Beta-gamma tail asymptotics

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Abstract

We compute the tail asymptotics of the product of a beta random variable and a generalized gamma random variable which are independent and have general parameters. A special case of these asymptotics were proved and used in a recent work of Bubeck, Mossel, and Rácz in order to determine the tail asymptotics of the maximum degree of the preferential attachment tree. The proof presented here is simpler and highlights why these asymptotics hold.

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1 Introduction

There has been a lot of recent interest in various urn schemes due to their appearance in many graph growth models (see, e.g., [15, 2, 3, 16, 4, 17, 5, 9, 18]). The limiting distributions arising in these urn schemes are often related to the beta and gamma distributions. Consequently, the computation of various statistics in random graph models often boils down to using algebraic properties of these distributions, commonly referred to as the beta-gamma algebra [6].

The purpose of this note is to simplify and demystify a recent computation done in [5] involving beta and generalized gamma random variables. Bubeck, Mossel, and Rácz [5] were interested in the influence of the seed graph in the preferential attachment model, which led them to study the tail asymptotics of the maximum degree of the preferential attachment tree. This, in turn, essentially reduces to computing the asymptotics of $\mathbb{P}\left(BZ > t\right)$ as $t \to \infty$, where $B \sim \mathrm{Beta}\left(a, b\right)$ and $Z \sim \mathrm{GGa}\left(a + b + 1, 2\right)$ are independent random variables and a and b are positive integers; here $\mathrm{Beta}\left(a, b\right)$ denotes the beta distribution with positive parameters a and b (with density $\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\mathbf{1}_{\{x \in [0,1]\}}$, where $B\left(a, b\right) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function), and $\mathrm{GGa}\left(c, p\right)$ denotes the generalized gamma distribution with density $\frac{p}{\Gamma(c/p)}x^{c-1}e^{-x^p}\mathbf{1}_{\{x>0\}}$ for $c, p > 0$. We refer to [5] for details on these connections; see also [12, 17].

The computation in [5] involves a few pages of alternating sums cancelling each other out in just the right way. Here, in contrast, we provide a short and simple proof of these asymptotics. The core calculation is only a few lines long, involving approximations at three points which are natural and which can be justified in a relatively straightforward

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manner. Moreover, the argument works for all positive values of the parameters a, b, c, and p. Throughout the paper we use standard asymptotic notation; for instance, $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t \to \infty} f(t) / g(t) = 1$.

Claim 1.1. *Let a, b, c, and p be positive, let $B \sim \mathrm{Beta}(a, b)$, and let $Z \sim \mathrm{GGa}(c, p)$, with B and Z independent. Then we have*

$$\mathbb{P}(BZ > t) \sim \frac{\Gamma(a+b)}{\Gamma(c/p)\,\Gamma(a)} p^{-b} t^{c-(b+1)p} e^{-t^p} \qquad \text{as } t \to \infty.$$

There has been lots of work on understanding the distribution and tail asymptotics of products of random variables; see, e.g., [19] for a paper from nearly half a century ago, and [10, 11] and references therein for recent developments. In particular, Claim 1.1 is a special case of [10, Theorem 4.1], where the authors prove a general result for any product BZ where $B \sim \mathrm{Beta}(a, b)$ and Z has a law which is in the maximum domain of attraction of the Gumbel distribution. Due to the generality of their result their proof is fairly involved. We thus believe that the simple proof we present here is useful in highlighting why these asymptotics hold.

The product BZ studied in Claim 1.1 has many nice properties, for instance it has moments of Gamma type [13] (and so its density can be written in terms of the Fox H-function [8]). When $p = 1$, Z is a gamma random variable and BZ has a so-called G distribution, with its moments described by Meijer's G-function [7]. The case of $p = 2$ appears in many settings, including the preferential attachment model as mentioned above, and see also [2, 18] for connections to critical random graphs, random walks, and various random trees, including Aldous's Brownian continuum random tree (CRT). In a very interesting recent work, Peköz, Röllin, and Ross [18] showed that generalized gamma random variables with p being an integer greater than 2 arise as limits in time inhomogeneous Pólya-type urn schemes. It would be interesting to find connections to urn schemes and random graph models for general values of p.

