties of $H^{\gamma}_{p,\theta}(G)$ (see [9] and [13] for detail). For $\nu \in (0,1]$, we denote

$$|u|_{C(X)} = \sup_X |u(x)|, \quad [u]_{C^{\nu}(X)} = \sup_{x \ne y} \frac{|u(x)-u(y)|}{|x-y|^{\nu}}.$$

Lemma 2.1. *(i) Assume that $\gamma - d/p = m + \nu$ for some $m = 0, 1, ...$ and $\nu \in (0,1]$. Let i, j be multi-indices such that $|i| \le m, |j| = m$. Then for any $u \in H^{\gamma}_{p,\theta}(G)$, we have*

$$\psi^{|i|+\theta/p} D^i u \in C(G), \quad \psi^{m+\nu+\theta/p} D^j u \in C^{\nu}_{loc}(G),$$

$$|\psi^{|i|+\theta/p} D^i u|_{C(G)} + [\psi^{m+\nu+\theta/p} D^j u]_{C^{\nu}(G)} \le N\|u\|_{H^{\gamma}_{p,\theta}(G)}.$$

(ii) $\psi D, D\psi : H^{\gamma}_{p,\theta}(G) \to H^{\gamma-1}_{p,\theta}(G)$ are bounded linear operators, and for any $u \in H^{\gamma}_{p,\theta}(G)$

$$\|u\|_{H^{\gamma}_{p,\theta}(G)} \le N\|\psi u_x\|_{H^{\gamma-1}_{p,\theta}(G)} + N\|u\|_{H^{\gamma-1}_{p,\theta}(G)} \le N\|u\|_{H^{\gamma}_{p,\theta}(G)}, \tag{2.10}$$

$$\|u\|_{H^{\gamma}_{p,\theta}(G)} \le N\|(\psi u)_x\|_{H^{\gamma-1}_{p,\theta}(G)} + N\|u\|_{H^{\gamma-1}_{p,\theta}(G)} \le N\|u\|_{H^{\gamma}_{p,\theta}(G)}. \tag{2.11}$$

(iii) For any $\nu, \gamma \in \mathbb{R}$, $\psi^{\nu} H^{\gamma}_{p,\theta}(G) = H^{\gamma}_{p,\theta-p\nu}(G)$ and

$$\|u\|_{H^{\gamma}_{p,\theta-p\nu}(G)} \le N\|\psi^{-\nu}u\|_{H^{\gamma}_{p,\theta}(G)} \le N\|u\|_{H^{\gamma}_{p,\theta-p\nu}(G)}.$$

(iv) Let $\{\xi_n\}$ be a sequence of $C^{\infty}_0(G)$ functions such that

$$|D^m \xi_n| \le Ne^{nm}, \quad supp\, \xi_n \subset \{x \in G : e^{-n-k_0} < \rho(x) < e^{-n+k_0}\}$$

for some $k_0 > 0$. Then for any $u \in H^{\gamma}_{p,\theta}(G)$

$$\sum_n \|\xi_{-n}(e^n x) u(e^n x)\|^p_{H^{\gamma}_{p,\theta}(G)} \le N\|u\|^p_{H^{\gamma}_{p,\theta}(G)}.$$

And, if in addition $\sum_n \xi_n(x) \ge \delta > 0$, then

$$\|u\|^p_{H^{\gamma}_{p,\theta}(G)} \le N \sum_n \|\xi_{-n}(e^n x) u(e^n x)\|^p_{H^{\gamma}_{p,\theta}(G)}.$$

Now we define stochastic Banach spaces. For any stopping time τ, denote $(\!(0,\tau]\!] = \{(\omega,t) : 0 < t \le \tau(\omega)\}$,

$$\mathbb{H}_p^\gamma(\tau) = L_p((\!(0,\tau]\!], \mathcal{P}, H_p^\gamma), \quad \mathbb{H}_{p,\theta}^\gamma(G,\tau) = L_p((\!(0,\tau]\!], \mathcal{P}, H_{p,\theta}^\gamma(G)),$$

$$\mathbb{L}_{\dots}(\dots) = \mathbb{H}_{\dots}^0(\dots), \quad U_p^\gamma = L_p(\Omega, \mathcal{F}_0, H_p^{\gamma-2/p}),$$

$$U_{p,\theta}^{\gamma;\alpha}(G) = \psi^{-\frac{2}{p}(1-\alpha)+1} L_p(\Omega, \mathcal{F}_0, H_{p,\theta}^{\gamma-2/p}(G)).$$

Definition 2.2. We write $u \in \mathfrak{H}_{p,\theta}^{\gamma+2,\alpha}(G,\tau)$ if $u \in \psi\mathbb{H}_{p,\theta}^{\gamma+2}(G,\tau)$, $u(0,\cdot) \in U_{p,\theta}^{\gamma+2,\alpha}(G)$ and for some $f \in \psi^{-1+2\alpha}\mathbb{H}_{p,\theta}^\gamma(G,\tau)$, $g \in \psi^\alpha\mathbb{H}_{p,\theta}^{\gamma+1}(G,\tau,\ell_2)$

$$du = f\,dt + g^k\,dw_t^k, \tag{2.12}$$

in the sense of distribution. In other words, for any $\phi \in C_0^\infty(G)$, the equality

$$(u(t,\cdot),\phi) = (u(0,\cdot),\phi) + \int_0^t (f(s,\cdot),\phi)\,ds + \sum_{k=1}^\infty \int_0^t (g^k(s,\cdot),\phi)\,dw_s^k$$

holds for all $t \le \tau$ with probability 1. In this situation we also write $f = \mathbb{D}u, g = \mathbb{S}u$. Let

$$\mathfrak{H}_{p,\theta,0}^{\gamma+2,\alpha}(G,\tau) = \mathfrak{H}_{p,\theta}^{\gamma+2,\alpha}(G,\tau) \cap \{u : u(0,\cdot) = 0\}.$$

The norm in $\mathfrak{H}_{p,\theta}^{\gamma+2,\alpha}(G,\tau)$ is introduced by

$$\|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2,\alpha}(G,\tau)} = [|u|]_{\mathfrak{H}_{p,\theta}^{\gamma+2,\alpha}(G,\tau)} + \|u(0,\cdot)\|_{U_{p,\theta}^{\gamma+2,\alpha}(G)},$$

where

$$[|u|]_{\mathfrak{H}_{p,\theta}^{\gamma+2,\alpha}(G,\tau)} := \|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^\gamma(G,\tau)} + \|\psi^{1-2\alpha}\mathbb{D}u\|_{\mathbb{H}_{p,\theta}^\gamma(G,\tau)} + \|\psi^{-\alpha}\mathbb{S}u\|_{\mathbb{H}_{p,\theta}^{\gamma+1}(G,\tau)},$$

$$\|u(0,\cdot)\|_{U_{p,\theta}^{\gamma+2,\alpha}(G)}^p = E\|\psi^{\frac{2}{p}(1-\alpha)-1}u(0,\cdot)\|_{H_{p,\theta}^{\gamma+2-2/p}(G)}^p.$$

Remark 2.3. Up to equivalent norms, the space $\mathfrak{H}_{p,\theta}^{\gamma+2,\alpha}(G,\tau)$ is independent of the choice of ψ, and for instance the norm $\|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^{\gamma+2}(G,\tau)}$ can be replaced by $\|u\|_{\mathbb{H}_{p,\theta-p}^{\gamma+2}(G,\tau)}$. Also note that if $u \in \psi\mathbb{H}_{p,\theta}^{\gamma+2}(G,\tau)$, then by Lemma 2.1

$$\psi^{2\alpha}\Delta u \in \psi^{-1+2\alpha}\mathbb{H}_{p,\theta}^\gamma(G,\tau), \quad \psi^\alpha Du \in \psi^\alpha\mathbb{H}_{p,\theta}^{\gamma+1}(G,\tau).$$

Thus considering equation (2.4), we find that the spaces for $\mathbb{D}u$ and $\mathbb{S}u$ are defined naturally.

