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On criteria of disconnectedness for Λ-Fleming-Viot support*

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Abstract

The totally disconnectedness of support for superBrownian motion in high dimensions is well known. In this paper, we prove that similar results also hold for Λ-Fleming-Viot processes with Brownian spatial motion provided that the associated Λ-coalescent comes down from infinity fast enough. Our proof is another application of the lookdown particle representation for Λ-Fleming-Viot process. We also discuss the disjointness of independent Λ-Fleming-Viot supports and ranges in high dimensions. The disconnectedness of the Λ-Fleming-Viot support remains open in certain low dimensions.

Keywords: superprocess; Λ-Fleming-Viot process; Λ-coalescent; lookdown representation; disconnectedness of support.
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1 Introduction

It is well known that in dimension one superBrownian motion is absolutely continuous with respect to Lebesgue measure, and in dimensions two and above it is a singular random measure with its support of Hausdorff dimension two. The connectedness of the superBrownian motion support at a fixed time is a question asked by Donald Dawson; see Section 1 of Tribe [24] for the remark.

For a d-dimensional superBrownian motion with binary branching, it is shown by Perkins [18] that at any fixed positive time its support is totally disconnected, i.e. the support contains no nontrivial connected component, in dimension four or above and its support is totally disconnected, uniformly for all positive times, in dimension six or above; also see Section III.6 of Perkins [19]. The proof in [19] uses the cluster decomposition and historical modulus of continuity for superBrownian motion.

The disconnectedness of support for superBrownian motion with general branching mechanism is studied in Delmas [10] using a snake representation and a subordination method. It is shown in [10] that at any fixed positive time the support of a d-dimensional superBrownian motion with $(1+\alpha)$-stable branching mechanism is totally disconnected if $d\alpha > 4$.

For superBrownian motion X with binary branching in dimension three, a partial disconnectedness result is obtained by [24] which states that for any $t > 0$, with probability one for X_t almost all x, the connected component of its support containing x is

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exactly $\{x\}$, the set consisting of a single element x; also see Abraham [1] for a proof of this result with the approach of Brownian snake representation. The disconnectedness of superBrownian motion support remains an open problem in certain low dimensions.

Fleming-Viot processes belong to another important class of probability-measure-valued superprocesses arising from population genetics. The Λ-Fleming-Viot process is a Fleming-Viot process with a more general reproduction mechanism that is associated with Λ-coalescent of multiple collisions. In this paper we want to study the disconnectedness of support for the Λ-Fleming-Viot processes with Brownian spatial motion in $\mathbb{R}^d$, which to our best knowledge has not been studied before. We are going to show that its support is totally disconnected in high dimensions when the associated Λ-coalescent comes down from infinity fast enough. In particular, if the associated Λ-coalescent is a Beta$(2-\beta,\beta)$-coalescent, our main results state that the corresponding Beta$(2-\beta,\beta)$-Fleming-Viot support is totally disconnected at any fixed positive time for $d(\beta-1)>4$ and it is totally disconnected uniformly in positive time for $d(\beta-1)>4+2\alpha$.

The celebrated lookdown representation, initially proposed by Donnelly and Kurtz [11, 12], is a powerful technique in the study of Fleming-Viot processes. Applying the lookdown representation, it is pointed out in Blath [8] that at any positive time the support of a Λ-Fleming-Viot process with d-dimensional Brownian spatial motion is always the whole space $\mathbb{R}^d$ provided the associated Λ-coalescent stays infinite. On the other hand, in Liu and Zhou [15, 16] this representation is applied to establish the compact support property for a class of Λ-Fleming-Viot processes with Brownian spatial motion whose associated Λ-coalescents come down from infinity. Hausdorff dimensions for supports of these Fleming-Viot processes are studied in [15, 16], which suggest that the faster the corresponding coalescent comes down from infinity, the smaller the dimension of the support is. A (one-sided) modulus of continuity for the ancestry process recovered from the lookdown representation is also found in [16] to describe the speed at which the support propagates locally in time.

Another interesting representation of motionless Λ-Fleming-Viot processes via stochastic flows of bridges is due to Bertoin and Le Gall. We refer to papers [3, 4, 5] for more details and its applications.

The proofs of this paper also rely on the above mentioned lookdown representation of Donnelly and Kurtz. Our approach, which is inspired by that for superBrownian motion, can be sketched as follows. Consider the lookdown particle representation of the Λ-Fleming-Viot process X. For any fixed current time $T>0$ and any $n>0$, by considering the ancestors of those particles present at previous times $T-n^{-1}$ and $T-n^{-\varepsilon}$ for small $0<\varepsilon<1/2$, respectively, we first group the particles at time T according to their respective ancestors at time $T-n^{-1}$ and then group those ancestors at time $T-n^{-1}$ together according to their respective ancestors at the earlier time $T-n^{-\varepsilon}$.

On one hand, in space $\mathbb{R}^d$ of higher dimension d any two ancestors at time $T-n^{-1}$ who belong to two different groups typically do not stay very close to each other since, by the lookdown representation, given positions of the ancestors at time $T-n^{-\varepsilon}$, positions of these two ancestors at time $T-n^{-1}$ are determined by two conditionally independent d-dimensional Brownian motions starting at positions of their respective ancestors at time $T-n^{-\varepsilon}$ and running up to time $n^{-\varepsilon}-n^{-1}$. For our purpose we only need to show that the pair of particles are more than a distance of $n^{-1/2}$ away from each other. But we want this to be true for all pairs of particles with distinct ancestors at time $T-n^{-\varepsilon}$.

On the other hand, a typical distance of a particle at time T from its respective ancestor at time $T-n^{-1}$ is of the order $n^{-1/2}$, and from its ancestor at time $T-n^{-\varepsilon}$ is of the order $n^{-\varepsilon/2}$ due to the modulus of continuity for the ancestry process obtained in

[16].

Therefore, if the associated Λ-coalescent comes down from infinity fast enough to allow us a control over the total numbers of ancestors at different previous times, and if the dimension d is high so that the d-dimensional Brownian spatial motion is transient enough, we can show that for large n and at time T the different groups of particles with different ancestors at time $T-n^{-\varepsilon}$ typically stay away from each other, and the maximal distance among particles at time T with the same ancestors at time $T-n^{-\varepsilon}$ are typically of the order $n^{-\varepsilon/2}$. Consequently each connected component of the support $\text{Supp}X(T)$, whenever exists, typically contains particles from the same ancestor at time $T-n^{-\varepsilon}$ and has a diameter at most of the order $n^{-\varepsilon/2}$. As a result, when dimension d is high enough, we can show the disconnectedness of support at time T via the Borel-Cantelli arguments.

