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On percolation in one-dimensional stable Poisson graphs

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Abstract

Equip each point x of a homogeneous Poisson point process $\mathcal{P}$ on $\mathbb{R}$ with D_x edge stubs, where the D_x are i.i.d. positive integer-valued random variables with distribution given by μ. Following the *stable multi-matching scheme* introduced by Deijfen, Häggström and Holroyd [1], we pair off edge stubs in a series of rounds to form the edge set of a graph G on the vertex set $\mathcal{P}$. In this note, we answer questions of Deijfen, Holroyd and Peres [2] and Deijfen, Häggström and Holroyd [1] on percolation (the existence of an infinite connected component) in G. We prove that percolation may occur a.s. even if μ has support over odd integers. Furthermore, we show that for any $\varepsilon > 0$, there exists a distribution μ such that $\mu(\{1\}) > 1 - \varepsilon$, but percolation still occurs a.s..

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1 Introduction

In this paper, we study certain matching processes on the real line. Let D be a random variable with distribution μ supported on the positive integers. Generate a set of vertices $\mathcal{P}$ by a Poisson point process of intensity 1 on $\mathbb{R}$. Equip each vertex $x \in \mathcal{P}$ with a random number D_x of edge stubs, where the $(D_x)_{x\in\mathcal{P}}$ are i.i.d. random variables with distribution given by D. Now form edges in rounds by matching edge stubs in the following manner. In each round, say that two vertices x, y are *compatible* if they are not already joined by an edge and both x and y still possess some unmatched edge stubs. Two such vertices form a *mutually closest compatible pair* if x is the nearest y-compatible vertex to y in the usual Euclidean distance and vice-versa. For each such mutually closest compatible pair (x, y), remove an edge stub from each of x and y to form the edge xy. Repeat the procedure indefinitely.

This matching scheme, known as *stable multi-matching*, was introduced by Deijfen, Häggström and Holroyd [1], who showed that it a.s. exhausts the set of edge stubs, yielding an infinite graph $G = G(\mu)$ with degree distribution given by μ. Note that the graph $G(\mu)$ arising from our multi-matching process is *stable* a.s.; for any pair of

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distinct points $x, y \in \mathcal{P}$, either $xy \in E(G)$ or at least one of x, y is incident to no edge in G of length greater than $|x - y|$. The concept of stable matchings was introduced in an influential paper of Gale and Shapley [3]; in the context of spatial point processes its study was initiated by Holroyd and Peres, and by Holroyd, Pemantle, Peres and Schramm [4, 5].

A natural question to ask is which degree distributions μ (if any) yield an infinite connected component in $G(\mu)$. For example if $\mu(\{1\}) = 1$, then no such component exists, while if $\mu(\{2\}) = 1$, Deijfen, Holroyd and Peres [2] suggest that percolation (the existence of an infinite component) occurs a.s.. Note that by (a version of) Kolmogorov's zero–one law, the probability of percolation occurring is zero or one. Also, as shown by Deijfen, Holroyd and Peres (see [2], Proposition 1.1), an infinite component in G, if it exists, is almost surely unique.

Taking the Poisson point process in $\mathbb{R}^d$ for some $d \geq 1$ and applying the stable multi-matching scheme mutatis mutandis, we obtain the d-dimensional Poisson graph G_d. Deijfen, Häggström and Holroyd proved the following result on percolation in G_d:

Theorem 1.1. *(Deijfen, Häggström and Holroyd [1, Theorem 1.2])*

(i) For all $d \geq 2$ there exists $k = k(d)$ such that if $\mu(\{n \in \mathbb{N} : \ n \geq k\}) = 1$, then a.s. G_d percolates.

(ii)) For all $d \geq 1$, if $\mu(\{1, 2\}) = 1$ and $\mu(\{1\}) > 0$, then a.s. G_d does not percolate.

Their proof of part (i) of Theorem 1.1 relies on a comparison of the d-dimensional stable multi-matching process with dependent site percolation on $\mathbb{Z}^d$. In particular, since the threshold for percolation in $\mathbb{Z}$ is trivial, their argument cannot say anything about percolation in the 1-dimensional Poisson graph $G = G_1$.

Related to part (ii) of Theorem 1.1, Deijfen, Häggström and Holroyd asked the following question.

Question 1 (Deijfen, Häggström and Holroyd)**.** *Does there exist some $\varepsilon > 0$ such that if $\mu(\{1\}) > 1 - \varepsilon$, then a.s. G_d contains no infinite component?*

In subsequent work on $G = G_1$, Deijfen, Holroyd and Peres [2] observed that simulations suggested percolation might not occur when $\mu(\{3\}) = 1$, and asked whether the presence of odd degrees kills off infinite components in general.

