DOI: 10.1214/EJP.v11-326

Electronic Journal of Probability

Vol. 11 (2006), Paper no. 17, pages 418–433.

Journal URL
http://www.math.washington.edu/~ejpecp/

The Metastability Threshold for Modified Bootstrap Percolation in d Dimensions

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Abstract

In the modified bootstrap percolation model, sites in the cube $\{1,\dots,L\}^d$ are initially declared active independently with probability p. At subsequent steps, an inactive site becomes active if it has at least one active nearest neighbour in each of the d dimensions, while an active site remains active forever. We study the probability that the entire cube is eventually active. For all $d \geq 2$ we prove that as $L \to \infty$ and $p \to 0$ simultaneously, this probability converges to 1 if $L \geq \exp\cdots\exp\frac{\lambda+\epsilon}{p}$, and converges to 0 if $L \leq \exp\cdots\exp\frac{\lambda-\epsilon}{p}$, for any $\epsilon > 0$. Here the exponential function is iterated $d-1$ times, and the threshold λ equals $\pi^2/6$ for all d.

Key words: bootstrap percolation, cellular automaton, metastability, finite-size scaling

AMS 2000 Subject Classification: Primary 60K35; Secondary: 82B43.

Submitted to EJP on March 30 2006, final version accepted June 2 2006.

*Funded in part by an NSERC (Canada) Discovery Grant, and by MSRI (Berkeley USA)

1 Introduction

Let $\mathbb{Z}^d := \{x = (x_1, \ldots, x_d) : x \in \mathbb{Z}\}$ be the d-dimensional integer lattice. We call the elements of $\mathbb{Z}^d$ **sites**. Let $e_1 := (1, 0, \ldots, 0), \ldots, e_d := (0, \ldots, 0, 1) \in \mathbb{Z}^d$ be the standard basic vectors. For a set of sites $W \subseteq \mathbb{Z}^d$, define

$$\beta(W) := W \cup \Big\{ x \in \mathbb{Z}^d : \forall i = 1, \ldots, d \text{ we have } x + e_i \in W \text{ or } x - e_i \in W \Big\},$$

and

$$\langle W \rangle := \lim_{t \to \infty} \beta^t(W),$$

where β^t denotes the t-th iterate of the function β. $\langle W \rangle$ is the final active set for the **modified bootstrap percolation model** starting with W active.

Now fix $p \in (0, 1)$ and let X be a random subset of $\mathbb{Z}^d$ in which each site is independently included with probability p. More formally, denote by $\mathbf{P}_p = \mathbf{P}$ the product probability measure with parameter p on the product σ-algebra of $\{0, 1\}^{\mathbb{Z}^d}$, and define the random variable X by $X(\omega) := \{x \in \mathbb{Z}^d : \omega(x) = 1\}$ for $\omega \in \{0, 1\}^{\mathbb{Z}^d}$. A site $x \in \mathbb{Z}^d$ is said to be **occupied** if $x \in X$.

We say that a set $W \subseteq \mathbb{Z}^d$ is **internally spanned**, or **i.s.**, if $\langle X \cap W \rangle = W$ (that is, if the model restricted to W fills W up). For a positive integer L we define the d-dimensional **cube** of side L to be

$$Q^d(L) := \{1, \ldots, L\}^d.$$

For convenience we also write $Q^d(L) = Q^d(\lfloor L \rfloor)$ when L is not an integer. We define

$$I^d(L) = I^d(L, p) := \mathbf{P}_p\big(Q^d(L) \text{ is internally spanned}\big).$$

Let $\exp^n$ denote the n-th iterate of the exponential function.

Theorem 1. *Let $d \geq 2$ and $\epsilon > 0$. For the modified bootstrap percolation model, as $L \to \infty$ and $p \to 0$ simultaneously we have*

(i) $I^d(L, p) \to 1$ *if* $L \geq \exp^{d-1} \dfrac{\lambda + \epsilon}{p}$;

(ii) $I^d(L, p) \to 0$ *if* $L \leq \exp^{d-1} \dfrac{\lambda - \epsilon}{p}$;

where

$$\lambda = \frac{\pi^2}{6}.$$

Remarks

The case $d = 2$ of Theorem 1 was proved in [8]. The modified bootstrap percolation model considered here is a minor variant of the **standard bootstrap percolation model**, which is defined in the same way except replacing the function β with

$$\beta'(W) := W \cup \Big\{ x \in \mathbb{Z}^d : \#\{y \in W : \|y - x\|_1 = 1\} \geq d \Big\},$$

(so a site becomes active if it has at least d active neighbours). In the case $d = 2$, the analogue of Theorem 1 was proved for the standard bootstrap percolation model in [8]; in this case the threshold λ becomes $\pi^2/18$. Similar results were obtained for a further family of two-dimensional models in [9]. The present work is the first proof of the existence of a sharp threshold λ for a bootstrap percolation model in 3 or more dimensions; in addition we determine the value $\pi^2/6$. The analogue of Theorem 1 but with two *different* constants c_1, c_2 in place of $\lambda + \epsilon,\ \lambda - \epsilon$ was proved earlier in [2] ($d = 2$), [4] ($d = 3$) and [5] ($d \geq 4$). These works apply to the standard model (among others), but can easily be adapted to the modified model considered here. It is a fascinating open problem to prove the existence of a sharp threshold for the standard model in 3 or more dimensions.

In [3] a different kind of "sharpness" is proved, by a general method, for various models including standard and modified bootstrap percolation: writing $p_\alpha = p_\alpha(L)$ for the value such that $I(L, p_\alpha) = \alpha$, then $p_{1-\epsilon} - p_\epsilon = o(p_{1/2})$ as $L \to \infty$, with a certain explicit bound. (However this result says nothing about the behaviour of $p_{1/2}$ as a function of L). Similar results with the roles of p and L exchanged may be obtained using the methods of [2].

There have been numerous other beautiful rigorous contributions to the study of bootstrap percolation models, initiated by [11]. For example see the references in [5],[8].

