DOI: 10.1214/ECP.v9-1081

Elect. Comm. in Probab. **9** (2004), 36–52

ELECTRONIC COMMUNICATIONS in PROBABILITY

LONG-TERM BEHAVIOR FOR SUPERPROCESSES OVER A STOCHASTIC FLOW

JIE XIONG[1]
Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, USA
email: `jxiong@math.utk.edu`

Submitted 10 September 2003, *accepted in final form* 1 April 2004

AMS 2000 Subject classification: 60G57, 60H15, 60J80
Keywords: Superprocess, stochastic flow, log-Laplace equation, long-term behavior

Abstract
We study the limit of a superprocess controlled by a stochastic flow as $t \to \infty$. It is proved that when $d \leq 2$, this process suffers long-time local extinction; when $d \geq 3$, it has a limit which is persistent. The stochastic log-Laplace equation conjectured by Skoulakis and Adler [7], and studied by this author [12], plays a key role in the proofs, similar to the one played by the log-Laplace equation in deriving long-term behavior for the standard super-Brownian motion.

1 Introduction and main results

Suppose that a branching system is affected by a Brownian motion $W(t)$ which applies to every individual in that system. Between branchings, the motion of the ith particle is governed by an individual Brownian motion $B_i(t)$ and the common Brownian motion $W(t)$:

$$d\eta_i(t) = b(\eta_i(t))dt + \sigma_1(\eta_i(t))dW(t) + \sigma_2(\eta_i(t))dB_i(t)$$

where $b: \mathbb{R}^d \to \mathbb{R}^d$, $\sigma_1, \sigma_2: \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable functions, $W, B_1, B_2, \cdots$ are independent d-dimensional Brownian motions. Each individual, independent of others, splits into 2 or dies with equal probabilities after its standard exponential time runs out. This system has been constructed by Skoulakis and Adler [7] (a similar model has been investigated by Wang [9] and Dawson et al [2]). It is indicated by [7] that there are situations in which a common background noise would be a natural effect to include in a stochastic model. In fact, it can be regarded as an outside force which applies to each individual of the system. Because of the introduction of this outside force, the process no longer has the multiplicative property which is the key to the successes in the study of the classical superprocesses. To overcome this difficulty, new tools have to be developed. The aim of this paper is to study the long-term behavior of this process.

[1]RESEARCH SUPPORTED PARTIALLY BY NSA, BY CANADA RESEARCH CHAIR PROGRAM AND BY ALEXANDER VON HUMBOLDT FOUNDATION

Let $\mathcal{M}_F(\mathbb{R}^d)$ be the collection of all finite Borel measures on $\mathbb{R}^d$. Let $C_0^2(\mathbb{R}^d)$ be the collection of functions of compact support and continuous derivatives up to order 2. Let $C_0^2(\mathbb{R}^d)^+$ consist of the nonnegative elements of $C_0^2(\mathbb{R}^d)$. It has been established by Skoulakis and Adler [7] that the scaling limit of the system is an $\mathcal{M}_F(\mathbb{R}^d)$-valued superprocess X_t which is uniquely characterized by the following martingale problem: $X_0 = \mu \in \mathcal{M}_F(\mathbb{R}^d)$ and for any $\phi \in C_0^2(\mathbb{R}^d)$,

$$M_t(\phi) \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, L\phi\rangle \, ds \tag{1.1}$$

is a continuous martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \left(\left\langle X_s, \phi^2 \right\rangle + \left| \left\langle X_s, \sigma_1^T \nabla \phi \right\rangle \right|^2 \right) ds. \tag{1.2}$$

Here

$$L\phi = \sum_{i=1}^d b^i \partial_i \phi + \frac{1}{2} \sum_{i,j=1}^d a^{ij} \partial_{ij}^2 \phi,$$

$a^{ij} = \sum_{k=1}^d \sum_{\ell=1}^2 \sigma_\ell^{ik} \sigma_\ell^{kj}$, ∂_i means the partial derivative with respect to the ith component of $x \in \mathbb{R}^d$, σ_1^T is the transpose of the matrix σ_1, $\nabla = (\partial_1, \cdots, \partial_d)^T$ is the gradient operator and $\langle \mu, f\rangle$ represents the integral of the function f with respect to the measure μ. It was conjectured in [7] that the conditional log-Laplace transform of X_t should be the unique solution to a nonlinear stochastic partial differential equation (SPDE). Namely

$$\mathbb{E}_\mu \left(e^{-\langle X_t, f\rangle} \middle| W \right) = e^{-\langle \mu, y_{0,t}\rangle} \tag{1.3}$$

and

$$\begin{aligned} y_{s,t}(x) &= f(x) + \int_s^t \left(L y_{r,t}(x) - y_{r,t}(x)^2 \right) dr \\ &\quad + \int_s^t \nabla^T y_{r,t}(x) \sigma_1(x) \hat{d}W(r) \end{aligned} \tag{1.4}$$

where $\hat{d}W(r)$ represents the backward Itô integral:

$$\int_s^t g(r) \hat{d}W(r) = \lim_{|\Delta| \to 0} \sum_{i=1}^n g(r_i) \left(W(r_i) - W(r_{i-1}) \right)$$

where $\Delta = \{r_0, r_1, \cdots, r_n\}$ is a partition of $[s,t]$ and $|\Delta|$ is the maximum length of the subintervals.

This conjecture was confirmed by Xiong [12] under the following conditions (BC) which will be assumed throughout this paper: $f \geq 0$, b, σ_1, σ_2 *are bounded with bounded first and second derivatives.* $\sigma_2^T \sigma_2$ *is uniformly positive definite,* σ_1 *has third continuous bounded derivatives.* f *is of compact support.*

We have proved in Theorem 1.2 in [12] that (1.4) has a unique $L^2(\mathbb{R}^d)^+$-valued solution in the following sense: $\forall\ \phi \in C_0^\infty(\mathbb{R}^d)$, $\forall\ s \leq t$,

$$\langle y_{s,t}, \phi\rangle = \langle f, \phi\rangle + \int_s^t \langle y_{r,t}, L^*\phi - y_{r,t}\phi\rangle \, dr + \int_s^t \left\langle y_{r,t}, \nabla^T(\sigma_1 \phi) \right\rangle \hat{d}W(r)$$

where L^* is the dual operator of L given by

$$L^*\phi = -\sum_{i=1}^d \partial_i(b^i\phi) + \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}^2(a^{ij}\phi).$$

Further, we have shown that (cf. Lemma 2.5 in [12])

$$\mathbb{E}\sup_{0\le r\le t}\|\partial_x y_{r,t}\|^2_{L^2(\mathbb{R}^d)} < \infty,$$

where $\partial_x y_{r,t}$ is the weak derivative. This then implies that for fixed r and t, $y_{r,t}(x)$ is a continuous function of x. Furthermore, by Lemma 2.2 in [12], we see that $|y_{r,t}(x)|$ is bounded by $\|f\|_\infty$, the supremum of f. Theorem 1.4 in [12] implies (1.3). As a consequence, we see that $y_{s,t}$ in (1.4) is nonnegative since $-y_{s,t}$ is the logarithm of a conditional Laplace transform of a nonnegative random variable.

