Electron. Commun. Probab. **20** (2015), no. 57, 1–6.
DOI: 10.1214/ECP.v20-4183
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Necessary and sufficient conditions for the continuity of permanental processes associated with transient Lévy processes*

Michael B. Marcus† Jay Rosen‡

Abstract

Let $u^\beta(x,y)$ be the β-potential density of a transient Lévy process $\overline{Y}$ and $X_\alpha = \{X_{\alpha,x}, x \in R\}$ be the α-permanental process determined by $u^\beta(x,y)$. Let $\overline{L} = \{\overline{L}_t^x, (t,x), \in R_+ \times R\}$ be the local time process of $\overline{Y}$ and let $G = \{G_x, x \in R\}$ be the stationary mean zero Gaussian process with covariance $u^\beta(x,y) + u^\beta(y,x)$. Then the processes X_α, $\overline{L}$ and G are either all continuous almost surely or all unbounded almost surely. Therefore, the well known necessary and sufficient condition for the continuity of $\overline{L}$ and G is also a necessary and sufficient condition for the continuity of X_α.

Keywords: permanental processes, path properties, Lévy processes.
AMS MSC 2010: 60G51; 60G99; 60G17.
Submitted to ECP on March 12, 2015, final version accepted on August 1, 2015.

1 Introduction

Let $Y = \{Y_t, t \in R_+\}$ be a Lévy process with state space R and continuous β-potential densities $u^\beta = \{u^\beta(x,y), x,y \in R\}$, $\beta > 0$. Consider the transient Lévy process $\overline{Y} = \{\overline{Y}_t, t \in R_+\}$ that is Y killed at an independent exponential time with mean $\beta > 0$. Note that u^β is also the zero potential density of $\overline{Y}$. We have

$$u^\beta(x,y) = u^\beta(0, y-x) =: u^\beta(y-x). \tag{1.1}$$

For all $x_1, \ldots, x_n$ in R_+ consider $U^\beta(x_1,\ldots,x_n)$, the $n\times n$ matrix with entries $\{u^\beta(x_i - x_j)\}_{i=1}^n$. Since $u^\beta(x,y)$ is the potential density of a transient Markov process, it follows from [3, Theorem 3.1] that for all $\alpha > 0$ and $x_1, \ldots, x_n \in R_+$ there exists a random variable $(X_{x_1}, \ldots, X_{x_n})$ with Laplace transform

$$E\left(e^{-\sum_{i=1}^n s_i X_{x_i}}\right) = \frac{1}{|I + U^\beta(x_1,\ldots,x_n)S|^\alpha} \tag{1.2}$$

where S is the diagonal matrix with entries s_i, $1 \le i \le n$. Therefore $\{u^\beta(y-x), x,y \in R\}$ determines a stochastic process $X = \{X_x, x \in R\}$ which is called the α-permanental process associated with $\overline{Y}$.

*Research partially supported by grants from the National Science Foundation and PSC CUNY.
†City College, CUNY, N.Y. USA. E-mail: mbmarcus@optonline.net
‡College of Staten Island, CUNY, N.Y. USA. E-mail: jrosen30@optimum.net

Vere-Jones defines and briefly considers permanental processes in [9]. Here is why we are interested in them. Local times of certain Markov processes with symmetric potential densities are related by isomorphism theorems to the squares of Gaussian processes. (Suppose that $u^\beta(x,y)$ is symmetric. Then by [5, Lemma 3.3.3] it is also positive definite and $(\eta^2_{x_1}/2,\dots,\eta^2_{x_n}/2)$, where $(\eta_{x_1},\dots,\eta_{x_n})$ is an n-dimensional normal random variable with mean zero and covariance matrix $\{u^\beta(x_i-x_j)\}_{i=1}^n$ satisfies (1.2) with $\alpha=1/2$.) When $\alpha\neq 1/2$ or $u^\beta(x,y)$ is not symmetric, the isomorphism theorems that hold for Gaussian squares can be generalized, by replacing the squares of the Gaussian processes by other permanental processes, so that they also hold for Markov processes with potential densities that are not symmetric. To apply these isomorphism theorems it is important to know sample path properties of permanental processes.