Bootstrap percolation models have important applications, both directly and as tools in the study of more complicated systems (see for example the references in [5],[8]). The models have been extensively studied via simulation, and it is a remarkable fact that the resulting asymptotic predictions often differ greatly from rigorous asymptotic results, apparently because the convergence as $(L, p) \to (\infty, 0)$ is extremely slow. See [10],[8] for examples. In the case of the modified bootstrap percolation model in $d = 2$, the value 0.47 ± 0.02 for λ was predicted numerically in [1], whereas the rigorous result from [8] is $\lambda = \pi^2/6 = 1.644934\cdots$. It would be worthwhile to compare simulations with our rigorous result that $\lambda = \pi^2/6$ for $d \geq 3$. It is of interest to understand this slow convergence phenomenon in more detail, and it is relevant to applications: a typical physical system might have $L^d \approx 10^{20}$ particles, which is much larger than current computer simulations allow, but potentially not large enough to exhibit a threshold close to the limiting value. See [6] for an interesting partly non-rigorous investigation of some these issues.

Proof outline

The proof of Theorem 1 is by induction on the dimension. The base case $d = 2$ is provided by the results in [8]. (The proof in [8] is quite involved, and very specific to the 2-dimensional model). It is interesting that the constant $\lambda = \pi^2/6$ enters only here. The inductive step follows closely the pioneering work of [4],[5], although since our result is more precise we need to be more careful with the estimates. As in [5], the case $d = 3$ is the most delicate.

The proof of the lower bound in Theorem 1(i) is relatively straightforward, and many of the ideas were already present in [10]. The fundamental observation is that if a cube is already entirely active, then the sites lying on its faces evolve according to the modified bootstrap percolation model in $d-1$ dimensions. Hence, by the inductive hypothesis, a cube of size L is likely to be internally spanned if it contains some internally spanned cube of size $m = \exp^{d-2}\frac{\lambda+\epsilon}{p}$ (sometimes called a "critical droplet" or "nucleation centre"), because such a cube will grow forever from its faces. For $d \geq 3$, straightforward arguments show that such a cube is internally spanned with probability at least (roughly) e^{-m}, and so in order to internally span the larger cube we should take approximately $L > 1/(e^{-m}) = \exp^{d-1}\frac{\lambda+\epsilon}{p}$, completing the induction.

The proof of the upper bound in Theorem 1(ii) is more challenging, and is based on the more subtle construction originating in [4]. The idea is to find an upper bound on the probability that a cube of size $m = \exp^{d-2}\frac{\lambda-\epsilon}{p}$ has a left-right crossing in its final configuration. (Such a crossing plays the role of a nucleation centre in this bound). The proof proceeds by dividing this cube into "slices", and running the $(d-1)$-dimensional model in each, to produce a configuration which dominates the d-dimensional model. By the inductive hypothesis, the probability that a slice becomes fully active is small, and, where a slice does not become fully active, its final configuration resembles subcritical percolation. Hence a connection of length m has probability at most (roughly) e^{-m}, and again we can complete the induction. A key point in the present proof is that for the modified model, we can use slices of thickness 1, whereas in [4],[5] (for the standard model) it was necessary to use slices of thickness 2, and to replace the parameter p with $2p$. Changing p in this way makes it impossible to obtain matching upper and lower bounds, and it is for this reason that our method cannot be adapted directly to prove an analogous result for the standard bootstrap percolation model. Another difference in the proof here as compared with [4],[5] is that (in the case $d = 3$) we need to carefully balance the probabilities of fully active slices with those of percolation connections. Equation (24) is the heart of this calculation.

Notation and conventions

It will be convenient to consider lower-dimensional versions of the model running on subsets $\mathbb{Z}^d$. Let $\delta \in \{1, \ldots, d\}$. We define the δ-dimensional cube

$$Q^\delta(L) := \{1, \ldots, L\}^\delta \times \{0\}^{d-\delta} \subseteq \mathbb{Z}^d.$$

By a **copy** of $Q^\delta(L)$ we mean an image of $Q^\delta(L)$ under any isometry of $\mathbb{Z}^d$. For a set $W \subseteq \mathbb{Z}^d$, which will always be a subset of some copy of a δ-dimensional cube, we define

$$\beta_\delta(W) := W \cup \Big\{x \in \mathbb{Z}^d : \#\big\{i : \{x+e_i, x-e_i\} \cap W \neq \emptyset\big\} \geq \delta\Big\},$$

and $\langle W \rangle_\delta := \lim_{t\to\infty} \beta_\delta^t(W)$. We say that W is **δ-internally-spanned** if $\langle X \cap W\rangle_\delta = W$. Let $I^\delta(L) := \mathbf{P}_p(Q^\delta(L)$ is δ-i.s.$)$, and note that this is consistent with the earlier definition.

Theorem 1 involves an asymptotic statement as $p \to 0$ with d and ϵ fixed. Many of the inequalities used in the proof will be valid "for p sufficiently small", be which we mean for all p less than some $q = q(d, \epsilon) > 0$, whose value may vary from one instance to another. In some of following proofs we use $C_1, C_2, \ldots$ to denote constants in $(0, \infty)$ which may depend on d and ϵ, but not on p.

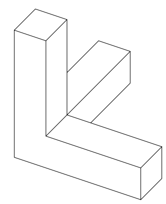

Figure 1: The occupied set in event A.

2 Lower Bound

Lemma 2. *For any $d \geq 3$ and for $r, \epsilon > 0$, if p is sufficiently small (depending on d, r, ϵ) then*

$$I^d\left(\exp^{d-2}\frac{r}{p}\right) \geq 1/\exp^{d-1}\frac{r+\epsilon}{p}.$$

Proof. If $d \geq 4$, note that a cube is internally spanned if *all* of its sites are occupied. Therefore

$$I^d\left(\exp^{d-2}\frac{r}{p}\right) \geq p^{\left[\exp^{d-2}(r/p)\right]^d} \geq 1/\exp^{d-1}\frac{r+\epsilon}{p}$$

for p sufficiently small. (To check the second inequality, take three successive logarithms of the reciprocal of both sides).