Note that in the study of the classical superprocess, the PDE satisfied by the log-Laplace transform played an important role. In this note, we shall demonstrate that the stochastic log-Laplace equation (1.4) plays a similar role in the study of the long-term behavior of the superprocess over a stochastic flow. The main idea is to show that $\mathbb{E}e^{-\langle\mu,y_{0,t}\rangle}$ has a limit by making use of (1.4) (see also (3.3)).

If the initial measure is finite, then the total mass of X_t is Feller's branching diffusion which reaches 0 in finite time. To obtain an interesting long-time limit, we need to consider the infinite measure case. In Section 3, we construct the process in the state space of tempered measures by making use of the conditional branching property of this process which is implied from the conditional log-Laplace formula (1.3). Throughout this paper, we shall assume that the initial measure μ is infinite.

This article is organized as follows: In Section 2, we consider a diffusion process driven by two Brownian motions. We shall prove that, given one of the Brownian motions, the conditional process is still a Markov process. Then, we give sufficient conditions for a σ-finite measure to be invariant for this conditional process with any realization of the given Brownian motion. In Section 3 we prove that X_t converges in law to a persistent distribution when the spatial dimension $d \ge 3$. In Section 4, we show that the process becomes extinct locally (eventually) when $d \le 2$.

The results of this paper (Theorems 10 and 11) are analogous to the corresponding classical results for super-Brownian motion. Although the proofs are adapted from the classical ones (cf. [10], [1]), the novelty of this article is its employment of the stochastic log-Laplace equation. Furthermore, as we point out in Remark 5, the σ-finite invariant measure is not unique. Therefore, even in the classical superprocess case, the long-term limit is not unique.

Throughout this paper, we use c to represent a constant which can vary from place to place. We use ξ_t and $\xi(t)$ to denote the same process whenever it is convenient to do so.

2 Conditional Markov processes and their infinite invariant measures

Let $\xi(t)$ be the diffusion process given by

$$d\xi(t) = b(\xi(t))dt + \sigma_1(\xi(t))dW(t) + \sigma_2(\xi(t))dB_1(t). \tag{2.1}$$

In this section, we consider the conditional process of $\xi(t)$ with given W. More specifically, we give sufficient conditions for an infinite measure to be invariant for this conditional process with any given W (cf. (2.5)). The existence of such a measure is crucial in next section. In Proposition 3 we give sufficient conditions for the existence of such invariant measures. In Remark 4, we give examples where such conditions are satisfied.
Let $\mathbb{E}^W$ denote the conditional expectation with W given. Let

$$\mathcal{F}_t^\xi = \sigma(\xi_s : \ s \leq t).$$

Lemma 1 *$\xi(t)$ is a conditional Markov process in the following sense: $\forall\ s < t$ and $f \in C_b(\mathbb{R}^d)$,*

$$\mathbb{E}^W(f(\xi(t))|\mathcal{F}_s^\xi) = \mathbb{E}^W(f(\xi(t))|\xi(s)), \qquad a.s.$$

Proof: For $s < t$ fixed, denote the process $\{W_r - W_s : \ r \in [s,t]\}$ by $W^{s,t}$. Since (2.1) has a unique strong solution, we see that $\xi(t)$ is a function of $\xi(s)$, $W^{s,t}$ and $B_1^{s,t}$. Namely $\xi(t) = G(s,t,\xi(s),W^{s,t},B_1^{s,t})$ for a measurable function G. Therefore

$$\begin{aligned} \mathbb{E}^W(f(\xi(t))|\mathcal{F}_s^\xi) &= \mathbb{E}(f(\xi(t))|\mathcal{F}_s^\xi \vee \mathcal{F}_t^W) \\ &= \mathbb{E}\left(\mathbb{E}(G(s,t,\xi(s),W^{s,t},B_1^{s,t})|\mathcal{F}_s^{W,B_1} \vee \sigma(W^{s,t}))\middle|\mathcal{F}_s^\xi \vee \mathcal{F}_t^W\right). \end{aligned} \tag{2.2}$$

Since $B_1^{s,t}$ is independent of $\mathcal{F}_s^{W,B_1} \vee \sigma(W^{s,t})$, we see that the conditional expectation

$$\mathbb{E}(G(s,t,\xi(s),W^{s,t},B_1^{s,t})|\mathcal{F}_s^{W,B_1} \vee \sigma(W^{s,t}))$$

is simply the expectation of $G(s,t,\xi(s),W^{s,t},B_1^{s,t})$ for $B_1^{s,t}$ with $\xi(s)$ and $W^{s,t}$ being fixed. Namely, it is a function of $\xi(s)$ and $W^{s,t}$, say $g(s,t,\xi(s),W^{s,t})$. Therefore, we can continue (2.2) with

$$\begin{aligned} \mathbb{E}^W(f(\xi(t))|\mathcal{F}_s^\xi) &= \mathbb{E}(g(s,t,\xi(s),W^{s,t})|\mathcal{F}_s^\xi \vee \mathcal{F}_t^W) \\ &= g(s,t,\xi(s),W^{s,t}). \end{aligned} \tag{2.3}$$

Similarly, we can show that

$$\mathbb{E}^W(f(\xi(t))|\xi(s)) = g(s,t,\xi(s),W^{s,t}). \tag{2.4}$$

The conclusion of the lemma then follows from (2.3) and (2.4). ∎

Given W, denote the conditional transition function by

$$p^W(s,x;t,\cdot) \equiv \mathbb{P}^W(\xi(t) \in \cdot|\xi(s) = x).$$

Note that for $A \in \mathcal{B}(\mathbb{R}^d)$ and $t > 0$ fixed, $p^W(s,x;t,A)$ is measurable in (s,x,W).
Throughout this paper, we assume that μ is an invariant measure of $\xi(t)$: $\forall\ s < t$, for almost all given W,

$$\int p^W(s,x;t,\cdot)\mu(dx) = \mu. \tag{2.5}$$

It is clear that

$$g(s,t,x,W^{s,t}) = \int_{\mathbb{R}^d} f(y)p^W(s,x;t,dy).$$

Note that for $t>0$ fixed, we can choose a version of g which is continuous in $s<t$. In fact, it can be proved that g satisfies an SPDE similar to (1.4) without the quadratic term which corresponding to the branching there. Therefore, we may and will take a version of p^W such that (with $t>0$ fixed) *for almost all W, (2.5) holds for all $s<t$.*

Since the condition (2.5) is not easy to verify, we seek sufficient conditions for it to hold. To this end, we write (2.1) in Stratonovich form:

$$d\xi(t)=\left(\bar{b}(\xi(t))dt+\sigma_2(\xi(t))dB_1(t)\right)+\sigma_1(\xi(t))\circ dW(t) \tag{2.6}$$

where $\circ dW(t)$ denotes the Stratonovich differential and $\bar{b}^i=b^i-\frac{1}{2}\sum_{j,k=1}^d\partial_k\sigma_1^{ij}\sigma_1^{kj}$.

