Electron. Commun. Probab. **19** (2014), no. 75, 1–7.
DOI: 10.1214/ECP.v19-3735
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A note on the times of first passage for 'nearly right-continuous' random walks*

Matija Vidmar†

Abstract

A natural extension of a right-continuous integer-valued random walk is one which can jump to the right by one or two units. First passage times above a given fixed level then admit — on each of the two events, which correspond to overshoot zero and one, separately — a tractable probability generating function. Some applications are considered.

Keywords: Random walks; first entrance/passage times; fluctuation theory.
AMS MSC 2010: 60G50.
Submitted to ECP on August 12, 2014, final version accepted on October 30, 2014.
Supersedes arXiv:1310.6661v2.

1 Introduction

It is well-known that, within the class of integer-valued random walks, those which can jump to the right by only one unit, are singled out in terms of having a more tractable fluctuation theory [2, 9, 7] [3, Section 4] [4, Section 7] [5, Section 9.3] [11, *passim*]. For their defining property, they are called 'right-continuous' or also 'skip-free to the right'. In particular, first passage times above a given level then admit (semi)explicit probability generating functions, at every point in terms of a single parameter. This is also by analogy to the spectrally negative class of Lévy processes [1, Chapter VII] [5, Section 9] [6, Chapter 8] [10, Section 9.46]. Indeed, if the right-continuous integer-valued random walk is embedded into continuous time as a compound Poisson process [13], then together (modulo trivial cases) these two types exhaust the class of Lévy processes having non-random overshoots [14], a property by and large responsible for the fluctuation theory then being more explicit.

It seems natural to ask, then, to what extent fluctuation theory remains (and, for that matter, does not remain) tractable when the demand of non-random overshoots is relaxed. In this paper only the simplest extension is considered, namely we allow the random walk to jump to the right by one or two units (making it 'nearly right-continuous', but not quite). Apart from such theoretical considerations, these 'nearly right-continuous' random walks also extend some applied queuing and branching models related to right-continuous random walks, lending further relevance to their study.

The mandate of this paper is restricted to characterizing the joint law of the times of first passage above a given fixed level and the overshoots at that level, for the type of processes just described. It emerges that the probability generating functions of these

*Support: Slovene Human Resources Development and Scholarship Fund contract number 11010-543/2011.
†University of Warwick, UK. E-mail: m.vidmar@warwick.ac.uk

times of first passage are given, on each of the two events corresponding to overshoot 0 and 1, respectively, and at every point, in terms of two parameters, them in turn being characterized precisely through the deterministic characteristics of the process.

As regards the presentation of the remainder of this paper, the main result of the paper is stated in Theorem 2.1 of Section 2, which also introduces the setting. Section 3 contains the (not too difficult, still non-trivial) proof. In Section 4 we briefly remark upon some applications. Section 5 gives a couple of concluding remarks.

2 Setting and statement of result

On a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, we consider given an integer-valued random walk [11], denoted $W = (W_k)_{k\in\mathbb{N}_0}$, with $W_0 = 0$, a.s., allowing $\mathsf{P}(W_1 = 0)$ to assume a non-zero value. We assume throughout that (i) W does not a.s. have monotone paths, i.e. $\mathsf{P}(W_1 > 0) \wedge \mathsf{P}(W_1 < 0) > 0$, and that (ii) $\mathsf{E}[\beta^{W_1}] < \infty$ for all $\beta \in [1,\infty)$; but specify to the 'nearly right-continuous' setting later on.