To state our assumptions on the coefficients, we take some notations from [2]. Denote $\rho(x,y) = \rho_G(x,y) = \rho(x) \wedge \rho(y)$. For $\delta \in (0,1)$, and $k = 0,1,2,\dots$, define

$$[f]_k^{(0)} = [f]_{k,G}^{(0)} = \sup_{x\in G} \rho^k(x)|D^k f(x)|,$$

$$[f]_{k+\delta}^{(0)} = [f]_{k+\alpha,G}^{(0)} = \sup_{\substack{x,y\in G \\ |\beta|=k}} \rho^{k+\alpha}(x,y)\frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha},$$

$$|f|_k^{(0)} = |f|_{k,G}^{(0)} = \sum_{j=0}^{k}[f]_{j,G}^{(0)}, \quad |f|_{k+\alpha}^{(0)} = |f|_{k+\alpha,G}^{(0)} = |f|_{k,G}^{(0)} + [f]_{k+\alpha,G}^{(0)}.$$

By $D^\beta f$ we mean either classical derivatives or Sobolev ones and in the latter case sup's in the above are understood as ess sup's. We also use the same notations for ℓ_2-valued functions.

Fix a function $\delta_0(\tau) \geq 0$ defined on $[0,\infty)$ such that $\delta_0(\tau) > 0$ unless $\tau \in \{0,1,2,...\}$. For $\tau \geq 0$ define

$$\tau+ = \tau + \delta_0(\tau),$$

and fix some constants

$$\delta_0, K \in (0,\infty), \quad \gamma \in \mathbb{R}.$$

Assumption 2.4. (i) For each $x \in G$, the coefficients $a^{ij}(t,x)$, $\bar{b}^i(t,x)$, $b^i(t,x)$ $c(t,x)$, $\sigma^{ik}(t,x)$ and $\nu^k(t,x)$ are predictable functions of (ω,t).

(ii) For any x,t, ω and $\lambda \in \mathbb{R}^d$,

$$\delta_0|\lambda|^2 \leq (a^{ij}(t,x) - \alpha^{ij}(t,x))\lambda^i\lambda^j \leq K|\lambda|^2, \tag{2.13}$$

where $\alpha^{ij} = \frac{1}{2}\sum_k \sigma^{ik}\sigma^{jk}$.

Assumption 2.5. For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{\omega,t}(|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_{\ell_2}) \leq \varepsilon$$

whenever $x,y \in G$ and $|x-y| \leq \delta(\varepsilon)\rho(x,y)$.

Assumption 2.6. For any $t > 0$ and $\omega \in \Omega$,

$$|a^{ij}(t,\cdot)|_{|\gamma|+}^{(0)} + |\psi^{1-\alpha}b^i(t,\cdot)|_{|\gamma|+}^{(0)} + |\psi^{2(1-\alpha)}c(t,\cdot)|_{|\gamma|+}^{(0)}$$

$$+|\sigma^i(t,\cdot)|_{|\gamma+1|+}^{(0)} + |\psi^{1-\alpha}\nu(t,\cdot)|_{|\gamma+1|+}^{(0)} \leq K.$$

Remark 2.7. Assumption 2.5 is much weaker than uniform continuity of a^{ij} and σ^i. For instance, let $G = (0,1)$ and $a(t,x) = 2 + \sin(\ln x(1-x))$. Then one can easily check that a satisfies Assumptions 2.5 and 2.6 for any $\gamma \in \mathbb{R}$.

Here are our main results. From this point on we assume that

$$\tau \leq T, \quad \alpha \in [1,\infty), \quad p \in [2,\infty).$$

Theorem 2.8. *Let Assumptions 2.4, 2.5 and 2.6 be satisfied. Then*

(i) for any $f \in \psi^{-1+2\alpha}\mathbb{H}^\gamma_{p,\theta}(G,\tau), g \in \psi^\alpha\mathbb{H}^{\gamma+1}_{p,\theta}(G,\tau)$ and $u_0 \in U^{\gamma+2,\alpha}_{p,\theta}(G)$, equation (2.4) with initial data u_0 admits a unique solution u (in the sense of distribution) in the class $\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,\tau)$,

(ii) for this solution

$$\|u\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,\tau)} \leq N(\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,\tau)} + \|\psi^{-\alpha}g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,\tau)} + \|u_0\|^p_{U^{\gamma+2,\alpha}_{p,\theta}(G)}), \tag{2.14}$$

where the constant N depends only on $d,\gamma,p,\theta,\delta_0,K$ and T.

Note that in the following theorem Assumption 2.6 is not assumed.

Theorem 2.9. *Let Assumptions 2.4 and 2.5 be satisfied, and*

$$|\psi^{1-\alpha}\bar{b}^i| + |\psi^{1-\alpha}b^i| + |\psi^{2(1-\alpha)}c| + |\psi^{1-\alpha}\nu| \leq K, \quad \forall \omega, t, x. \tag{2.15}$$

Then

(i) for any $\bar{f}^i \in \psi^{2\alpha}\mathbb{L}_{p,\theta}(G,\tau)$, $f \in \psi^{-1+2\alpha}\mathbb{H}^{-1}_{p,\theta}(G,\tau)$, $g \in \psi^{\alpha}\mathbb{L}_{p,\theta}(G,\tau)$ and $u_0 \in U^{1,\alpha}_{p,\theta}(G)$, equation (2.5) with initial data u_0 admits a unique solution u (in the sense of distribution) in the class $\mathfrak{H}^{1,\alpha}_{p,\theta}(G,\tau)$,

(ii) for this solution

$$\begin{aligned}\|u\|^p_{\mathfrak{H}^{1,\alpha}_{p,\theta}(G,\tau)} \leq N(\|\psi^{-2\alpha}\bar{f}\|^p_{\mathbb{L}_{p,\theta}(G,\tau)} + \|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^{-1}_{p,\theta}(G,\tau)} \\ + \|\psi^{-\alpha}g\|^p_{\mathbb{L}_{p,\theta}(G,\tau)} + \|u_0\|^p_{U^{1,\alpha}_{p,\theta}(G)}),\end{aligned} \tag{2.16}$$

where the constant N depends only on $d, \gamma, p, \theta, \delta_0, K$ and T.

Now we state the regularity of the solutions in terms of Hölder continuity in time and space, both inside the domain and near the boundary. The following results are immediate consequences of Lemma 2.1, Remark 3.2 and Theorem 3.3.

Corollary 2.10. *Let $u \in \mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,\tau)$ be the solution of Theorem 2.8 or Theorem 2.9. Let*

$$2/p < \mu < \beta < 1, \quad \gamma + 2 - \beta - d/p = m + \nu,$$

for some $m = 0, 1, ...,$ $\nu \in (0, 1]$. Then for any multi-indices i, j such that $|i| \leq m, |j| = m$

$$E \sup_{0\leq s<t\leq\tau} \frac{|\psi^{|i|-1+\theta/p}D^i(u(t)-u(s))|^p_{C(G)}}{|t-s|^{p\mu/2-1}} < \infty, \tag{2.17}$$

$$E \sup_{0\leq s<t\leq\tau} \frac{[\psi^{m+\nu-1+\theta/p}D^j(u(t)-u(s))]^p_{C^{\nu}(G)}}{|t-s|^{p\mu/2-1}} < \infty. \tag{2.18}$$

Remark 2.11. In particular, if $\gamma \geq -1$ and

$$\kappa_0 := 1 - 2/p - d/p > 0,$$

then for any $\kappa \in (0, \kappa_0)$, we have

$$E \sup_{t\geq 0} \sup_{x,y\in G} \frac{|\psi^{\kappa-1+\theta/p}(x)u(t,x) - \psi^{\kappa-1+\theta/p}(y)u(t,y)|^p}{|x-y|^{\kappa p}} < \infty, \tag{2.19}$$

$$E \sup_{x\in G} \sup_{t\neq s} \frac{|\psi^{-1+\theta/p}(x)(u(t,x)-u(s,x))|^p}{|t-s|^{\kappa p/2}} < \infty. \tag{2.20}$$

Indeed, to estimate the first term take $\beta = \kappa_0 - \kappa + 2/p$, then $1 - \beta - d/p = \nu = \kappa$ and (2.18) implies (2.19). For the second estimate, take $\mu = \kappa + 2/p$ and $\beta = 1 - d/p$, then $p\mu/2 - 1 = \kappa p/2$ and (2.17) implies (2.20). Obviously (2.19) and (2.20) yield that if $\theta \leq p$,

$$E \sup_{t\geq 0} \sup_{x,y\in G} \frac{|u(t,x)-u(t,y)|^p}{|x-y|^{\kappa p}} + E \sup_{x\in G} \sup_{t\neq s} \frac{|(u(t,x)-u(s,x)|^p}{|t-s|^{\kappa p/2}} < \infty.$$

Remark 2.12. The condition $\alpha \geq 1$ in the previous theorems is crucial in our proof. More precisely, our scaling argument fails if $\alpha < 1$. The case $\alpha < 1$ will be treated differently elsewhere under some additional conditions.