Question 2 (Deijfen, Holroyd and Peres)**.** *Is it true that percolation in $G = G_1$ occurs a.s., if and only if, μ has support only on the even integers?*

In this paper we prove the following theorem:

Theorem 1.2. *Let μ be a degree distribution such that*

$$\mu(\{n \in \mathbb{N} : \ n \geq 20 \cdot 3^i\}) \geq \frac{1}{2^i}$$

for all but finitely many i, then a.s. the one-dimensional stable Poisson graph $G = G_1(\mu)$ will contain an infinite path.

Since Theorem 1.2 does not assume anything about μ besides its heavy tail, our result implies a negative answer to both Question 1 and Question 2:

Corollary 1.3. *For any $\varepsilon > 0$, there exist degree distributions μ with $\mu(\{1\}) > 1 - \varepsilon$ such that the one-dimensional stable Poisson graph $G = G_1(\mu)$ a.s. contains an infinite connected component.* □

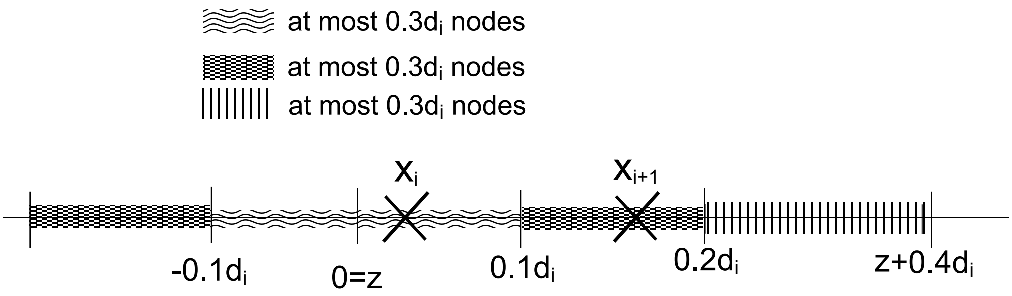

Figure 1: Restrictions on the number of nodes in various intervals when the event $E_i(z)$ occurs.

Corollary 1.4. *There exist degree distributions μ with support on the odd integers, such that the one-dimensional stable Poisson graph $G = G_1(\mu)$ a.s. contains an infinite connected component.* □

We note however that the degree distributions μ satisfying the assumptions of Theorem 1.2 have unbounded support; it would be interesting to find a distribution μ with bounded support only that still gives a negative answer to Questions 1 and 2 (see the discussion of this problem in Section 3).

2 Proof of Theorem 1.2

To prove Theorem 1.2, we construct a degree distribution μ for which $G_1(\mu)$ a.s. contains an infinite path, and then show that for any degree distribution μ' stochastically dominating μ, $G_1(\mu')$ also a.s. contains an infinite path.

The idea underlying our construction of μ is to set $\mu(\{d_i\}) = 1/2^i$ for a sharply increasing sequence of integers $(d_i)_{i\in\mathbb{N}}$. Suppose that we are given a vertex x_i with degree $D_{x_i} = d_i$. By choosing d_i large enough we can ensure that with probability close to 1, there exists some vertex x_{i+1} with $D_{x_{i+1}} = d_{i+1}$ that is connected to x_i by an edge of G. Let U_i, $i \geq 1$, be the event that a given vertex x_i of degree d_i is connected to some vertex x_{i+1} of degree d_{i+1}. Starting from a vertex x_1 of degree d_1, we see that if $\bigcap_{i=1}^{\infty} U_i$ occurs, then there is an infinite path $x_1x_2x_3\ldots$ in G. If the events $(U_i)_{i\in\mathbb{N}}$ were independent of each other, then $\mathbb{P}(\bigcap_{i=1}^{\infty} U_i) = \prod_{i\in\mathbb{N}} \mathbb{P}(U_i)$, which we could make strictly positive by letting the sequence $(d_i)_{i\in\mathbb{N}}$ grow sufficiently quickly, ensuring in turn that percolation occurs a.s.. Of course the events $(U_i)_{i\in\mathbb{N}}$ as we have loosely defined them above are highly dependent. We circumvent this problem by working with a sequence of slightly more restricted events, for which we do have full independence.

Before we begin the proof, let us introduce the following notation. Given $x \in \mathcal{P}$, let $B(x, r)$ be the collection of all vertices in $\mathcal{P}$ within distance at most r of x. We say that a pair of vertices (x, y) with degrees (D_x, D_y) is *strongly connected* if $|B(x, |y - x|)| \leq D_x$ and $|B(y, |y - x|)| \leq D_y$. Observe that if a pair of vertices (x, y) is strongly connected, then, by the stability property of the multi-matching scheme, there will a.s. be an edge of $G(\mu)$ joining x and y.