Notation-wise, the first passage times are defined as $T_n := \inf\{k \in \mathbb{N}_0 : W_k \geq n\}$, $n \in \mathbb{Z}$; $\mathcal{L}(\gamma) := \mathsf{E}[\gamma^{W_1}]$, $\gamma \in \mathbb{R}\backslash(-1,1)$, is the probability generating function of W_1; $\lambda := (W_1)_\star\mathsf{P}$ is the jump measure. It is trivial that $\mathcal{L}$ is continuous (dominated convergence); $\mathcal{L}|_{[1,\infty)}$ is strictly convex (differentiation under the integral sign & $\mathsf{P}(W_1 < 0) > 0$); $\lim_\infty \mathcal{L} = \infty$ ($\mathsf{P}(W_1 > 0) > 0$); $\mathsf{E}[\gamma^{W_k}] = \mathcal{L}(\gamma)^k$ for all $\gamma \geq 0$, $k \in \mathbb{N}_0$ (stationary independent increments of W). Letting $\alpha(1)$ be the largest zero of $\mathcal{L} - 1$ on $[1,\infty)$ (where it has at most one in addition to 1), $\mathcal{L}|_{[\alpha(1),\infty)} : [\alpha(1),\infty) \to [1,\infty)$ is an increasing continuous bijection; we may define $\alpha := (\mathcal{L}|_{[\alpha(1),\infty)})^{-1}$.

Here is now our result.

Theorem 2.1. *Suppose W is 'nearly skip-free to the right', i.e. $\mathrm{supp}(\lambda|_{2^{\mathbb{N}}}) \subset \{1,2\}$. Assume furthermore that $2 \in \mathrm{supp}(\lambda)$ (so we are excluding the skip-free version) and $\mathrm{supp}(\lambda) \not\subset 2\mathbb{Z}$ (which is again the skip-free version but on double the lattice). Then for all $\gamma \in [1,\infty)$ and $n \in \mathbb{N}_0$:*

$$\mathsf{E}[\gamma^{-T_n}, T_n < \infty, W(T_n) - n = 0] = \frac{1}{\lambda_+(\gamma) - \lambda_-(\gamma)}(\lambda_+(\gamma)^{n+1} - \lambda_-(\gamma)^{n+1}) \text{ and} \tag{2.1}$$

$$\mathsf{E}[\gamma^{-T_n}, T_n < \infty, W(T_n) - n = 1] = \frac{-\lambda_+(\gamma)\lambda_-(\gamma)}{\lambda_+(\gamma) - \lambda_-(\gamma)}(\lambda_+(\gamma)^n - \lambda_-(\gamma)^n), \tag{2.2}$$

where:

(a) If $\alpha(\gamma) > 1$, $1/\lambda_-(\gamma)$ is the unique zero of $\mathcal{L} - \gamma$ on $(-\infty,-1)$ and $1/\lambda_+(\gamma) = \alpha(\gamma)$ is the unique zero of $\mathcal{L} - \gamma$ on $(1,\infty)$; further, the following inequalities hold: $-\lambda_+(\gamma) < \lambda_-(\gamma) < 0 < \lambda_+(\gamma) < 1$.

(b) $1/\lambda_+(1) = \alpha(1)$ is the largest zero of $\mathcal{L} - 1$ on $[1,\infty)$ and $1/\lambda_-(1)$ is the unique zero of $\mathcal{L} - 1$ on $(-\infty,-1)$. In addition: $-\lambda_+(1) < \lambda_-(1) < 1 < \lambda_+(1) \leq 1$.

In particular, for each $\gamma \in [1,\infty)$,

$$\mathsf{E}[\gamma^{-T_n}, T_n < \infty] = \frac{1 - \lambda_-(\gamma)}{\lambda_+(\gamma) - \lambda_-(\gamma)}\lambda_+(\gamma)^{n+1} - \frac{1 - \lambda_+(\gamma)}{\lambda_+(\gamma) - \lambda_-(\gamma)}\lambda_-(\gamma)^{n+1}.$$

Remark 2.2.

1. It follows from Theorem 2.1(a)-(b) and the continuity of $\mathcal{L}$, that $\mathcal{L}|_{(-\infty,1/\lambda_-(1)]} - 1 : (-\infty, 1/\lambda_-(1)] \to [0,\infty)$ is a decreasing bijection, $\mathcal{L}|_{(1/\lambda_-(1),-1]\cup(1,1/\lambda_+(1))} - 1$ is strictly negative, $\mathcal{L}|_{[1/\lambda_+(1),\infty)} - 1 : [1/\lambda_+(1),\infty) \to [0,\infty)$ is an increasing bijection, $1/\lambda_- = (\mathcal{L}_{(-\infty,1/\lambda_-(1)]})^{-1}$ and $1/\lambda_+ = (\mathcal{L}_{[1/\lambda_+(1),\infty)})^{-1}$. See Figure 1 for an illustration.