Intuitively, μ is an invariant measure for $\xi(t)$ with each given realization of W if and only if it is invariant for both parts of (2.6). Namely, it should be invariant for the diffusion process

$$d\eta(t)=\bar{b}(\eta(t))dt+\sigma_2(\eta(t))dB_1(t)$$

and, formally, for the dynamical system

$$\dot{\zeta}(t)=\sigma_1(\zeta(t))\dot{W}_t$$

with each given realization of W.

Let

$$\bar{L}\phi=\sum_{i=1}^d\bar{b}^i\partial_i\phi+\frac{1}{2}\sum_{i,j=1}^d\bar{a}^{ij}\partial_{ij}^2\phi,$$

where $\bar{a}^{ij}=\sum_{k=1}^d\sigma_2^{ik}\sigma_2^{kj}$.

If μ is finite, it is well-known (cf. Varadhan [8], and Ethier and Kurtz [3], Theorem 9.17) that μ is invariant for $\eta(t)$ if and only if μ is absolutely continuous with respect to Lebesgue measure and $\bar{L}^*\mu=0$ (denote the Radon-Nickodym derivative by the same notation as the original measure), where $\bar{L}^*$ is the dual operator of $\bar{L}$ given by

$$\bar{L}^*\phi=-\sum_{i=1}^d\partial_i(\bar{b}^i\phi)+\frac{1}{2}\sum_{i,j=1}^d\partial_{ij}^2(\bar{a}^{ij}\phi).$$

Under suitable conditions, it was proved in Xiong [13] that the same statement is true for μ being a σ-finite measure.

Formally, the second part leads to $\nabla(\sigma_1^T\mu)=0$. Therefore, we conjecture that under a suitable growth condition, μ is a σ-finite invariant measure for p^W for each W if and only if $\bar{L}^*\mu=0$ and $\nabla(\sigma_1^T\mu)=0$.

To investigate this conjecture, we need to study the Wong-Zakai approximation $\xi^\epsilon(t)$ for the process $\xi(t)$:

$$d\xi^\epsilon(t)=\left(\bar{b}(\xi^\epsilon(t))+\sigma_1(\xi^\epsilon(t))\dot{W}_t^\epsilon\right)dt+\sigma_2(\xi^\epsilon(t))dB_1(t)$$

where $\dot{W}_t^\epsilon=\epsilon^{-1}(W_{(k+1)\epsilon}-W_{k\epsilon})$ if $k\epsilon\le t\le(k+1)\epsilon$, $k=0,1,\cdots$.

Lemma 2 *For any $c_1>0$, there exists a constant $c=c(t)$ such that for any $\epsilon>0$,*

$$\mathbb{E}_x\exp\left(-c_1|\xi^\epsilon(t)|\right)\le ce^{-c_1|x|}.$$

Proof: Note that

$$|\xi^\epsilon(t)| \geq |x| - Kt - \left|\int_0^t \sigma_1(\xi^\epsilon(s))\dot{W}_s^\epsilon ds\right| - \left|\int_0^t \sigma_2(\xi^\epsilon(s))dB_1(s)\right|. \tag{2.7}$$

By the martingale representation theorem, there is a real-valued Brownian motion B such that

$$\int_0^t \sigma_2(\xi^\epsilon(s))dB_1(s) = B(\tau_t)$$

where

$$\tau_t = \int^t |\sigma_2(\xi^\epsilon(s))|^2 ds \leq Kt.$$

It is well-known that for any $K_1 > 0$ and $T > 0$,

$$\mathbb{E}\exp\left(K_1 \sup_{s\leq T}|B(s)|\right) < \infty.$$

Therefore,

$$\mathbb{E}\exp\left(2c_1\left|\int_0^t \sigma_2(\xi^\epsilon(s))dB_1(s)\right|\right) \leq \mathbb{E}\exp\left(2c_1 \sup_{s\leq Kt}|B_s|\right) < \infty. \tag{2.8}$$

Now we consider $\int_0^t \sigma_1(\xi^\epsilon(s))\dot{W}_s^\epsilon ds$. To simplify the notation, we take $d = 1$. Let $\pi_\epsilon(s) = k\epsilon$ for $k\epsilon \leq s < (k+1)\epsilon$. By Itô's formula, we have

$$\begin{aligned}
& \int_0^t (\sigma_1(\xi^\epsilon(s)) - \sigma_1(\xi^\epsilon(\pi_\epsilon(s))))\dot{W}_s^\epsilon ds \\
= \ & \sum_k \int_{k\epsilon}^{(k+1)\epsilon} (\sigma_1(\xi^\epsilon(s)) - \sigma_1(\xi^\epsilon(k\epsilon)))ds\epsilon^{-1}(W_{(k+1)\epsilon} - W_{k\epsilon}) \\
= \ & \sum_k \int_{k\epsilon}^{(k+1)\epsilon} \int_{k\epsilon}^{s} \bar{L}\sigma_1(\xi^\epsilon(r))drds\epsilon^{-1}(W_{(k+1)\epsilon} - W_{k\epsilon}) \\
& + \sum_k \int_{k\epsilon}^{(k+1)\epsilon} \int_{k\epsilon}^{s} \sigma_1'(\xi^\epsilon(r))\sigma_1(\xi^\epsilon(r))drds\epsilon^{-2}(W_{(k+1)\epsilon} - W_{k\epsilon})^2 \\
& + \sum_k \int_{k\epsilon}^{(k+1)\epsilon} \int_{k\epsilon}^{s} \sigma_1'(\xi^\epsilon(r))\sigma_2(\xi^\epsilon(r))dB_1(r)ds\epsilon^{-1}(W_{(k+1)\epsilon} - W_{k\epsilon}) \\
\equiv \ & I_1 + I_2 + I_3.
\end{aligned}$$

As

$$\begin{aligned}
|I_1| & \leq \sum_k c\epsilon|W_{(k+1)\epsilon} - W_{k\epsilon}| \\
& \leq c\epsilon\left(\sum_k |W_{(k+1)\epsilon} - W_{k\epsilon}|^2\right)^{1/2} (t/\epsilon)^{1/2} \\
& \leq ct\sqrt{\epsilon},
\end{aligned}$$

$$|I_2| \leq \sum_k c|W_{(k+1)\epsilon} - W_{k\epsilon}|^2 \leq ct$$

and

$$\begin{aligned}
|I_3|^2 &= \left| \sum_k \int_{k\epsilon}^{(k+1)\epsilon} \epsilon^{-1}((k+1)\epsilon - r)\sigma_1'(\xi^\epsilon(r))\sigma_2(\xi^\epsilon(r))dB_1(r)(W_{(k+1)\epsilon} - W_{k\epsilon}) \right|^2 \\
&\leq \sum_k \left(\int_{k\epsilon}^{(k+1)\epsilon} \epsilon^{-1}((k+1)\epsilon - r)\sigma_1'(\xi^\epsilon(r))\sigma_2(\xi^\epsilon(r))dB_1(r) \right)^2 \sum_k (W_{(k+1)\epsilon} - W_{k\epsilon})^2 \\
&\leq t \int_0^t |\epsilon^{-1}(\pi_\epsilon(r) + \epsilon - r)\sigma_1'(\xi^\epsilon(r))\sigma_2(\xi^\epsilon(r))|^2 dr \leq c.
\end{aligned}$$

we see that

$$\left| \int_0^t (\sigma_1(\xi^\epsilon(s)) - \sigma_1(\xi^\epsilon(\pi_\epsilon(s))))\dot{W}_s^\epsilon ds \right| \leq c. \tag{2.9}$$