3 Auxiliary Results

In this section, we introduce an embedding theorem and few results about partitions of unity and point-wise multipliers.

A similar version of the following lemma can be found in [6] and [14].

Lemma 3.1. *There exists a constant $N = N(d, p, \gamma, |\gamma|+)$ such that*

$$\|af\|_{H^\gamma_{p,\theta}(G)} \leq N|a|^{(0)}_{|\gamma|+}\|f\|_{H^\gamma_{p,\theta}(G)}. \tag{3.21}$$

Proof. By Lemma 5.2 in [8],

$$\|af\|^p_{H^\gamma_{p,\theta}(G)} \leq N\sum_n e^{n\theta}\|a(e^n x)\zeta^2_{-n}(e^n x)f(e^n x)\|^p_{H^\gamma_p}$$

$$\leq N\sup_n |a(e^n x)\zeta_{-n}(e^n x)|_{B^{|\gamma|+}}\sum_n e^{n\theta}\|\zeta_{-n}(e^n x)f(e^n x)\|^p_{H^\gamma_p},$$

where B^ν is a natural Hölder's norm in $\mathbb{R}^d$. Therefore, it is enough to show

$$|a(e^n x)\zeta_{-n}(e^n x)|_{B^{|\gamma|+}} \leq N|a|^{(0)}_{|\gamma|+}. \tag{3.22}$$

Let $|\gamma|+ = m + \delta$, $\delta \in [0, 1)$. Assume that $\delta = 0$. Observe that

$$\sup_n \sup_x |D^k(\zeta_{-n}(e^n x))| < \infty, \qquad \forall k > 0. \tag{3.23}$$

If $k \leq m$ and $e^n x \in \text{supp}\zeta_{-n}(e\cdot)$, then (since $\rho(e^n x) \sim e^n$),

$$|e^{nk}(D^k a)(e^n x)| \leq N\rho^k(e^n x)|(D^k a)(e^n x)| \leq N|a|^{(0)}_{|\gamma|+}. \tag{3.24}$$

Obviously, (3.23) and (3.24) prove (3.22). Next let $\delta \neq 0$. To show

$$|D^m a(e^n x)\zeta_{-n}(e^n x) - D^m a(e^n y)\zeta_{-n}(e^n)| \leq N|x-y|^\delta, \forall x, y \in \mathbb{R}^d,$$

we may assume that $|x - y| \leq e^{-4}$ and $e^n x \in \text{supp}\zeta_{-n}(e\cdot)$. In this case, $e^n y \in \bar{B}_{e^{-4+n}}(e^n x) \subset G$ and $\rho(e^n x) \sim \rho(e^n x, e^n y) \sim e^n$. Thus, due to (3.23),

$$|D^m a(e^n x)\zeta_{-n}(e^n x) - D^m a(e^n y)\zeta_{-n}(e^n)|$$

$$\leq N\sum_{k\leq m}\rho^k(e^n x, e^n y)|(D^k a)(e^n x) - (D^k a)(e^n y)||D^{m-k}(\zeta_{-n}(e^n x))|$$

$$+N\sum_{k\leq m}e^{nk}|(D^k a)(e^n y)||D^{m-k}(\zeta_{-n}(e^n x)) - D^{m-k}(\zeta_{-n}(e^n y))|$$

$$\leq N|a|^{(0)}_{|\gamma|+}(e^{-n\delta}|e^n x - e^n y|^\delta + |x-y|) \leq N|x-y|^\delta.$$

The lemma is proved.

□

Remark 3.2. Let $\theta_1 \leq \theta_2$. By Lemmas 2.1 and 3.1

$$\|u\|_{H^{\gamma}_{p,\theta_2}(G)} \leq N\|\psi^{(\theta_2-\theta_1)/p}u\|_{H^{\gamma}_{p,\theta_1}(G)} \leq N\|u\|_{H^{\gamma}_{p,\theta_1}(G)}.$$

Consequently, if $\alpha_1 \leq \alpha_2$ then

$$\|u\|_{\mathfrak{H}^{\gamma,\alpha_1}_{p,\theta}(G,\tau)} \leq N\|u\|_{\mathfrak{H}^{\gamma,\alpha_2}_{p,\theta}(G,\tau)}.$$

The following results are due to Lototsky ([12]).

Theorem 3.3. *(i) For any $t \leq T$,*

$$\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,t)} \leq N(d,\gamma,p,T)\int_0^t \|u\|^p_{\mathfrak{H}^{\gamma+2,1}_{p,\theta}(G,s)}\,ds.$$

(ii) Let

$$2/p < \mu < \beta < 1.$$

Then

$$E\|u\|^p_{C^{\mu/2-1/p}([0,\tau],H^{\gamma+2-\beta}_{p,\theta-p}(G))} \leq N(\mu,\beta,d,\gamma,p,T)\|u\|^p_{\mathfrak{H}^{\gamma+2,1}_{p,\theta}(G)}.$$

We choose and fix smooth functions ξ_n such that $|D^m\xi_n| \leq N(m)e^{nm}$, $\mathrm{supp}\xi_n \subset (G_{n-1}\cup G_n \cup G_{n+1})$ and $\xi_n = 1$ on the support of ζ_n.

Lemma 3.4. *Let Assumptions 2.4(ii) and 2.5 be satisfied. By I we denote $d\times d$ identity matrix. Define*

$$a^{ij}_n(t,x) = e^{-2n\alpha}\psi^{2\alpha}(e^n x)\xi^2_{-n}(e^n x)a^{ij}(e^{2n(1-\alpha)}t, e^n x) + (1-\xi^2_{-n}(e^n x))I,$$
$$\sigma^{ik}_n(t,x) = e^{-n\alpha}\psi^{\alpha}(e^n x)\xi_{-n}(e^n x)\sigma^{ik}(e^{2n(1-\alpha)}t, e^n x).$$

Then

(i) For any $\lambda \in \mathbb{R}^d$,

$$e^{-4\alpha}\delta_0|\lambda|^2 \leq (a^{ij}_n - 1/2\sigma^{ik}_n\sigma^{jk}_n)\lambda^i\lambda^j \leq e^{4\alpha}K|\lambda|^2.$$

(ii) For any $\varepsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\sup_n \sup_{\omega,t}(|a^{ij}_n(t,x) - a^{ij}_n(t,y)| + |\sigma^i_n(t,x) - \sigma^i_n(t,y)|) < \varepsilon,$$

whenever $x, y \in \mathbb{R}^d$ and $|x-y| < \delta$.

(iii)

$$\sup_n \sup_{\omega,t}(|a^{ij}_n(t,\cdot)|_{B^{|\gamma|+}} + |\sigma^i_n|_{B^{|\gamma+1|+}}) < \infty. \tag{3.25}$$