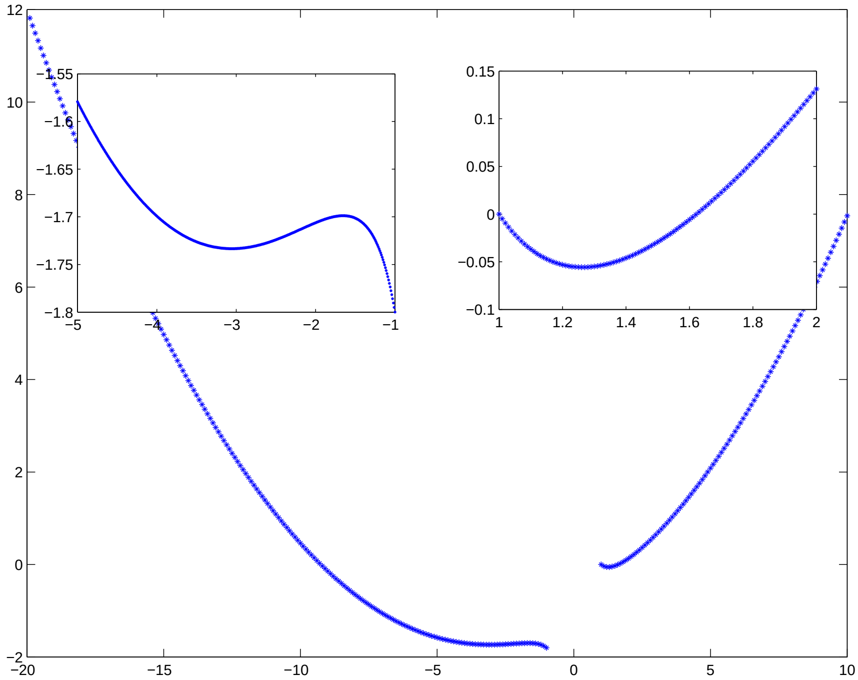

Figure 1: The function $\mathcal{L}-1$ for the random walk with jump measure $\lambda = 0.05\delta_2 + 0.35\delta_1 + 0.4\delta_{-1} + 0.05\delta_{-2} + 0.15\delta_{-3}$. On $[1,\infty)$, $\mathcal{L}$ is strictly convex. Its behavior on the interval $(-\infty,-1]$ is not trivial (left inset).

2. From Eq. (2.1)-(2.2) we identify the right Wiener-Hopf factor: if $\tau := T_1$ is the first strict ascending ladder time and $H := W(\tau)$ is the corresponding ascending ladder height (on $\{\tau<\infty\}$), then $\mathsf{E}[\gamma^{-\tau}\theta^{-H}, \tau<\infty] = (\lambda_+(\gamma)+\lambda_-(\gamma)-\lambda_+(\gamma)\lambda_-(\gamma)/\theta)/\theta$ for all $\{\gamma,\theta\}\subset[1,\infty)$.

3 Proof of theorem

For notational convenience we first introduce some relevant notation pertaining to the joint laws of the first passage times and the overshoots (with $\gamma\in[1,\infty)$): $\mu_n^\gamma(A) := \mathsf{E}[\gamma^{-T_n}, T_n<\infty, W(T_n)-n\in A]$ (for $n\in\mathbb{Z}$ and $A\subset\mathbb{Z}$), whilst $p_n^i(\gamma) := \mu_n^\gamma(\{i\})$ and $p_n(\gamma) := \mu_n^\gamma(\mathbb{Z})$ (for $\{n,i\}\subset\mathbb{N}_0$).

Next, the following proposition will give all the necessary ingredients towards the proof of Theorem 2.1.