As

$$\int_0^t \sigma_1(\xi^\epsilon(\pi_\epsilon(s))))\dot{W}_s^\epsilon ds = \int_0^t \sigma_1(\xi^\epsilon(\pi_\epsilon(s))))dW_s,$$

similar to (2.8), we have

$$\mathbb{E} \exp \left(2c_1 \left| \int_0^t \sigma_1(\xi^\epsilon(\pi_\epsilon(s))))\dot{W}_s^\epsilon ds \right| \right) < \infty. \tag{2.10}$$

The conclusion of the lemma then follows from (2.7-2.10) and a simple Cauchy-Schwarz argument. ∎

The following proposition proves the sufficiency of the conditions in our conjecture. It remains open whether these conditions are necessary.

Proposition 3 *Suppose that μ is a nonnegative function and is of derivatives up to order 2 on $\mathbb{R}^d$ such that*

$$|\nabla \log \mu(x)| \leq K(1 + |x|), \quad \forall x \in \mathbb{R}^d. \tag{2.11}$$

If $\bar{L}^*\mu = 0$ and $\nabla(\sigma_1^T \mu) = 0$, then (2.5) holds.

Proof: Let $p_\epsilon^W(s, x; t, \cdot)$ be the transition probabilities of the Markov process $\xi^\epsilon(t)$ with given W. Note that the generator of $\xi^\epsilon(t)$ is

$$L_t^\epsilon \phi = \bar{L}\phi + (\dot{W}_t^\epsilon)^T \sigma_1 \nabla \phi.$$

Now we fix W and ϵ, and show that μ is a σ-finite invariant measure for p_ϵ^W by adapting the proof of [13] to the present time-dependent case.
For any $f \in C_0^\infty(\mathbb{R}^d)^+$, take r large enough such that the support of f is contained in $S \equiv \{x \in \mathbb{R}^d : \ |x| < r\}$. Let

$$U_S(t, x) = \mathbb{E}_x^W f(\xi^\epsilon(t)) 1_{\tau_S > t}$$

where τ_S is the first exit time of $\xi^\epsilon(t)$ from S. Then

$$\begin{cases} \frac{\partial U_S}{\partial t} = L_t^\epsilon U_S & (t, x) \in (0, \infty) \times S \\ U_S(0, x) = f(x) & x \in \bar{S} \\ U_S(t, x) = 0 & x \in \partial S. \end{cases}$$

Note that

$$\begin{aligned}\frac{\partial}{\partial t}\int_S U_S(t,x)\mu(x)dx &= \int_S L_t^\epsilon U_S(t,x)\mu(x)dx \\ &= -\int_{\partial S}\mu(x)\nabla^T U_S(t,x)\bar{a}(x)\vec{n}dx \\ &= -\int_{\partial S}\mu(x)|\bar{a}\vec{n}|\frac{\partial U_S}{\partial \vec{e}}dx \qquad (2.12)\end{aligned}$$

where $\vec{n}$ is the inner normal vector, $\vec{e} = |\bar{a}\vec{n}|^{-1}(\bar{a}\vec{n})$ and $\frac{\partial U_S}{\partial \vec{e}}$ is the directional derivative. Note that

$$\vec{e}\cdot\vec{n} = |\bar{a}\vec{n}|^{-1}\vec{n}^T\bar{a}\vec{n} > 0,$$

so that $\vec{e}$ points to the interior of S. As $U_S(t,x)\ge 0$ for $x\in S$ and $U_S(t,x) = 0$ for $x\in\partial S$, we have $\frac{\partial U_S}{\partial \vec{e}}\ge 0$. Hence, we can continue (2.12) with

$$\frac{\partial}{\partial t}\int_S U_S(t,x)\mu(x)dx \le 0.$$

Thus

$$\int_S U_S(t,x)\mu(x)dx \le \int_S f(x)\mu(x)dx.$$

Taking $r\to\infty$, we have

$$\int_{\mathbb{R}^d}\mathbb{E}_x^W f(\xi^\epsilon(t))\mu(x)dx \le \int_{\mathbb{R}^d} f(x)\mu(x)dx < \infty.$$

Let ρ_n be a smooth function on $\mathbb{R}^d$ such that $\rho_n(x) = 1$ for $|x|\le n$, $\rho_n(x) = 0$ for $|x|\ge 2n$ and

$$\sup_{x\in\mathbb{R}^d}|\nabla\rho_n(x)| \le cn^{-1}, \qquad \sup_{x\in\mathbb{R}^d,\ 1\le i,j\le d}\left|\partial_{ij}^2\rho_n(x)\right| \le cn^{-2}.$$

By (2.11) and the condition (BC), we have

$$|\bar{L}^*(\mu\rho_n)(x)| + |\nabla^T(\sigma_1\mu\rho_n)(x)| \le c\mu(x).$$

Define

$$u_n(t) = \int_{\mathbb{R}^d}\mu(x)\rho_n(x)\mathbb{E}_x^W f(\xi^\epsilon(t))dx \text{ and } u(t) = \int_{\mathbb{R}^d}\mu(x)\mathbb{E}_x^W f(\xi^\epsilon(t))dx.$$

Then

$$\begin{aligned}|u_n'(t)| &= \left|\int_{\mathbb{R}^d}\mu(x)\rho_n(x)L_t^\epsilon\mathbb{E}_x^W f(\xi^\epsilon(t))dx\right| \\ &= \left|\int_{\mathbb{R}^d}\left(\bar{L}^*(\mu\rho_n)(x) - \nabla^T(\sigma_1\mu\rho_n)(x)\dot{W}_t^\epsilon\right)\mathbb{E}_x^W f(\xi^\epsilon(t))dx\right| \\ &\le c\int_{|x|\ge 2n}\mu(x)\mathbb{E}_x^W f(\xi^\epsilon(t))dx \equiv v_n(t).\end{aligned}$$

Then $v_n\in C([0,T])$ decreases to 0 as $n\to\infty$. By Dini's theorem, $v_n\to 0$ uniformly for $t\in[0,T]$. Therefore, $u_n'(t)\to 0$ as $n\to\infty$ uniformly for $t\in[0,T]$. Note that $u_n(t)\to u(t)$. Therefore,

$$u'(t) = \lim_{n\to\infty} u_n'(t) = 0.$$

Namely,

$$\int_{\mathbb{R}^d} \mathbb{E}_x^W f(\xi^\epsilon(t))\mu(x)dx = \int_{\mathbb{R}^d} f(x)\mu(x)dx.$$