Proof. (i) is obvious and (3.25) follows from the same arguments as in the proof of Lemma 3.1. Thus we only give a proof of the second assertion. Let $\delta \le e^{-4}$ and $|x-y|<\delta$. Without loss of generality, we assume that $\xi_{-n}(e^n x)\neq 0$. Observe that

$$e^n y \in \bar{B} := \bar{B}_{e^n\delta}(e^n x) \subset G, \quad |e^n x - e^n y| \le \delta e^n \le N_0 \delta \rho(e^n x, e^n y),$$

and for any $z \in B_{e^n\delta}(e^n x)$ we have $\rho(z) \sim e^n$. Thus,

$$|\xi_{-n}(e^n x) - \xi_{-n}(e^n y)| \le |x-y| e^n \sup_z |D(\xi_{-n})(z)| \le N_1 |x-y|,$$

$$|\psi^{2\alpha}(e^n x) - \psi^{2\alpha}(e^n y)| \le \sup_{z\in B} |D\psi^{2\alpha}(z)||e^n x - e^n y| \le N e^{2n\alpha}|x-y|,$$

and

$$\begin{aligned}
&|e^{-2n\alpha}\psi^{2\alpha}(e^n x)a(e^n x)\xi_{-n}(e^n x) - e^{-2n\alpha}\psi^{2\alpha}(e^n y)a(e^n y)\xi_{-n}(e^n y)| \\
&\le e^{-2n\alpha}\psi^{2\alpha}(e^n x)\xi_{-n}(e^n x)|a(e^n x) - a(e^n y)| \\
&+ |a(e^n y)| e^{-2n\alpha}\psi^{2\alpha}(e^n x)|\xi_{-n}(e^n x) - \xi_{-n}(e^n y)| \\
&+ |a(e^n y)\xi_{-n}(e^n y)| e^{-2n\alpha}|\psi^{2\alpha}(e^n x) - \psi^{2\alpha}(e^n y)| \\
&\le N_2(|a(e^n x) - a(e^n y)| + \delta + \delta).
\end{aligned}$$

Note that the constant N_i are independent of x, y and n. So, if $\varepsilon > 0$ is given, then it is enough to take $\delta > 0$ such that $(N_1 + 2N_2)\delta < \varepsilon/2$ and $N_2|a(t,x) - a(t,y)| \le \varepsilon/3$ whenever $|x-y| < N_0\delta\rho(x,y)$.

We handle σ_n^i similarly. The lemma is proved.

□

The following lemma is taken from [13].

Lemma 3.5. *Let $\{\phi_k : k = 1, 2, ...\}$ be a collection of $C_0^\infty(G)$ functions such that for each $m > 0$*

$$\sup_{x\in G} \sum_k \rho^m(x)|D^m \phi_k(x)| \le M(m) < \infty.$$

Then there exists a constant $N = N(d, \gamma, M)$ such that for any $f \in \mathbb{H}^\gamma_{p,\theta}(G)$,

$$\sum_k \|\phi_k f\|^p_{H^\gamma_{p,\theta}(G)} \le N \|f\|^p_{H^\gamma_{p,\theta}(G)}.$$

If in addition

$$\sum_k |\phi_k(x)|^p \ge c > 0,$$

then

$$\|f\|^p_{H^\gamma_{p,\theta}(G)} \le N(d, \gamma, M, c) \sum_k \|\phi_k f\|^p_{H^\gamma_{p,\theta}(G)}.$$

4 Proof of Theorem 2.8

As usual, we may assume that $\tau \equiv T$ (see [8]). For a moment, we assume that $b^i = c = \nu^k = 0$. Take a_n^{ij} and σ_n^{ik} from Lemma 3.4. Denote

$$c_n := e^{-2n(1-\alpha)}, \quad w_t^k(n) := e^{-n(1-\alpha)} w^k_{e^{2n(1-\alpha)}t}$$

Then for each n, $w_t^k(n)$ are independent one dimensional Wiener processes. By Theorem 5.1 in [8], for any $f \in \mathbb{H}_p^\gamma(c_nT)$, $g \in \mathbb{H}_p^{\gamma+1}(c_nT)$ and $u_0 \in U_p^{\gamma+2}$ the equation

$$du = (a_n^{ij} u_{x^ix^j} + f)dt + (\sigma_n^{ik} u_{x^i} + g^k) dw_t^k(n) \quad u(0,\cdot) = u_0, \tag{4.26}$$

has a unique solution $u \in \mathbb{H}_p^{\gamma+2}(c_nT)$ and u satisfies

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(c_nT)} \le N(\|f\|_{\mathbb{H}_p^\gamma(c_nT)} + \|g\|_{\mathbb{H}_p^{\gamma+1}(c_nT)} + \|u_0\|_{U_p^{\gamma+2}}), \tag{4.27}$$

where the constant N depends only $d, p, \gamma, \delta_0, K, c_nT, |a_n|_{B^{|\gamma|+}}, |\sigma_n|_{B^{|\gamma+1|+}}$ and uniform continuity of a_n, σ_n.

By $S_n(f,g,u_0)$ we denote the the solution of (4.26). Define

$$\bar{S}_n(f,g,u_0)(t,x) = S_n(f,g,u_0)(c_nt, e^{-n}x).$$

From now on, without loss of generality, we assume that

$$\sum_n \zeta_{-n}^2(x) = 1, \quad \forall x \in G.$$

Remember the fact that a function v satisfies

$$dv = f\,dt + g^k\,dw_t^k, \quad t \le T$$

if and only if $v_c(t,x) := v(c^2t, cx)$ $(c > 0)$ satisfies

$$dv_c = c^2 f(c^2t, cx)\,dt + cg(c^2t, cx) d(c^{-1} w^k_{c^2t}), \quad t \le c^{-2}T.$$

It follows that if $u \in \mathfrak{H}_{p,\theta}^{\gamma+2,\alpha}(G,T)$ is a solution of equation (2.4), then $v_n(t,x) := (\zeta_{-n}u)(c_n^{-1}t, e^nx)$ satisfies

$$dv_n = (a_n^{ij} v_{nx^ix^j} + A_nu + f_n)dt + (\sigma_n^{ik} v_{nx^i} + B_n^k u + g_n^k) dw_t^k(n),$$

where

$$\begin{aligned} A_nu(t,x) &:= -2a_n^{ij} e^{2n} u_{x^i}(c_n^{-1}t, e^nx)\zeta_{-nx^j}(e^nx) \\ &\quad -a_n^{ij} e^{2n} u(c_n^{-1}t, e^nx)\zeta_{-nx^ix^j}(e^nx), \\ B_n^k u(t,x) &:= -\sigma_n^{ik} e^n u(c_n^{-1}t, e^nx)\zeta_{-nx^i}(e^nx), \\ f_n(t,x) &:= e^{2n(1-\alpha)} f(e^{2n(1-\alpha)}t, e^nx)\zeta_{-n}(e^nx), \\ g_n^k(t,x) &:= e^{n(1-\alpha)} g(e^{2n(1-\alpha)}t, e^nx)\zeta_{-n}(e^nx), \\ u_{0n} &:= u_0(e^nx)\zeta_{-n}(e^nx). \end{aligned} \tag{4.28}$$

Consequently,

$$v_n(t,x) := (\zeta_{-n}u)(c_n^{-1}t, e^n x) = S_n(A_n u + f_n, B_n u + g_n, u_{0n})$$

and

$$u = \sum_n \zeta_{-n}(\zeta_{-n}u) = \sum_n \zeta_{-n}\bar{S}_n(A_n u + f_n, B_n u + g_n, u_{0n}). \tag{4.29}$$

To proceed further, we need the following lemma.

Lemma 4.1. *Fix $f \in \psi^{-1+2\alpha}\mathbb{H}^{\gamma}_{p,\theta}(G,T), g \in \psi^{\alpha}\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)$ and $u_0 \in U^{\gamma+2}_{p,\theta}(G)$. Then a sufficiently high power of the operator*

$$\mathcal{R} : u \to \sum_n \zeta_{-n}\bar{S}_n(A_n u + f_n, B_n u + g_n, u_{0n}) \tag{4.30}$$

is a contraction in $\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T) \cap \{u : u(0,\cdot) = u_0\}$, and the unique solution $u \in \mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)$ of (4.29) satisfies the estimate (2.14).

