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On harmonic functions of killed random walks in convex cones

Jetlir Duraj*

Abstract

We prove the existence of uncountably many nonnegative harmonic functions for random walks in the euclidean space with non-zero drift, killed when leaving general convex cones with vertex in 0. We also make the natural conjecture about the Martin boundary for lattice random walks in general convex cones in two dimensions. Proving that the set of harmonic functions found is the full Martin boundary for these processes is an open problem.

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1 Introduction and statement of result

We prove that for random walks of non zero drift on the euclidean space $\mathbb{R}^d, d \geq 2$, killed when leaving a convex cone with vertex in 0, there are uncountably many non-negative harmonic functions. The main assumption is finiteness of the jump generating function of the step of the random walk in a neighborhood of its preimage of 1. The proof is constructive and an adaptation of the similar proof in [Ignatiouk-Robert, Loree], which considers the special case of lattice random walks in the two-dimensional positive quadrant. We also make a conjecture about the Martin boundary of two-dimensional random walks, killed when leaving convex cones of $\mathbb{R}^2$ and comment on the difficulties in translating the [Ignatiouk-Robert, Loree] proof to the more general setting we are considering.

We consider a convex cone in $\mathbb{R}^d, d \geq 2$ with vertex in 0, denote by K its interior, which we assume to be nonempty throughout the paper and also a random walk on the euclidean space $\mathbb{R}^d$ with steps $X_i, i \in \mathbb{N}$ and step distribution γ. We also set $\Sigma = \overline{K} \cap \mathbb{S}^{d-1}$. We will study the random walk when some or all of the following assumptions are fulfilled.

A1 The step distribution has

$$m := \mathbb{E}[X_1] \neq 0 \quad \text{and} \quad \mathbb{E}[|X_1|] \neq 0.$$

A2 The jump generating function $\varphi(a) := \mathbb{E}[e^{a \cdot X_1}]$ fulfills

$$D = \{a \in \mathbb{R}^2 | \varphi(a) \leq 1\} \subset int(\{a \in \mathbb{R}^2 | \varphi(a) < \infty\}).$$

*Department of Economics, Harvard University, USA. E-mail: duraj@g.harvard.edu

A3 The random walk is lattice-valued, irreducible in $\mathbb{Z}^d$ and the killed random walk when leaving K is irreducible in K. Moreover, the angle between every two points in $\partial\Sigma$ is strictly smaller than π.

We will denote $\mathbb{E}_z$ for the measure describing the distribution of random walks started at z, i.e. with $S(0) = z$.
Assumption **A2** is the standard assumption made for the study of the Martin boundary of lattice random walks in the euclidean lattice (see[Ney, Spitzer]). It implies in particular that X_1 has all moments.

Under assumptions **A1** and **A2** it is well-known, that

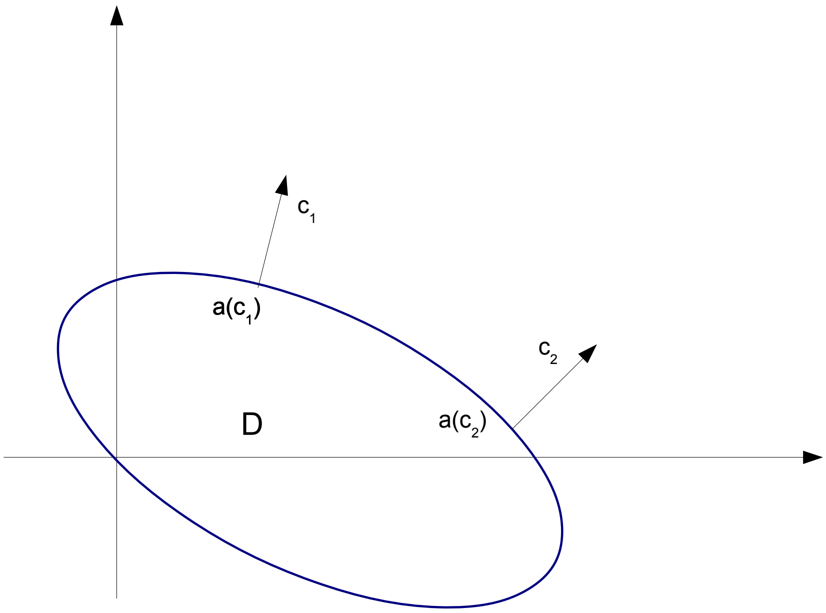
$$D = \{a \in \mathbb{R}^d : \varphi(a) \leq 1\}$$