The Lévy processes we consider have local times. Therefore the potential density u^β can be associated with a local time process and a permanental process. It can also be associated with a stationary Gaussian process, that we define below. In Theorem 1.1 we show that these processes are either all continuous almost surely or else all unbounded almost surely.

We write the characteristic function of Y as

$$Ee^{i\lambda Y_t}=e^{-t\psi(\lambda)}. \tag{1.3}$$

When $u^\beta(y-x)$ is not symmetric, $\psi(\lambda)$ is complex.

It follows from the assumption that $u^\beta(x,y)$ is continuous, that

$$u^\beta(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{R}e\left(1/(\beta+\psi(\lambda))\right)\,d\lambda \tag{1.4}$$

and

$$\begin{aligned}\sigma^2(z)\quad&:=\quad 2u^\beta(0)-(u^\beta(z)+u^\beta(-z)) \\ &=\quad \frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos(\lambda z))\mathcal{R}e\left(1/(\beta+\psi(\lambda))\right)\,d\lambda;\end{aligned} \tag{1.5}$$

see [2, Theorem 19]. Clearly

$$u^\beta(z)+u^\beta(-z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\cos(\lambda z)\mathcal{R}e\left(1/(\beta+\psi(\lambda))\right)\,d\lambda. \tag{1.6}$$

Since $\mathcal{R}e\left(1/(\beta+\psi(\lambda))\right)$ is positive and in $L^1(R)$, $\Gamma(x,y)=u^\beta(y-x)+u^\beta(x-y)$ is the covariance function of a stationary Gaussian process, say $G=\{G_x,x\in R\}$ and

$$\left(E(G_x-G_0)^2\right)^{1/2}=\sigma(x). \tag{1.7}$$

We refer to G as the stationary Gaussian process associated with $\overline{Y}$.

Another consequence of the assumption that the potential densities $u^\beta(x,y)$ are continuous is that $\overline{Y}$ has a local time which we denote by $\overline{L}=\{\overline{L}^x_t,(t,x),\in R_+\times R\}$, which we can normalize so that $E^x(\overline{L}^y_\infty)=u^\beta(x,y)$.

Let $\overline{\sigma}(u)$ denote the non-decreasing rearrangement of $\sigma(u)$ on [0,1]. We note that the well known criteria

$$I_\sigma(1):=\int_0^1\frac{\overline{\sigma}(u)}{u(\log 2/u)^{1/2}}\,du<\infty \tag{1.8}$$

is a necessary and sufficient condition for there to exist a version of $\{G_x,x\in R\}$ that is continuous almost surely; see e.g. [5, Corollary 6.4.4].

We have the following theorem:

Theorem 1.1. *Let Let $\overline{Y} = \{\overline{Y}_t, t \in R_+\}$ be a transient Lévy process with continuous potential densities $u^\beta(x,y)$. Then the following are equivalent:*

- *(i) The local time process $\overline{L} = \{\overline{L}_t^x, (t,x), \in R_+ \times R\}$ has continuous sample paths almost surely,*
- *(ii) The associated α-permanental processes $X_\alpha = \{X_{\alpha,x}, x \in R\}$ have continuous sample paths almost surely,*
- *(iii) The associated stationary Gaussian process $G = \{G_x, x \in R\}$ has a continuous version,*
- *(iv) (1.8) holds.*

Furthermore, if the processes in (i), (ii) and (iii) are not continuous almost surely, they are unbounded almost surely.

Under additional conditions on the Lévy exponent ψ in (1.3) the condition in (1.8) can be expressed in a more transparent form. It follows from [5, Section 6.4] that

$$\int_2^\infty \frac{\left(\int_\lambda^\infty \mathcal{R}e\,(1/\psi(s))\,ds\right)^{1/2}}{\lambda(\log\lambda)^{1/2}}\,d\lambda < \infty \tag{1.9}$$

implies (1.8) and when $\mathcal{R}e\,(1/\psi(\lambda))$ is asymptotic to a monotonic function at infinity it is equivalent to (1.8).