2   The core calculation

In this section we prove Claim 1.1 modulo some approximations whose validity is justified later in Section 3. Put $W = Z^p$, $z = w^{1/p}$, and $dz = \frac{1}{p} w^{1/p-1} dw$. The density of W is thus

$$f_W(w) = \frac{1}{\Gamma(c/p)} w^{c/p-1} e^{-w} \mathbf{1}_{\{w > 0\}}.$$

In other words, W has a gamma distribution: $W \sim \mathrm{Gamma}(c/p, 1)$. We have

$$\mathbb{P}(BZ > t) = \mathbb{P}(B^p Z^p > t^p) = \mathbb{P}(B^p W > t^p),$$

and so the claim is equivalent to

$$\mathbb{P}(B^p W > w) \sim \frac{\Gamma(a+b)}{\Gamma(c/p)\,\Gamma(a)} p^{-b} w^{c/p-b-1} e^{-w} \qquad \text{as } w \to \infty.$$

We can write

$$\mathbb{P}(B^p W > w) = \mathbb{P}(W > w)\,\mathbb{P}(B^p W > w \mid W > w).$$

The first factor has well-known asymptotics: $\mathbb{P}(W > w) \sim f_W(w)$ as $w \to \infty$ [1, formula 6.5.32]. Also, it is well known that for any random variable W which has a $\mathrm{Gamma}(r, 1)$ distribution, we have

$$(W - w \mid W > w) \xrightarrow{d} \mathcal{E} \qquad \text{as } w \to \infty,$$

where $\mathcal{E}$ is a standard exponential random variable. This convergence is rather strong, e.g., convergence of densities. So

$$\mathbb{P}\left(B^p W > w \,|\, W > w\right) = \mathbb{P}\left(B^p\left(w + W - w\right) > w \,|\, W > w\right) \tag{2.1}$$

$$\sim \mathbb{P}\left(B^p\left(w + \mathcal{E}\right) > w\right) = \mathbb{P}\left(B > \left(\frac{1}{1 + \mathcal{E}/w}\right)^{1/p}\right) \tag{2.2}$$

$$\sim \mathbb{P}\left(B > 1 - \frac{\mathcal{E}}{pw}\right) = \mathbb{P}\left(1 - B < \frac{\mathcal{E}}{pw}\right) \tag{2.3}$$

$$\sim \int_0^\infty \frac{1}{B(a,b)} u^{b-1} e^{-pwu} du = \frac{1}{B(a,b)} \left(\frac{1}{pw}\right)^b \Gamma(b) \tag{2.4}$$

and the conclusion follows. The only thing that remains is to rigorously justify the three points where asymptotic equivalence was used in the line of reasoning above; see Section 3 for details.

3 Justifying the approximations

Here we justify why the expressions in (2.1), (2.2), (2.3), and (2.4) are all asymptotically equivalent as $w \to \infty$.

Asymptotic equivalence of (2.3) **and** (2.4)**.** By definition we have

$$\mathbb{P}\left(1 - B < \frac{\mathcal{E}}{pw}\right) = \int_0^\infty \mathbb{P}\left(1 - B < \frac{z}{pw}\right) e^{-z} dz = \int_0^{pw} \mathbb{P}\left(1 - B < \frac{z}{pw}\right) e^{-z} dz + e^{-pw}.$$

Since $\int_0^\infty u^{b-1} e^{-pwu} du = \Theta\left(w^{-b}\right)$ as $w \to \infty$, we can neglect all terms that are $o\left(w^{-b}\right)$ as $w \to \infty$, and so we only have to deal with the integral term on the right hand side of the display above. Using that $1 - B \sim \mathrm{Beta}(b,a)$ and also the change of variables $z = pwu$, we have

$$\int_0^{pw} \mathbb{P}\left(1 - B < \frac{z}{pw}\right) e^{-z} dz = \int_0^{pw} \int_0^{\frac{z}{pw}} \frac{1}{B(a,b)} x^{b-1} (1-x)^{a-1} dx\, e^{-z} dz$$

$$= \frac{pw}{B(a,b)} \int_0^1 \int_0^u x^{b-1} (1-x)^{a-1} dx\, e^{-pwu} du.$$

The integral from $1/2$ to 1 is negligible, since

$$pw \int_{1/2}^1 \frac{1}{B(a,b)} \int_0^u x^{b-1} (1-x)^{a-1} dx\, e^{-pwu} du \leq \int_{1/2}^1 (pw) e^{-pwu} du \leq e^{-pw/2}.$$