is a strictly convex and closed set, the gradient $\nabla\varphi(a)$ exists everywhere and does not vanish on $\partial D = \{a \in \mathbb{R}^d | \varphi(a) = 1\}$. Moreover, the mapping

$$a \to q(a) = \frac{\nabla\varphi(a)}{|\nabla\varphi(a)|}$$

is a homeomorphism between ∂D and an open set of $\mathbb{S}^{d-1}$. D does not need to be bounded as the case $d = 2, \gamma = \frac{1}{3}\delta_{(1,-1)} + \frac{1}{3}\delta_{(-1,1)} + \frac{1}{3}\delta_{(-1,-1)}$ shows. If **A3** is additionally fulfilled then D is additionally compact and the image of $q(\cdot)$ is the whole sphere in d dimensions $\mathbb{S}^{d-1}$ (see[Ney, Spitzer] and the references therein). The inverse mapping is

Figure 1: A typical D in $\mathbb{R}^2$.

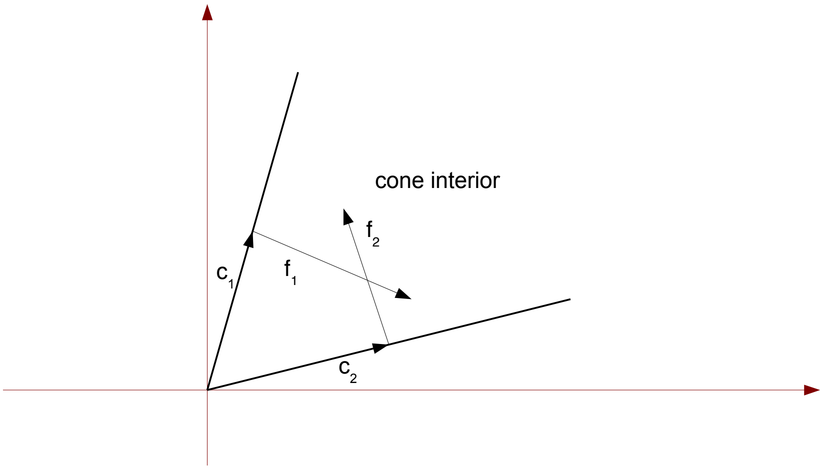


denoted by $q \to a(q)$ and, whenever possible, we extend this map to nonzero $q \in \mathbb{R}^d$ by setting $a(q) := a\left(\frac{q}{|q|}\right)$. This definition implies that $a(q)$ is the only point in ∂D where q is normal to D. See figure 1 for a typical picture of D in the case $d = 2$ and that **A1**-**A3** hold.

If **A3** is not fulfilled, we make the following weaker assumption to avoid trivialities.

A4 $\Gamma = \{a \in \partial D \; | q(a) \in \Sigma = \overline{K} \cap \mathbb{S}^{d-1}\}$ is nonempty.

Figure 2: A convex cone in $\mathbb{R}^2$.



For the case $d = 2$ we encode the cone K as follows: take Λ to be a set of two points in the unit circle $\mathbb{S}^1$, $\Lambda = \{c_1, c_2\}$, so ordered that the angle ϕ between them is in $(0, \pi)$. The rays from (0,0) to infinity going through the $\mathbb{S}^1$-sector between the two vectors in Λ enclose a convex cone. Its interior K depends on the vectors we chose, i.e. $K = K(c_1, c_2)$. **A4** implies that at least one of the c_i is normal to ∂D if not both. We also note the unit vectors f_1 and f_2, respectively perpendicular to c_1 and c_2, pointing inwards. See figure 2 for a typical example. Defining additionally the stopping time

$$\tau = \inf\{n \geq 0 | S(n) \notin K\}$$

we want to prove the following.

Proposition 1.1. *(a) Under assumptions A1, A2 and A4 for every a such that $q(a) \in int(\Sigma)$ and $z \in K$*

$$h_a(z) = \exp(a \cdot z) - \mathbb{E}_z[\exp(a \cdot S(\tau)), \tau < \infty].$$

are nonnegative and harmonic for the random walk, killed when leaving the cone.
(b) If in (a) $d = 2$ and a fulfills $q(a) = c_i,\ i \in \{1, 2\}$ the function

$$h_a(z) = z \cdot f_i \exp(a \cdot z) - \mathbb{E}_z[f_i \cdot S(\tau) \exp(a \cdot S(\tau)), \tau < \infty]$$

is nonnegative and harmonic for the random walk, killed when leaving the cone.
(c) The harmonic functions from (a)-(b) are strictly positive if A3 is additionally fulfilled.