The only part of Theorem 1.1 that is new is $(ii) \implies (iv)$. Finding necessary conditions for permanental processes to be continuous seemed very difficult because so little is known about them compared to what is known about Gaussian processes or the local times of Lévy processes. In [7] we make progress in this direction in the general case, i.e., when u is the potential density of a transient Markov process that does not have to be a Lévy process. However, the necessary conditions obtained in [7] are not best possible. While working on [7] we realized that actually all the ingredients for obtaining $(ii) \iff (iv)$, when u is the potential density of a transient Lévy process, are already contained in the literature in results of R. Dudley and X. Fernique, M. Barlow and J. Hawkes, N. Eisenbaum and H. Kaspi and ourselves. References are given in Section 2 in which we give the proof of Theorem 1.1.

We conjecture that Theorem 1.1, with obvious modifications, holds when u is the potential density of a general transient Markov process. In Remark 2.4 we point out that this is the case if $u(x,y)$ is symmetric.

2 Proofs

The next theorem shows that the necessary and sufficient condition for the continuity of a stationary Gaussian process in (1.8) is also a sufficient condition for continuity of α-permanental processes associated with Lévy processes.

Theorem 2.1. *Let $\theta_\alpha = \{\theta_{t,\alpha}, t \in R\}$ be an α-permanental process associated with a transient Lévy process with potential density $u(x,y) = u(y-x,0) =: u(y-x)$ satisfying $u(0) < \infty$. For σ, defined in (1.5), let $\overline{\sigma}(u)$ denote the non-decreasing rearrangement of $\sigma(u)$ on [0,1]. Then $I_\sigma(1) < \infty$ implies that there exists a version $\theta'_\alpha = \{\theta'_{x,\alpha}, x \in R\}$ of θ_α that is continuous on (R,σ).*

Proof Theorem 2.1 is proved in [4, Theorem 3.1] with σ replaced by

$$\widetilde{\sigma}(z) := \left(2(u(0) - (u(z)u(-z))^{1/2}\right)^{1/2}. \tag{2.1}$$

We can replace $\widetilde{\sigma}$ by σ since it follows from [6, Lemma 5.5] and [5, Theorem 3.4.3] that

$$\frac{1}{\sqrt{2}}\widetilde{\sigma}(z) \le \sigma(z) \le \widetilde{\sigma}(z). \tag{2.2}$$

Note that [4, Theorem 3.1] is proved in the general case of associated processes determined by potential densities $u(x,y)$ that need not be symmetric nor functions of $y-x$. Consequently the sufficient condition for continuity [4, (3.25)] is given in terms of majorizing measures. Since [4, (3.25)] is necessary and sufficient for the continuity of all Gaussian processes and (1.8) is necessary and sufficient for the continuity of stationary Gaussian processes the two conditions for continuity must either both hold or both not hold when σ can be expressed as it is in (1.7). $\square$

Let $Y=(\Omega, Y_t, P^x)$ be a Borel right process with state space S and 0-potential densities $u(x,y)$ that satisfy $u(x,y)>0$, for all $x,y\in S$. Let $L=\{L_t^y\,;\,(y,t)\in S\times R_+\}$ denote the local times of Y, normalized so that

$$E^x\left(L_\infty^y\right)=u(x,y) \tag{2.3}$$

and let $X=\{X_x, x\in S\}$ be the α-permanental process associated with Y. The proof that $(ii) \implies (iv)$ in Theorem 1.1 uses the following lemma:

Lemma 2.2. *Let C be a countable subset of S and z denote a fixed element of S. If*

$$P^z\left(\sup_{x\in C} L_t^x=\infty\right)=1 \tag{2.4}$$

for all $t>0$, then

$$P_X\left(\sup_{x\in C} X_x=\infty\right)=1. \tag{2.5}$$

This lemma is proved using an isomorphism theorem relating permanental processes and local times obtained by Eisenbaum and Kaspi, [3], when $\alpha=1/2$. For the general case, see [8, Section 7.3].

Let $h_x(z)=u(z,x)$ and assume that $h_x(z)>0$ for all $x,z\in S$. The expectation operator E^{z/h_x} is defined by

$$E^{z/h_x}(F1_{\{t<\zeta\}})=\frac{1}{h_x(z)}E^z(Fh_x(Y_t)), \tag{2.6}$$

for all bounded $\mathcal{F}_t^0$ measurable functions F, where $\mathcal{F}_t^0$ is the σ-algebra generated by $\{Y_r,\,0\le r\le t\}$. (See e.g. [5, (3.211)].) Here, as usual, E^z denotes the expectation operator for Y started at z. The next theorem is the isomorphism theorem we use in the proof of Lemma 2.2.