The case $d = 3$ is a little more delicate. Write $L = \lfloor e^{r/p} \rfloor$ and $k = \lfloor p^{-3} \rfloor$ (so $k \ll L$ for p sufficiently small). Let A be the event that every site having two of its coordinates in $\{1, \ldots, k\}$ and one coordinate in $\{1, \ldots, L\}$ is occupied (see Figure 1). Let B be the event that every copy of $Q^1(k)$ in $Q^3(L)$ contains at least one occupied site. It is straightforward to check that if A and B both occur then $Q^3(L)$ is internally spanned. Since A and B are increasing events, the Harris-FKG inequality (see e.g. [7]) yields

$$I^3(L) \geq \mathbf{P}(A)\mathbf{P}(B).$$

We now estimate

$$\mathbf{P}(A) \geq p^{3Lk^2} \geq 1/\exp^2\frac{r+\epsilon}{p}$$

for p sufficiently small (to check the second inequality, take two logarithms of the reciprocals), while

$$\mathbf{P}(B) \geq 1 - 3L^3(1-p)^k \geq 1 - 3\exp(\lfloor r/p \rfloor - \lfloor p^{-2} \rfloor) \geq e^{-1}$$

for p sufficiently small. Therefore we have

$$I^3(L) \geq 1/\exp^2\frac{r+2\epsilon}{p}$$

for p sufficiently small, and since ϵ was arbitrary this proves the result for $d = 3$. □

The following result from [2] states that $I^d(L)$ increases rapidly with L once it is large enough.

Lemma 3. *For each $d \geq 1$ there exist $c = c(d) < 1$ and $C = C(d) < \infty$ such that provided $I^d(\ell) \geq c$, we have for all $L \geq \ell$ that*

$$I^d(L) \geq 1 - Ce^{-L/\ell}.$$

Proof. See [2],[10] or [5]. The idea is to divide $Q^d(L)$ into disjoint or nearly-disjoint copies of $Q^d(\ell)$. If $Q^d(L)$ is not i.s. then it is crossed by a path of non-i.s. copies of $Q^d(\ell)$. The probability of this event can be estimated using standard percolation methods. □

Proof of Theorem 1(i). First note that it is enough to prove the required statement if $L \to \infty$ and $p \to 0$ with $L = \exp^{d-1} \frac{\lambda+\epsilon}{p}$. For then if $L \geq \exp^{d-1} \frac{\lambda+\epsilon}{p}$, we may take $p' \leq p$ such that $L = \exp^{d-1} \frac{\lambda+\epsilon}{p'}$, and since $I^d(L,p)$ is monotone in p we have $I^d(L,p) \geq I^d(L,p') \to 1$.

The proof is by induction on d. The required statement holds in the case $d = 2$ by Theorems 4 and 1(i) of [8]. Now let $d \geq 3$, and suppose that for all $\delta = 2, \ldots, d-1$ we have

$$\text{for every } \epsilon > 0, \quad I^\delta\Big(\exp^{\delta-1} \frac{\lambda+\epsilon}{p},\ p\Big) \to 1 \text{ as } p \to 0. \tag{4}$$

We shall deduce that (4) holds for $\delta = d$ also.

Fix $\epsilon > 0$. We first claim that

$$I^d\big(\exp^{d-2} p^{-2}\big) \geq 1/\exp^{d-1} \frac{\lambda+4\epsilon}{p} \tag{5}$$

for p sufficiently small. To prove this, write

$$\ell = \Big\lfloor \exp^{d-2} \frac{\lambda+\epsilon}{p} \Big\rfloor; \quad a = \Big\lfloor \exp^{d-2} \frac{\lambda+2\epsilon}{p} \Big\rfloor; \quad b = \Big\lfloor \exp^{d-2} p^{-2} \Big\rfloor.$$

By Lemma 2 we have

$$I^d(a) \geq 1/\exp^{d-1} \frac{\lambda+3\epsilon}{p}. \tag{6}$$

We will deduce the claimed lower bound on $I^d(b)$ using the fact that an internally spanned cube will grow if each of its faces (of all possible dimensions) is internally spanned in the model of the appropriate lower dimension.

More precisely, for $L \geq 1$ and a proper subset $S \subsetneq \{1, \ldots, d\}$, define the **face**

$$F_S(L) := \big\{x \in \mathbb{Z}^d : x_i \in [1, L]\ \forall\, i \in S,\ \text{and } x_i = L+1\ \forall\, i \notin S\big\}.$$

Thus $F_S(L)$ is a copy of $Q^{|S|}(L)$, and we have the disjoint union

$$Q^d(L+1) = Q^d(L) \sqcup \bigsqcup_{S \subsetneq \{1,\ldots,d\}} F_S(L).$$

It is straightforward to check that if $Q^d(L)$ is d-i.s. and the face $F_S(L)$ is $|S|$-i.s. for every $S \subsetneq \{1, \ldots, d\}$ then $Q^d(L+1)$ is d-i.s. Hence we have

$$I^d(b) \geq I^d(a)\, \mathbf{P}(G_a^b), \tag{7}$$

where G_a^b is the event that $F_S(j)$ is $|S|$-i.s. for every $j \in [a,b)$ and every $S \subsetneq \{1,\ldots,d\}$.

In order to bound $\mathbf{P}(G_a^b)$, first note that we may take p sufficiently small that

$$I^\delta(\ell) \geq c(\delta) \quad \text{ for all } \delta \in [1, d-1], \tag{8}$$

where $c(\delta)$ is as in Lemma 3. (The case $\delta = d-1$ follows directly from (4); the cases $\delta \in [2, d-2]$ follow from (4) by an additional application of Lemma 3; the case $\delta = 1$ is trivial.) Therefore writing $C' = \max_{\delta \in [1,d-1]} C(\delta)$, Lemma 3 yields

$$\mathbf{P}(G_a^b) \geq 1 - 2^d C' \sum_{j=a}^{b-1} e^{-j/\ell} \geq 1 - 2^{d+1} C' \ell e^{-a/\ell} \geq e^{-1}$$

for p sufficiently small. Combining this with (6),(7) proves the claim (5).