These harmonic functions are just a generalization of the functions found in [Ignatiouk-Robert, Loree]. Intuitively, a look at figure 2 and at their paper suggests, that these functions must be the harmonic functions for general cones in the two-dimensional case.

For the case $q(a) \notin int(\Sigma)$ our proof method doesn't work in general for $d \geq 3$. The difficulty lies in proving a general version of Corollary 3.4, whose proof here uses the fact that for $d = 2$ the event *{the random walk doesn't leave K from a specific supporting hyperplane of the cone}* can be encoded easily through the *unique* opposite supporting hyperplane. This simple characterization generalizes to $d \geq 3$ only if the cone is defined as intersection of finitely many halfspaces, which we don't pursue here since we are interested in the class of general cones. We remark also the following.

Remark 1.2. In the formulation of Proposition 1.1 the event $\{\tau < \infty\}$ can be left out when $m \notin K$.

Remark 1.3. For $d = 1$ the only cone to consider is $(0, \infty)$. [Doney] fully characterizes the Martin boundary in the lattice case. His result can be used direcly in special cases even in $\mathbb{R}^d, d \geq 2$, when our assumptions are not fulfilled. For example, it shows that random walks which are cartesian products of one-dimensional lattice random walks with drift $-\infty$ and such that $\partial D = \{0\}$, killed when leaving $K = (0, \infty)^d$ have no nontrivial nonnegative harmonic function.

Finally, one can see how the harmonicity result in [Ignatiouk-Robert, Loree] immediately follows from our proposition by taking $c_1 = (0, 1)$ and $c_2 = (1, 0)$ in (b).

Proposition 1.4 ([Ignatiouk-Robert, Loree]-Harmonic functions for the positive quadrant)**.** *Assume A1-A3. For every* $a \in \Gamma_+ := \{a \in \partial D : q(a) \in \mathbb{R}^2_+,\ |q(a)| = 1\}$ *and* $z = (x_1, x_2) \in \mathbb{N}^* \times \mathbb{N}^*$

$$h_a(z) = \begin{cases} x_1 \exp(a \cdot z) - \mathbb{E}_z[S_1(\tau) \exp(a \cdot S(\tau)), \tau < \infty], & \textit{if } q(a) = (0,1), \\ x_2 \exp(a \cdot z) - \mathbb{E}_z[S_2(\tau) \exp(a \cdot S(\tau)), \tau < \infty], & \textit{if } q(a) = (1,0) \\ \exp(a \cdot z) - \mathbb{E}_z[\exp(a \cdot S(\tau)), \tau < \infty], & \textit{otherwise} \end{cases}$$

are strictly positive and harmonic for the random walk, killed when leaving the positive quadrant.

The rest of the paper is organized as follows. The next section states the natural conjecture about the Martin boundary of random walk, killed when leaving a two-dimensional convex cone, when A1-A3 are all fulfilled. We also underline where the proof in [Ignatiouk-Robert, Loree], which considers only the positive quadrant, breaks down for the general case. In the last section, Proposition 1.1 is proven by adapting the proof of Proposition 1.4, contained in [Ignatiouk-Robert, Loree], to the general setting we are considering.

2 A Conjecture: Martin boundary for general convex cones in two dimensions

For this section only we assume that A1-A3 are fulfilled and that $d = 2$. In [Ney, Spitzer] the authors show that every positive harmonic function h for the random walk can be expressed as

$$h(z) = \int_C e^{c \cdot z} d\gamma(c).$$

Here γ is a positive Borel measure on some suitable set C. These types of functions and the types considered in Remark 3.2 of the next section are not harmonic for killed random walk on the quadrant. To "make" them harmonic, one has to consider the correction term. Therefore the form of the functions in Proposition 1.4.

The main contribution of [Ignatiouk-Robert, Loree] is to show that these functions are the whole Martin boundary for the case of the positive quadrant (see Theorem 1 there).

Judging from the analogy between Proposition 1.1 and 1.4, we conjecture the following (stated analoguously to Theorem 1 in [Ignatiouk-Robert, Loree]).

Conjecture For the cone encoded by c_1 and c_2 as in section 1 and under the assumptions A1 - A4 made there, we have that :

1. A sequence of points z_n in K with $\lim_{n\to} |z_n| = +\infty$ converge to a point of the Martin boundary for the killed random walk when leaving the cone, if and only if $\frac{z_n}{|z_n|} \to q$ for some $q \in \Gamma$.