Let $F(W)$ be a bounded continuous function of W. Then

$$\int_{\mathbb{R}^d} \mathbb{E}_x(f(\xi^\epsilon(t))F(W))\mu(x)dx = \int_{\mathbb{R}^d} f(x)\mu(x)dx\mathbb{E}(F(W)). \tag{2.13}$$

By the Wong-Zakai theorem (cf. [11] or [5], P410, Theorem 7.2), we have $\xi^\epsilon(t) \to \xi(t)$ as $\epsilon \to 0$. Note that $|f(x)| \leq ce^{-c_1|x|}$ for any $c_1 > 0$. By Lemma 2, apply the dominated convergence theorem to (2.13), we have

$$\int_{\mathbb{R}^d} \mathbb{E}_x(f(\xi(t))F(W))\mu(x)dx = \int_{\mathbb{R}^d} f(x)\mu(x)dx\mathbb{E}(F(W)).$$

This implies the conclusion of the proposition. ∎

Remark 4 *1) If b, σ_1 and σ_2 are constants, then $\mu = \lambda$, the Lebesgue measure, satisfies the conditions of Proposition 3 and hence, (2.5) holds.*
2) Suppose that $\sigma_1(x) = \bar{\sigma}_1(x)I$, where $\bar{\sigma}_1$ is a real-valued function bounded away from 0 and I is the identity matrix. If $\mu(dx) \equiv \frac{1}{\bar{\sigma}(x)}dx$ satisfies $\bar{L}^*\mu = 0$, then the conditions of Proposition 3 hold for μ and hence, μ is an invariant measure for the conditional process.

Remark 5 *In general, the σ-finite invariant measure is not unique. Suppose that $\sigma_2 = I$ and b is a constant vector. As being pointed out in [13], $\mu_1(x) = 1$ and $\mu_2(x) = e^{2b^Tx}$ are two solutions to $\bar{L}^*\mu = 0$. For the second condition, we seek $\sigma_1 = (\sigma_1^{ij})_{d\times d}$ such that*

$$\sum_{i=1}^d \partial_i \sigma_1^{ij} = 0, \quad \sum_{i=1}^d \partial_i(\sigma_1^{ij} e^{2b^Tx}) = 0$$

for $j = 1, 2, \cdots, d$. The existence of such σ_1 is clear if $d > 2$ since there are d^2 entries of σ_1 and $2d < d^2$ equations.

3 Non-trivial limit when $d \geq 3$

In this section, we extend the process X_t to the space of infinite measures and consider the long-time behavior of X_t in high spatial dimensions. We shall prove that X_t has a non-trivial limit in distribution which is, in fact, persistent. The proof is adopted from Wang [10].
Let $P^W(\cdot) \equiv P(\cdot|W)$ be the conditional probability measure. First, we establish the equivalence between the martingale problem (1.1-1.2) and the conditional martingale problem defined below which is more natural and is easier to handle.

Definition 6 *A real valued process U_t (adapted to σ-field $\mathcal{F}_t$) is a P^W-martingale if for any $t > s$,*

$$\mathbb{E}(U_t|\mathcal{F}_t \vee \sigma(W)) = U_s, \qquad a.s.$$

Lemma 7 *X_t is a solution to the martingale problem (1.1-1.2) if and only if it is a solution to the following conditional martingale problem (CMP): For all $\phi \in C_0^2(\mathbb{R}^d)$,*

$$N_t(\phi) \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, L\phi\rangle\, ds - \int_0^t \left\langle X_s, \nabla^T \phi\sigma_1\right\rangle dW(s) \tag{3.1}$$

is a continuous P^W-martingale with quadratic variation process

$$\langle N(\phi)\rangle_t = \int_0^t \left\langle X_s, \phi^2\right\rangle ds. \tag{3.2}$$

Proof: Suppose that X_t is a solution to the martingale problem (1.1-1.2). Similar to the martingale representation Theorem 3.3.6 in Kallianpur and Xiong [6] there exist processes W and B such that W is a $\mathbb{R}^d$-valued Brownian motion, B is an $L^2(\mathbb{R}^d)$-cylindrical Brownian motion independent of W, and

$$M_t(\phi) = \int_0^t \left\langle X_s, \nabla^T \phi\sigma_1\right\rangle dW(s) + \int_0^t \langle f(s, X_s)^*\phi, dB_s\rangle_{L^2(\mathbb{R}^d)},$$

where $f(s, X_s)$ is a linear map from $L^2(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, the space of Schwartz distributions such that

$$\langle X_t, \phi_1\phi_2\rangle = \langle f(t, X_t)^*\phi_1, f(t, X_t)^*\phi_2\rangle_{L^2(\mathbb{R}^d)}, \qquad \forall \phi_1,\ \phi_2 \in \mathcal{S}(\mathbb{R}^d).$$

It is then easy to see that X_t solves the CMP (3.1-3.2).
On the other hand, suppose that X_t is a solution to the CMP (3.1-3.2). As $N_t(\phi)$ is a P^W-martingale, for $s < t$, we have

$$\begin{aligned}
\mathbb{E}(N_t(\phi)W_t|\mathcal{F}_s^X) &= \mathbb{E}(\mathbb{E}(N_t(\phi)|\sigma(W) \vee \mathcal{F}_s)W_t|\mathcal{F}_s^X) \\
&= \mathbb{E}(N_s(\phi)W_t|\mathcal{F}_s^X) \\
&= N_s(\phi)W_s
\end{aligned}$$

where $\mathcal{F}_t^X$ is the σ-field generated by X. Hence the quadratic covariation process $\langle N(\phi), W\rangle_t = 0$. Therefore,

$$M_t(\phi) = N_t(\phi) + \int_0^t \left\langle X_s, \nabla^T \phi\sigma_1\right\rangle dW(s)$$

is a martingale with quadratic variation process

$$\begin{aligned}
\langle M(\phi)\rangle_t &= \langle N(\phi)\rangle_t + \int_0^t \left|\left\langle X_s, \nabla^T\phi\sigma_1\right\rangle\right|^2 ds \\
&= \int_0^t \left(\left\langle X_s, \phi^2\right\rangle + \left|\left\langle X_s, \nabla^T\phi\sigma_1\right\rangle\right|^2\right) ds.
\end{aligned}$$

This proves that X_t is a solution to the MP (1.1-1.2). ∎

Now, we extend the state space of the superprocess to the space of infinite measures. Let $\phi_a(x) = e^{-a|x|}$. Define the space of tempered measures of as:

$$M_{tem}(\mathbb{R}^d) = \{\mu : \exists a > 0,\ \langle \mu, \phi_a\rangle < \infty\}.$$