Now write

$$L = \Big\lfloor \exp^{d-1} \frac{\lambda + 5\epsilon}{p} \Big\rfloor.$$

Let E be the event that $Q^d(L)$ contains some d-i.s. copy of $Q^d(b)$, and let F be the event that for each $\delta \in [1, d-1]$, every copy of $Q^\delta(b)$ in $Q^d(L)$ is δ-i.s. It is straightforward to check that if E and F both occur then $Q^d(L)$ is d-i.s. Hence by the Harris-FKG inequality,

$$I^d(L) \geq \mathbf{P}(E)\, \mathbf{P}(F). \tag{9}$$

By tiling $Q^d(L)$ with disjoint copies of $Q^d(b)$, we have

$$\begin{aligned} \mathbf{P}(E) &\geq 1 - \big(1 - I^d(b)\big)^{\lfloor L/b \rfloor^d} \\ &\geq 1 - \exp\big[- I^d(b)(L/b)^d \big] \end{aligned}$$

But by (5), for p sufficiently small we have

$$I^d(b)(L/b)^d \geq I^d(b) L/b \geq \frac{\Big\lfloor \exp^{d-1} \frac{\lambda+5\epsilon}{p} \Big\rfloor}{\exp^{d-1} \frac{\lambda+4\epsilon}{p} \exp^{d-2} p^{-2}} \geq \exp^{d-1} \frac{\lambda + 4.9\epsilon}{p} \to \infty$$

as $p \to 0$, hence $\mathbf{P}(E) \to 1$.

On the other hand, again taking for p sufficiently small to satisfy (8), we have by Lemma 3,

$$\mathbf{P}(F) \geq 1 - 2^d L^d C' e^{-b/\ell}.$$

But we have

$$\log(L^d e^{-b/\ell}) \leq d \exp^{d-2} \frac{\lambda + 5\epsilon}{p} - \frac{\lfloor \exp^{d-2} p^{-2} \rfloor}{\exp^{d-2} \frac{\lambda+\epsilon}{p}} \leq -\exp^{d-2}(p^{-2}/2)$$

for p sufficiently small. Hence $\mathbf{P}(F) \to 1$ as $p \to 0$.

Thus by (9) we have proved that $I^d(\exp^{d-1} \frac{\lambda+5\epsilon}{p}) \to 1$ as $p \to 0$. Since ϵ was arbitrary this is (4) with $\delta = d$, and the induction is complete. □

3 Upper Bound

The main step in the proof of Theorem 1(ii) will be Theorem 10 below, which states that within a cube of appropriate size, the final configuration of the model resembles highly subcritical percolation.

We call a set of sites $W \subseteq \mathbb{Z}^d$ **connected** if it induces a connected graph in the nearest-neighbour hypercubic lattice. A **component** is a maximal connected subset. For sites $x, y \in \mathbb{Z}^d$ and a (random) set $W \subseteq \mathbb{Z}^d$ we write "$x \stackrel{W}{\longleftrightarrow} y$" for the event that W has a component containing x and y. (Note that $x \stackrel{W}{\longleftrightarrow} x$ is equivalent to $x \in W$). For sites $x, y \in Q^d(m)$, we define

$$f_m^d(x,y) = f_m^d(x,y,p) := \mathbf{P}_p\Big(x \stackrel{\langle X \cap Q^d(m)\rangle}{\longleftrightarrow} y\Big).$$

Theorem 10. *Let $d \geq 3$ and $\epsilon > 0$. Let*

$$m = \left\lfloor \exp^{d-2} \frac{\lambda - \epsilon}{p} \right\rfloor,$$

where $\lambda = \pi^2/6$. There exist $\gamma = \gamma(d,\epsilon) > 0$ and $q = q(d,\epsilon) > 0$ such that for all $p < q$ and all $x, y \in Q^d(m)$,

$$f_m^d(x,y) \leq p^{\gamma(\|x-y\|_\infty + 1)}.$$

The "+1" term in the exponent is important, since for the induction we need a bound which is $o(1)$ as $p \to 0$ even in the case $x = y$.

The following result from [2] is very useful. The **diameter** of a set $W \subseteq \mathbb{Z}^d$ is $\operatorname{diam} W := \sup_{x,y\in W} \|x-y\|_\infty$.

Lemma 11. *If $S \subseteq \mathbb{Z}^d$ is connected and internally spanned then for every real $a \in [1, \operatorname{diam} S]$ there exists a connected, internally spanned set $T \subseteq S$ with $\operatorname{diam} T \in [a, 2a]$.*

Proof. See [2],[8] or [5]. The idea is to realize the bootstrap percolation model by an iterative algorithm. We keep track of a collection of disjoint connected i.s. sets S_j. At each step, if there is a site in $\langle \cup_j S_j \rangle \setminus \cup_j S_j$ then we unite it with at most d of the sets to form a new set. In the case of the modified model, $\max_j(\operatorname{diam} S_j)$ is at most doubled at each step. □

Proof of Theorem 1(ii). As in part (i), by monotonicity of I^d we may assume that $L = \exp^{d-1} \frac{\lambda-\epsilon}{p}$. The case $d = 2$ was proved in [8]. Therefore fix $d \geq 3$ and $\epsilon > 0$, and let

$$L = \left\lfloor \exp^{d-1} \frac{\lambda - \epsilon}{p} \right\rfloor \quad \text{and} \quad m = \left\lfloor \exp^{d-2} \frac{\lambda - \epsilon}{p} \right\rfloor.$$