Proof. For simplicity, we use the notations S_n and $\bar{S}_n$ instead of $S_n(A_n u + f_n, B_n u + g_n, u_{0n})$ and $\bar{S}_n(A_n u + f_n, B_n u + g_n, u_{0n})$, respectively.
Note that $\zeta_n\zeta_m = 0$ if $|n-m| > 1$. By Lemmas 3.5 and 3.1,

$$\|\mathcal{R}u\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)} \le N\sum_n \|\zeta_{-n}\mathcal{R}u\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)} \le N\sum_n \|\zeta_{-n}\bar{S}_n\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)}. \tag{4.31}$$

By definition,

$$\|\zeta_{-n}\bar{S}_n\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)} = \|\psi^{-1}\zeta_{-n}\bar{S}_n\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(G,T)} + \|\psi^{1-2\alpha}\mathbb{D}(\zeta_{-n}\bar{S}_n)\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(G,T)}$$
$$+\|\psi^{-\alpha}\mathbb{S}(\zeta_{-n}\bar{S}_n)\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)} + \|\zeta_{-n}u_0\|^p_{U^{\gamma+2,\alpha}_{p,\theta}(G)}.$$

Remember that $\|u(e^{\pm 1}x)\|_{H^\nu} \sim \|u(x)\|_{H^\nu}$ and $\sup_n |\zeta_{-n}(e^n x)|_{B^\nu} < \infty$ for each $\nu > 0$. Thus (cf. Lemma 5.2 in [8]),

$$\sum_n \|\zeta_{-n}\bar{S}_n\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta-p}(G,T)} \le N\sum_n e^{n(\theta-p)}\|\zeta_{-n}(e^n x)\bar{S}_n(t,e^n x)\|^p_{\mathbb{H}^{\gamma+2}_p(T)}$$
$$\le N\sum_n e^{n(\theta-p+2-2\alpha)}\|S_n(t,x)\|^p_{\mathbb{H}^{\gamma+2}_p(c_n T)}. \tag{4.32}$$

By writing the equation for $\zeta_{-n}(e^n x)S_n$, we find that $\bar{v}_n := \zeta_{-n}\bar{S}_n$ satisfies

$$d\bar{v}_n = \mathbb{D}(\zeta_{-n}\bar{S}_n)\,dt + \mathbb{S}(\zeta_{-n}\bar{S}_n)\,dw^k_t$$
$$= [\psi^{2\alpha}a^{ij}\bar{v}_{nx^ix^j} - 2\psi^{2\alpha}a^{ij}(\bar{S}_n\zeta_{-nx^i})_{x^j} + \psi^{2\alpha}a^{ij}\bar{S}_n\zeta_{-nx^ix^j}$$
$$-2\psi^{2\alpha}a^{ij}u_{x^i}\zeta_{-nx^j}\zeta_{-n} - \psi^{2\alpha}a^{ij}u\zeta_{-nx^ix^j}\zeta_{-n} + \zeta^2_{-n}f]\,dt$$
$$+[\psi^{\alpha}\sigma^{ik}\bar{v}_{nx^i} - \psi^{\alpha}\sigma^{ik}\bar{S}_n\zeta_{-nx^i} - \psi^{\alpha}\sigma^{ik}u\zeta_{-nx^i}\zeta_{-n} + \zeta^2_{-n}g]\,dw^k_t.$$

Thus, by Lemmas 2.1 and 3.1,

$$\sum_n \|\psi^{1-2\alpha}\mathbb{D}(\zeta_{-n}\bar{S}_n)\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}$$

$$\leq N\sum_n \|\psi(\zeta_{-n}\bar{S}_n)_{xx}\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)} + N\sum_n \|\psi(\bar{S}_n\zeta_{-nx})_x\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}$$

$$+N\sum_n \|\psi^{-1}\bar{S}_n\psi^2\zeta_{-nxx}\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)} + N\sum_n \|u_x\psi\zeta_{-nx}\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}$$

$$+N\sum_n \|\psi^{-1}u\psi^2\zeta_{-nxx}\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)} + N\sum_n \|\psi^{1-2\alpha}f\zeta_{-n}\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}$$

$$\leq N\sum_n \|\zeta_{-n}\bar{S}_n\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta-p}(G,T)} + N\sum_n \|\bar{S}_n\zeta_{-nx}\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)}$$

$$+\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)} + N\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}.$$

Here,

$$\sum_n \|\zeta_{-nx}\bar{S}_n\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)}$$

$$= \sum_{n,m} e^{m(\theta-p)}\|\bar{S}_n(e^m x)e^m\zeta_{-nx}(e^m x)\zeta_{-m}(e^m x)\|^p_{\mathbb{H}^{\gamma+1}_p(T)}$$

$$\leq N\sum_n e^{n(\theta-p)}\|\bar{S}_n(t,x)(e^n x)\|^p_{\mathbb{H}^{\gamma+2}_p(T)}$$

$$= N\sum_n e^{n(\theta-p+2-2\alpha)}\|S_n(t,x)\|^p_{\mathbb{H}^{\gamma+2}_p(c_nT)}.$$

We estimate $\sum_n \|\psi^{-\alpha}\mathbb{S}(\zeta_{-n}\bar{S}_n)\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)}$ similarly, and conclude that

$$\|\mathcal{R}u\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)} \leq N\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)} + N\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}$$

$$+\|\psi^{-\alpha}g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta}(G)}$$

$$+\sum_n e^{n(\theta-p+2-2\alpha)}\|S_n(t,x)\|^p_{\mathbb{H}^{\gamma+2}_p(c_nT)}. \tag{4.33}$$

Since G is bounded, we may assume that $\zeta_{-n}=0$ for all $n>1$. For each $n\leq 0$, we have

$$c_nT := e^{-2n(1-\alpha)}T \leq T. \tag{4.34}$$

Thus by (4.27), there exists a constant N independent of n (due to Lemma 3.4 and (4.34)) such that for each $n\leq 0$,

$$\|S_n(t,x)\|^p_{\mathbb{H}^{\gamma+2}_p(c_nT)} \leq N\|A_nu+f_n\|^p_{\mathbb{H}^\gamma_p(c_nT)}$$

$$+N\|B_nu+g_n\|^p_{\mathbb{H}^{\gamma+1}_p(c_nT)} + N\|u_{0n}\|^p_{U^{\gamma+2}_p}.$$

Also, by Lemma 2.1,

$$\sum_{n\le 0} e^{n(\theta-p+2-2\alpha)}(\|A_n u\|^p_{\mathbb{H}^\gamma_p(c_nT)}+\|B_n u\|^p_{\mathbb{H}^{\gamma+1}_p(c_nT)})$$

$$=\sum_{n\le 0} e^{n(\theta-p)}\|(A_n u)(c_nt,x)\|^p_{\mathbb{H}^\gamma_p(T)}+\|(B_n u)(c_nt,x)\|^p_{\mathbb{H}^{\gamma+1}_p(T)}$$

$$\le N\sum_{n\le 0} e^{n\theta}\|u_x(e^n x)e^n\zeta_{-nx}(e^n x)\|^p_{\mathbb{H}^\gamma_p(T)}$$

$$+N\sum_{n\le 0} e^{n(\theta-p)}\|u(e^n x)e^{2n}\zeta_{-nxx}(e^n x)\|^p_{\mathbb{H}^\gamma_p(T)}$$

$$+N\sum_{n\le 0} e^{n(\theta-p)}\|u(e^n x)e^{n}\zeta_{-nx}(e^n x)\|^p_{\mathbb{H}^{\gamma+1}_p(T)}$$

$$\le N\|u_x\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}+N\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta-p}(G,T)}\le N\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)}.$$

Similarly,

$$\sum_{n\le 0} e^{n(\theta-p+2-2\alpha)}(\|f_n\|^p_{\mathbb{H}^\gamma_p(c_nT)}+\|g_n\|^p_{\mathbb{H}^{\gamma+1}_p(c_nT)})$$

$$\le N\sum_{n\le 0} e^{n(\theta+p(1-2\alpha))}\|f(t,e^n x)\zeta_{-n}(e^n x)\|^p_{\mathbb{H}^\gamma_p(T)}$$

$$+N\sum_{n\le 0} e^{n(\theta-p\alpha)}\|g(t,e^n x)\zeta_{-n}(e^n x)\|^p_{\mathbb{H}^{\gamma+1}_p(T)}$$

$$\le N\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}+N\|\psi^{-\alpha}g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)},$$

and

$$\sum_{n} e^{n(\theta+p(\frac{2}{p}(1-\alpha)-1))}\|u_0(e^n x)\zeta_{-n}(e^n x)\|^p_{U^{\gamma+2}_p}\le N\|u_0\|^p_{U^{\gamma+2}_{p,\theta}(G)}.$$

Hence, coming back to (4.33), we get

$$\|\mathcal{R}u\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,\tau)}\le N(\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_p(T)}+\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^\gamma_p(T)}$$
$$+\|\psi^{-\alpha}g\|^p_{\mathbb{H}^{\gamma+1}_p(T)}+\|u_0\|^p_{U^{\gamma+2}_{p,\theta}(G)}). \tag{4.35}$$

