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On the result of Doney

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Abstract

Let X denote a spectrally positive stable process of index $\alpha \in (1, 2)$ whose Lévy measure has density $cx^{-\alpha-1}$, $x > 0$ and let $S = \sup_{0 \le t \le 1} X_t$. Doney [4] proved that the density of S say s behaves as $s(x) \sim cx^{-\alpha-1}$ as $x \to \infty$. The proof given was nearly four pages long. Here, we: i) give a shorter and a more general proof of the same result; ii) derive the first known closed form expressions for $s(x)$ and the corresponding cumulative distribution function; iii) derive the order of the remainder in the asymptotic expansion for $s(x)$.

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1 Introduction

Let X denote a spectrally positive stable process of index $\alpha \in (1, 2)$ whose Lévy measure has density $cx^{-\alpha-1}$, $x > 0$ and let $S = \sup_{0 \le t \le 1} X_t$. Let f denote the density of X and s denote the density of S. It is well known that

$$f(x) \sim cx^{-\alpha-1}$$

and

$$\Pr(S > x) \sim \Pr(X > x) \sim c\alpha^{-1}x^{-\alpha}$$

as $x \to \infty$, see Sato [6], Proposition 4, page 221 and Bertoin [2], equation (14.34), page 88. Doney [4], Theorem 2.3 proved that

$$s(x) \sim cx^{-\alpha-1} \tag{1.1}$$

as $x \to \infty$. The proof given was nearly four pages long.

The aim of this short note is to: i) provide a shorter proof of (1.1); ii) derive a closed form expression for $s(x)$, thought to be the first such expression for $s(x)$. An expression for $s(x)$ is given in Bernyk [1], but that is an infinite series of terms involving the gamma function, and not a closed form expression. Closed form expressions are preferred for computational purposes and for deriving properties of $s(x)$ like the corresponding

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cumulative distribution function $S(x) = \int_0^x s(t)dt$ and moments, among others; iii) give the order of the remainder in the asymptotic expansion of $s(x)$ as $x \to \infty$.

The proof uses the complex parameter Wright generalized hypergeometric function, ${}_p\Psi_q(\cdot)$, with p numerator and q denominator parameters (Kilbas *et al.* [5], equation (1.9)) defined by the series

$$ {}_p\Psi_q \left[\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!} \tag{1.2} $$

for $z \in \mathbb{C}$, where $\alpha_j \in \mathbb{C}$, $\beta_k \in \mathbb{C} \backslash \mathbb{Z}_0^-$, A_j, $B_k \neq 0$, $j = \overline{1,p}$ and $k = \overline{1,q}$. The series converges for bounded values of $|z|$ when $\Delta = 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$. This function was originally introduced by Wright [7]. The most complete exposition of the asymptotic behavior of the Wright generalized hypergeometric Ψ function can be found in Braaksma [3], page 327 *et seq.*, Section 12.

Hypergeometric functions are included as in-built functions in most mathematical software packages, so the special function in (1.2) can be easily evaluated by the software packages **Maple**, Matlab and $\mathcal{M}athematica$ using known procedures.

2 Main results

Our main results are Theorems 2.1 and 2.2.

Theorem 2.1 derives a closed form expression for $s(x)$ involving the Wright generalized hypergeometric Ψ function. As an illustration of the use of the Wright generalized hypergeometric Ψ function, Theorem 2.1 also derives a closed form expression for the cumulative distribution function $S(x)$. Of course many other properties of $s(x)$ can be derived in closed form by using the Wright generalized hypergeometric Ψ function. Theorem 2.2 derives the asymptote of $s(x)$ using a known property of the special function. Theorem 2.2 also gives the order of the remainder in the asymptotic expansion.

Theorem 2.1. *For all $\alpha \in (1,2)$ and for all $x > 0$, we have*

$$ s(x) = \frac{\sin\left(\pi\alpha^{-1}\right) x^{\alpha-2}}{\pi \left(c\Gamma(-\alpha)\right)^{1-\alpha^{-1}}} {}_2\Psi_1 \left[\begin{array}{c} \left(1-\alpha^{-1}, 1\right), (1,1) \\ (\alpha-1, \alpha) \end{array} \middle| -\frac{x^\alpha}{c\Gamma(-\alpha)} \right] \tag{2.1} $$

and

$$ S(x) = \frac{\sin\left(\pi\alpha^{-1}\right) x^{\alpha-1}}{\pi \left(c\Gamma(-\alpha)\right)^{1-\alpha^{-1}}} {}_2\Psi_1 \left[\begin{array}{c} \left(1-\alpha^{-1}, 1\right), (1,1) \\ (\alpha, \alpha) \end{array} \middle| -\frac{x^\alpha}{c\Gamma(-\alpha)} \right]. \tag{2.2} $$

Proof. By Bernyk *et al.* [1], Theorem 1, page 1779,

$$ s(x) = \sum_{n \geq 1} \frac{x^{\alpha n - 2}}{\left(c\Gamma(-\alpha)\right)^{n-\alpha^{-1}} \Gamma(\alpha n - 1) \Gamma\left(-n + 1 - \alpha^{-1}\right)}. $$

By Euler's reflection formula for the gamma function, $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we can write

$$ \frac{1}{\Gamma\left(-n+1+\alpha^{-1}\right)} = \frac{1}{\pi} (-1)^{n-1} \sin\left(\pi\alpha^{-1}\right) \Gamma\left(n - \alpha^{-1}\right). $$