To show the disconnectedness of support uniformly in time we need to, under a stronger condition on the rates of coalescence and for higher dimensions, carry out the above argument simultaneously for a collection of positive times and we need an estimate obtained in [16] on the number of ancestors present in a time interval.

In some sense the lookdown particle representation for the Fleming-Viot process plays the role of the cluster decomposition for superBrownian motion.

These ideas are also implemented to obtain results on the disjointness of independent Λ-Fleming-Viot supports.

The paper is structured as follows. After the introduction in Section 1, we first present the lookdown representation of Λ-Fleming-Viot processes with Brownian spatial motion and then present the associated Λ-coalescent. We later introduce the ancestry process and the result on its modulus of continuity. In Section 3 we prove the two main theorems. The first theorem concerns the disconnectedness of the Fleming-Viot support at a fixed positive time. The second theorem concerns the uniform (in time) disconnectedness of the support. Additional results on the disjointness of independent Fleming-Viot supports and ranges are also proved in this section.

2 The Donnelly-Kurtz lookdown representation and Λ-coalescent

The lookdown particle representation for Fleming-Viot process was first introduced by Donnelly and Kurtz [11]. A modified lookdown particle representation for Fleming-Viot process with general reproduction mechanism was later proposed in Donnelly and Kurtz [12]. Intuitively, in the lookdown model each particle is attached a level taking a value from $\{1, 2, \ldots\}$. The evolution of a particle at level n only depends on the evolution of a finite number of particles at lower levels. This projective property allows us to construct approximating particle systems, and their limit in the same probability space. Such coupled particle systems naturally result in a genealogy for the Fleming-Viot process, which turns out to be very useful in studying Fleming-Viot support properties.

Following Birkner and Blath [6] and Blath [8], we briefly introduce the modified lookdown representation of the Λ-Fleming-Viot process with underlying Brownian motion. Let $(X_1(t), X_2(t), X_3(t), \ldots)$, $t \geq 0$, be an $(\mathbb{R}^d)^\infty$-valued stochastic process with exchangeable initial values $(X_1(0), X_2(0), \ldots)$, where for any $i = 1, 2, \ldots$, $X_i(t)$ represents the spatial location of the particle at level i and at time t.

Let Λ be a finite measure on $[0, 1]$. The reproduction of the particle system consists of two kinds of birth events: the events of single birth that are determined by measure $\Lambda(\{0\})\delta_0$ and the events of multiple births that are determined by the measure Λ restricted to $(0, 1]$, which is denoted by Λ_0. The particle system undergoes lookdowns and spatial motions.

To describe the evolution of the system during events of single birth, let $\{\mathbf{N}_{ij}(t) :$

$1 \le i < j < \infty\}$ be independent Poisson processes with common rate $\Lambda(\{0\})$. At a jump time t of the process $\mathbf{N}_{ij}$, the particle at level j looks down at the particle at level i and assumes its location, and values of particles at levels above j are shifted accordingly (another way of looking at it is that the particle at level i gives birth to a new particle at level j and levels of the other particles are rearranged according to their pre-reproduction levels). More precisely, for any i, j and t with $\mathbf{N}_{ij}(t) - \mathbf{N}_{ij}(t-) = 1$, we have

$$X_k(t) = \begin{cases} X_k(t-), & \text{if } k < j, \\ X_i(t-), & \text{if } k = j, \\ X_{k-1}(t-), & \text{if } k > j. \end{cases}$$

For those events of multiple births we can construct an independent Poisson point process $\tilde{\mathbf{N}}$ on $\mathbb{R}^+ \times (0,1]$ with intensity measure $dt \otimes x^{-2}\Lambda_0\,(dx)$. For $[\infty] := \{1, 2, \ldots\}$, let $\{U_{ij},\, i, j \in [\infty]\}$ be i.i.d. random variables uniformly distributed on $[0,1]$. Write $\{(t_i, x_i)\}$ for the collection of all the jump points of $\tilde{\mathbf{N}}$. Put $[n] := \{1, \ldots, n\}$. For any $t \ge 0, n$ and $J \subset [n]$ with $|J| \ge 2$, define a counting process

$$\mathbf{N}_J^n(t) \equiv \sum_{i: t_i \le t} \prod_{j \in J} \mathbf{1}_{\{U_{ij} \le x_i\}} \prod_{j \in [n] \setminus J} \mathbf{1}_{\{U_{ij} > x_i\}}. \tag{2.1}$$

Then $\mathbf{N}_J^n(t)$ counts the number of the kind of birth events up to time t among the particles at levels $\{1, 2, \ldots, n\}$ such that exactly those at levels in J are involved. Intuitively, t_i represents the coalescing time and x_i stands for the proportion of particles that coalesce into one particle at time t_i. More precisely, all the particles at levels j with $U_{ij} \le x_i$ participate in the lookdown event. Those particles involved jump to the location of the particle at the lowest level involved, and the spatial locations of particles on the other levels, keeping their original order, are shifted accordingly (another way of looking at it is that the particle at the lowest level involved gives births to particles at all the other levels involved and the pre-reproduction levels are rearranged). If $t = t_i$ is the jump time and j is the lowest level involved, then

$$X_k(t) := \begin{cases} X_k(t-), \text{ for } k \le j, \\ X_j(t-), \text{ for } k > j \text{ with } U_{ik} \le x_i, \\ X_{k-J_t^k}(t-), \text{ otherwise}, \end{cases}$$

where $J_{t_i}^k := \#\{m < k : U_{im} \le x_i\} - 1$.

Between the jump times of the above mentioned Poisson or Poisson point processes, particles at different levels move independently according to Brownian motions in $\mathbb{R}^d$.

We assume that the above-mentioned lookdown representation is carried out in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration generated by the processes $(N_{ij}), \tilde{\mathbf{N}}$ and the Brownian motions of the particles in the lookdown representation.

For each $t > 0$, $X_1(t), X_2(t), \ldots$ are known to be exchangeable random variables in $\mathbb{R}^d$ so that the random measure

$$X(t) = \lim_{n\to\infty} X^{(n)}(t) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)}$$

exists almost surely by the de Finetti theorem and follows the probability law of the Λ-Fleming-Viot process with Brownian spatial motion. Further, we have that $X^{(n)}$ converges to X in the path space $D_{M_1(\mathbb{R}^d)}([0,\infty))$ equipped with the Skorohod topology, where $M_1(\mathbb{R}^d)$ denotes the space of probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence; see Theorem 3.2 of [12] and Section 5 of [8]. In the sequel we always write X for such a Λ-Fleming-Viot process with Brownian spatial motion.