By Lemma 11, if $Q^d(L)$ is internally spanned then it contains some connected internally spanned set T with $\operatorname{diam} T \in [m/2, m]$. This implies that there exist $v \in \{0, \ldots, L-m\}^d$ and $x, y \in v + Q^d(m) \subset Q^d(L)$ with $\|x - y\|_\infty \geq m/2$ such that $x \stackrel{\langle X \cap (v + Q^d(m))\rangle}{\longleftrightarrow} y$. Hence by Theorem 10 we obtain

$$I^d(L) \leq L^d m^d m^d p^{\gamma(m/2+1)} \leq 1/\exp^{d-1} \frac{\lambda - \epsilon}{p} \to 0 \quad \text{as } p \to 0 \tag{12}$$

as required. Here $L^d m^d m^d$ is a bound on the number of choices for v, x, y, and the second inequality holds for p sufficiently small, by taking the logarithm thus:

$$\begin{aligned}\log\left(L^d m^{2d} p^{\gamma(m/2+1)}\right) &\le \left(d - \frac{\gamma}{2}\log\frac{1}{p}\right)\exp^{d-2}\frac{\lambda-\epsilon}{p} + 2d\exp^{d-3}\frac{\lambda-\epsilon}{p}\\ &\le (-1)\exp^{d-2}\frac{\lambda-\epsilon}{p}.\end{aligned}$$

□

The proof of Theorem 10 is by induction on the dimension. The key estimate is Lemma 13 below, for which we need to define two more quantities. Let

$$\chi_n^d = \chi_n^d(p) := \sum_{y\in Q^d(2n+1)} f_{2n+1}^d(z,y)$$

where $z := (n+1, \dots, n+1)$ is the site at the centre of $Q^d(2n+1)$. (Thus χ_n^d is the expected volume of the component at z in the final configuration of the model on $Q^d(2n+1)$). For $n \le m$ define

$$F_{m,n}^d = F_{m,n}^d(p) := \mathbf{P}_p\Big(\langle X \cap Q^d(m)\rangle \text{ has a component of diameter } \ge n\Big).$$

Lemma 13. *For any $n \le m$ and $x, y \in Q^d(m)$ with $\ell = \|x-y\|_\infty$ we have*

$$f_m^d(x,y) \le \sum_{k=0}^{\ell} \sum_{0<i_1<\dots<i_k<i_{k+1}=\ell+1} \left(H(i_1+1) + \sum_{a=0}^{m} F_{m,n}^{d-1}\left[1\wedge m^{d-1}H(i_1+a)\right]\right)\prod_{j=1}^{k} F_{m,n}^{d-1}\left[1\wedge m^{d-1}H(i_{k+1}-i_k)\right]$$

where

$$H(r) := \frac{1}{2}\sum_{s=r-1}^{\infty}\left(2\chi_n^{d-1}\right)^s.$$

(Perhaps the easiest way to understand Lemma 13 is to read the proof as far as (14),(15), and look at Figures 2,3.)

Proof. The following construction is based on that of [4],[5]. Without loss of generality suppose that $\|x-y\|_\infty = (y)_d - (x)_d$; if not we reorder the coordinates and/or reverse the direction of the dth coordinate. Write $(x)_d = u$ and $(y)_d = u+\ell$. Divide the cube $Q^d(m)$ into the **slices**

$$T_j := Q^{d-1}(m) + (0,\dots,0,j+u), \quad j = -u+1,\dots,m-u,$$

so that $x \in T_0$ and $y \in T_\ell$. Let $Y_j := \langle X \cap T_j\rangle_{d-1}$ be the final configuration of the $(d-1)$-dimensional model restricted to T_j. Let

$$Z_j := \begin{cases} T_j & \text{if } Y_j \text{ has a component of diameter } \ge n;\\ Y_j & \text{otherwise.}\end{cases}$$

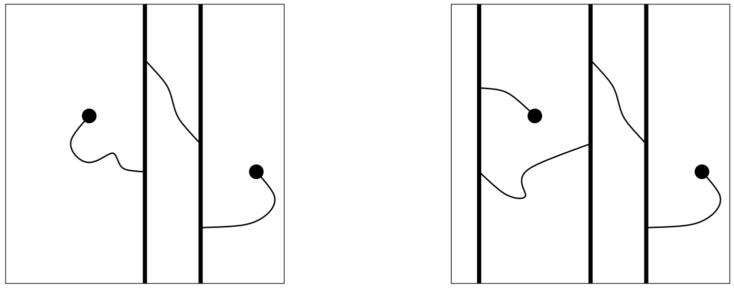

Figure 2: Two possibilities for the connection from x to y (black dots). Vertical bars indicate full slices. In the left picture $\mathcal{W}$ does not occur; in the right picture it does.

In the former case we say that the slice T_j is **full**. Now let

$$Z := \bigcup_{j=-u+1}^{m-u} Z_j.$$

The point of this construction is that $Z \supseteq \langle X \cap Q^d(m) \rangle_d$. To see this note that $Z_j \supseteq Y_j \supseteq \langle X \cap Q^d(m) \rangle_d \cap T_j$; the latter inclusion holds because running the $(d-1)$-dimensional model in T_j is equivalent to running the d-dimensional model with the boundary condition that every site in $Q^d(m) \setminus T_j$ is occupied – hence it must result in a larger configuration in T_j than running the d-dimensional model in $Q^d(m)$. (Note that the argument would not work in this form for the standard bootstrap percolation model, since the boundary condition adds *two* extra neighbours to each site in the slice.) Therefore

$$f_m^d(x,y) \le \mathbf{P}\Big(x \stackrel{Z}{\longleftrightarrow} y\Big).$$

We shall bound the above probability by splitting the event up according to which slices are full. Let $I_1 < \cdots < I_K$ be the random indices of those slices among $T_1, \ldots, T_\ell$ that are full. Also let $\mathcal{W}$ be the event than every path in Z from x to y intersects some full slice among $T_{-u}, \ldots, T_0$, and let $-A$ be the index of the last full slice among $T_{-u}, \ldots, T_0$ (or $A = \infty$ if there is none). Then

$$f_m^d(x,y) \le \sum_{k=0}^{\ell} \sum_{0<i_1<\cdots<i_k<\ell+1} \sum_{a=0,\ldots,m,\infty} \Big[\mathbf{P}\Big(x \stackrel{Z}{\longleftrightarrow} y,\ (I_1,\ldots,I_K)=(i_1,\ldots,i_k),\ A=a,\ \mathcal{W}^C\Big) \qquad (14)$$

$$+\,\mathbf{P}\Big(x \stackrel{Z}{\longleftrightarrow} y,\ (I_1,\ldots,I_K)=(i_1,\ldots,i_k),\ A=a,\ \mathcal{W}\Big)\Big]. \qquad (15)$$