Proposition 3.1. *Assume the setting as described in Section 2, prior to stating Theorem 2.1.*

(i) $\int\alpha(\gamma)^m\mu_n^\gamma(dm) = \alpha(\gamma)^{-n}$ *(for $n\in\mathbb{N}_0$, $\gamma\in(1,\infty)$, and also for $\gamma=1$ if $\mathrm{supp}(\lambda)$ is bounded from above).*

(ii) $\mu_n^\gamma(A) = \gamma^{-1}\int\mu_{n-m}^\gamma(A)\lambda(dm)$ *(for $n\in\mathbb{N}$, $\gamma\in[1,\infty)$, $A\subset\mathbb{Z}$).*

Moreover, if X is ‘nearly right-continuous’, i.e. $\mathrm{supp}(\lambda|_{2^{\mathbb{N}}})\subset\{1,2\}$*, then:*

(iii) For each $\gamma \in [1,\infty)$, the sequences $(p_n^0(\gamma))_{n\in\mathbb{N}_0}$ and $(p_n^1(\gamma))_{n\in\mathbb{N}_0}$ satisfy the following system of linear difference equations with constant coefficients (in $n \in \mathbb{N}_0$):

$$p_{n+1}^0(\gamma) = p_n^0(\gamma)p_1^0(\gamma) + p_n^1(\gamma); \tag{3.1}$$
$$p_{n+1}^1(\gamma) = p_n^0(\gamma)p_1^1(\gamma). \tag{3.2}$$

Proof. Suppose $\gamma \in (1,\infty)$, in the first instance. Applying Optional Sampling Theorem to the bounded stopping time $T_n \wedge N$ ($N \in \mathbb{N}_0$ fixed) and the martingale $(\alpha(\gamma)^{W_k}\gamma^{-k})_{k\in\mathbb{N}_0}$, it follows that $\mathsf{E}[\alpha(\gamma)^{W(T_n\wedge N)}\gamma^{-(N\wedge T_n)}] = 1$. Let $N \to \infty$. Then by monotone convergence $\mathsf{E}[\alpha(\gamma)^{W(T_n\wedge N)}\gamma^{-(N\wedge T_n)}, T_n \le N] \uparrow \mathsf{E}[\alpha(\gamma)^{W(T_n)}\gamma^{-T_n}, T_n < \infty]$, whilst bounded convergence yields $\mathsf{E}[\alpha(\gamma)^{W(T_n\wedge N)}\gamma^{-(N\wedge T_n)}, T_n > N] \le \mathsf{E}[\alpha(\gamma)^{n-1}\gamma^{-N}] \to 0$. The case $\gamma = 1$ follows by taking limits, $\gamma \downarrow 1$, using the continuity of α. This concludes the proof of item (i).

(ii) is the Markov property of W at time 1. For, we have:

$$\mathsf{E}[\gamma^{-T_n}, T_n < \infty, W(T_n) - n \in A] = \sum_{m\in\mathbb{Z}} \mathsf{E}\left[\gamma^{-(1+\overset{\triangle}{T}_{n-m})}, W_1 = m, \overset{\triangle}{T}_{n-m} < \infty, \overset{\triangle}{W}(\overset{\triangle}{T}_{n-m}) - (n-m) \in A\right],$$

where $\overset{\triangle}{W} := (W_{k+1} - W_1)_{k\in\mathbb{N}_0}$ is the incremental process after time 1, and for $l \in \mathbb{Z}$, $\overset{\triangle}{T}_l$ is its first entrance time into $[l,\infty)$. Use the facts that W_1 is independent of $\overset{\triangle}{W}$, and that $\overset{\triangle}{W} \overset{(\mathrm{d})}{=} W$.