The lookdown representation induces a Λ-coalescent of multiple collisions that we briefly introduce in the following. An ordered *partition* of $D \subset [\infty]$ is a countable collection $\pi = \{\pi_i, i = 1, 2, \dots\}$ of disjoint *blocks* such that $\cup_i \pi_i = D$ and $\min \pi_i < \min \pi_j$ for $i < j$. Denote by $\mathcal{P}_n$ the set of ordered partitions of $[n]$ and by $\mathcal{P}_\infty$ the set of ordered partitions of $[\infty]$. Write $\mathbf{0}_{[n]} \equiv \{\{1\}, \dots, \{n\}\}$ for the partition of $[n]$ consisting of singletons and $\mathbf{0}_{[\infty]}$ for the partition of $[\infty]$ consisting of singletons.

Kingman's coalescent, a special case of the Λ-coalescent, is a $\mathcal{P}_\infty$-valued time homogeneous Markov process such that all the different pairs of the existing blocks independently merge at the same rate. Pitman [20], Sagitov [22] and Donnelly and Kurtz [12] generalize the Kingman's coalescent to the Λ-*coalescent* which allows multiple collisions, i.e., more than two blocks could merge at a time. The Λ-coalescent is defined as a $\mathcal{P}_\infty$-valued Markov process $\Pi \equiv (\Pi(t))_{t\geq 0}$ such that for each $n \in [\infty]$, its restriction to $[n]$, $\Pi_n \equiv (\Pi_n(t))_{t\geq 0}$ is a $\mathcal{P}_n$-valued Markov process whose transition rates are described as follows: if there are currently b blocks in the partition, then each k-tuple of blocks ($2 \leq k \leq b$) independently merges to form a single block at rate

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\Lambda(\mathrm{d}x), \tag{2.2}$$

where Λ is a finite measure on $[0, 1]$, which we call the *coalescence measure*. Consequently, $(\lambda_{b,k})$ satisfies the following *consistency condition*

$$\lambda_{b,k} = \lambda_{b+1,k} + \lambda_{b+1,k+1}. \tag{2.3}$$

The Kingman coalescent is the Λ-coalescent with $\Lambda = \delta_0$, the delta measure at point 0. We call a Λ-coalescent Beta$(2-\beta, \beta)$-coalescent with parameter $\beta \in (0, 2)$ if the coalescence measure Λ on $[0, 1]$ is given by

$$\Lambda(\mathrm{d}x) = \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta}(1-x)^{\beta-1}\mathrm{d}x.$$

For $n = 2, 3, \dots$, denote by

$$\lambda_n = \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \tag{2.4}$$

the total coalescence rate starting with n blocks. From the consistency condition (2.3) it is clear that $(\lambda_n)_{n\geq 2}$ is an increasing sequence, i.e., $\lambda_n \leq \lambda_{n+1}$ for any $n \geq 2$. In addition, denote by

$$\gamma_n = \sum_{k=2}^{n} (k-1) \binom{n}{k} \lambda_{n,k}$$

the rate at which the number of blocks decreases.

Given any Λ-coalescent $\Pi \equiv (\Pi(t))_{t\geq 0}$ with $\Pi(0) = \mathbf{0}_{[\infty]}$, let $\#\Pi(t)$ be the number of blocks in the partition $\Pi(t)$. The Λ-coalescent Π *comes down from infinity* if

$$\mathbb{P}\{\#\Pi(t) < \infty\} = 1$$

for all $t > 0$ and it *stays infinite* if

$$\mathbb{P}\{\#\Pi(t) = \infty\} = 1$$

for all $t > 0$. It is shown by Schweinsberg [23] that if $\Lambda(\{1\}) = 0$, the Λ-coalescent comes down from infinity if and only if $\sum_{n=2}^{\infty} \gamma_n^{-1} < \infty$ and stays infinite if and only if $\sum_{n=2}^{\infty} \gamma_n^{-1} = \infty$.

Consequently, Kingman's coalescent comes down from infinity, and the Beta$(2-\beta, \beta)$-coalescent comes down from infinity if and only if $1 < \beta < 2$.

For $q > 0$ put

$$\psi_\Lambda(q) := \int_{[0,1]} (e^{-qx} - 1 + qx)\Lambda(\mathrm{d}x).$$

It is found in Bertoin and Le Gall [5] that the Λ-coalescent comes down from infinity if and only if

$$\int_a^\infty \frac{1}{\psi_\Lambda(q)}\mathrm{d}q < \infty$$

for some (and then for all) $a > 0$; see Section 4 of [5] and Section 2 of Berestycki et al. [2].

The genealogy of the particles recovered from the lookdown representation naturally leads to the partition-valued Λ-coalescent. Given $T > 0$, for any $0 \le s \le T$ and $i \in [\infty]$, let $L_i^T(s)$ denote the ancestor's level at time s of the particle with level i at time T. Then $X_{L_i^T(s)}(s-)$ represents that ancestor's location. Write $\left(\Pi^T(t)\right)_{0\le t\le T}$ for the $\mathcal{P}_\infty$-valued process such that i and j belong to the same block of $\Pi^T(t)$ if and only if $L_i^T(T-t) = L_j^T(T-t)$, i.e., i and j belong to the same block if and only if the two particles with levels i and j, respectively, at time T share a common ancestor at time $T-t$. It is known from the lookdown representation that the process $\left(\Pi^T(t)\right)_{0\le t\le T}$ follows the same probability law as that of a Λ-coalescent with coalescence measure Λ running up to time T. We thus call $\left(\Pi^T(t)\right)_{0\le t\le T}$ the associated Λ-coalescent.

For any $T > 0$ and $m > 0$ write

$$T_m \equiv T(m) := \inf\{0 \le t \le T : \#\Pi^T(t) \le m\}$$

with the convention $\inf\emptyset := T$. To prove the results in this paper we introduce two conditions concerning the rates of coalescence. The first condition is that there exists a constant $\alpha > 0$ such that

$$\limsup_{m\to\infty} m^\alpha \mathbb{E}T_m < \infty. \tag{2.5}$$

The second condition is that there exists a constant $\alpha > 0$ such that

$$\limsup_{m\to\infty} m^\alpha \sum_{i\ge m} \frac{1}{\lambda_i} < \infty. \tag{2.6}$$

It is known that condition (2.6) implies condition (2.5), which is sufficient for the coalescent to come down from infinity; see [16].