2. The full Martin Compactification of $K \cap \mathbb{Z}^2$ is homeomorphic to the closure of the set $\{w = \frac{z}{1+|z|} | z \in K \cap \mathbb{Z}^2\}$ in $\mathbb{R}^2$.

In short, Proposition 1.1 (a)-(b) fully characterizes the Martin boundary of random walks on the two dimensional euclidean lattice, killed when leaving convex cones.

If one tries to carry over the methods of [Ignatiouk-Robert, Loree] to this general case, one sees that the *communication condition* contained there and the *large deviations result* can be modified to work for the more general setting as well. We will not give details how this is done, but we mention shortly that both can be proven if one augments assumption **A3** by the following.

"Strong local" irreducibility: There exists some uniform $R > 0$ such that for every $z \in K, e \in \mathbb{Z}^2, \quad |e| = 1$ such that $z + e \in K$ we have: there exists a path of measure non zero within $K \cap B_R(z)$ from z to $z + e$.

This assumption is neccessary, if one wants to work with the communication condition as [Ignatiouk-Robert, Loree] do and is fulfilled in the positive quadrant setting due to irreducibility. The obstacle for generalizing the proof in the case of the positive quadrant is the lack of Markov-additivity for local processes for the general case. We recall that a Markov Chain $\mathcal{Z}_n = (A(n), M(n))$ on a countable space $\mathbb{Z}^d \times E$ is called *Markov-additive* if for its transition matrix p it holds:

$$p((x, y), (x', y')) = p((0, y), (x' - x, y')) \text{ for all } x, x' \in \mathbb{Z}^d, y, y' \in E$$

[Ignatiouk-Robert, Loree] make extensive use of this property when showing the above conjecture for the case of the positive quadrant. The idea for the general case of convex cones would be to look at local processes "deep" inside the cone, where the random walk is Markov-additive in two directions. But approaching the boundary of the cone, this property disappears in general in both directions. For the positive quadrant this happens only for one direction and this is crucial for the proof in [Ignatiouk-Robert, Loree]. Without Markov-additivity it seems impossible to come to a usable Ratio Limit theorem as is done in [Ignatiouk-Robert, Loree]. On the other hand, the proof of Proposition 1.1 does not use Markov-additivity. This suggests the existence of more general methods than those of [Ignatiouk-Robert, Loree] for proving the conjecture made in this section or a similar conjecture in higher dimensions.

3 Proof of Proposition 1.1.

We assume throughout that when considering random walks in general dimensions $d \geq 2$ **A1**,**A2** and **A4** are fulfilled and when $d = 2$ instead of **A4** the stronger assumption **A3** is fulfilled. Before starting with a series of Lemmas, which will lead to the proof of Proposition 1.1 we introduce the family of twisted random walks S_a with (substochastic) transition matrix

$$p_a(z, z') = \gamma(z' - z) e^{a \cdot (z' - z)} \ , \ a \in D$$

and τ_a the respective exit time from K. Note that these are equivalent measures to γ. In particular, S_a is irreducible (in K) if and only if S is and the stopped random walk $S_a(\cdot \wedge \tau)$ is irreducible (in K) if and only if $S(\cdot \wedge \tau)$ is.

We start the proof of Proposition 1.1 by proving the following simple Lemma.

Lemma 3.1. *For every $a \in D, z \in K : \mathbb{E}_z[e^{a \cdot (S(\tau) - z)}, \tau < \infty] = \mathbb{P}_z(\tau_a < \infty)$. In particular, $z \to 1 - \mathbb{E}_z[e^{a \cdot (S(\tau) - z)}, \tau < \infty]$ is a nonnegative function.*

Proof. For every $n \in \mathbb{N}$, $A \subset K^c$ measurable one sees easily

$$\begin{aligned} \mathbb{P}_z(S_a(n) \in A, \tau_a = n) &= \mathbb{P}_z(S_a(i) \in K, i \leq n - 1, S_a(n) \in A) \\ &= \mathbb{E}[e^{a \cdot (S(n) - z)}, S(n) \in A, \tau = n] \end{aligned}$$

and with this

$$\begin{aligned}\mathbb{P}_z(\tau_a < \infty) &= \sum_{n\geq 0}\int_{K^c} \mathbb{P}_z(\tau_a = n, S_a(n) \in dz') \\ &= \sum_{n\geq 0}\int_{K^c} e^{a\cdot(S(n)-z)}\mathbb{P}_z(S(n) \in dz', \tau = n) \\ &= \mathbb{E}_z[e^{a\cdot(S(\tau)-z)}, \tau < \infty].\end{aligned}$$

□

We note the following simple Remark.