Let S_i, $i = 1, 2, \cdots$, be bounded disjoint subsets of $\mathbb{R}^d$ such that $\mathbb{R}^d = \cup_{i=1}^\infty S_i$, and $\mu^i(\cdot) = \mu(\cdot \cap S_i)$. Let X^i be a sequence of $M_F(\mathbb{R}^d)$-valued processes which are, given W, conditionally independent and for each i, X_t^i is a solution to the CMP (3.1-3.2) with μ^i in place of μ. Let $X_t = \sum_{i=1}^\infty X_t^i$. For any $a > 0$,

$$\mathbb{E}\left\langle X_t, e^{-a|x|}\right\rangle = \sum_{i=1}^\infty \mathbb{E}\left\langle X_t^i, e^{-a|x|}\right\rangle = \sum_{i=1}^\infty \mathbb{E}\int \mu^i(dx)\mathbb{E}_x e^{-a|\xi(t)|}$$

where the last equality follows from Theorem 5.1 in [12]. By Lemma 2, we have

$$\mathbb{E}_x e^{-a|\xi(t)|} \leq ce^{-a|x|}.$$

Therefore, we can continue (3.3) with

$$\mathbb{E}\left\langle X_t, e^{-a|x|}\right\rangle \leq c\int \mu(dx)e^{-a|x|} < \infty.$$

Hence, X_t is a well-defined $M_{tem}(\mathbb{R}^d)$-valued process. It is easy to show that X_t solves the CMP (3.1-3.2), and hence, the MP (1.1-1.2). It is clear that (1.3) remains true for $\mu \in M_{tem}(\mathbb{R}^d)$. Next, we consider the following SPDE:

$$\begin{aligned} y_s(x) &= f(x) + \int_0^s \left(Ly_r(x) - y_r(x)^2\right)dr \\ &\quad + \int_0^s \nabla^T y_r(x)\sigma_1 dW(r). \end{aligned} \tag{3.3}$$

Lemma 8

$$y_t(x) = \int p^W(0, x; t, du)f(u) - \int_0^t dr \int p^W(r, x; t, du)y_r(u)^2. \tag{3.4}$$

Proof: Note that the existence of a solution to (3.4) follows from Picard iteration. Since the solution to (3.3) is unique, we only need to show that (3.4) implies (3.3). Suppose z_t is the solution to (3.4). Let

$$T_{s,t}^W f(x) = \int p^W(s, x; t, du)f(u).$$

Then

$$\begin{aligned} z_t(x) &= T_{0,t}^W f(x) - \int_0^t dr T_{r,t}^W(z_r^2)(x) \\ &= f(x) + \int_0^t ds L T_{0,s}^W f(x) + \int_0^t \nabla^T T_{0,s}^W f(x)\sigma_1 dW(s) \\ &\quad - \int_0^t dr\left(z_r^2(x) + \int_r^t ds L T_{r,s}^W(z_r^2)(x) + \int_r^t \nabla^T T_{r,s}^W(z_r^2)(x)\sigma_1 dW(s)\right). \end{aligned}$$

By the stochastic Fubini's theorem (cf. [5], P116, Lemma 4.1), we can continue with

$$\begin{aligned} z_t(x) &= f(x) + \int_0^t ds L T^W_{0,s} f(x) - \int_0^t ds \int_0^s dr L T^W_{r,s}(z_r^2)(x) \\ &\quad - \int_0^t dr z_r^2(x) + \int_0^t \nabla^T T^W_{0,s} f(x) \sigma_1 dW(s) \\ &\quad - \int_0^t \left(\int_0^s dr \nabla^T T^W_{r,s}(z_r^2)(x) \sigma_1 \right) dW(s) \\ &= f(x) + \int_0^t ds L z_s(x) - \int_0^t dr z_r^2(x) + \int_0^t \sigma_1^T \nabla z_s(x) \cdot dW(s). \end{aligned}$$

This finishes the proof of (3.4). ∎

Denote the first term on the right hand side of (3.4) by $T_t^W f(x)$. Then, it satisfies (3.3) without the square term. Namely, $\forall \phi \in C_0^\infty(\mathbb{R}^d)$,

$$\left\langle T_t^W f, \phi \right\rangle = \langle f, \phi \rangle + \int_0^t \left\langle T_s^W f, L^* \phi \right\rangle ds - \int_0^t \left\langle T_s^W f, \nabla^T(\sigma_1 \phi) \right\rangle dW(s).$$

Lemma 9

$$\mathbb{E}(T_t^W f(x)^2) \leq c t^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(z)| dz \int_{\mathbb{R}^d} |f(z)| p_0(t,x,z) dz$$

where c is a constant and p_0 is the transition function of the Brownian motion.

Proof: By Itô's formula, it is easy to see that $\forall \phi,\ \psi \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} d\left(\left\langle T_t^W f, \phi \right\rangle \left\langle T_t^W g, \psi \right\rangle \right) &= \bigg(\left\langle T_t^W f, L^* \phi \right\rangle \left\langle T_t^W g, \psi \right\rangle + \left\langle T_t^W f, \phi \right\rangle \left\langle T_t^W g, L^* \psi \right\rangle \\ &\qquad + \left\langle T_t^W f, \nabla^T(\sigma_1 \phi) \right\rangle \left\langle T_t^W g, \nabla^T(\sigma_1 \psi) \right\rangle \bigg) dt \\ &\quad + d(mart.) \end{aligned}$$

Denote $(f * g)(x,y) = f(x)g(y)$. Then

$$\frac{d}{dt} \left\langle \mathbb{E}(T_t^W f * T_t^W g), \phi * \psi \right\rangle = \left\langle \mathbb{E}(T_t^W f * T_t^W g), \mathbb{L}^*(\phi * \psi) \right\rangle \tag{3.5}$$

where $\mathbb{L}^*$ is the dual operator of $\mathbb{L}$ given by

$$\begin{aligned} \mathbb{L}F(x,y) &= \frac{1}{2} \sum_{i,j=1}^d \left(a_{ij}(x) \frac{\partial^2 F(x,y)}{\partial x_i \partial x_j} + a_{ij}(y) \frac{\partial^2 F(x,y)}{\partial y_i \partial y_j} + \sum_{k=1}^d \sigma_1^{ik}(x) \sigma_1^{jk}(y) \frac{\partial^2 F(x,y)}{\partial x_i \partial y_j} \right) \\ &\quad + \sum_{i=1}^d \left(b_i(x) \frac{\partial F(x,y)}{\partial x_i} + b_i(y) \frac{\partial F(x,y)}{\partial y_i} \right). \end{aligned}$$

Let $p(t,(x,y),(z_1,z_2))$ be the transition function of the Markov process generated by $\mathbb{L}$. By (3.5), we see that

$$\mathbb{E}(T_t^W f * T_t^W g)(x,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z_1) g(z_2) p(t,(x,y),(z_1,z_2)) dz_1 dz_2.$$

By Theorem 4.5 in Friedman [4], there exists a constant c such that

$$p(t,(x,y),(z_1,z_2)) \le cp_0(t,x,z_1)p_0(t,y,z_2).$$

The conclusion of the lemma then follows from the facts that $p_0(t,x,z_1) \le ct^{-\frac{d}{2}}$ and

$$\mathbb{E}(T_t^W f(x)^2) = \mathbb{E}(T_t^W f * T_t^W f)(x,x).$$

∎

Here is our main result.