Condition (2.6) holds for Kingman's coalescent with $\alpha = 1$. For $\beta > 1$ it holds for the Beta$(2-\beta, \beta)$-coalescent with $\alpha = \beta - 1$. Further, for the class of Beta coalescents both conditions (2.5) and (2.6) are sufficient and necessary conditions for coming down from infinity; see [16] for more details. More generally, if the coalescence measure Λ allows a density function $f(x)$ for x near 0 that is *regularly varying* at 0 with index $-\gamma$ for a constant $\gamma \in (0,1)$, i.e. $f(x) = x^{-\gamma}L(x)$ for x close to 0 for some function $L(x)$ that is slowly varying at 0, then one can show that the condition (2.6) holds for any $\alpha < \gamma$; see a related estimate on the so called Λ-coalescent with the (c, ϵ, γ)-property in Lemma 4.13 of [15].

A useful observation from Lemma 3.1 of [15] is that given $t \in [0, T]$ and the ordered random partition

$$\Pi^T(t) := \left\{\pi_l^T(t) : l = 1, \ldots, \#\Pi^T(t)\right\},$$

where $\#\Pi^T(t)$ denotes the number of blocks in $\Pi^T(t)$ and the blocks are ordered by their least elements, we have

$$L_j^T(T-t) = l \quad \text{for any} \quad j \in \pi_l^T(t). \tag{2.7}$$

In the sequel we often write $\pi_i(t)$ for $\pi_i^T(t)$ for ease of notation.

For any $T > 0$, denote by

$$(X_{1,t}, X_{2,t}, X_{3,t}, \ldots)_{0\le t\le T}$$

a $C(\mathbb{R}^d)^\infty$-valued *ancestry process* with each coordinate $X_{i,t}$ defined by

$$X_{i,t}(s) := X_{L_i^t(s)}(s-) \quad \text{for} \quad 0 \le s \le t.$$

Intuitively, the process $X_{i,t}$, running backwards in time, keeps track of locations for all the ancestors of the particle with level i at the present time t.

For any $0 \le s < t$ let $H(s,t)$ be the maximal dislocation between the particles at time t and their respective ancestors at time s. More precisely, we have

$$\begin{aligned} H(s,t) &:= \max_{i\in[\infty]} |X_i(t) - X_{i,t}(s)| \\ &= \max_{i\in[\infty]} \left| X_i(t) - X_{L_i^t(s)}(s-) \right| \\ &= \max_{1\le l\le N(s,t)} \max_{j\in\pi_l(t-s)} |X_j(t) - X_l(s-)|, \end{aligned} \tag{2.8}$$

where

$$N(s,t) := \#\Pi^t(t-s),$$
$$\{\pi_l(t-s) : 1 \le l \le \#\Pi^t(t-s)\} = \Pi^t(t-s)$$

and we need property (2.7) for the last equation of (2.8).

A modulus of continuity for the ancestry process under condition (2.5) is found in Theorem 4.1 of [16], which states that for any $T > 0$ there exist a constant $C(d,\alpha)$ and a positive random variable $\theta(T,d,\alpha)$ such that $\mathbb{P}$-a.s.

$$H(s,t) \le C(d,\alpha)\sqrt{(t-s)\log(1/(t-s))} \tag{2.9}$$

for all $0 \le s < t \le T$ with $t - s \le \theta$.

The above modulus of continuity for the ancestry process leads to a modulus of continuity at a fixed time for the Λ-Fleming-Viot support process in Theorem 4.2 of [16] with the same modulus function. One can compare it with Theorem 1.3 of Dawson and Vinogradov [9], where a similar modulus of continuity is obtained for the support process of the superBrownain motion with a stable branching mechanism. Note that the modulus functions from both [16] and [9] are of the same order.

3 Main results

For any nonempty compact sets A and B in $\mathbb{R}^d$, let

$$\rho_1(A,B) := \sup_{x\in A} d(x,B) \wedge 1$$

and

$$\rho(A,B) := \rho_1(A,B) + \rho_1(B,A),$$

where $d(x,B)$ denotes the distance from x to B; i.e. ρ is the *Hausdorff metric* on the space of compact subsets of $\mathbb{R}^d$.

Until the end of this paper we always assume that the coalescence measure Λ for the associated Λ-coalescent has no atom at 1, i.e. $\Lambda(\{1\}) = 0$.

Write $\text{Supp}\,\mu$ for the closed support of measure μ. We first present a result on the disconnectedness of support at a fixed time for the Λ-Fleming-Viot process X with Brownian spatial motion in $\mathbb{R}^d$.

Theorem 3.1 (Disconnectedness at a fixed time)**.** *Suppose that condition (2.5) holds for the associated Λ-coalescent with α satisfying $d\alpha > 4$. Then for any $T > 0$, Supp $X(T)$ is totally disconnected $\mathbb{P}$-a.s.*

Proof of Theorem 3.1. Note that the associated Λ-coalescent Π^T comes down from infinity under condition (2.5). Fix time $T > 0$. For any $0 < \varepsilon < 1/2$, any $m > 1$ large enough such that $m^{-\varepsilon} < T$ and $i = 1, 2, \dots$ write

$$x_i \equiv x_i(m^{-\varepsilon}) := X_i((T - m^{-\epsilon})-).$$

Intuitively, by observation (2.7), $\{x_i, i = 1, \dots, N(T - m^{-\epsilon}, T)\}$ is the finite collection of all the positions of ancestral particles at time $T - m^{-\epsilon}$ for those particles at time T.

Recall that

$$\Pi^T(m^{-1}) = \left(\pi_1(m^{-1}), \dots, \pi_{N(T-m^{-1},T)}(m^{-1})\right)$$

and

$$\Pi^T(m^{-\epsilon}) = \left(\pi_1(m^{-\varepsilon}), \dots, \pi_{N(T-m^{-\epsilon},T)}(m^{-\varepsilon})\right).$$

For each $i = 1, 2, \dots, N(T - m^{-\epsilon}, T)$ write

$$A_i \equiv A_i\,(m) := \{j : \pi_j(m^{-1}) \subset \pi_i(m^{-\varepsilon})\}$$

and

$$x_{ij} \equiv x_{ij}(m) := X_j((T - m^{-1})-),\ j \in A_i.$$

Intuitively, because of property (2.7), $\{x_{ij}, j \in A_i\}$ stands for the finite collection of positions of the ancestors of $X(T)$ at time $T - m^{-1}$ who, at the same time, are descendants of the ancestor situated at position x_i at time $T - m^{-\varepsilon}$.