For the integral from 0 to $1/2$ we first drop the factor $(1-x)^{a-1}$ and we justify the validity of this later. Integration by parts then tells us that

$$\int_0^{1/2} \frac{u^b}{b} (pw) e^{-pwu} du = -\frac{1}{2^b b} e^{-pw/2} + \int_0^{1/2} u^{b-1} e^{-pwu} du$$

$$= \int_0^\infty u^{b-1} e^{-pwu} du - \frac{1}{2^b b} e^{-pw/2} - \int_{1/2}^\infty u^{b-1} e^{-pwu} du.$$

Using that $(1/2 + v)^{b-1} \leq 2 + (2v)^{b-1}$ for all $b > 0$ and $v > 0$, we have that the integral from $1/2$ to ∞ in the display above is negligible:

$$\int_{1/2}^\infty u^{b-1} e^{-pwu} du = e^{-pw/2} \int_0^\infty (1/2 + v)^{b-1} e^{-pwv} dv$$

$$\leq e^{-pw/2} \int_0^\infty \left(2 + (2v)^{b-1}\right) e^{-pwv} dv = \left(\frac{2}{pw} + \frac{2^{b-1}\Gamma(b)}{(pw)^b}\right) e^{-pw/2}.$$

We have thus shown that

$$\frac{pw}{B\left(a,b\right)}\int_0^{1/2}\int_0^u x^{b-1}dxe^{-pwu}du \sim \int_0^\infty \frac{1}{B\left(a,b\right)}u^{b-1}e^{-pwu}du \qquad\qquad \text{as } w\to\infty.$$

Finally, to justify dropping the $(1-x)^{a-1}$ factor, note that $\left|1-(1-x)^{a-1}\right| \leq \max\{a,2\}\,x$ for all $a>0$ and $x\in(0,1/2)$, and so

$$\left|\frac{pw}{B\left(a,b\right)}\int_0^{1/2}\int_0^u x^{b-1}\left(1-x\right)^{a-1}dxe^{-pwu}du - \frac{pw}{B\left(a,b\right)}\int_0^{1/2}\int_0^u x^{b-1}dxe^{-pwu}du\right|$$

$$\leq \max\{a,2\}\frac{pw}{B\left(a,b\right)}\int_0^{1/2}\int_0^u x^b dxe^{-pwu}du = O\left(w^{-(b+1)}\right).$$

Asymptotic equivalence of (2.2) **and** (2.3)**.** We need to show that, as $w\to\infty$,

$$\left|\mathbb{P}\left(B > 1-\frac{\mathcal{E}}{pw}\right) - \mathbb{P}\left(B > \left(\frac{1}{1+\mathcal{E}/w}\right)^{1/p}\right)\right| = o\left(w^{-b}\right). \qquad (3.1)$$

Using that $1-x/p \leq (1+x)^{-1/p} < 1-x/p+\frac{p+1}{2p^2}x^2$ for all $x>0$, we have that the left hand side of (3.1) is at most

$$\mathbb{P}\left(B\in\left[1-\frac{\mathcal{E}}{pw},1-\frac{\mathcal{E}}{pw}+\frac{p+1}{2p^2}\frac{\mathcal{E}^2}{w^2}\right],\mathcal{E}\leq\frac{p}{p+1}w\right) + \mathbb{P}\left(\mathcal{E}>\frac{p}{p+1}w\right).$$

We know that $\mathbb{P}\left(\mathcal{E}>\frac{p}{p+1}w\right) = e^{-\frac{p}{p+1}w}$, so we only have to deal with the first term in the sum above. Using the change of variables $z=pwu$ we have

$$\mathbb{P}\left(B\in\left[1-\frac{\mathcal{E}}{pw},1-\frac{\mathcal{E}}{pw}+\frac{p+1}{2p^2}\frac{\mathcal{E}^2}{w^2}\right],\mathcal{E}\leq\frac{p}{p+1}w\right)$$

$$= \int_0^{\frac{p}{p+1}w}\int_{\frac{z}{pw}-\frac{p+1}{2p^2}\frac{z^2}{w^2}}^{\frac{z}{pw}}\frac{1}{B\left(a,b\right)}x^{b-1}\left(1-x\right)^{a-1}dxe^{-z}dz$$