Proof of Theorem 1.2. Set $d_i = 20 \cdot 3^i$ and $\mu(\{d_i\}) = \frac{1}{2^i}$ for each $i \in \mathbb{N}$. Let $z \in \mathbb{R}$ be arbitrary. Suppose that we condition on a particular vertex x_i of degree d_i belonging to the point process $\mathcal{P}$ and lying inside the interval $[z, z + 0.1d_i]$, and further condition on there being at most $0.3d_i$ points of $\mathcal{P}$ in the interval of length $0.2d_i$ centered at z. Write $F_i(z)$ for the event that we are conditioning on. By the standard properties of Poisson point processes, conditioning on $F_i(z)$ does not affect the probability of any event defined outside the interval $[z-0.1d_i, z+0.1d_i]$.

Let $A_i(z)$ be the event that there is a vertex $x_{i+1} \in \mathcal{P}$ with degree d_{i+1} such that $0.1d_i < x_{i+1} - z < 0.2d_i$. Viewing $\mathcal{P}$ as the union of two thinned Poisson point processes, one of intensity $2^{-(i+1)}$ giving us the vertices of degree d_{i+1} and another of intensity $1-2^{-(i+1)}$ giving us the rest of the vertices, we see that $\mathbb{P}((A_i(z))^c) = e^{-\frac{0.1d_i}{2^{i+1}}} = e^{-\left(\frac{3}{2}\right)^i}$. If $A_i(z)$ occurs, let x_{i+1} denote the a.s. unique vertex of degree d_{i+1} which is nearest to x_i among those degree d_{i+1} vertices lying at distance at least $0.1d_i$ to the right of z.

Let $B_i(z)$ be the event that there are at most $0.3d_i$ vertices $x \in \mathcal{P}$ with $0.1d_i < |x-z| < 0.2d_i$. Furthermore, let $C_i(z)$ be the event that there are at most $0.3d_i$ vertices $x \in \mathcal{P}$ lying in the interval $[z+0.2d_i, z+0.4d_i]$. A quick calculation (using the Chernoff bound, see e.g., [6]) yields that $\mathbb{P}(B_i(z)^c) = \mathbb{P}(C_i(z)^c) = e^{-2\left(3\log\left(\frac{3}{2}\right)-1\right)3^i + O(i)}$.

Finally, let $E_i(z) = A_i(z) \cap B_i(z) \cap C_i(z)$. If $E_i(z)$ occurs, then the vertices x_i and x_{i+1} are strongly connected, since our initial assumption $F_i(z)$ together with $B_i(z)$ tells us that

$$|B(x_i, |x_i - x_{i+1}|)| \le |B(z, 0.2d_i)| \le 0.6d_i,$$

while $F_i(z)$ together with $B_i(z) \cap C_i(z)$ yield that

$$|B(x_{i+1}, |x_{i+1} - x_i|)| \le |B(z+0.1d_i, 0.3d_i)| \le 0.9d_i = 0.3d_{i+1}$$

(see Figure 1). This last inequality (together with the fact that $x_{i+1} \in [z+0.1d_i, z+0.2d_i]$) also gives our initial conditioning $F_i(z)$ with i replaced by $i+1$ and z replaced by $z+0.1d_i$; hence $E_i(z) \cap F_i(z) \subseteq F_{i+1}(z+0.1d_i)$.

By the union bound, we have

$$\begin{aligned} \mathbb{P}\big(E_i(z)|F_i(z)\big) &\ge 1 - \mathbb{P}\big((A_i(z))^c\,|F_i(z)\big) - \mathbb{P}\big((B_i(z))^c\,|F_i(z)\big) \\ &\quad - \mathbb{P}\big((C_i(z))^c\,|F_i(z)\big) \\ &> 1 - e^{-\left(\frac{3}{2}\right)^i}(1+o(1)). \end{aligned}$$

Selecting i_0 sufficiently large and some arbitrary vertex $z_{i_0} = x_{i_0}$ of degree d_{i_0} as a starting point, we may define events $E_{i_0}(z_{i_0}), E_{i_0+1}(z_{i_0+1}), E_{i_0+2}(z_{i_0+2}), \dots$ inductively, each conditional on its predecessors, with $z_{i+1} = z_i + 0.1d_i$ for all $i \ge i_0$, and

$$\begin{aligned} \mathbb{P}\Big(\bigcap_{i\ge i_0} E_i(z_i)|F_{i_0}(z_{i_0})\Big) &= \prod_{i\ge i_0} \mathbb{P}\big(E_i(z_i)|\cap_{j<i} E_j(z_j) \cap F_{i_0}(z_{i_0})\big) \\ &= \prod_{i\ge i_0} \mathbb{P}\big(E_i(z_i)|F_i(z_i)\big) > 1 - 2\sum_{i\ge i_0} e^{-\left(\frac{3}{2}\right)^i} > 0. \end{aligned}$$