Remark 3.2. For every $q \in \mathbb{S}^{d-1} \cap Im(q), d \geq 2$ and $\tilde{q} \in \mathbb{S}^{d-1}$ perpendicular to q we have that

$$f_q(z) = \tilde{q} \cdot z e^{a(q)\cdot z}$$

is harmonic for the original random walk $S(n)$.
Indeed we use that $a(q) \in \partial D$ in the following calculation.

$$\begin{aligned}&\mathbb{E}_z[f_q(S(1))] = \mathbb{E}_z[\tilde{q} \cdot S(1)e^{a(q)\cdot S(1)}] \\ &= \mathbb{E}_z[\tilde{q} \cdot (S(1) - z)e^{a(q)\cdot(S(1)-z)+a(q)\cdot z} + \tilde{q} \cdot z e^{a(q)\cdot(S(1)-z)+a(q)\cdot z}] \\ &= e^{a(q)\cdot z}\tilde{q} \cdot (\nabla\varphi(a)|_{a=a(q)} + z) = f_q(z),\end{aligned}$$

since $\nabla\varphi(a)|_{a=a(q)} = q$ for $q \in \mathbb{S}^{d-1} \cap Im(q)$.

Returning to our main task, we define the following for the case $d = 2$:

$$H_i = \{z \in \mathbb{R}^2 | z \cdot f_i > 0\}$$

and

$$\tau_i = \inf\{n \geq 0 | S(n) \notin H_i\}.$$

Then of course $\tau = \tau_1 \wedge \tau_2$ since $K = H_1 \cap H_2$. With this, we note the following remark for the case $d = 2$.

Remark 3.3. In the case $d = 2$, for $z \in K$ and $a \in D$

$$\begin{aligned}&\mathbb{E}_z[e^{a\cdot S(\tau)}, \tau = \tau_2 < \tau_1] \\ &= \mathbb{E}_z[e^{a\cdot(S(\tau)-z))}, \tau = \tau_2 < \tau_1]e^{a\cdot z} \leq e^{a\cdot z},\end{aligned}$$

since the expectation in the second line is just $\mathbb{P}_z(\tau_a = \tau_{a_2} < \tau_{a_1}) \leq 1$ with the same reasoning as in Lemma 3.1.

From this last remark the following is immediate.

Corollary 3.4. *In the case $d = 2$ for $z \in K$ and $i, j \in \{1, 2\}$ so that $c_i \in Im(q)$ and $i \neq j$*

$$z \to \mathbb{E}_z[|f_i \cdot S(\tau)|e^{a(c_i)S(\tau)}, \tau = \tau_j < \tau_i]$$

is finite.

Proof. Take w.l.o.g. $i = 1$ and $j = 2$. Then in the event that $\tau = \tau_2 < \tau_1$ we have $f_1 \cdot S(\tau) > 0$ and $f_2 \cdot S(\tau) \leq 0$. Also (look again at figure 1) for small enough $\delta > 0$ there always exists some suitable $\epsilon > 0$ so that $a(c_1) + \delta f_1 - \epsilon f_2$ lies in D. This yields

$$\begin{aligned}&\mathbb{E}_z[|f_1 \cdot S(\tau)e^{(a(c_1)\cdot S(\tau)}, \tau = \tau_2 < \tau_1] \leq \frac{1}{\delta}\mathbb{E}_z[e^{(a(c_1)+\delta f_1)\cdot S(\tau)}, \tau = \tau_2 < \tau_1] \\ &\leq \frac{1}{\delta}\mathbb{E}_z[e^{(a(c_1)+\delta f_1-\epsilon f_2)S(\tau)}, \tau = \tau_2 < \tau_1]\end{aligned}$$

since $-\epsilon f_2 S(\tau_2) \geq 0$. Now the result follows from Remark 3.3. □

Before going on with the next step in the proof of Proposition 1.1, we need an auxiliary lemma, which is part of the folklore now in probability. The proof for the case $x = 0$ can be found in [Feller] and for $x > 0$ we give an easy version here due to lack of a definite reference.