Note that $\bar{\mathfrak{H}}^{\gamma+2,\alpha}_{p,\theta}(G,T):=\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)\cap\{u:u(0,\cdot)=u_0\}$ is a complete Banach space and contains $\mathcal{R}0$ (thus not empty), where

$$\mathcal{R}0:=\sum_n \zeta_{-n}\bar{S}_n(f_n,g_n,u_{0n}).$$

By (4.35) and Theorem 3.3, for any $u,v\in\bar{\mathfrak{H}}^{\gamma+2,\alpha}_{p,\theta}(G,T)$,

$$\mathcal{R}u-\mathcal{R}v=\sum_n \zeta_{-n}\bar{S}_n(A_n(u-v),B_n(u-v),0),$$

$$\begin{aligned}\|\mathcal{R}u-\mathcal{R}v\|_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)} &\leq N\|\psi^{-1}(u-v)\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)}\\ &\leq N\int_0^T \|u-v\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,s)}\,ds. \end{aligned}\tag{4.36}$$

(4.36) shows that there exists $m_0>0$ such that $\mathcal{R}^m$ is a contraction in $\bar{\mathfrak{H}}^{\gamma+2,\alpha}_{p,\theta}(G,T)$, and all the assertions of the lemma follow from this and (4.35). For more technical details, we refer to the proof of Theorem 6.4 in [8]. The lemma is proved.

□

Let u be the solution of (4.29). We will show that u satisfies equation (2.4). Obviously $u(0,\cdot)=u_0$, and (by definition) for some $f_0\in\psi^{-1+2\alpha}\mathbb{H}^{\gamma}_{p,\theta}(G,T)$ and $g_0\in\psi^{\alpha}\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)$

$$du=f_0\,dt+g^k_0\,dw^k_t.$$

Observe that u satisfies equation (2.4) with $\bar{f}:=f_0-\psi^{2\alpha}a^{ij}u_{x^ix^j}$ and $\bar{g}^k:=g^k_0-\psi^{\alpha}\sigma^{ik}u_{x^i}$ instead of f and g^k, respectively. By the above arguments (see (4.29))

$$u=\sum_n\zeta_{-n}\bar{S}_n(A_nu+\bar{f}_n,B_nu+\bar{g}_n,u_{0n}),$$

where $\bar{f}_n,\bar{g}_n,u_{0n}$ are defined from $\bar{f},\bar{g},u_0$ as in (4.28). Also,

$$0=\sum_n\zeta_{-n}\bar{S}_n(\tilde{f}_n,\tilde{g}_n,0),\tag{4.37}$$

where $\tilde{f}=f-\bar{f}$, $\tilde{g}=g-\bar{g}$ and $\tilde{f}_n$, $\tilde{g}_n$ are defined as before.
Define the operators $\bar{A}_n$ and $\bar{B}_n$ such that

$$\bar{A}_nu=2a^{ij}\psi^{2\alpha}u_{x^i}\zeta_{-nx^j}-a^{ij}\psi^{2\alpha}u\zeta_{-nx^ix^j},$$
$$\bar{B}_nu=\psi^{\alpha}\sigma^{ik}u\zeta_{-nx^i}.$$

From (4.37),

$$0=\mathbb{D}\sum_n\zeta_{-n}\bar{S}_n(\tilde{f}_n,\tilde{g}_n,0)=\tilde{f}-\sum_n\bar{A}_n\bar{S}_n(\tilde{f}_n,\tilde{g}_n,0),$$
$$0=\mathbb{S}\sum_n\zeta_{-n}\bar{S}_n(\tilde{f}_n,\tilde{g}_n,0)=\tilde{g}-\sum_n\bar{B}_n\bar{S}_n(\tilde{f}_n,\tilde{g}_n,0).$$

Therefore, to show $\tilde{f}=\tilde{g}^k=0$, we only need to prove that a sufficiently high power of the operator

$$\bar{\mathcal{R}}:(f,g)\to\Big(\sum_n\bar{A}_n\bar{S}_n(f_n,g_n,0),\sum_n\bar{B}_n\bar{S}_n(f_n,g_n,0)\Big)$$

is a contraction in $\mathcal{F}^{\gamma,\alpha}_{p,\theta}(G,T):=\psi^{-1+2\alpha}\mathbb{H}^{\gamma}_{p,\theta}(G,T)\times\psi^{\alpha}\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)$.
By Lemma 3.5,

$$\|\sum_n\psi\zeta_{-nx}\bar{S}_{nx}\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(G,T)}\leq N\sum_{\substack{m,n\\|m-n|\leq 1}}\|\psi\zeta^2_{-m}\zeta_{-nx}\bar{S}_{nx}\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(G,T)}$$

$$= N \sum_{\substack{m,n \\ |m-n|\le 1}} \|\psi\zeta_{-nx}(\bar{S}_n\zeta_{-m}^2)_x - 2\psi^{-1}\bar{S}_n\zeta_{-m}\psi^2\zeta_{-mx}\zeta_{-nx}\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}$$

$$\le N \sum_{\substack{m,n \\ |m-n|\le 1}} \|\psi^{-1}\zeta_{-m}\bar{S}_n\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)} \le N\int_0^T \sum_{\substack{m,n \\ |m-n|\le 1}} \|\zeta_{-m}\bar{S}_n\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,s)}\,ds.$$

As in the proof of Lemma 4.1 (see (4.31) and (4.35)),

$$\sum_{\substack{m,n \\ |m-n|\le 1}} \|\zeta_{-m}\bar{S}_n\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,s)} \le N\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,s)} + N\|\psi^{-\alpha}g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,s)}.$$

We estimate other terms in $\sum_n \bar{A}_n\bar{S}_n$ and $\sum_n \bar{B}_n\bar{S}_n$ similarly (actually much easily) and get

$$\|\bar{\mathcal{R}}(f,g)\|^p_{\mathcal{F}^\gamma_{p,\theta}(G,T)} \le N\int_0^T \|(f,g)\|^p_{\mathcal{F}^\gamma_{p,\theta}(G,s)}\,ds.$$

This shows that a sufficiently high power of $\bar{\mathcal{R}}$ is a contraction and $\bar{f} = f, \bar{g} = g$.

For general case (previously we assumed that $b^i = c = \nu^k = 0$), having the method of continuity in mind, we only show that (2.14) holds true given that a solution $u \in \mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)$ already exists. Let $\bar{u} \in \mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)$ be the solution of

$$d\bar{u} = \psi^{2\alpha}\Delta\bar{u}\,dt, \quad \bar{u}(0,\cdot) = u_0.$$

Then

$$\|\bar{u}\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)} \le N\|u_0\|^p_{U^{\gamma+2}_{p,\theta}(G)}.$$

Thus by considering $u - \bar{u}$, as usual, we may assume that $u_0 = 0$.

By the previous results (when $b^i = c = \nu^k = 0$),

$$\|u\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)} \le N(\|\psi^{1-2\alpha}\tilde{f}\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)} + \|\psi^{-\alpha}\tilde{g}\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)}),$$

where

$$\tilde{f} = \psi^\alpha b^i u_{x^i} + cu + f, \quad \tilde{g}^i k = \nu^k u + g^k.$$

By Lemma 3.1,

$$\|\psi^{1-2\alpha}\tilde{f}\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)} + \|\psi^{-\alpha}\tilde{g}\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)}$$

$$\le N\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)} + N\|\psi^{-\alpha}g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)} + N\|\psi^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)}.$$

Thus, by Theorem 3.3 for each $t \le T$,

$$\|u\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)} \le N\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^\gamma_{p,\theta}(G,T)}$$

$$+N\|\psi^{-\alpha}g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)} + \int_0^t \|u\|^p_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,s)}\,ds.$$

This and Gronwall's inequality lead to (2.14). The theorem is proved.