(See Figure 2 for an illustration).

Using independence of the slices, the probability (15) above is at most

$$\mathbf{P}\big(T_{-a} \text{ is full}\big)\mathbf{P}\big(\mathcal{E}(-a+1, i_1-1)\big) \prod_{j=1}^{k} \mathbf{P}\big(T_{i_j} \text{ is full}\big)\mathbf{P}\big(\mathcal{E}(i_j+1, i_{j+1}-1)\big), \qquad (16)$$

where we have written for convenience $i_{k+1} := \ell + 1$, and where

$$\mathcal{E}(i,i') := \Big\{ T_i, \dots, T_{i'} \text{ are not full, and } v \overset{\bigcup_{j=i}^{i'} Z_j}{\longleftrightarrow} v' \text{ for some } v \in T_i,\ v' \in T_{i'} \Big\}$$

(and taking $\mathcal{E}(i,i')$ to be an event of probability 1 if $i > i'$).

Similarly, the probability (14) is at most

$$\mathbf{P}\big(\mathcal{G}(x,-a+1,i_1-1) \cap \{A=a\}\big) \prod_{j=1}^{k} \mathbf{P}\big(T_{i_j} \text{ is full}\big)\mathbf{P}\big(\mathcal{E}(i_j+1, i_{j+1}-1)\big), \tag{17}$$

where

$$\mathcal{G}(x,i,i') := \Big\{ T_i, \dots, T_{i'} \text{ are not full, and } x \overset{\bigcup_{j=i}^{i'} Z_j}{\longleftrightarrow} v' \text{ for some } v' \in T_{i'} \Big\}.$$

Next we bound the factors in (16),(17). For any slice T_j we have

$$\mathbf{P}(T_j \text{ is full}) = F_{m,n}^{d-1}, \tag{18}$$

so it remains only to bound the probabilities of $\mathcal{E}(i,i')$ and $\mathcal{G}(x,i,i')$.

Suppose that the event $\mathcal{E}(i,i')$ occurs. Then there is a self-avoiding nearest-neighbour path in $\bigcup_{j=i}^{i'} Z_j$ from some site in T_i to some site in $T_{i'}$. Given such a path, we say that there is a **changeover** whenever two consecutive sites in the path lie in different slices. Let α be such a path chosen so that the number of changeovers is a minimum. Define sites $v_1, w_1, v_2, w_2, \dots, w_s$ along the path α as follows (see Figure 3 for an illustration). Let $v_1 \in T_i$ be the first site of α. Given $v_1, \dots, v_t$, let v_{t+1} be the first site after v_t at which the path enters a slice different from that of v_t. Iterate this until when we reach a site $v_s \in T_{i'}$. Let w_t be the site preceding v_{t+1} in α for each $t < s$, and let $w_s \in T_{i'}$ be the last site of α. Thus α consists of a sequence of s sub-paths $(v_1, \dots, w_1)$, $\dots$, $(v_s, \dots, w_s)$, each one lying entirely within one slice, and with a changeover occurring between each sub-path and the next. (Note that two non-consecutive sub-paths may lie in the same slice). More precisely we have the following facts.

For each $t = 1, \dots, s$:

- (i) $v_t, w_t \in T_{j(t)}$ for some $j(t) \in [i, i']$, with $j(1) = i$ and $j(s) = i'$;
- (ii) $|j(t) - j(t+1)| = 1$ and $\|w_t - v_{t+1}\|_1 = 1$ for $t < s$;
- (iii) $\|v_t - w_t\|_\infty \le n$;
- (iv) the component of $Z_{j(t)}$ at v_t has diameter $\le n$;
- (v) $v_t \overset{Z_{j(t)}}{\longleftrightarrow} w_t$ occurs;
- (vi) $v_t \overset{Z_{j(t)}}{\longleftrightarrow} v_{t'}$ does not occur for any $t' \neq t$.

Properties (iii),(iv) hold because the slices $T_i, \dots, T_{i'}$ are not full, and property (vi) holds because of the minimality assumption on α.

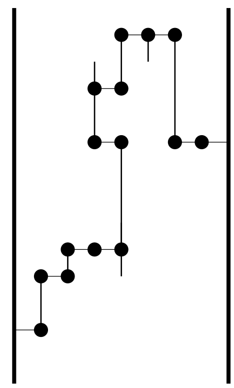

Figure 3: An illustration of the event $\mathcal{E}(i,i')$ – there is a path connecting the two full slices. The sites $v_1, w_1, v_2, w_2, \dots$ are shown as black dots in order along the path from left to right. In this example $v_3 = w_3$, while v_4, v_6 lie in the same slice, but (necessarily) in distinct components within the slice.