By condition (2.5),

$$\begin{aligned} \mathbb{P}\{N(T - m^{-1}, T) > m^{\frac{1+\varepsilon}{\alpha}}\} &= \mathbb{P}\{T(m^{\frac{1+\varepsilon}{\alpha}}) > m^{-1}\} \\ &\le m\mathbb{E}T(m^{\frac{1+\varepsilon}{\alpha}}) \\ &\le C(m^{-\frac{1+\varepsilon}{\alpha}})^{\alpha} m \\ &\le Cm^{-\varepsilon} \end{aligned} \tag{3.1}$$

for m large enough.

Given $X(T - m^{-\varepsilon})$, $N(T - m^{-\epsilon}, T) = k_\varepsilon \ge 2, N(T - m^{-1}, T) = k$ and values in $\mathcal{P}_{k_\varepsilon}$ and $\mathcal{P}_k$, respectively, of the random partitions

$$(\pi_1(m^{-\varepsilon}), \dots, \pi_k(m^{-\varepsilon})) = (\pi_1^\varepsilon, \dots, \pi_{k_\varepsilon}^\varepsilon)$$

and

$$(\pi_1(m^{-1}), \dots, \pi_k(m^{-1})) = (\pi_1, \dots, \pi_k),$$

for any $i \neq i'$ with $i \vee i' \le k_\varepsilon$, by the lookdown representation x_i and $x_{i'}$ are independent samples from the probability measure $X(T - m^{-\varepsilon})$; in addition, for any $j \in A_i$ and $j' \in A_{i'}$, the points x_{ij} and $x_{i'j'}$ are connected to x_i and $x_{i'}$, respectively, by independent

Brownian motion paths. Using an estimate on d-dimensional Brownian motion, for the following event specifying the configurations of the random partitions

$$\begin{aligned}&D(k_\varepsilon, k, (\pi_l^\varepsilon), (\pi_l))\\ &\quad := \left\{N(T-m^{-\epsilon}, T) = k_\varepsilon, N(T-m^{-1}, T) = k, (\pi_l(m^{-\varepsilon})) = (\pi_l^\varepsilon), (\pi_l(m^{-1})) = (\pi_l)\right\},\end{aligned}$$

we have

$$\begin{aligned}
&\mathbb{P}\left\{D(k_\varepsilon, k, (\pi_l^\varepsilon), (\pi_l)), |x_{ij} - x_{i'j'}| \le m^{-\frac{1}{2}+\varepsilon}\right\}\\
&= \int \mathbb{P}\{D(k_\varepsilon, k, (\pi_l^\varepsilon), (\pi_l)), X_{T-m^{-\varepsilon}} \in \mathrm{d}(\mu)\} \int \mu(\mathrm{d}x_i) \int \mu(\mathrm{d}x_{i'})\\
&\qquad \times \mathbb{P}\left\{\left|W_{x_i}(m^{-\varepsilon} - m^{-1}) - W_{x_{i'}}(m^{-\varepsilon} - m^{-1})\right| \le m^{-\frac{1}{2}+\varepsilon}\right\}\\
&= \mathbb{P}\{D(k_\varepsilon, k, (\pi_l^\varepsilon), (\pi_l))\} \int \mathbb{P}\{X(T-m^{-\varepsilon}) \in \mathrm{d}(\mu)\} \int \mu(\mathrm{d}x_i) \int \mu(\mathrm{d}x_{i'})\\
&\qquad \times \int_{|y| \le m^{-\frac{1}{2}+\varepsilon}} \left(4\pi(m^{-\varepsilon} - m^{-1})\right)^{-d/2} e^{-\frac{|y-(x_i - x_{i'})|^2}{4(m^{-\varepsilon}-m^{-1})}} \mathrm{d}y\\
&\le C(d) m^{\frac{d\varepsilon}{2}} m^{d(-\frac{1}{2}+\varepsilon)} \mathbb{P}\{D(k_\varepsilon, k, (\pi_l^\varepsilon), (\pi_l))\},
\end{aligned} \tag{3.2}$$

where W_{x_i} and $W_{x_{i'}}$ denote two independent d-dimensional Brownian motions with initial values x_i and $x_{i'}$, respectively, and we have used the independence between the random measure $X(T-m^{-\varepsilon})$ and the collection of random variables together with random partitions $\{N(T-m^{-\epsilon}, T), N(T-m^{-1}, T), (\pi_i(m^{-\varepsilon})), (\pi_i(m^{-1}))\}$ that are determined by the Poisson processes $(\mathbf{N}_{ij})$ and the Poisson point process $\tilde{\mathbf{N}}$ over the time interval $[T-m^{-\varepsilon}, T)$ in the lookdown representation.

Therefore, writing

$$D_m := \bigcup_{i,i' \le N(T-m^{-\epsilon}, T), i \ne i'} \;\bigcup_{j \in A_i, j' \in A_{i'}} \left\{|x_{ij}(m) - x_{i'j'}(m)| \le m^{-\frac{1}{2}+\varepsilon}\right\}$$

for the event that there exist two ancestors from different groups at time $T-m^{-1}$ that are within a distance of $m^{-\frac{1}{2}+\varepsilon}$ from each other, we have by (3.1) and (3.2)