So,

$$\begin{aligned} s(x) &= -\left(c\Gamma(-\alpha)\right)^{\alpha^{-1}} \frac{\sin\left(\pi\alpha^{-1}\right)}{\pi x^2} \sum_{n\geq 1} \frac{\Gamma\left(n-\alpha^{-1}\right)}{\Gamma(\alpha n-1)} \left[-\frac{x^\alpha}{c\Gamma(-\alpha)}\right]^n \\ &= \frac{\sin\left(\pi\alpha^{-1}\right) x^{\alpha-2}}{\pi\left(c\Gamma(-\alpha)\right)^{1-\alpha^{-1}}} \sum_{n\geq 0} \frac{\Gamma\left(1-\alpha^{-1}+n\right)\Gamma(1+n)}{\Gamma\left(\alpha-1+\alpha n\right) n!} \left[-\frac{x^\alpha}{c\Gamma(-\alpha)}\right]^n . \end{aligned} \tag{2.3}$$

The result in (2.1) follows from (1.2) since $\Delta = \alpha - 1 > 0$. The derivation of (2.2) is only a routine exercise. Namely integrating (2.3) from 0 to x, we obtain

$$\begin{aligned} S(x) &= \frac{\sin\left(\pi\alpha^{-1}\right)}{\pi\left(c\Gamma(-\alpha)\right)^{1-\alpha^{-1}}} \sum_{n\geq 0} \frac{\Gamma\left(1-\alpha^{-1}+n\right)\Gamma(1+n)}{\Gamma\left(\alpha-1+\alpha n\right)\ n!\ \left[-c\Gamma(-\alpha)\right]^n} \frac{x^{\alpha-1+\alpha n}}{\alpha-1+\alpha n} \\ &= \frac{\sin\left(\pi\alpha^{-1}\right)\ x^{\alpha-1}}{\pi\left(c\Gamma(-\alpha)\right)^{1-\alpha^{-1}}} \sum_{n\geq 0} \frac{\Gamma\left(1-\alpha^{-1}+n\right)\Gamma(1+n)}{\Gamma\left(\alpha+\alpha n\right)\ n!\ \left[c\Gamma(-\alpha)\right]^n} \left[-\frac{x^\alpha}{c\Gamma(-\alpha)}\right]^n , \end{aligned}$$

which is actually the asserted Fox-Wright function formula. □

Note too that $s(x)$ is entire since $\Delta > 0$ and there are no non-negative integer poles.

Theorem 2.2. *For all* $\alpha \in (1,2)$*, we have* $s(x) = cx^{-\alpha-1} + O\left(x^{-\alpha-2}\right)$ *as* $x \to \infty$*.*

Proof. The proof is based on Theorem 19 in Braaksma [3], page 330. Adapting its statement, we have

$${}_2\Psi_1\left[\begin{array}{c} \left(1-\alpha^{-1},1\right),(1,1) \\ (\alpha-1,\alpha) \end{array}\middle| z\right] = \sum_{p=1,2;n\geq 0} \operatorname{Res}\left[(-z)^s\Gamma(-s)\frac{\Gamma\left(1-\alpha^{-1}+s\right)\Gamma(1+s)}{\Gamma\left(\alpha-1+\alpha s\right)}; s_n^{(p)}\right],$$

where $s_n^{(1)} = -\left(1-\alpha^{-1}+n\right)$ and $s_n^{(2)} = -(1+n)$, $n \in \mathbb{N}_0$. Since $\operatorname{Res}\left[\Gamma(a+s); -a-n\right] = (-1)^n/n!$, we see

$$\begin{aligned} {}_2\Psi_1\left[\begin{array}{c} \left(1-\alpha^{-1},1\right),(1,1) \\ (\alpha-1,\alpha) \end{array}\middle| z\right] &= \frac{-\Gamma\left(\alpha^{-1}-1\right)\Gamma\left(2-\alpha^{-1}\right)}{\Gamma(-\alpha)}(-z)^{-2+\alpha^{-1}} + O\left(z^{-3+\alpha^{-1}}\right) \\ &\quad + \frac{\Gamma\left(-\alpha^{-1}-1\right)}{\Gamma\left(-\alpha-1\right)} z^{-2} + O\left(z^{-3}\right) \\ &= \frac{-\Gamma\left(\alpha^{-1}-1\right)\Gamma\left(2-\alpha^{-1}\right)}{\Gamma(-\alpha)}(-z)^{-2+\alpha^{-1}} + O\left(z^{-2}\right) \end{aligned} \tag{2.4}$$

as $z \to \infty$. Setting $z = -x^\alpha/\left[c\Gamma(-\alpha)\right]$ and combining (2.4) and (2.1), we arrive at

$$s(x) = -\frac{c}{\pi}\sin\left(\pi\alpha^{-1}\right)\Gamma\left(\alpha^{-1}-1\right)\Gamma\left(2-\alpha^{-1}\right)x^{-\alpha-1} + O\left(x^{-\alpha-2}\right).$$

The proof is complete on noting that $-\frac{1}{\pi}\sin\left(\pi\alpha^{-1}\right)\Gamma\left(\alpha^{-1}-1\right)\Gamma\left(2-\alpha^{-1}\right) = 1$. □

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