The occurrence of $\mathcal{E}(i,i')$ implies the existence of $v_1, \dots, w_s$ satisfying the above properties. For $v, w \in T_j$ define the event

$$\Big\{v \overset{\le n}{\longleftrightarrow} w\Big\} := \Big\{\text{there exists a connected, } (d-1)\text{-internally-spanned set } S \subseteq T_j \text{ with } \operatorname{diam} S \le n \text{ and } v, w \in S\Big\}.$$

(The definition of this event is delicate, and corrects a small error in [5]). Then we have

$$\begin{aligned}\mathbf{P}\big(\mathcal{E}(i,i')\big) &\le \sum_{s \ge i'-i+1} \sum_{v_1,\dots,w_s} \mathbf{P}\Big(\{v_1 \overset{\le n}{\longleftrightarrow} w_1\} \circ \cdots \circ \{v_s \overset{\le n}{\longleftrightarrow} w_s\}\Big) \\ &\le \sum_{s \ge i'-i+1} \sum_{v_1,\dots,w_s} \mathbf{P}\big(v_1 \overset{\le n}{\longleftrightarrow} w_1\big) \cdots \mathbf{P}\big(v_s \overset{\le n}{\longleftrightarrow} w_s\big) \end{aligned} \tag{19}$$

Here the second sum is over all possible choices of $v_1, w_1, v_2, \dots, w_s$ satisfying properties (i)–(iii) above, the symbol $\circ$ denotes disjoint occurrence (which holds because of property (vi)), and the second inequality follows from the Van den Berg-Kesten inequality (see e.g. [7]).

In order to bound the above, consider choosing $v_1, w_1, v_2, \dots, w_s$ in order. There are $\#T_i = m^{d-1}$ possible choices for v_1. Once v_t is chosen, the possible choices for w_t lie in the cube

$$V(v_t) := v_t - z + Q^{d-1}(2n+1)$$

centred at v_t, where $z := (n+1, \dots, n+1, 0)$. Furthermore the event $\big\{v_t \overset{\le n}{\longleftrightarrow} w_t\big\}$ is contained in the event $\big\{v_t \overset{\langle X \cap V(v_t)\rangle_{d-1}}{\longleftrightarrow} w_t\big\}$. Once w_t is chosen, there are (at most) 2 possible choices for v_{t+1}, corresponding to the two neighbouring slices. Hence we obtain

$$\sum_{v_1,\dots,w_s} \mathbf{P}(v_1 \overset{\le n}{\longleftrightarrow} w_1) \cdots \mathbf{P}(v_s \overset{\le n}{\longleftrightarrow} w_s) \le m^{d-1} \Bigg(\sum_{w \in V(z)} f_{2n+1}^{d-1}(z,w) \Bigg)^s 2^{s-1}$$

Substituting into (19) we obtain

$$\mathbf{P}\big(\mathcal{E}(i,i')\big) \leq 1 \wedge m^{d-1} H(i'-i+2), \tag{20}$$

where $H(r)$ is as in the statement of Lemma 13.

We use an almost identical argument to bound the probability of $\mathcal{G}(x,i,i')$. In this case the path starts at the fixed site x, so there is no need for the factor m^{d-1}. We obtain

$$\sum_a \mathbf{P}\Big(\mathcal{G}(x,-a+1,i-1) \cap \{A=a\}\Big) \leq H(i+1). \tag{21}$$

Finally, substituting (18),(20),(21) into (16),(17), and substituting these into (14),(15) we obtain the conclusion of Lemma 13. □

In the following proofs we use $C_1, C_2, \ldots$ to denote constants in $(0,\infty)$ which may depend on d and ϵ, but not on p.

Proof of Theorem 10 (case $d=3$). Let $d=3$ and fix $\epsilon>0$. Since ϵ is arbitrary we can take for convenience

$$m = \Big\lfloor \exp \frac{\lambda - 2\epsilon}{p} \Big\rfloor.$$

We shall bound $f^3_m(x,y)$ using Lemma 13; for this we need to choose n and find upper bounds on $F^2_{m,n}$ and χ^2_n.

We first consider $F^2_{m,n}$. If $\langle X \cap Q^2(m)\rangle_2$ has a component of diameter $\geq n$ then by Lemma 11, $Q^2(m)$ contains a connected, 2-i.s. set T with diameter in $[n/2, n]$. By Theorems 4 and 2(ii) of [8] we may find $B = B(\epsilon) \in (0,\infty)$ such that $I^2(B/p) \leq 1/\exp\frac{2\lambda-\epsilon}{p}$ for p sufficiently small (note that the factor of 2 in the exponent is important). Indeed, by equation (11) in [8] and the proof of Theorem 1(ii) in [8], we can choose B sufficiently large that, for p sufficiently small, for any connected $T \subset \mathbb{Z}^2$ with $\operatorname{diam} T \in \big[\lfloor B/(2p)\rfloor, \lfloor B/p \rfloor\big]$ we have

$$\mathbf{P}(T \text{ is 2-i.s.}) \leq 1/\exp\frac{2\lambda - 2\epsilon}{p}.$$

Therefore let

$$n = \Big\lfloor \frac{B}{p} \Big\rfloor.$$

Then by the above remarks we have

$$\begin{aligned} F^2_{m,n} &\leq m^2 n^2 / \exp \frac{2\lambda - 2\epsilon}{p} \\ &\leq \left(\frac{B}{p}\right)^2 \exp \frac{2\lambda - 4\epsilon - 2\lambda + 2\epsilon}{p} \\ &\leq \exp -\frac{\epsilon}{p} \end{aligned} \tag{22}$$

for p sufficiently small.

We now turn to χ^2_n, which is the expected volume of the component at $(n+1,n+1)$ in $\langle X \cap Q^2(2n+1)\rangle_2$. This component is a rectangle, R say. R is 2-i.s., and $\#R \leq (\operatorname{diam} R)^2$.

We bound $\chi_n^2 = \mathbf{E}_p \# R$ by considering two cases. If $1 \leq \operatorname{diam} R \leq 10$, then some site within distance 10 of z must be occupied. If $\operatorname{diam} R > 10$ then by Lemma 11, $Q^2(2n+1)$ must contain some 2-i.s. rectangle S with diameter in $[5, 10]$; and a 2-i.s. rectangle has at least one occupied site in each row and each column. Hence we have

$$\begin{aligned}\chi_n^2 &\leq 10^2(21^2 p) + (2n+1)^2[(2n+1)^2 10^2](10p)^5 \\ &\leq C_1 p \leq \sqrt{p},\end{aligned} \tag{23}$$

for p sufficiently small. (Here $(2n+1)^2 10^2$ is a bound on the number of possible choices for the rectangle S, $(10p)^5$ is a bound on the probability S is internally spanned, and we have used the definition of n for the second inequality).