$$\begin{aligned}
\mathbb{P}\{D_m\} &\le \mathbb{P}\Bigg\{\bigcup_{k_\varepsilon \ge 2} \bigcup_{k \le m^{\frac{1+\varepsilon}{\alpha}}} \bigcup_{(\pi_l(m^{-\varepsilon}))} \bigcup_{(\pi_l(m^{-1}))} D(k_\varepsilon, k, (\pi_l^\varepsilon), (\pi_l))\\
&\qquad\qquad \bigcap \left\{\bigcup_{i,i' \le k_\varepsilon, i \ne i'} \;\bigcup_{j \in A_i, j' \in A_{i'}} \left\{|x_{ij}(m) - x_{i'j'}(m)| \le m^{-\frac{1}{2}+\varepsilon}\right\}\right\}\Bigg\}\\
&\qquad + \mathbb{P}\{N(T-m^{-1}, T) > m^{\frac{1+\varepsilon}{\alpha}}\}\\
&\le \sum_{k_\varepsilon \ge 2} \sum_{k \le m^{\frac{1+\varepsilon}{\alpha}}} \sum_{(\pi_l(m^{-\varepsilon}))} \sum_{(\pi_l(m^{-1}))} \sum_{i,i' \le k_\varepsilon, i \ne i'} \sum_{j \in A_i, j' \in A_{i'}}\\
&\qquad C(d) m^{\frac{d\varepsilon}{2}} m^{d(-\frac{1}{2}+\varepsilon)} \mathbb{P}\{D(k_\varepsilon, k, (\pi_l^\varepsilon), (\pi_l))\}\\
&\qquad + Cm^{-\varepsilon}\\
&\le m^{\frac{2(1+\varepsilon)}{\alpha}} C(d) m^{\frac{d\varepsilon}{2}} m^{d(-\frac{1}{2}+\varepsilon)} \sum_{k_\varepsilon \ge 2} \sum_{k \le m^{\frac{1+\varepsilon}{\alpha}}} \sum_{(\pi_l(m^{-\varepsilon}))} \sum_{(\pi_l(m^{-1}))} \mathbb{P}\{D(k_\varepsilon, k, (\pi_l^\varepsilon), (\pi_l))\}\\
&\qquad + Cm^{-\varepsilon}\\
&\le C(d) m^{\frac{2(1+\varepsilon)}{\alpha}} m^{\frac{d\varepsilon}{2}} m^{d(-\frac{1}{2}+\varepsilon)} + Cm^{-\varepsilon}\\
&\le C(d) m^{-\varepsilon}
\end{aligned}$$

for $d > 4/\alpha$ and for ε small enough. Applying the Borel-Cantelli lemma to a sequence of events $\{D_{n^{2/\varepsilon}}, n = 1, 2, \ldots\}$ we have $\mathbb{P}$-a.s. for n big enough,

$$|x_{ij}(n^{2/\varepsilon}) - x_{i'j'}(n^{2/\varepsilon})| > (n^{2/\varepsilon})^{-\frac{1}{2}+\varepsilon} \tag{3.3}$$

for all $i, i' \le N(T - n^{-2}, T)$ with $i \neq i'$ and all $j \in A_i(n^{2/\varepsilon}), j' \in A_{i'}(n^{2/\varepsilon})$.

On the other hand, by the modulus of continuity (2.9) for the ancestry process, $\mathbb{P}$-a.s. for m big enough, for any i and for $j \in A_i(m)$, we have

$$|X_k(T) - x_{ij}(m)| < m^{-\frac{1}{2}+\frac{\varepsilon}{2}} \text{ for all } k \in \pi_j(m^{-1}). \tag{3.4}$$

Put

$$B_i(n) := \left\{ X_k(T) : k \in \pi_i(n^{-2}) = \bigcup_{j \in A_i(n^{2/\varepsilon})} \pi_j(n^{-2/\varepsilon}) \right\}$$

for the set of positions of all the descendants at time T from the i-th particle at time $T - n^{-2}$. It then follows from the previous arguments for (3.3) and (3.4) that $\mathbb{P}$-a.s. for all $i, i' \le N(T - n^{-2}, T)$ with $i \neq i'$, we have

$$\rho(B_i(n), B_{i'}(n)) > (n^{2/\varepsilon})^{-\frac{1}{2}+\frac{\varepsilon}{2}}$$

for all n large enough. Consequently, each connected component of Supp$X(T)$ can at most result from the offspring of the same ancestor at time $T - n^{-2}$ for a large n.

By the modulus of continuity (2.9) again, for large n the diameter of the set $B_i(n)$ is uniformly bounded from above by

$$C(d, \alpha)\sqrt{n^{-2} \ln n^2} = C(d, \alpha) n^{-1} \sqrt{\ln n},$$

which converges to 0 as $n \to \infty$. Therefore, Supp $X(T)$ is totally disconnected $\mathbb{P}$-a.s. □

The arguments in the proof for Theorem 3.1 can be applied to study the intersection of two independent Fleming-Viot supports. Let X and Y be independent Λ_X-Fleming-Viot process and Λ_Y-Fleming-Viot process, respectively, both with Brownian spatial motion.

Proposition 3.2. *Suppose that condition (2.5) holds with constants α_X and α_Y, respectively, for the associated Λ_X-coalescent and Λ_Y-coalescent. If $d > \frac{2}{\alpha_X} + \frac{2}{\alpha_Y}$, we then have for any $T > 0$,*

$$\textit{Supp}\, X(T) \cap \textit{Supp}\, Y(T) = \emptyset \quad \mathbb{P}\text{-}a.s.$$

Proof of Proposition 3.2. Similar to the proof of Theorem 3.1, for $n > 0$ define from their respective lookdown representations

$$N_X := N_X(T - 1/n, T),\ N_Y := N_Y(T - 1/n, T)$$

and

$$x_i := X_i(T - 1/n),\ i = 1, \ldots, N_X,$$

$$y_i := Y_i(T - 1/n),\ i = 1, \ldots, N_Y.$$

Given $\varepsilon > 0$, for n large enough we have

$$\mathbb{P}\left\{N_X > n^{\frac{1+\varepsilon}{\alpha_X}}\right\} \vee \mathbb{P}\left\{N_Y > n^{\frac{1+\varepsilon}{\alpha_Y}}\right\} \le C n^{-\varepsilon}.$$

Since X_i and Y_i are independent Brownian motions with initial distributions $X_0 \equiv X(0)$ and $Y_0 \equiv Y(0)$, respectively, we have

$$\begin{aligned}
&\mathbb{P}\left\{\cup_{i=1}^{N_X} \cup_{j=1}^{N_Y} \{|x_i - y_j| < n^{-1/2+\varepsilon}\}\right\} \\
&\leq n^{\frac{1+\varepsilon}{\alpha_X}} n^{\frac{1+\varepsilon}{\alpha_Y}} \mathbb{E}\int_{|x-y|<n^{-1/2+\varepsilon}} \int X_0(\mathrm{d}x_0) P_{T-1/n}(x_0, x)\mathrm{d}x \int Y_0(\mathrm{d}y_0) P_{T-1/n}(y_0, y)\mathrm{d}y + Cn^{-\varepsilon} \\
&\leq n^{\frac{1+\varepsilon}{\alpha_X}+\frac{1+\varepsilon}{\alpha_Y}} \mathbb{E}\int Y_0(\mathrm{d}y_0) \sup_x \int_{|x-y|<n^{-1/2+\varepsilon}} P_{T-1/n}(y_0, y)\mathrm{d}y + Cn^{-\varepsilon} \\
&\leq n^{\frac{1+\varepsilon}{\alpha_X}+\frac{1+\varepsilon}{\alpha_Y}} C(d, T) n^{-d(1/2-\varepsilon)} + Cn^{-\varepsilon},
\end{aligned}$$

where $P_t(x, y)$ denotes the Brownian transition function. Then the desired result follows from the modulus of continuity and the Borel-Cantelli lemma. □

Increasing the dimension allows us to show a result on the disconnectedness uniform in time for the Λ-Fleming-Viot support.