Theorem 10 *Suppose that $d \ge 3$, (2.5) holds and μ has density which is bounded by $c_1 e^{c_2|x|}$, where c_1 and c_2 are two constants. Then X_t converges in distribution to a limit X_∞ as $t \to \infty$. Furthermore, $\mathbb{E}X_\infty = \mu$.*

Proof: By (1.4), we have

$$\begin{aligned} y_{t-s,t}(x) &= f(x) + \int_{t-s}^t \left(Ly_{r,t}(x) - y_{r,t}(x)^2\right) dr + \int_{t-s}^t \nabla^T y_{r,t}(x)\sigma_1 \hat{d}W(r) \\ &= f(x) + \int_0^s \left(Ly_{t-r,t}(x) - y_{t-r,t}(x)^2\right) dr + \int_0^s \nabla^T y_{t-r,t}(x)\sigma_1 d\bar{W}^t(r), \end{aligned} \tag{3.6}$$

where $\bar{W}^t(r) = W(t) - W(t-r)$ and the stochastic integral above is the usual Itô integral. Recall that y_s is given by (3.3). Since W and $\bar{W}^t$ are both Brownian motions, $\{y_s :\ 0 \le s \le t\}$ and $\{y_{t-s,t} :\ 0 \le s \le t\}$ have the same distribution as stochastic processes. Therefore,

$$\mathbb{E}e^{-\langle \mu, y_{0,t}\rangle} = \mathbb{E}e^{-\langle \mu, y_t\rangle}. \tag{3.7}$$

Note that $y_{s,t}$ and y_s are nonnegative (when $f \ge 0$), the above expectations are finite. Taking integral on both sides of (3.4) with respect to the measure μ, by (2.5), we have

$$\langle \mu, y_t\rangle = \langle \mu, f\rangle - \int_0^t \left\langle \mu, y_r^2\right\rangle dr. \tag{3.8}$$

Let $t \to \infty$ in (3.8), we obtain

$$\lim_{t\to\infty} \langle \mu, y_t\rangle = \langle \mu, f\rangle - \int_0^\infty \left\langle \mu, y_r^2\right\rangle dr. \tag{3.9}$$

Then, as $t \to \infty$,

$$\begin{aligned} \mathbb{E}_\mu e^{-\langle X_t, f\rangle} &= \mathbb{E}e^{-\langle \mu, y_{0,t}\rangle} = \mathbb{E}e^{-\langle \mu, y_t\rangle} \\ &\to \mathbb{E}\exp\left(-\langle \mu, f\rangle + \int_0^\infty \left\langle \mu, y_r^2\right\rangle dr\right). \end{aligned} \tag{3.10}$$

Note that, $\forall\ f \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned} \mathbb{E}_\mu \langle X_t, f\rangle &= \mathbb{E}\left(\mathbb{E}_\mu^W \langle X_t, f\rangle\right) \\ &= \mathbb{E}\langle \mu, y_{0,t}\rangle \\ &\le \mathbb{E}\int \mu(dx) \int p^W(0,x;t,du) f(u) \\ &= \int \mu(du) f(u) < \infty, \end{aligned} \tag{3.11}$$

where the second equality follows from Theorem 5.1 in [12], the inequality follows from (3.4) and the last equality from (2.5). By approximation, we can show that (3.11) still hold if $f(x) = e^{-a|x|}$. Therefore, $\{X_t\}$ is tight in $M_{tem}(\mathbb{R}^d)$. Let X_∞ be a limit point. Then, the Laplace transform of X_∞ is given by the limit on the right hand side of (3.10). Therefore, the limit distribution is unique and hence, X_t converges to X_∞ in distribution.
By Fatou's lemma, we have

$$\mathbb{E}\left\langle X_\infty, f\right\rangle \leq \liminf_{t\to\infty} \mathbb{E}_\mu \left\langle X_t, f\right\rangle \leq \left\langle \mu, f\right\rangle,$$

where the second inequality follows from (3.11). On the other hand, by Jensen's inequality

$$e^{-\mathbb{E}\langle X_\infty, f\rangle} \leq \mathbb{E}e^{-\langle X_\infty, f\rangle} = \mathbb{E}\exp\left(-\left\langle \mu, f\right\rangle + \int_0^\infty \left\langle \mu, y_r^2\right\rangle dr\right)$$

and hence

$$\mathbb{E}\left\langle X_\infty, f\right\rangle \geq -\log \mathbb{E}\exp\left(-\left\langle \mu, f\right\rangle + \int_0^\infty \left\langle \mu, y_r^2\right\rangle dr\right).$$

Replace f by ϵf, we have

$$\begin{aligned} \left\langle \mu, f\right\rangle &\geq \mathbb{E}\left\langle X_\infty, f\right\rangle \\ &\geq -\epsilon^{-1}\log \mathbb{E}\exp\left(-\epsilon\left\langle \mu, f\right\rangle + \int_0^\infty \left\langle \mu, y_r^2(\epsilon f)\right\rangle dr\right) \\ &= \left\langle \mu, f\right\rangle - \epsilon^{-1}\log \mathbb{E}\exp\left(\int_0^\infty \left\langle \mu, y_r^2(\epsilon f)\right\rangle dr\right) \end{aligned}$$

here $y_r(\epsilon f)$ is defined as in (3.3) with f replaced by ϵf. We only need to show that

$$\epsilon^{-1}\log \mathbb{E}\exp\left(\int_0^\infty \left\langle \mu, y_r^2(\epsilon f)\right\rangle dr\right) \to 0 \qquad \text{as } \epsilon \to 0. \tag{3.12}$$

By (3.9), we have

$$\int_0^\infty \left\langle \mu, y_r^2(\epsilon f)\right\rangle dr \leq \epsilon \left\langle \mu, f\right\rangle. \tag{3.13}$$

Hence

$$\begin{aligned} &\lim_{\epsilon\to 0} \epsilon^{-1}\log \mathbb{E}\exp\left(\int_0^\infty \left\langle \mu, y_r^2(\epsilon f)\right\rangle dr\right) \qquad (3.14) \\ \leq\ &\lim_{\epsilon\to 0} \mathbb{E}\epsilon^{-1}\left(\exp\left(\int_0^\infty \left\langle \mu, y_r^2(\epsilon f)\right\rangle dr\right) - 1\right) \\ =\ &\mathbb{E}\lim_{\epsilon\to 0} \epsilon^{-1}\left(\exp\left(\int_0^\infty \left\langle \mu, y_r^2(\epsilon f)\right\rangle dr\right) - 1\right) \end{aligned}$$

where the last equality follows from (3.13) and the dominated convergence theorem.
By (3.4), we have

$$\int_0^\infty \left\langle \mu, y_r^2(\epsilon f)\right\rangle dr \leq \epsilon^2 \int_0^\infty \left\langle \mu, (T_r^W f)^2\right\rangle dr.$$