Finally, (iii) is the strong Markov property at the time T_n. Specifically, we have:

$$\begin{aligned}
\mathsf{E}[\gamma^{-T_{n+1}}, T_{n+1} < \infty, W(T_{n+1}) = n+1] &= \mathsf{E}[\gamma^{-T_n - \overset{\triangle}{T}_1}, T_n < \infty, W(T_n) = n, \overset{\triangle}{T}_1 < \infty, \overset{\triangle}{W}(\overset{\triangle}{T}_1) = 1] + \\
&\quad \mathsf{E}[\gamma^{-T_n}, T_n < \infty, W(T_n) = n+1] \\
\mathsf{E}[\gamma^{-T_{n+1}}, T_{n+1} < \infty, W(T_{n+1}) = n+2] &= \mathsf{E}[\gamma^{-T_n - \overset{\triangle}{T}_1}, T_n < \infty, W(T_n) = n, \overset{\triangle}{T}_1 < \infty, \overset{\triangle}{W}(\overset{\triangle}{T}_1) = 2],
\end{aligned}$$

where now $\overset{\triangle}{W} := (W_{T_n+1} - W_{T_n})_{k\in\mathbb{N}_0}$ is the incremental process after T_n on $\{T_n < \infty\}$, and for $l \in \mathbb{Z}$, $\overset{\triangle}{T}_l$ is its first entrance time into $[l,\infty)$. Use the facts that the history up to T_n, $\sigma(W^{T_n})$, when traced on $\{T_n < \infty\}$, is independent of $\overset{\triangle}{W}$ under $\mathsf{P}(\cdot|T_n < \infty)$, and that $(\overset{\triangle}{W})_\star \mathsf{P}(\cdot|T_n < \infty) = W_\star \mathsf{P}$. $\square$

Let us now apply the above to gain understanding of the 'nearly right-continuous' random walk. We assume henceforth $\mathrm{supp}(\lambda|_{2^{\mathbb{N}}}) \subset \{1,2\}$.

Remark 3.2. Suppose furthermore $\lambda(\{2\}) = 0$, for the right-continuous case. Then Proposition 3.1(i) yields at once $\mathsf{E}[\gamma^{-T_n}\mathbb{1}(T_n < \infty)] = \alpha(\gamma)^{-n}$, for all $n \in \mathbb{N}_0$, $\gamma \in [1,\infty)$.

We now assume $\lambda(\{2\}) > 0$. If λ is supported by $2\mathbb{Z}$, this is just the right-continuous case, but on the lattice $2\mathbb{Z}$. So without loss of generality take the converse case. In particular, it follows that $p_1^1(\gamma)p_1^0(\gamma) > 0$ for all $\gamma \in [1,\infty)$.

In the first step towards the proof of Theorem 2.1, we solve the recursion system of Proposition 3.1(iii) (with $\gamma \in [1,\infty)$ fixed). Simply plug (3.2) into (3.1) to get:

$$p_{n+2}^0(\gamma) - p_{n+1}^0(\gamma)p_1^0(\gamma) - p_n^0(\gamma)p_1^1(\gamma) = 0, \quad n \in \mathbb{N}_0.$$

The characteristic polynomial of this last recursion is (in the dummy variable ζ) $\zeta^2 - p_1^0(\gamma)\zeta - p_1^1(\gamma)$, with the zeros:

$$\lambda_\pm(\gamma) := \frac{p_1^0(\gamma)}{2} \pm \sqrt{\left(\frac{p_1^0(\gamma)}{2}\right)^2 + p_1^1(\gamma)}. \tag{3.3}$$

Note that $-\lambda_+(\gamma) < \lambda_-(\gamma) < 0 < \lambda_+(\gamma) \leq 1$ (the last inequality follows from $p_1^0(\gamma) + p_1^1(\gamma) \leq 1$). From the general theory of linear difference equations with constant coefficients, it now follows that, for some $\{A_-(\gamma), A_+(\gamma)\} \subset \mathbb{R}$, and then for all $n \in \mathbb{N}_0$, $p_n^0(\gamma) = A_+(\gamma)\lambda_+^n + A_-(\gamma)\lambda_-(\gamma)^n$. The two initial values are $p_0^0(\gamma) = 1$ and $p_1^0(\gamma) = \lambda_+(\gamma) + \lambda_-(\gamma)$. From this we obtain immediately, for all $n \in \mathbb{N}_0$:

$$p_n^0(\gamma) = \frac{1}{\lambda_+(\gamma) - \lambda_-(\gamma)}(\lambda_+(\gamma)^{n+1} - \lambda_-(\gamma)^{n+1}), \text{ hence (using (3.2) \& } p_0^1(\gamma) = 0) \tag{3.4}$$

$$p_n^1(\gamma) = \frac{-\lambda_+(\gamma)\lambda_-(\gamma)}{\lambda_+(\gamma) - \lambda_-(\gamma)}(\lambda_+(\gamma)^n - \lambda_-(\gamma)^n), \text{ hence (by summing)} \tag{3.5}$$

$$p_n(\gamma) = \frac{-\lambda_+(\gamma)\lambda_-(\gamma) + \lambda_+(\gamma)}{\lambda_+(\gamma) - \lambda_-(\gamma)}\lambda_+(\gamma)^n - \frac{-\lambda_+(\gamma)\lambda_-(\gamma) + \lambda_-(\gamma)}{\lambda_+(\gamma) - \lambda_-(\gamma)}\lambda_-(\gamma)^n. \tag{3.6}$$

In the second step, we characterize the values of $\lambda_+(\gamma)$ and $\lambda_-(\gamma)$, $\gamma \in [1, +\infty)$. First, Proposition 3.1(i) implies $p_n^0(\gamma) + p_n^1(q)\alpha(\gamma) = \alpha(\gamma)^{-n}$ (for all $n \in \mathbb{N}_0$). In this relation plug in (3.4) and (3.5), divide by $\lambda_+(\gamma)^n$ and send $n \to \infty$. Since $|\lambda_-(\gamma)/\lambda_+(\gamma)| < 1$, the left-hand side has the limit $\lambda_+(\gamma)(1 - \lambda_-(\gamma)\alpha(\gamma))/(\lambda_+(\gamma) - \lambda_-(\gamma)) \in (0, \infty)$, so necessarily $\lambda_+(\gamma) = \alpha(\gamma)^{-1}$. If so, then the relation appearing in Proposition 3.1(i) is *a priori* satisfied and does not yield $\lambda_-(\gamma)$, which we shall have to identify by other means.

Suppose first $\alpha(\gamma) > 1$ and hence $\lambda_+(\gamma) < 1$ (i.e. $1/\lambda_+(\gamma) \in (1, \infty)$). In this instance we resort to Proposition 3.1(ii) with $A = \mathbb{R}$, which tells us that (for all $n \in \mathbb{N}\backslash\{1\}$):

$$\gamma p_n(\gamma) = \sum_{k \in \mathbb{Z}} \lambda(\{k\}) p_{n-k}(\gamma).$$

Plugging in (3.6), this implies (since $\mathcal{L}(1/\lambda_+(\gamma)) = \mathcal{L}(\alpha(\gamma)) = \gamma$ and $-\lambda_+(\gamma)\lambda_-(\gamma) + \lambda_-(\gamma) \neq 0$):

$$\mathcal{L}(1/\lambda_-(\gamma)) = \gamma, \tag{3.7}$$

where we know $1/\lambda_-(\gamma) \in (-\infty, -1)$.

Even if $\alpha(\gamma) = 1$ (hence $\gamma = 1$), however, still (3.7) holds (and, of course, $1/\lambda_-(1) \in (-\infty, -1)$), since one may pass to the limit $\gamma \downarrow 1$ in it, exploiting the continuity of $\mathcal{L}$, and of $(\gamma \mapsto \lambda_-(\gamma))$ on $[1, \infty)$ (which fact follows from (3.3), using bounded convergence in the definition of the quantities $p_1^0(\gamma)$ and $p_1^1(\gamma)$).

Now, from the Introduction, it is clear that $\mathcal{L} - \gamma$ has a unique zero on $(1, \infty)$, namely $\alpha(\gamma)$, and that $\alpha(1)$ is the largest root of $\mathcal{L} - 1$ on $[1, \infty)$. It remains to argue then that $\mathcal{L} - \gamma$ has at most one zero on $(-\infty, -1)$ for each $\gamma \in [1, \infty)$. Fix a $\gamma \in [1, \infty)$; let R be any such zero.