5 Proof of Theorems 2.9

Consider the operators

$$L_0 u := \psi^{2\alpha}\Delta u = D_i(\psi^{2\alpha}u_{x^i}) - 2\alpha\psi^{2\alpha-1}\psi_{x^i}u_{x^i},$$

$$L_1 u := D_i(\psi^{2\alpha}a^{ij}u_{x^j} + \bar{b}^i u) + b^i u_{x^i} + cu, \quad \Lambda_1^k u : \psi^\alpha\sigma^{ik}u_{x^i} + \nu^k u.$$

One can easily check that the coefficients of the operators $L_\lambda := (1-\lambda)L_0 + \lambda L_1$ and $\Lambda_\lambda := \lambda\Lambda_1$ satisfy Assumptions 2.4, 2.5 and (2.15). Also note that

$$\|\psi^{1-2\alpha}\bar{f}^i_x\|_{H^{-1}_{p,\theta}(G)} \le N\|\bar{f}^i\|_{L_{p,\theta-2p\alpha}(G)} \le N\|\psi^{-2\alpha}\bar{f}^i\|_{L_{p,\theta}(G)}.$$

By Theorem 2.8, the equation

$$du = (L_0 u + \bar{f}^i_{x^i} + f)dt + (\Lambda_0^k u + g^k)dw^k_t$$

has a unique solution $u \in \mathfrak{H}^{1,\alpha}_{p,\theta}(G,\tau)$. Thus by the method of continuity, we only need to prove that the estimate (2.16) holds true given that a solution $u \in \mathfrak{H}^{1,\alpha}_{p,\theta}(G,\tau)$ of equation (2.5) already exists.

Again without loss of generality we assume that $\tau \equiv T$ and $\zeta_{-n} = 0$ for all $n > 0$. Also as in the proof of Theorem 2.8, we assume that $u_0 = 0$.

Step 1. We will show that there exists a constant $\varepsilon_0 = \varepsilon_0(d,p,\delta_0,K) > 0$ such that the theorem holds true if $T \le \varepsilon_0 \le 1$. As before, denote $c_n := e^{-2n(1-\alpha)}$. By Lemma 2.1,

$$\|\psi^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}(G,T)} \le N\sum_{n\le 0} e^{n(\theta-p)}\|u(e^n x)\zeta_{-n}(e^n x)\|^p_{\mathbb{H}^1_p(T)}$$

$$= N\sum_{n\le 0} e^{n(\theta-p+2-2\alpha)}\|u(e^{2n(1-\alpha)}t, e^n x)\zeta_{-n}(e^n x)\|^p_{\mathbb{H}^1_p(c_n T)}. \tag{5.38}$$

Denote $v_n(t,x) = u(e^{2n(1-\alpha)}t, e^n x)\zeta_{-n}(e^n x)$. Then v_n satisfies

$$dv_n = (D_i(a^{ij}_n v_{nx^j} + \bar{b}^i_n v_n + \bar{f}^i_n) + b_n v_{nx^i} + c_n v_n + f_n)\,dt$$
$$+ (\sigma^{ik}_n v_{nx^i} + \nu^k_n v_n + g^k_n)\,dw^k_t(n), \tag{5.39}$$

where $a^{ij}_n, \sigma^{ik}_n, w^k_t(n)$ are defined as before and

$$\bar{b}^i_n(t,x) = e^{n-2n\alpha}\psi^\alpha(e^n x)\xi_{-n}(e^n x)\bar{b}^i(c_n^{-1}t, e^n x),$$

$$b^i_n(t,x) = e^{n-2n\alpha}\psi^\alpha(e^n x)\xi_{-n}(e^n x)b^i(c_n^{-1}t, e^n x),$$

$$c_n(t,x) = e^{2n(1-\alpha)}c(c_n^{-1}t, e^n x)\xi_{-n}(e^n x),$$

$$\nu^k_n(t,x) = e^{n(1-\alpha)}\nu^k(c_n^{-1}t, e^n x)\xi_{-n}(e^n x),$$

$$\bar{f}^i_n(t,x) = -a^{ij}_n e^n\zeta_{-nx^j}(e^n x)u(c_n^{-1}t, e^n x) + e^{n-2n\alpha}\bar{f}^i(c_n^{-1}t, e^n x)\zeta_{-n}(e^n x),$$

$$g^k_n(t,x) = -\sigma^{ik}_n u(c_n^{-1}t, e^n x)e^n\zeta_{-nx^i}(e^n x) + e^{n(1-\alpha)}g^k(c_n^{-1}t, e^n x)\zeta_{-n}(e^n x),$$

$$f_n(t,x) = -e^n a_n^{ij} u_{x^j}(c_n^{-1}t, e^n x) e^n \zeta_{-nx^i}(e^n x)$$
$$+\bar{b}_n u(c_n^{-1}t, e^n x) e^n \zeta_{-nx^i}(e^n x) + e^{n-2n\alpha} \bar{f}^i(c_n^{-1}t, e^n x) e^n \zeta_{-nx^i}(e^n x)$$
$$-b_n^i e^n \zeta_{-nx^i}(e^n x) u(c_n^{-1}t, e^n x) + e^{2n(1-\alpha)} f(c_n^{-1}t, e^n x) \zeta_{-n}(e^n x).$$

Note that $\psi(e^n x) \sim e^n$ on the support of $\xi_{-n}(e^n x)$. It follows from (2.15) that

$$\sup_n \sup_{\omega,t,x} (|\bar{b}_n^i| + b_n^i| + |c_n| + |\nu_n|) < \infty.$$

By Theorem 2.12 in [4],

$$\|v_n\|^p_{\mathbb{H}^1_p(c_nT)} \leq N(\|\bar{f}_n\|^p_{\mathbb{L}_p(c_nT)} + N\|f_n\|^p_{\mathbb{H}^{-1}_p(c_nT)} + \|g_n\|^p_{\mathbb{L}_p(c_nT)}). \tag{5.40}$$

Actually, due to the term $-e^n a_n^{ij} u_{x^j}(c_n^{-1}t, e^n x) e^n \zeta_{-nx^i}(e^n x)$ of f_n, (5.40) only yields

$$\|\psi^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}(G,T)} \leq N\|\psi^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}(G,T)} + \cdots.$$

Of course, this estimate is useless unless $N < 1$. The following argument below is to avoid estimating $\|e^n a_n^{ij} u_{x^j}(c_n^{-1}t, e^n x) e^n \zeta_{-nx^i}(e^n x)\|_{\mathbb{H}^{-1}_p(c_nT)}$.

Denote

$$\tilde{f}_n(t,x) = -e^n a_n^{ij} u_{x^j}(c_n^{-1}t, e^n x) e^n \zeta_{-nx^i}(e^n x) \in \mathbb{L}_p(c_nT).$$

By Theorem 5.1 in [8], the equation

$$du = (\Delta u + \tilde{f}_n)dt, \quad u(0,\cdot) = 0$$

has a unique solution $u_n \in \mathbb{H}^2_p(c_nT)$, and (see Theorems 7.1 and 7.2 in [8])

$$\|u_n\|^p_{\mathbb{H}^1_p(c_nT)} \leq N(T)\|\tilde{f}_n\|^p_{\mathbb{L}_p(c_nT)}, \tag{5.41}$$

where $N(T)$ is independent of n (since $c_nT \leq T$), and $N(T) \downarrow 0$ as $T \to 0$.

$\hat{v}_n := v_n - u_n$ satisfies (5.39) with

$$\hat{\bar{f}}_n := \bar{f}_n + \bar{b}_n u_n + (a_n^{ij} - \delta^{ij}) u_{nx^i},$$

$$\hat{f}_n := f_n - \tilde{f}_n + b_n u_{nx} + c_n u_n, \quad \hat{g}_n := g_n + \sigma^i u_{nx^i} + \nu u_n$$

instead of $\bar{f}_n, f_n$ and g_n, respectively. Thus by Theorem 2.12 in [4], there exists a constant N depending only on d, p, δ_0 and K (remember $c_nT \leq T \leq 1$) such that

$$\|\hat{v}_n\|^p_{\mathbb{H}^1_p(c_nT)} \leq N(\|\hat{\bar{f}}_n\|^p_{\mathbb{L}_p(c_nT)} + \|\hat{f}_n\|^p_{\mathbb{H}^{-1}_p(c_nT)} + \|\hat{g}_n\|^p_{\mathbb{L}_p(c_nT)}). \tag{5.42}$$