Lemma 3.5. *For a random walk with jump X_1 of mean zero, $\mathbb{E}[|X_1|] > 0$ and $\mathbb{E}[X_1^2] < \infty$ and $T_0 = \inf\{n \geq 1 | S(n) \leq 0\}$ we have $\mathbb{E}_x[|S(T_0)|] < \infty$ for $x > 0$.*

Proof. The problem is the same as proving that for the positive ladder heights $\{\chi_+^{(n)}\}_n$ of the random walk $\{-S(n)|n \geq 1\}$ and $\sigma_x = \inf\{k > 0 | T_k := \sum_{i=1}^k \chi_+^{(i)} > x\}$ we have $\mathbb{E}[T(\sigma_x)] < \infty$. Since we are in the driftless case for the original random walk, we know that the ladder heights are proper and that $\mathbb{E}[\sigma_x] < \infty$. The assumption $\mathbb{E}[X_1^2] < \infty$ and results in [Chow] imply that here Wald's identity can be applied on $\{T(n)|n \geq 1\}$ to give $\mathbb{E}[T(\sigma_x)] = \mathbb{E}[\chi_+^{(1)}]\mathbb{E}[\sigma_x] < \infty$. □

Returning to our main task we prove the following.

Lemma 3.6. *For $d = 2$, $z \in K$, $i = 1, 2$*

$$z \to \mathbb{E}_z[|f_i \cdot S(\tau)|e^{a(c_i)\cdot S(\tau)}, \tau < \infty]$$

is a finite well-defined function if $c_i \in Im(q)$.

Proof. Take $i = 1$ w.l.o.g. Then

$$\begin{aligned}&\mathbb{E}_z[|f_1 \cdot S(\tau)|e^{a(c_1)\cdot S(\tau)}, \tau < \infty] = \mathbb{E}_z[|f_1 \cdot S(\tau)|e^{a(c_1)\cdot S(\tau)}, \tau = \tau_2 < \tau_1] \\ &+ \mathbb{E}_z[|f_1 \cdot S(\tau)|e^{a(c_1)\cdot S(\tau)}, \tau = \tau_1 < \infty]\end{aligned}$$

Note that the first term in the sum above is finite due to Corollary 3.4. The second one is smaller than

$$\mathbb{E}_z[|f_1 \cdot S(\tau)|e^{a(c_1)\cdot S(\tau)}, \tau_1 < \infty] = -\mathbb{E}_z[f_1 \cdot S(\tau_1)e^{a(c_1)\cdot S(\tau_1)}, \tau_1 < \infty]$$

Now we have that

$$\begin{aligned}&\mathbb{E}_0[f_1 \cdot S(1)e^{a(c_1)\cdot S(1)}] = f_1 \cdot \mathbb{E}_0[S(1)e^{a(c_1)\cdot S(1)}] \\ &= f_1 \cdot \nabla\varphi(a)|_{a=a(c_1)} = f_1 \cdot c_1 = 0,\end{aligned}$$

which means in short

$$\mathbb{E}_0[f_1 \cdot S_a(1)] = 0$$

Now the real-valued random walk $f_1 \cdot S_a(n)$ fulfills

$$\mathbb{E}_0[|f_1 \cdot S_a(1)|^2] < \infty.$$

With this and

$$\mathbb{E}_z[|f_1 \cdot S(\tau_1)|e^{a(c_1)\cdot S(\tau_1)}, \tau_1 < \infty] = \mathbb{E}_{f_1\cdot z}[|f_1 \cdot S_a(\tau_{a_1})|]$$

we can use lemma 3.5 and finish the proof. □

We also prove the following lemma.

Lemma 3.7. *For $a \in \Gamma$ and $q(a) \in int(\Gamma)$*

$$z \to 1 - \mathbb{E}_z[e^{a\cdot(S(\tau)-z)}, \tau < \infty]$$

*is strictly positive in K if **A3** holds. If $d = 2$, **A3** holds and $a = a(c_i)$ for $c_i \in Im(q)$ then it is zero.*

Proof. Consider first $d = 2$, $i \in \{1, 2\}$ fixed and $a = a(c_i)$. We have due to Lemma 3.1

$$\mathbb{E}_z[e^{a\cdot(S(\tau)-z)}, \tau < \infty] = \mathbb{P}_z(\tau_a < \infty) = 1$$

since also $\mathbb{E}_0[f_i \cdot S_a(1)] = 0$ i.e. the respective one dimensional random walk is recurrent with the same calculation as in the previous Lemma.