It does not seem immediately clear analytically why R should be unique (cf. Figure 1); so we use a probabilistic method.[1] Indeed, the argument is essentially verbatim that of the proof of Proposition 3.1(i). First, $(R^{W_k}\gamma^{-k})_{k \in \mathbb{N}_0}$ is a martingale. For, $\mathsf{E}[|R^{W_k}|] = \mathsf{E}[|R|^{W_k}] = \mathcal{L}(|R|)^k < \infty$ and $\mathsf{E}[R^{W_k}] = \mathcal{L}(R)^k = \gamma^k$ ($k \in \mathbb{N}_0$). The assertion then follows by stationary independent increments of W. Further, for any $\{n, N\} \subset \mathbb{N}_0$, Optional Sampling Theorem yields $\mathsf{E}[R^{W(T_n \wedge N)}\gamma^{-(T_n \wedge N)}] = 1$. Letting $N \to \infty$ we deduce by dominated convergence (as $W(T_n \wedge N) \leq n + 1$ and since a.s. on the event $\{T_n = \infty\}$, the process W limits to $-\infty$): $\mathsf{E}[R^{W(T_n)}\gamma^{-T_n}\mathbb{1}(T_n < \infty)] = 1$, i.e. $p_n^0(\gamma) + p_n^1(\gamma)R = R^{-n}$. Since the left-hand side is a linear combination of $(n \mapsto \lambda_+(\gamma)^n)$ and $(n \mapsto \lambda_-(\gamma)^n)$, it follows that:

$$R \in \{1/\lambda_-(\gamma), 1/\lambda_+(\gamma)\} \text{ and hence } R = 1/\lambda_-(\gamma).$$

This establishes that R is indeed unique and completes the proof of Theorem 2.1. □

[1] However, for numerical reasons, note that $\lim_{-\infty} \mathcal{L} = \infty$ and $\mathcal{L}(-1) < 1$.

4 Applications in queues and branching processes

Right-continuous random walks are related (at least in the distributional sense) to certain quantities in the theory of queues, and branching processes, see e.g. [8, Section 5] for a nice exposition. For the ‘nearly right-continuous’ case, we offer the following two examples in applications.

Consider first a single queue of customers with *two* equally capable servers, the latter attending to the former simultaneously, two at a time, per service. There are a total of $k \geq 2$ individuals in the queue at the start, $Q_0 := k$. The time the servers are working consists of idle and busy periods, where we define an idle period as a period in which at least one of the servers has no customer to attend to (and then only one server performs the service, say, while the other one rests). Thus, each busy period consists of one or more services; one service per two customers. Let Q_n denote the number of customers in the queue at the end of the n-th service. Assume that the number of individuals which arrive during each service is distributed according to the distribution function F ($\mathrm{d}F$ supported by $\mathbb{N}_0$) and that arrivals during each service period are independent (in their number). Let $(X_i)_{i\geq 1}$ be an independency of random variables distributed according to F and $(S_n)_{n\geq 0}$ be their partial sums. Consider the process $P_n := k + S_n - 2n$ ($n \geq 0$), which is to model $(Q_n)_{n\geq 0}$ up to the idle period. If we let $T_k := \inf\{n \geq 0 : P_n \leq 1\}$, then (with equality in distribution) T_k is the total number of services during the first busy period; accordingly it is also the time to the first idle period, and $2T_k$ is the number of customers served during the first busy period. Crucially, $(S_n - 2n)_{n\geq 0}$ is nothing else than the negative of a ‘nearly right-continuous’ random walk, and T_k is precisely one of its first passage times.

As our second application, note that we can also find in the above queue an example of what is essentially (but not quite) a Galton-Watson branching process for *paired* individuals, in the following precise sense. Consider having a totality of $k \geq 2$ initial ancestors in the 0-th generation, which reproduce in pairs, each pair giving young to a certain number of descendants of the next generation, independently (in offspring number), and according to the distribution F. All the pairs are assumed disjoint. Note in each generation there is of course the possibility of having an individual, which cannot be paired up. How precisely to treat him will soon become clear, once the connection to the above has been established.