$$= \frac{pw}{B\left(a,b\right)}\int_0^{\frac{1}{p+1}}\int_{u-\frac{p+1}{2}u^2}^u x^{b-1}\left(1-x\right)^{a-1}dxe^{-pwu}du$$

$$\leq \frac{\max\left\{1,(p/p+1)^{a-1}\right\}}{B\left(a,b\right)}(pw)\int_0^{\frac{1}{p+1}}\int_{u-\frac{p+1}{2}u^2}^u x^{b-1}dxe^{-pwu}du.$$

Using simple estimates we have

$$\int_{u-\frac{p+1}{2}u^2}^u x^{b-1}dx = \tfrac{1}{b}\left(u^b - \left(u-\tfrac{p+1}{2}u^2\right)^b\right) \leq \frac{(p+1)(b+1)}{2b}u^{b+1}$$

for all $u\in\left(0,\frac{1}{p+1}\right)$ and so

$$(pw)\int_0^{\frac{1}{p+1}}\int_{u-\frac{p+1}{2}u^2}^u x^{b-1}dxe^{-pwu}du \leq \frac{(p+1)(b+1)}{2b}(pw)\int_0^{\frac{1}{p+1}}u^{b+1}e^{-pwu}du$$

$$\leq \frac{(p+1)(b+1)}{2b}(pw)\int_0^\infty u^{b+1}e^{-pwu}du = \frac{(p+1)(b+1)\Gamma(b+2)}{2bp^{b+1}}w^{-(b+1)},$$

which concludes the proof of (3.1).

Asymptotic equivalence of (2.1) **and** (2.2). We justify this via a direct calculation, though there might be a more elegant way to do this. To abbreviate notation, let $r := c/p$. By definition we have

$$
\mathbb{P}\left(B^p\left(w+W-w\right) > w \,\middle|\, W > w\right) = \frac{\int_w^\infty \mathbb{P}\left(B^p\left(w+y-w\right) > w\right) \frac{1}{\Gamma(r)} y^{r-1} e^{-y} dy}{\frac{1}{\Gamma(r)} \int_w^\infty x^{r-1} e^{-x} dx}
$$

$$
= \frac{w^{r-1} e^{-w}}{\int_w^\infty x^{r-1} e^{-x} dx} \int_0^\infty \mathbb{P}\left(B^p\left(w+z\right) > w\right) \left(1 + \frac{z}{w}\right)^{r-1} e^{-z} dz.
$$

For $r = 1$ this is exactly equal to $\mathbb{P}\left(B^p\left(w+\mathcal{E}\right) > w\right)$. Since $\mathbb{P}\left(W > w\right) \sim f_W\left(w\right)$ as $w \to \infty$, the fraction in front of the integral in the display above goes to 1 as $w \to \infty$, so what remains to show is that

$$
\int_0^\infty \mathbb{P}\left(B^p\left(w+z\right) > w\right) \left| \left(1 + \frac{z}{w}\right)^{r-1} - 1 \right| e^{-z} dz = o\left(\mathbb{P}\left(B^p\left(w+\mathcal{E}\right) > w\right)\right) \quad \text{as } w \to \infty.
$$

We partition the integral into two parts: from 0 to $\sqrt{w}$, and from $\sqrt{w}$ to ∞. For the first term note that for $r \geq 1$ and $z \in (0, \sqrt{w})$ we have $(1 + z/w)^{r-1} - 1 \leq e^{r/\sqrt{w}} - 1$, while for $r \in (0,1)$ and $z \in (0, \sqrt{w})$ we have $1 - (1 + z/w)^{r-1} \leq 1 - (1 + 1/\sqrt{w})^{-1} \leq 1/\sqrt{w}$, and so

$$
\int_0^{\sqrt{w}} \mathbb{P}\left(B^p\left(w+z\right) > w\right) \left| \left(1 + \frac{z}{w}\right)^{r-1} - 1 \right| e^{-z} dz
$$

$$
\leq \max\left\{\frac{1}{\sqrt{w}}, e^{\frac{r}{\sqrt{w}}} - 1\right\} \int_0^{\sqrt{w}} \mathbb{P}\left(B^p\left(w+z\right) > w\right) e^{-z} dz
$$

$$
\leq \max\left\{\frac{1}{\sqrt{w}}, e^{\frac{r}{\sqrt{w}}} - 1\right\} \mathbb{P}\left(B^p\left(w+\mathcal{E}\right) > w\right) = o\left(\mathbb{P}\left(B^p\left(w+\mathcal{E}\right) > w\right)\right).
$$