Let now $d \geq 2$ and $a \in int(\Gamma)$. It holds

$$m(a) = \int z e^{a\cdot z}\gamma(z) = \nabla\varphi(a) = |\nabla\varphi(a)|q(a).$$

This means that $m(a) \in K$. The Strong Law of Large Numbers implies

$$\frac{S_a(n)}{n} \to m(a), \text{ for } n \to \infty \tag{3.1}$$

regardless of the starting point z. Note that there exists some $N > 0$ and $\epsilon > 0$, so that $\{z \in \mathbb{R}^d | \, |\frac{z}{n} - m(a)| < \epsilon \text{ for some } n \geq N\}$ is contained in K, since $dist(m(a), \partial K) > 0$. Together with (3.1) this implies the existence of some random $N_{z,\epsilon} > N$ such that for $n \geq N_{z,\epsilon}$ we have $S_a(n) \in K$, if $S_a(0) = z$. Define for each point $b \in \partial\Sigma$ the vector $f(b)$ to be a unit normal vector to ∂K, perpendicular to b and pointing in the interior of K. Without loss of generality we will assume that $\partial\Sigma$ is smooth (otherwise just restrict the cone accordingly, so that the ball of radius ϵ around $m(a)$ is still contained in the interior of the restricted smooth open cone). We have from the discussion above: if $S_a(0) = 0$

$$\min_{n\in\mathbb{N}} \min_{b\in\partial\Sigma} f(b) \cdot S_a(n) > -\infty \quad \text{almost surely.}$$

For some fixed and suitable $\hat{z} \in K$ we get therefore with help of Lemma 3.1

$$\begin{aligned} 1 - \mathbb{E}_{\hat{z}}[e^{a\cdot(S(\tau)-\hat{z})}, \tau < \infty] &= \mathbb{P}_{\hat{z}}(\tau_a = \infty) \\ &= \mathbb{P}_0(\min_{n\in\mathbb{N}} \min_{b\in\partial\Sigma} f(b) \cdot S_a(n) > -\min_{b\in\partial\Sigma} f(b) \cdot \hat{z}) > 0. \end{aligned}$$

Now we use **A3** to get through the Markov property for general $z \in K$

$$\begin{aligned} 1 - \mathbb{E}_z[e^{a\cdot(S(\tau)-z)}, \tau < \infty] &= \mathbb{P}_z(\tau_a = \infty) \\ &\geq \mathbb{P}_z(S_a(t) = \hat{z}, \tau_a > t)\mathbb{P}_{\hat{z}}(\tau_a = \infty) > 0, \end{aligned}$$

if t is chosen such that the first probability is not zero. □

Just before proving Proposition 1.1, we prove the following.

Lemma 3.8. *If $d = 2$, for $z \in K$ and $i \in \{1, 2\}$ so that $c_i \in Im(q)$*

$$z \to f_i \cdot z e^{a(c_i)\cdot z} - \mathbb{E}_z[f_i \cdot S(\tau) e^{a(c_i)\cdot S(\tau)}]$$

is well-defined and nonnegative in K.

Proof. Due to Remark 3.2 we have that $f_i \cdot S(n) e^{a(c_i)\cdot S(n)}$ is a martingale and by optional stopping theorem for every $z \in K$ we have

$$\begin{aligned} &\mathbb{E}_z[f_i \cdot S(\tau) e^{a(c_i)\cdot S(\tau)}, \tau \leq n] \\ &= \mathbb{E}_z[f_i \cdot S(\tau \wedge n) e^{a(c_i)\cdot S(\tau\wedge n)}] - \mathbb{E}_z[f_i \cdot S(n) e^{a(c_i)\cdot S(\tau)}, \tau > n] \\ &= f_{c_i}(z) - \mathbb{E}_z[f_i \cdot S(n) e^{a(c_i)\cdot S(\tau)}, \tau > n] \leq f_{c_i}(z) \end{aligned}$$

with the notation of Remark 3.2. Now Lemma 3.6 justifies dominated convergence and the result follows. □

Proof of Proposition 1.1. (a) Take first $a \in int(\Gamma)$. By Lemma 3.1 h_a is nonnegative in K. Set

$$f(z) = \mathbb{E}_z[e^{a\cdot S(\tau)}, \tau < \infty].$$

For $z \notin K$ one has $f(z) = e^{a\cdot z}$ which implies $h_a(z) = 0$ and with it $\mathbb{E}_z[h_a(S(1)), \tau > 1] = 0$. For $z \in K$ we have