Theorem 2.3. *For any countable subset $D\subseteq S$,*

$$\left\{L_\infty^y+X_y\,;\,y\in D\,,\,P^{x/h_x}\times P_X\right\}\stackrel{law}{=}\left\{X_y\,;\,y\in D\,,\,\frac{X_x}{\alpha u(x,x)}P_X\right\}. \tag{2.7}$$

Equivalently, for all $x_1,\ldots,x_n$ in S and bounded measurable functions F on R_+^n, for all n,

$$E^{x/h_x}E_X\left(F\left(L_\infty^{x_i}+\frac{X_{x_i}}{2}\right)\right)=E_X\left(\frac{X_x}{\alpha u(x,x)}F\left(\frac{X_{x_i}}{2}\right)\right). \tag{2.8}$$

(Here we use the notation $F(f(x_i)):=F(f(x_1),\ldots,f(x_n))$.)

Proof of Lemma 2.2 Let

$$B=\Big\{\sup_{x\in C}X_x=\infty\Big\} \tag{2.9}$$

and assume that

$$P_X(B)<1. \tag{2.10}$$

Hence

$$E_X\left(1_B\,\frac{X_z}{\alpha u(z,z)}\right)<1. \tag{2.11}$$

This uses the fact that $E_X\left(X_z\right)=\alpha u(z,z)$ and $P(X_z=0)=0$.

Using (2.11) and the isomorphism theorem, Theorem 2.3, we get

$$E^{z/h_z}P_X\left(\sup_{x\in C}X_x+L^x_\infty=\infty\right)<1. \tag{2.12}$$

This implies that

$$c_0:=E^{z/h_z}\left(\sup_{x\in C}L^x_\infty=\infty\right)<1. \tag{2.13}$$

Therefore, since L^x_t is continuous and increasing in t, we see that for all $t>0$

$$E^{z/h_z}\left(\sup_{x\in C}L^x_t=\infty\right)\le c_0<1. \tag{2.14}$$

Therefore, by (2.6)

$$\int_{\{\sup_{x\in C}L^x_t=\infty\}}\frac{u(Y_t,z)}{u(z,z)}\,dP^z\le c_0<1. \tag{2.15}$$

It then follows from (2.4) that for all $t>0$

$$E^z\left(\frac{u(Y_t,z)}{u(z,z)}\right)\le c_0<1. \tag{2.16}$$

However, for z fixed, $\frac{u(x,z)}{u(z,z)}$ is bounded and continuous in x, and under P^z we have $\lim_{t\to 0}Y_t=z$. Hence $\lim_{t\to 0}E^z\left(\frac{u(Y_t,z)}{u(z,z)}\right)=1$. Thus we can choose $t_0>0$ such that $E^z\left(\frac{u(Y_t,z)}{u(z,z)}\right)>c_0$ for all $t\le t_0$. This contradiction proves the lemma. □

Proof of Theorem 1.1 We first prove the four equivalencies regarding continuity. It is well known that $(iii)\iff(iv)$. This is due to R. Dudley and X. Fernique, see [5, Section 6.5] for references.

We complete the proof by showing $(iv)\implies(i)$ and $(iv)\implies(ii)$ and $(iv)^c\implies(i)^c$ and $(iv)^c\implies(ii)^c$, where $(iv)^c$ is not (iv), etc.

$(iv)\implies(i)$ This is a result of Barlow and Hawkes, see [1, Theorem B].

$(iv)\implies(ii)$ This follows fromTheorem 2.1.

$(iv)^c\implies(i)^c$ This is a result of Barlow, [1, Theorem 1].

$(iv)^c\implies(ii)^c$ Since $(iv)^c\implies(i)^c$ it suffices to show that $(i)^c\implies(ii)^c$. To show this we first note that by [1, Theorem 1], if the local time process $\{L^x_t,(t,x),\in R_+\times R\}$ for Y does not have continuous sample paths almost surely, then for each $t>0$

$$P^0\left(\sup_{x\in Q\cap[-1,1]}L^x_t=\infty\right)=1, \tag{2.17}$$

where Q denotes the rational numbers.