Theorem 3.3 (Disconnectedness uniform in time)**.** *Suppose that condition (2.6) holds for the associated* Λ*-coalescent with* α *satisfying* $d\alpha > 4 + 2\alpha$*. Then* $\mathbb{P}$*-a.s. Supp* $X(t)$ *is totally disconnected for all* $t > 0$*.*

Proof of Theorem 3.3. We proceed with the argument of proof for Theorem 3.1 uniformly in time. For any $0 < \delta < 1$ and any positive integer n, let

$$I_n := \{k : \delta - 2^{-n+1} \leq k2^{-n} \leq \delta^{-1}\}.$$

Recall that $N(\frac{k-1}{2^n}, \frac{k}{2^n})$ denotes the number of ancestors at dyadic time $(k-1)/2^n$ for the particles at time $k/2^n$. By Lemma 5.8 of [16], T_n is stochastically dominated by a sum of independent exponential random variables with rate (λ_i) where λ_i is specified in (2.4). It then follows from condition (2.6) and the proof for Lemma 5.9 of [16] that

$$\mathbb{P}\left\{\max_{k\in I_n} N(\frac{k-1}{2^n}, \frac{k}{2^n}) \geq 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right\} \leq C(\alpha, \delta)(\frac{2}{e})^n. \tag{3.5}$$

Further, for small $0 < \varepsilon < 1$ let

$$I_n^\varepsilon := \{k : \delta - 2^{-\varepsilon n} \leq k2^{-\varepsilon n} \leq \delta^{-1}\}.$$

Similar to the proof of Theorem 3.1, for any $k_\varepsilon \in I_n^\varepsilon$ and integer k satisfying $k_\varepsilon 2^{-\varepsilon n} \leq k2^{-n} < (k_\varepsilon + 1)2^{-\varepsilon n}$, write

$$x_i^{k_\varepsilon} := X_i((k_\varepsilon - 1)/2^{\varepsilon n}-),\ i = 1, \dots, N(\frac{k_\varepsilon - 1}{2^{\varepsilon n}}, \frac{k+1}{2^n}).$$

Then by (2.7), $\{x_i^{k_\varepsilon}\}$ is the collection of locations of all those ancestors at time $(k_\varepsilon - 1)/2^{\varepsilon n}$ of the particles at time $(k+1)/2^n$. In addition, at the risk of abusing notation, in the present proof we also write

$$\Pi^{\frac{k+1}{2^n}}\left(\frac{k+1}{2^n} - \frac{k_\varepsilon - 1}{2^{\varepsilon n}}\right) := \left(\pi_1^{\varepsilon,k}, \dots, \pi^{\varepsilon,k}_{N(\frac{k_\varepsilon-1}{2^{\varepsilon n}}, \frac{k+1}{2^n})}\right),$$

$$\Pi^{\frac{k+1}{2^n}}(1/2^n) := \left(\pi_1^k, \dots, \pi^k_{N(\frac{k}{2^n}, \frac{k+1}{2^n})}\right)$$

and

$$A_i^k := \{j : \pi_j^k \subset \pi_i^{\varepsilon,k}\},\ i = 1, \dots, N(\frac{k_\varepsilon - 1}{2^{\varepsilon n}}, \frac{k+1}{2^n}).$$

For each $i = 1, \dots, N(\frac{k_\varepsilon - 1}{2^{\varepsilon n}}, \frac{k+1}{2^n})$ write

$$x_{ij}^k := X_j(k/2^n-),\ j \in A_i^k.$$

Then by (2.7), $\{x_{ij}^k, j \in A_i^k\}$ is the collection of locations of particles at time $k/2^n$ that are both the offspring of the ith ancestor at time $(k_\varepsilon - 1)/2^{\varepsilon n}$ and the ancestors of some particles at the later time $(k+1)/2^n$.

For any $k_\varepsilon/2^{\varepsilon n} \le k/2^n < (k_\varepsilon + 1)/2^{\varepsilon n}$, given the probability measure $X((k_\varepsilon - 1)/2^{\varepsilon n})$, the random variables $N(\frac{k_\varepsilon - 1}{2^{\varepsilon n}}, \frac{k+1}{2^n})$ and $N(\frac{k}{2^n}, \frac{k+1}{2^n})$, and the random partitions

$$\Pi^{\frac{k+1}{2^n}}\left(\frac{k+1}{2^n} - \frac{k_\varepsilon - 1}{2^{\varepsilon n}}\right)$$

and $\Pi^{\frac{k+1}{2^n}}(1/2^n)$, by the lookdown representation again for $i \neq i'$ and $j \in A_i^k, j' \in A_{i'}^k$, the values of x_{ij}^k and $x_{i'j'}^k$ are obtained by two independent d-dimensional Brownian motions starting at $x_i^{k_\varepsilon}$ and $x_{i'}^{k_\varepsilon}$, respectively, and evaluated at time $k/2^n - (k_\varepsilon - 1)/2^{\varepsilon n}$, where $x_i^{k_\varepsilon}$ and $x_{i'}^{k_\varepsilon}$ are chosen independently according to the probability measure $X((k_\varepsilon - 1)/2^{\varepsilon n})$ on $\mathbb{R}^d$. By the above description, similar to (3.2) we have