For the second term we can bound the factor $\mathbb{P}\left(B^p\left(w+z\right) > w\right)$ in the integral by 1. For $r \in (0,1)$ we have $1 - (1 + z/w)^{r-1} \leq 1 - (1 + z/w)^{-1} \leq z/w$ and so we get the upper bound of

$$
\frac{1}{w} \int_{\sqrt{w}}^\infty z e^{-z} dz = \frac{1}{w} e^{-\sqrt{w}} \int_0^\infty \left(\sqrt{w} + u\right) e^{-u} du = \frac{\sqrt{w} + 1}{w} e^{-\sqrt{w}}.
$$

For $r \geq 1$ we use the bound $(1 + z/w)^{r-1} - 1 \leq e^{rz/w}$ to obtain the upper bound of

$$
\int_{\sqrt{w}}^\infty e^{(r/w - 1)z} dz = \frac{1}{1 - r/w} e^{(r/w - 1)\sqrt{w}} \leq 2 e^{-\sqrt{w}/2},
$$

where we assumed that $w \geq 2r$.

4 An alternative calculation

After posting this note on the arXiv, Thomas Simon pointed out to us an alternative calculation which we reproduce here. We use the same notation as in Section 2.

The density of B^p is $f_{B^p}\left(x\right) = \frac{\Gamma(a+b)}{p\Gamma(a)\Gamma(b)} x^{a/p-1} \left(1 - x^{1/p}\right)^{b-1} \mathbf{1}_{\{x \in [0,1]\}}$ and so by multiplicative convolution the density of $B^p W$ is

$$
f_{B^p W}\left(x\right) = \frac{\Gamma\left(a+b\right)}{p\Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(c/p\right)} x^{c/p-1} \int_0^1 t^{(a-c)/p-1} \left(1 - t^{1/p}\right)^{b-1} e^{-x/t} dt \, \mathbf{1}_{\{x > 0\}}.
$$

Making the change of variable $t = 1/\left(1 + y\right)$ we get

$$
f_{B^p W}\left(x\right) = \frac{\Gamma\left(a+b\right) x^{\frac{c}{p}-1} e^{-x}}{p\Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(\frac{c}{p}\right)} \int_0^\infty \left(1 + y\right)^{\frac{c-a}{p}-1} \left(1 - \left(1 + y\right)^{-\frac{1}{p}}\right)^{b-1} e^{-xy} dy \, \mathbf{1}_{\{x > 0\}}.
$$

Let $\varphi(y) := (1+y)^{(c-a)/p-1}\left(1-(1+y)^{-1/p}\right)^{b-1}$. Since $\varphi(y) = (1+o(1))\,p^{1-b}y^{b-1}$ as $y \to 0$, it follows by direct integration that the integral in the display above is asymptotically equivalent to $\Gamma(b)\,p^{1-b}x^{-b}$ as $x \to \infty$. Therefore

$$f_{B^pW}(x) \sim \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(c/p)}p^{-b}x^{c/p-b-1}e^{-x} \qquad \text{as } x \to \infty,$$

and the result follows, again by integration. We omit the details. It should be possible to get the full asymptotic expansion of f_{B^pW} after deriving that of φ, although a general formula for the coefficients in terms of the parameters might be hard to find.

We refer the reader to [14] for a similar calculation in the context of stable processes. More generally, if the densities of random variables (and in particular their asymptotics) are known, then the approach above should work to compute the tail asymptotics of products of the random variables. However, if the densities are not known, then limit theorems like the one used in Section 2 can be useful in deriving tail asymptotics. Indeed, at the core of the proof of the much more general result presented in [10, Theorem 4.1] lies their formula (4.4), which is the appropriate generalization of the fact that when W has a gamma distribution, then $(W - w \,|\, W > w)$ converges to a standard exponential random variable as $w \to \infty$.

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