Now, we say the population becomes extinct if there are no longer two individuals present (in the current generation), which can pair up and reproduce. Then in the above (with equality in distribution, and as an approximation) $2T_k$ represents the total progeny (modulo, possibly one member) until extinction has occurred (interpret customer j a child of $j'j''$, if j has arrived during the service of $j'j''$). The approximation is in that if the total number of individuals in a generation K is odd, then the left-over specimen j' in generation K can reproduce with a member j'' of generation $K+1$ (provided, of course, the left over pairs of generation K have produced any progeny). In that case, if any progeny occurs between j' and j'', it is assumed to be added to the generation $K+2$, which follows the oldest generation of this pair. Of course then j'' is no longer available for reproduction in generation $K+1$. This continues until there are still individuals available to reproduce.

Indeed, if the branching process is defined in this latter sense (so a Galton-Watson branching for pairs, with suitable boundary conditions dealing with the possibility of having an odd number of individuals available (left over) for reproduction in the current generation), then the correspondence is exact and $2T_k$ represents the total progeny (modulo, possibly, one member) until extinction has occurred.

5 Concluding remarks

It would be interesting to see what (if anything definitive) can be said, when the jumps of W are allowed upwards up to a certain (fixed, but arbitrary) threshold $N \in \mathbb{N}$ (we had $N = 2$, $N = 1$ being the skip-free case). This remains open to future research.

On the other hand, embedding W into continuous time as a compound Poisson process, presents, of course, no (essential) further difficulty to the above analysis – see the arXiv version of this paper [12] for this continuous-time analogue of the ‘nearly right-continuous’ random walk.

References

[1] Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR-1406564

[2] Mark Brown, Erol A. Peköz, and Sheldon M. Ross, *Some results for skip-free random walk*, Probab. Engrg. Inform. Sci. **24** (2010), no. 4, 491–507. MR-2725345

[3] F. De Vylder and M. J. Goovaerts, *Recursive calculation of finite-time ruin probabilities*, Insurance Math. Econom. **7** (1988), no. 1, 1–7. MR-971858

[4] David C. M. Dickson and Howard R. Waters, *Recursive calculation of survival probabilities*, ASTIN Bulletin **21** (1991), no. 2, 199–221.

[5] Ronald A. Doney, *Fluctuation theory for Lévy processes*, Lecture Notes in Mathematics, vol. 1897, Springer, Berlin, 2007, Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005, Edited and with a foreword by Jean Picard. MR-2320889

[6] Andreas E. Kyprianou, *Introductory lectures on fluctuations of Lévy processes with applications*, Universitext, Springer-Verlag, Berlin, 2006. MR-2250061

[7] Philippe Marchal, *A combinatorial approach to the two-sided exit problem for left-continuous random walks*, Combin. Probab. Comput. **10** (2001), no. 3, 251–266. MR-1841644

[8] Jim Pitman, *Enumerations of trees and forests related to branching processes and random walks*, Microsurveys in discrete probability (Princeton, NJ, 1997), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 41, Amer. Math. Soc., Providence, RI, 1998, pp. 163–180. MR-1630413

[9] M. P. Quine, *On the escape probability for a left or right continuous random walk*, Ann. Comb. **8** (2004), no. 2, 221–223. MR-2079932

[10] Ken-iti Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999, Translated from the 1990 Japanese original, Revised by the author. MR-1739520

[11] Frank Spitzer, *Principles of random walk*, second ed., Springer-Verlag, New York-Heidelberg, 1976, Graduate Texts in Mathematics, Vol. 34. MR-0388547

[12] Matija Vidmar, *A note on the times of first passage for ‘nearly right-continuous’ random walks*, arXiv:1310.6661v2, 2014.

[13] ______, *Fluctuation theory for upwards skip-free Lévy chains*, arXiv:1309.5328v2, 2014.

[14] ______, *Non-random overshoots of Lévy processes*, arXiv:1301.4463v2, 2014.

Acknowledgments. The author thanks an anonymous Referee for his input, which has helped significantly to improve the presentation of this paper.