Thus, from any vertex $x_{i_0} \in \mathcal{P}$ of degree d_{i_0} there is, with strictly positive probability, an infinite sequence of vertices from $\mathcal{P}$, $x_{i_0}, x_{i_0+1}, \dots$, with increasing degrees $d_{i_0}, d_{i_0+1}, \dots$, such that (x_i, x_{i+1}) is strongly connected for every $i \ge i_0$. By the stability property of our multi-matching scheme, there is a.s. an infinite path in G through these vertices. It follows that G a.s. contains an infinite path. We now only need to make two remarks about the proof to obtain the full statement of Theorem 1.2.

Remark 2.1. The pairs $(x_{i_0}, x_{i_0+1}), (x_{i_0+1}, x_{i_0+2}), \dots$ remain strongly connected if we increase the degrees. Also, our proof of Theorem 1.2 does not use any information about d_i for $i < i_0$. Thus, for any measure μ' which agrees with (or stochastically dominates) μ on $\{n \in \mathbb{N} : n \ge d_{i_0}\}$, $G_1(\mu')$ will percolate a.s..

Remark 2.2. Note that we could replace the distribution in the proof of Theorem 1.2 by any distribution μ such that $\mu(\{x : x \ge d_i\}) \ge 2^{-i}$. Instead of obtaining a (strongly connected) sequence x_i such that x_i has exactly degree d_i, we get a (strongly connected) sequence x_i such that x_i has at least degree d_i.

□

3 Concluding remarks

Remark 3.1. The existence of degree distributions that a.s. result in an infinite component in dimensions $d \geq 2$ was established in [1, Theorem 1.2 a)]. Our proof of Theorem 1.2 for $G = G_1(\mu)$ easily adapts to higher dimensions $d \geq 2$ (with d-dimensional balls and annuli replacing intervals and punctured intervals, and the sequence $(d_i)_{i\in\mathbb{N}}$ being scaled accordingly), giving a different approach to the construction of examples in that setting.

The distribution μ we construct in Theorem 1.2 has unbounded support, and the expected degree of a vertex in $G(\mu)$ is infinite. We believe however that the answer to Questions 1 and 2 should still remain negative if μ is required to have bounded support. Indeed we conjecture the following:

Conjecture 3.1. *For every $\varepsilon > 0$, there exists $k = k(\varepsilon)$ such that if $\mu(\{n \in \mathbb{N} : n \geq k\}) > \varepsilon$, then percolation occurs a.s. in $G = G_1(\mu)$.*

One might expect that there is a *critical value* $d_\star$ of the expected degree for percolation. We believe however that no such critical value exists:

Conjecture 3.2. *There is no critical value $d_\star$, such that if $\mathbb{E}(D) < d_\star$, then a.s. percolation does not occur, while if $\mathbb{E}(D) > d_\star$, then a.s. percolation occurs in the stable multi-matching scheme on $\mathbb{R}$.*

Let us give some motivation for this conjecture. By [1, Theorem 1.2 b)], for any μ with support on $\{1, 2\}$ and $\mu(\{1\}) > 0$, $G_1(\mu)$ a.s. does not percolate. So any putative critical value must satisfy $d_\star \geq 2$. Now, pick $\varepsilon > 0$ and choose $\delta \gg d_\star$. Let μ be a degree distribution with support on $\{1, \delta\}$, such that the expected degree satisfies $\mathbb{E}(D) < d_\star - \varepsilon$. By the definition of $d_\star$ this would imply that $G(\mu)$ a.s. does not percolate. Assign degrees independently at random to the vertices of $G(\mu)$. Perform the first $\delta/2$ stages of the stable multi-matching process. By then most degree 1 vertices have been matched (and in fact matched to other degree 1 vertices). Now force the remaining degree 1 vertices to match to their future partners. Consider the vertices that had originally been assigned δ edge stubs. A number of these edge stubs will have been used up by the process so far, and the number of edge stubs left at each vertex is not independent; nevertheless we expect most degree δ vertices will have at least $\delta/4$ edge stubs left, and that the number of stubs left will be almost independently distributed. Thus, we believe that the stable multi-matching scheme on the remaining edge stubs of the degree δ vertices will contain as a subgraph the edges of a stable multi-matching scheme on a thinned Poisson point process on $\mathbb{R}$ corresponding to the degree δ vertices, and with degrees given by some random variable D' with $\mathbb{E}(D') > \delta/4 \gg d_\star$. Since rescaling a Poisson point process does not affect the stable multi-matching process, this would imply that $G(\mu)$ a.s. percolates (by definition of $d_\star$), a contradiction.

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