Consequently,

$$\|v_n\|^p_{\mathbb{H}^1_p(c_nT)} \leq N(\|\hat{\bar{f}}_n\|^p_{\mathbb{L}_p(c_nT)} + \|\hat{f}_n\|^p_{\mathbb{H}^{-1}_p(c_nT)} + \|\hat{g}_n\|^p_{\mathbb{L}_p(c_nT)} + \|u_n\|_{\mathbb{H}^1_p(c_nT)})$$

$$\leq N(T)\|e^n u_x(c_n^{-1}t, e^n x) e^n \zeta_{-nx}(e^n x)\|^p_{\mathbb{L}_p(c_nT)}$$

$$+N\|e^{n}\zeta_{-nx}(e^{n}x)u(c_n^{-1}t,e^{n}x)\|^p_{\mathbb{L}_p(c_nT)}$$

$$+N\|e^{n(1-2\alpha)}\bar{f}(c_n^{-1}t,e^{n}x)[\zeta_{-n}(e^{n}x)+e^{n}\zeta_{-nx}(e^{n}x)]\|^p_{\mathbb{L}_p(c_nT)}$$

$$+N\|e^{2n(1-\alpha)}f(c_n^{-1}t,e^{n}x)\zeta_{-n}(e^{n}x)\|^p_{\mathbb{H}^{-1}_p(c_nT)}$$

$$+N\|e^{n(1-\alpha)}g(c_n^{-1}t,e^{n}x)\zeta_{-n}(e^{n}x)\|^p_{\mathbb{L}_p(c_nT)}.$$

Coming back to (5.38), by Lemma 2.1, we get

$$\|\psi^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}(G,T)}\le N\|\psi^{-1}u\|^p_{\mathbb{L}_{p,\theta}(G,T)}+N\|\psi^{-2\alpha}\bar{f}\|^p_{\mathbb{L}_{p,\theta}(G,T)}$$

$$+N\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^{-1}_{p,\theta}(G,T)}+N\|\psi^{-\alpha}g\|^p_{\mathbb{L}_{p,\theta}(G,T)}+NN(T)\|\psi^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}(G,T)}.$$

Now fix ε_0 such that $NN(T)\le 1/2$ for each $T\le\varepsilon_0$, then by Theorem 3.3 for each $t\le T$

$$\|u\|^p_{\mathfrak{H}^{1,\alpha}_{p,\theta}(G,t)}\le N\int_0^t\|u\|^p_{\mathfrak{H}^{1,\alpha}_{p,\theta}(G,s)}\,ds$$

$$+N(\|\psi^{-2\alpha}\bar{f}\|^p_{\mathbb{L}_{p,\theta}(G,T)}+\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^{-1}_{p,\theta}(G,T)}+\|\psi^{-\alpha}g\|^p_{\mathbb{L}_{p,\theta}(G,T)}).$$

Gronwall's inequality leads to (2.16).

Step 2. Consider the case $T>\varepsilon_0$. To proceed further, we need the following lemma.

Lemma 5.1. *Let $\tau\le T$ be a stopping time. Let $u\in\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta,0}(\tau)$, and*

$$du(t)=f(t)dt+g^k(t)dw^k_t.$$

Then there exists a unique $\tilde{u}\in\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta,0}(T)$ such that $\tilde{u}(t)=u(t)$ for $t\le\tau(a.s)$ and, on $(0,T)$,

$$d\tilde{u}=(\psi^{2\alpha}\Delta\tilde{u}(t)+\tilde{f}(t))dt+g^kI_{t\le\tau}dw^k_t, \tag{5.43}$$

where $\tilde{f}=(f(t)-\psi^{2\alpha}\Delta u(t))I_{t\le\tau}$. Furthermore,

$$\|\tilde{u}\|_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,T)}\le N\|u\|_{\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta}(G,\tau)}, \tag{5.44}$$

where N is independent of u and τ.

Proof. Note that $\tilde{f}\in\psi^{-1+2\alpha}\mathbb{H}^{\gamma}_{p,\theta}(G,T)$ and $gI_{t\le\tau}\in\psi^{\alpha}\mathbb{H}^{\gamma+1}_{p,\theta}(G,T)$, so that, by Theorem 2.8, equation (5.43) has a unique solution $\tilde{u}\in\mathfrak{H}^{\gamma+2,\alpha}_{p,\theta,0}(G,T)$ and (5.44) holds. To show that $\tilde{u}(t)=u(t)$ for $t\le\tau$, notice that, for $t\le\tau$, the function $v(t)=\tilde{u}(t)-u(t)$ satisfies the equation

$$dv=\psi^{2\alpha}\Delta v\,dt,\quad v(0,\cdot)=0.$$

Theorem 2.8 shows that $v(t)=0$ for $t\le\tau\,(a.e)$. □

Now, to complete the proof, we repeat the arguments in [5]. Take an integer $M \geq 2$ such that $T/M \leq \varepsilon_0$, and denote $t_m = Tm/M$. Assume that, for $m = 1, 2, ..., M-1$, we have the estimate (2.16) with t_m in place of τ (and N depending only on d, p, δ_0, K and T). We are going to use the induction on m. Let $u_m \in \mathfrak{H}^{1,\alpha}_{p,\theta,0}$ be the continuation of u on $[t_m, T]$, which exists by Lemma 5.1 with $\gamma = -1$ and $\tau = t_m$. Denote $v_m := u - u_m$, then $(a.s)$ for any $t \in [t_m, T]$, $\phi \in C_0^\infty(G)$ (since $du_m = \psi^{2\alpha}\Delta u_m dt$ on $[t_m, T]$ and $v_m(t_m, \cdot) = 0$)

$$(v_m(t), \phi) = -\int_{t_m}^t (\psi^{2\alpha} a^{ij} v_{mx^j} + \psi^\alpha \bar{b}^i v_m + \bar{f}^i_m, \phi_{x^i})(s)ds$$

$$+\int_{t_m}^t (\psi^\alpha b^i v_{mx^i} + cv_m + f_m, \phi)(s)ds + \int_{t_m}^t (\psi^\alpha \sigma^{ik} v_{mx^i} + \nu^k v_m + g^k_m, \phi)(s)dw^k_s,$$

where

$$\bar{f}^i_m = \psi^{2\alpha}(a^{ij} - \delta^{ij})u_{mx^j} + \psi^\alpha \bar{b}^i u_m + \bar{f}^i,$$

$$f_m = \psi^\alpha b^i u_{mx^i} + cu_m + f, \quad g^k_m = \psi^\alpha \sigma^{ik} u_{mx^i} + \nu^k u_m + g^k.$$

Next instead of random processes on $[0, T]$ we consider processes given on $[t_m, T]$ and, in a natural way, introduce spaces $\mathfrak{H}^{\gamma,\alpha}_{p,\theta}(G, [t_m, T])$, $\mathbb{L}_{p,\theta}(G, [t_m, t])$, $\mathbb{H}^\gamma_{p,\theta}(G, [t_m, T])$. Then we get a counterpart of the result of step 1 and conclude that

$$E\int_{t_m}^{t_{m+1}} \|\psi^{-1}(u - u_m)(s)\|^p_{H^1_{p,\theta}(G)} ds$$

$$\leq NE\int_{t_m}^{t_{m+1}} \|\psi^{-2\alpha} \bar{f}^i_m(s)\|^p_{L_{p,\theta}(G)}\, ds$$

$$+NE\int_{t_m}^{t_{m+1}} \|\psi^{1-2\alpha} f_m(s)\|^p_{H^{-1}_{p,\theta}(G)} + \|\psi^{-\alpha} g_m(s)\|^p_{L_{p,\theta}(G)} ds.$$

Thus by the induction hypothesis,

$$E\int_0^{t_{m+1}} \|\psi^{-1} u(s)\|^p_{H^1_{p,\theta}(G)} ds \leq NE\int_0^T \|\psi^{-1} u_m(s)\|^p_{H^1_{p,\theta}(G)} ds$$

$$+NE\int_{t_m}^{t_{m+1}} \|\psi^{-1}(u - u_m)(s)\|^p_{H^1_{p,\theta}(G)} ds$$

$$\leq N(\|\psi^{-2\alpha} \bar{f}^i\|^p_{\mathbb{L}_{p,\theta}(G, t_{m+1})} + \|\psi^{1-2\alpha} f\|^p_{\mathbb{H}^{-1}_{p,\theta}(G, t_{m+1})} + \|\psi^\alpha g\|^p_{\mathbb{L}_{p,\theta}(G, t_{m+1})}).$$

We see that the induction goes through and thus the theorem is proved.

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