Now we use (22),(23) to bound the terms in Lemma 13. We have for p sufficiently small

$$H(r) \leq \frac{1}{2} \sum_{s=r-1}^{\infty} (2\sqrt{p})^s \leq \frac{(2\sqrt{p})^{r-1}}{2(1-2\sqrt{p})} \leq p^{C_2(r-1)}.$$

Writing $C = C_2$ we now bound the following expression from Lemma 13 by considering two possible cases for the value of r:

$$\begin{aligned}
&F_{m,n}^2[1 \wedge m^2 H(r)] \leq e^{-\epsilon/p}\left[1 \wedge e^{2\lambda/p}\, p^{C(r-1)}\right] \\
&\leq \begin{cases} e^{-\epsilon/p} \leq p\, e^{-\epsilon/(2p)} \leq p\, p^{\left[\frac{\epsilon C}{8\lambda}(r-1)\right]} & \text{if } p^{C(r-1)} \geq e^{-4\lambda/p} \\ p\left[e^{2\lambda/p}\, p^{C(r-1)/2}\, p^{C(r-1)/2}\right] \leq p\, p^{C(r-1)/2} & \text{if } p^{C(r-1)} < e^{-4\lambda/p} \end{cases} \\
&\leq \; p^{C_3 r}
\end{aligned} \tag{24}$$

for p sufficiently small. Looking again at Lemma 13 we therefore have

$$H(i_1+1) + \sum_{a=0}^{\infty} F_{m,n}^2\left[1 \wedge m^2 H(i_1+a)\right] \leq p^{C_2 i_1} + C_4 p^{C_3 i_1} \leq p^{C_5 i_1} \tag{25}$$

for p sufficiently small, where we can drop the initial multiplicative constant because $i_1 \geq 1$. Finally, substituting (24),(25) into Lemma 13 we obtain for p sufficiently small

$$\begin{aligned}f_m^3(x,y) &\leq \sum_{k=0}^{\ell} \sum_{0<i_1<\cdots<i_k<i_{k+1}=\ell+1} p^{C_5 i_1} p^{C_3(i_2-i_1)} \cdots p^{C_3(i_{k+1}-i_k)} \\ &\leq 2^\ell p^{C_5(\ell+1)} \leq p^{\gamma'(\ell+1)}\end{aligned}$$

for some $\gamma' = \gamma'(d, \epsilon) > 0$, as required. □

Proof of Theorem 10 (case $d \geq 4$). The proof is by induction on dimension. Fix $d \geq 4$ and $\epsilon > 0$ and suppose the case $d-1$ is proved. Let

$$m = \left\lfloor \exp^{d-2} \frac{\lambda - \epsilon}{p} \right\rfloor \quad \text{and} \quad n = \left\lfloor \frac{1}{2} \exp^{d-3} \frac{\lambda - \epsilon}{p} \right\rfloor - 1,$$

so that the inductive hypothesis gives for $x, y \in Q^{d-1}(n)$ that $f_n^{d-1}(x,y) \le f_{2n+1}^{d-1}(x,y) \le p^{\gamma(\|x-y\|_\infty+1)}$.

We shall apply Lemma 13. If $\langle X \cap Q^{d-1}(m)\rangle_{d-1}$ has a component of diameter $\ge n$ then by Lemma 11, $Q^{d-1}(m)$ contains a connected, $(d-1)$-i.s. subset T with diameter in $[n/2, n]$. Hence by the inductive hypothesis together with the reasoning used to obtain (12) in the proof of Theorem 1(ii), we have

$$F_{m,n}^{d-1} m^{d-1} \le m^{2(d-1)} n^{2(d-1)} p^{\gamma(n/2+1)} \le 1/\exp^{d-2}\frac{\lambda-\epsilon}{p} \le p$$

for p sufficiently small. (To check the second inequality, take the logarithm).

Using the inductive hypothesis again we have for p sufficiently small

$$\chi_n^{d-1} \le \sum_{r=0}^{n} (2r+1)^{d-1} p^{\gamma(r+1)} \le C_6 p^\gamma \left(1 + \int_1^\infty r^{d-1} p^{\gamma r}\, dr\right) \le p^{C_7},$$

since for p sufficiently small, $r^{d-1}p^{\gamma r}$ is decreasing in $r \ge 1$. So for p sufficiently small we have

$$H(r) \le p^{C_8(r-1)} \quad \text{and} \quad F_{m,n}^{d-1}\left[1 \wedge m^{d-1} H(r)\right] \le p^{C_9 r}.$$

Hence, as in the case $d = 3$, substituting into Lemma 13 gives for p sufficiently small

$$f_m^d(x,y) \le 2^\ell p^{C_9(\ell+1)} \le p^{\gamma'(\ell+1)},$$

as required. □

Open Problems

(i) Prove the analogue of Theorem 1 for the standard bootstrap percolation model in 3 or more dimensions. What is the value of the threshold λ in this case?

(ii) Currently all proofs of the existence of a sharp threshold (in the sense of Theorem 1 as opposed to [3]) for bootstrap percolation models involve calculating its value. (See [8],[9] and the present work). Is there a simpler method of proving existence without determining the value?

(iii) What is the "second order" asymptotic behaviour of the model? Specifically, for example, if $p = p_{1/2}(L)$ is such that $I^d(L,p) = 1/2$, what is the asymptotic growth rate of $p \log^{d-1} L - \lambda$ as $L \to \infty$?

Acknowledgement

I thank Raphael Cerf for stimulating conversations.

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