$$\begin{aligned} &\mathbb{E}_z[f(S(1))] = \mathbb{E}_z\left[\mathbb{E}_{S(1)}[e^{a\cdot S(\tau)}, \tau < \infty]\right] \\ &= \mathbb{E}_z[e^{a\cdot S(1)}, \tau = 1] + \mathbb{E}_z\left[\mathbb{E}_z[e^{a\cdot S(\tau)}, \tau < \infty | \mathcal{F}_1], \tau > 1\right] \\ &= f(z), \end{aligned} \tag{3.2}$$

as one can easily see. This implies for $h_a(z) = e^{a\cdot z} - f(z)$ the equality $\mathbb{E}_z[h_a(S(1))] = \mathbb{E}_z[h_a(S(1)), \tau > 1] = h_a(z)$. Here we have implicitly used that $\mathbb{E}_z[e^{a\cdot S(1)}] = e^{a\cdot z}$ since $a \in \partial D$. With this, the case $a \in int(\Gamma)$ is solved.
(b) Take w.l.o.g. $a = a(c_1)$. We know from Lemma 3.8 that h_a is well-defined and nonnegative in K. Take first $z \notin K$. Then, it is clear that $h_a(z) = 0$ as is $\mathbb{E}_z[h_a(S(1)), \tau > 1]$. Take now $z \in K$. We have first $\mathbb{E}_z[h_a(S(1)), \tau = 1] = 0$ and therefore

$$\begin{aligned} &\mathbb{E}_z[h_a(S(1)), \tau > 1] = \mathbb{E}_z[h_a(S(1))] \\ &= f_1 \cdot z e^{a\cdot z} - \mathbb{E}_z\left[\mathbb{E}_{S(1)}[f_1 \cdot S(\tau) e^{f_1\cdot S(\tau)}, \tau < \infty]\right] = h_a(z) \end{aligned}$$

since the second term in the sum after the second equality is equal to

$$\mathbb{E}_z[f_1 \cdot S(\tau) e^{a\cdot S(\tau)}, \tau < \infty]$$

by the similar reasoning as in (3.2). With this, harmonicity of h_a is proved.
(c) We only have to consider the case of (b), since Lemma 3.7 deals with the other case. We have

$$\begin{aligned} &h_a(z) e^{-a\cdot z} = f_1 \cdot z - \mathbb{E}_z[f_1 \cdot S(\tau) e^{a\cdot(S(\tau)-z)}, \tau = \tau_1 < \infty] \\ &- \mathbb{E}_z[f_1 \cdot S(\tau) e^{a\cdot(S(\tau)-z)}, \tau = \tau_2 < \tau_1 < \infty] = f_1 \cdot z - A - B \end{aligned}$$

where of course $f_1 \cdot z - A \geq f_1 \cdot z > 0$ since $z \in K$. For B and $\delta > 0$ we have

$$\begin{aligned} B &\leq \frac{1}{\delta} \mathbb{E}_z[e^{a\cdot(S(\tau)-z)+\delta f_1\cdot S(\tau)}, \tau = \tau_2 < \tau_1] \\ &\leq \frac{1}{\delta} \mathbb{E}_z[e^{a\cdot(S(\tau)-z)+\delta f_1\cdot S(\tau)-\epsilon f_2\cdot S(\tau_2)}, \tau = \tau_2 < \tau_1] \end{aligned}$$

where $\delta, \epsilon > 0$ are chosen such that $\tilde{c} := a + \delta f_1 - \epsilon f_2 \in D$ (note that this is possible, see figure 1 to get a grasp of this) and therefore due to Lemma 3.1

$$B \leq \frac{1}{\delta} \mathbb{E}_z[e^{\tilde{c}\cdot(S(\tau)-z)}, \tau < \infty] e^{(\epsilon f_2 - \delta f_1)\cdot z} \leq \frac{1}{\delta} e^{(\epsilon f_2 - \delta f_1)\cdot z}$$

Note now that there exists some $z \in K$ such that $(\epsilon f_2 - \delta f_1) \cdot z < 0$. Fix such a z and set $z_n = nz$ and the respective B and A evaluated at z_n with B_n and A_n. It follows that there certainly exists $\hat{z} \in K$ such that $h_a(\hat{z}) > 0$. Now for arbitrary z in the cone use **A3** to find some $n \in \mathbb{N}$ such that the probability the random walk reaches z from $\hat{z}$ within the cone in n steps is positive to see that

$$h_a(z) \geq h_a(\hat{z}) \mathbb{P}_{\hat{z}}(\textit{Random Walk reaches } z \textit{ in n steps within the cone}) > 0,$$

by harmonicity and nonnegativity of h_a. This yields the positivity result for all $z \in K$. □

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