Let λ denote an independent exponential random variable with mean $1/\beta$. The local time $\{\overline{L}_t^x,(t,x),\in R_+\times R\}$ of $\overline{Y}$ can be obtained from the local time of Y by setting $\overline{L}_t^x=L_{t\wedge\lambda}^x$. See [5, Remark 3.6.4 (3)]. It follows from (2.17) that

$$\overline{P}^0\left(\sup_{x\in Q\cap[-1,1]}\overline{L}_t^x=\infty\right)=\int_0^\infty P^0\left(\sup_{x\in Q\cap[-1,1]}L_{t\wedge r}^x=\infty\right)e^{-\beta r}\beta\,dr=1. \tag{2.18}$$

Using this and Lemma 2.2 we see that $(i)^c\implies(ii)^c$.

We now consider the statement that if the processes in (i), (ii) and (iii) are not continuous almost surely, they are unbounded almost surely. This is a well known result for stationary Gaussian processes, see [5, Theorem 5.3.10]. For the local times processes it is proved by Barlow and Hawkes and is what we show above in $(iv)\implies(i)$ and $(iv)^c\implies(i)^c$. For the permanental process we show above that $(ii)^c\iff(i)^c$. Furthermore, in the proof that $(i)^c\implies(ii)^c$ we actually show that $(i)^c$ implies that the permanental process is unbounded almost surely. □

Remark 2.4. When the potential densities $u=\{u(s,t),s,t\in S\}$ of a transient Markov process determine an associated α-permanental process $X_\alpha=\{X_\alpha(t),t\in S\}$ for some $\alpha>0$, they determine α-permanental processes for all $\alpha>0$. Therefore, by (1.2), X_α is infinitely divisible in α. It follows from this and the fact that X_α is positive, that for $\alpha<\alpha'<m\alpha$,

$$\|X_\alpha\|_\infty\le\|X_{\alpha'}\|_\infty\le m\|X_\alpha\|_\infty. \tag{2.19}$$

This shows that if $\|X_\alpha\|_\infty<\infty$ almost surely for some $\alpha>0$, it is finite almost surely for all $\alpha>0$. Therefore, for all $\alpha>0$, X_α is finite (infinite) almost surely if and only if $X_{1/2}$ is finite (infinite) almost surely. Suppose that u is symmetric. Then, as we point out on page 2, $X_{1/2}$ is the square of a Gaussian process. Consequently the necessary and sufficient condition for boundedness of Gaussian processes is also the necessary and sufficient condition for boundedness of α-permanental processes. A similar argument involving inequalities for other norms shows that this also holds for continuity.

References

[1] M. Barlow, Necessary and sufficient conditions for the continuity of local time of Lévy processes, Annals of Probability, (1988), Vol. 16, No. 4, 1389-1427. MR-0958195

[2] J. Bertoin, *Lévy Processes*, Cambridge University Press, Cambridge, (1996). MR-1406564

[3] N. Eisenbaum and H. Kaspi, On permanental processes, *Stochastic Processes and their Applications*, *119*, (2009), 1401-1415. MR-2513113

[4] H. Kogan, M. B. Marcus and J. Rosen, Permanental processes, Comm. on Stochastic Analysis, 5 (2011), 81-102. MR-2808537

[5] M. B. Marcus and J. Rosen, *Markov Processes, Gaussian Processes and Local Times*, Cambridge University Press, New York, (2006). MR-2250510

[6] M. B. Marcus and J. Rosen, A sufficient condition for the continuity of permanental processes with applications to local times of Markov processes, Annals of Probability, (2013), Vol. 41, No. 2, 671–698. arXiv:1005.5692 MR-3077522

[7] M. B. Marcus and J. Rosen, Conditions for permanental processes to be unbounded, submitted.

[8] J. Rosen, Isomorphism Theorems: Markov processes, Gaussian processes and beyond. In: Correlated Random Systems - Five Different Methods, editors: Véronique Gayrard, Nicola Kistler. Springer, to appear. arXiv:1407.1559

[9] D. Vere-Jones, *Alpha-permanents and their applications to multivariate gamma, negative binomial and ordinary binomial distributions*, New Zealand J. of Math., (1997), 26, 125–149. MR-1450811