Therefore, by (3.14), we only need to show that

$$\int_0^\infty \left\langle \mu, \left(T_t^W f(x)\right)^2\right\rangle dt < \infty, \qquad a.s. \tag{3.15}$$

Note that

$$\int_0^1 \left\langle \mu, \left(T_t^W f(x)\right)^2 \right\rangle dr \leq \int_0^1 \left\langle \mu, T_t^W f(x) \|f\|_\infty \right\rangle dt$$
$$= \langle \mu, f \rangle \|f\|_\infty < \infty. \tag{3.16}$$

On the other hand,

$$\begin{aligned} & \mathbb{E}\int_1^\infty \left\langle \mu, \left(T_t^W f(x)\right)^2 \right\rangle dt \\ \leq & \int_1^\infty ct^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(z)|dz \int_{\mathbb{R}^d} |f(z)| \int_{\mathbb{R}^d} e^{c_2|x|} p_0(t,x,z) dxdzdt \\ \leq & \; c\int_1^\infty t^{-\frac{d}{2}} dt \int_{\mathbb{R}^d} |f(z)|dz \int_{\mathbb{R}^d} |f(z)| e^{c_2|z|} dz < \infty \end{aligned} \tag{3.17}$$

where the first inequality follows from Lemma 9 and the second inequality follows from the well-known fact that

$$\int_{\mathbb{R}^d} e^{c_2|x|} p_0(t,x,z)dx \leq ce^{c_2|z|},$$

the finiteness in the last step of (3.17) follows from the the compact support property imposed on f in the condition (BC). This, together with (3.16), implies the almost sure finiteness in (3.15). ∎

4 Long-time local extinction when $d \leq 2$

In this section, we prove the long-term local extinction when $d \leq 2$. We adapt the proof of Dawson *et al* [1] to our present setup.

Theorem 11 *Suppose that $d \leq 2$ and (2.5) holds. Further, we assume that*

$$\mu << \lambda \text{ and } 0 < c_1 \leq \frac{d\mu}{d\lambda} \leq c_2 < \infty.$$

For any bounded Borel set B in $\mathbb{R}^d$, we have

$$\lim_{t\to\infty} X_t(B) = 0, \qquad \text{in probability.}$$

Proof: By (1.3) and (3.7), we see that it is sufficient to show

$$\lim_{t\to\infty} \langle \mu, y_t \rangle = 0 \qquad a.s. \tag{4.1}$$

By (3.9), the left hand side of (4.1) exists. By Fatou's lemma, we only need to show that

$$\liminf_{t\to\infty} \mathbb{E} \langle \mu, y_t \rangle = 0.$$

For $\epsilon > 0$, choose K such that

$$\int_{|x|^2 > K} p_1(x)dx < \epsilon, \tag{4.2}$$

where $p_t(x)$ is the density of the normal random vector with mean 0 and covariance matrix tI. Let c and τ be such that $f \le cp_\tau$. For $t > 0$, set

$$S_t = \{x \in \mathbb{R}^d : \ |x|^2 \le K(t+\tau)\}.$$

Note that by (3.3),

$$\mathbb{E}y_t(x) \le f(x) + \int_0^t \mathbb{E}(Ly_r(x))dr.$$

It is well-known that the above inequality yields

$$\mathbb{E}y_t(x) \le c \int p_t(x-u)f(u)du. \tag{4.3}$$

By (4.3) and (4.2), since $f \le cp_\tau$, we have

$$\int_{S_t^c} \mathbb{E}y_t(x)\mu(dx) \le c\int_{S_t^c} p_{t+\tau}(x)dx = c\int_{|x|^2>K} p_1(x)dx < c\epsilon. \tag{4.4}$$

By Jensen's inequality and (3.8), we have

$$\begin{aligned} \int_0^t |S_r|^{-1}g^2(r)dr &\le c\mathbb{E}\int_0^t \int_{S_r} y_r(x)^2 dxdr \\ &\le c\mathbb{E}\int_0^t \left\langle \mu, y_r^2 \right\rangle dr \\ &\le \left\langle \mu, f\right\rangle, \end{aligned} \tag{4.5}$$

here $|S_r|$ denotes the Lebesgue measure of S_r and $g(r) = \int_{S_r} \mathbb{E}y_r(x)\mu(dx)$. As

$$\int_0^\infty |S_r|^{-1}dr = \infty,$$

it follows from (4.5) that

$$\liminf_{t\to\infty} g(t) = 0, \qquad a.s. \tag{4.6}$$

By (4.4) and (4.6), we have

$$\lim_{t\to\infty} \mathbb{E}\left\langle \mu, y_t \right\rangle \le c\epsilon, \qquad a.s..$$

Since ϵ is arbitrary, the proof of the statement is complete. ∎

Acknowledgment: I would like to thank Klaus Fleischmann, Xiaowen Zhou, the referee and the associated editor for helpful comments and suggestions which improved the presentation of this paper. This work is started when I am in the University of Alberta supported by Canada Research Chair Program and is completed when I am in WIAS supported by Alexander von Humboldt Foundation. Hospitality from both institutes are appreciated.

References

[1] D.A. Dawson, K. Fleischmann, R.D. Foley and L.A. Peletier (1986). A critical measure-valued branching process with infinite mean. *Stoch. Anal. Appl.* **4**, 117-129.

[2] D. A. Dawson, Z. Li and H. Wang (2001). Superprocesses with dependent spatial motion and general branching densities. *Electronic Journal of Probability,* **6**, 1-33.

[3] S.N. Ethier and T.G. Kurtz (1986). *Markov processes. Characterization and convergence.* John Wiley & Sons, Inc., New York.

[4] A. Friedman, *Stochastic Differential Equations and Applications*, Volume 1. Academic Press, 1975.

[5] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusions*, North Holland Publishing Company, 1989.

[6] G. Kallianpur and J. Xiong (1995): Stochastic Differential Equations in Infinite Dimensional Spaces, *IMS Lecture Notes-Monograph Series* **26**. Institute of Mathematical Statistics.

[7] G. Skoulakis and R.J. Adler (2001). Superprocesses over a stochastic flow. *Ann. Appl. Probab.* **11**, no. 2, 488-543.

[8] S.R.S. Varadhan (1980). Lectures on diffusion problems and partial differential equations. With notes by Pl. Muthuramalingam and Tara R. Nanda. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 64.

[9] H. Wang (1998). A class of measure-valued branching diffusions in a random medium. *Stochastic Anal. Appl.* **16**, 753-786

[10] Y.J. Wang (1997). A proof of the persistence criterion for a class of superprocesses. *J. Appl. Prob.* **34**, 559-563.

[11] E. Wong and M. Zakai, On the relation between ordinary and stochastic differential equations, Intern. J. Engng. Sci., 3 (1965), 213-229.

[12] J. Xiong (2004). A stochastic log-Laplace equation. *To appear in Ann. Probab.*.

[13] J. Xiong (1988). The σ-finite invariant measure for diffusion processes, *Beijing Daxue Xuebao,* **24**, 287-290 (in Chinese).