$$\begin{aligned}
&\mathbb{P}\left\{\bigcup_{k_\varepsilon \in I_n^\varepsilon}\ \bigcup_{k_\varepsilon 2^{(1-\varepsilon)n} \le k < (k_\varepsilon+1)2^{(1-\varepsilon)n}}\ \bigcup_{i \neq i'}\bigcup_{j,j'}\{|x_{ij}^k - x_{i'j'}^k| \le 2^{n(-\frac{1}{2}+\varepsilon)}\}\right\} \\
&\le \sum_{k_\varepsilon \in I_n^\varepsilon}\ \sum_{k_\varepsilon 2^{(1-\varepsilon)n} \le k < (k_\varepsilon+1)2^{(1-\varepsilon)n}} \mathbb{P}\left\{\bigcup_{i \neq i'}\ \bigcup_{j,j' < 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}}\{|x_{ij}^k - x_{i'j'}^k| \le 2^{n(-\frac{1}{2}+\varepsilon)}\}\right\} \\
&\qquad + \mathbb{P}\left\{\max_{k \in I_n} N(\frac{k-1}{2^n}, \frac{k}{2^n}) \ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right\} \\
&\le \sum_{k_\varepsilon \in I_n^\varepsilon}\ \sum_{k_\varepsilon 2^{(1-\varepsilon)n} \le k < (k_\varepsilon+1)2^{(1-\varepsilon)n}} C(d,\alpha)\left(2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right)^2 (2^{\varepsilon n})^{\frac{d}{2}}\left(2^{n(-\frac{1}{2}+\varepsilon)}\right)^d + C(\alpha,\delta)(\frac{2}{e})^n \\
&\le C(d,\alpha,\delta)2^n \times \left(2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right)^2 2^{d(-\frac{1}{2}+\frac{3\varepsilon}{2})n} + C(\alpha,\delta)(\frac{2}{e})^n,
\end{aligned} \tag{3.6}$$

where in the above expressions $i, i' = 1, \dots, N(\frac{k_\varepsilon - 1}{2^{\varepsilon n}}, \frac{k+1}{2^n})$ and the union is taken for $j \in A_i^k$ and $j' \in A_{i'}^k$.

Clearly, for $d > 2 + 4/\alpha$ and ε small enough, the right hand side of inequality (3.6) is summable in n. By the Borel-Cantelli lemma, $\mathbb{P}$-a.s. for all n large enough we have

$$|x_{ij}^k - x_{i'j'}^k| > 2^{n(-\frac{1}{2}+\varepsilon)} \tag{3.7}$$

for all $k_\varepsilon \in I_n^\varepsilon$, all k with $k_\varepsilon 2^{(1-\varepsilon)n} \le k < (k_\varepsilon+1)2^{(1-\varepsilon)n}$, all $i, i' = 1, \dots, N(\frac{k_\varepsilon - 1}{2^{\varepsilon n}}, \frac{k+1}{2^n})$ with $i \neq i'$, and all $j \in A_i^k, j' \in A_{i'}^k$.

Given $t \in (\delta, \delta^{-1})$, for n large enough there exist $k \in I_n$ and $k_\varepsilon \in I_n^\varepsilon$ such that

$$t - 1/2^n \in [k/2^n, (k+1)/2^n) \subset [k_\varepsilon/2^{\varepsilon n}, (k_\varepsilon + 1)/2^{\varepsilon n}). \tag{3.8}$$

For any $m \in \pi_i^t(t - (k_\varepsilon - 1)/2^{\varepsilon n})$ and $m' \in \pi_{i'}^t(t - (k_\varepsilon - 1)/2^{\varepsilon n})$ with $i \neq i'$, observe that

$$N((k_\varepsilon - 1)/2^{\varepsilon n}, t) \le N((k_\varepsilon - 1)/2^{\varepsilon n}, (k+1)/2^n),$$

$$X_{m,t}(k/2^n) := X_{L_m^t(k/2^n)}(k/2^n-) \in \{x_{ij}^k, j \in A_i^k\}$$

and

$$X_{m',t}(k/2^n) = X_{L_{m'}^t(k/2^n)}(k/2^n-) \in \{x_{i'j}^k, j \in A_{i'}^k\}.$$

It follows from estimate (3.7) that $\mathbb{P}$-a.s. for n large enough we have

$$\begin{aligned}
&\sup_{t\in(\delta,\delta^{-1})}|X_m(t)-X_{m'}(t)| \\
&\geq \sup_{t\in(\delta,\delta^{-1})}|X_{m,t}(k/2^n)-X_{m',t}(k/2^n)| \\
&\quad -\sup_{t\in(\delta,\delta^{-1})}|X_m(t)-X_{m,t}(k/2^n)| - \sup_{t\in(\delta,\delta^{-1})}|X_{m'}(t)-X_{m',t}(k/2^n)| \\
&> 2^{n(-\frac{1}{2}+\frac{\varepsilon}{2})},
\end{aligned}$$

where we have used the fact that

$$\sup_{t\in(\delta,\delta^{-1})}(|X_m(t)-X_{m,t}(k/2^n)|\vee|X_{m'}(t)-X_{m',t}(k/2^n)|) < Cn2^{-n}$$

due to the modulus of continuity (2.9) for the ancestry process. Therefore, for all $t \in (\delta,\delta^{-1})$ satisfying (3.8), writing

$$B_i(t) := \left\{X_m(t) : m \in \pi_i^t(t-(k_\varepsilon-1)/2^{\varepsilon n})\right\},\ i = 1,\dots N((k_\varepsilon-1)/2^{\varepsilon n},t),$$

we have $\mathbb{P}$ a.s.

$$\rho(B_i(t),B_{i'}(t)) > 2^{n(-\frac{1}{2}+\frac{\varepsilon}{2})}$$

for all $i,i' = 1,\dots,N((k_\varepsilon-1)/2^{\varepsilon n},t)$ and $i\neq i'$.

Applying the modulus of continuity again, the diameter of $B_i(t)$ converges to 0 as $n\to\infty$ uniformly for all $t\in(\delta,\delta^{-1})$ and all i. So, similar to the proof for Theorem 3.1 one can conclude that $\mathbb{P}$-a.s. for all $t\in(\delta,\delta^{-1})$, Supp $X(t)$ cannot contain any connected component with positive diameter.

Letting $\delta\to 0+$, the desired result follows. □

The speed of coming down from infinity for the Λ-coalescent is discussed in [2]. It is shown that for any Λ-coalescent Π that comes down from infinity, there exists a deterministic speed function $\nu:(0,\infty)\to(0,\infty)$ such that $\#\Pi(t)/\nu(t)\to 1$ as $t\to 0$ both almost surely and in L^p for $p\geq 1$. The speed of coming down from infinity is certainly relevant to the estimate (3.5). Since (3.5) concerns the probability that the number of blocks is off from the speed function, the results of [2] do not seem to be directly applicable in showing (3.5). Perhaps the martingale arguments in [2] could be a useful alternative in proving estimate (3.5).

Corollary 3.4. *For the d-dimensional Beta$(2-\beta,\beta)$-Fleming-Viot process X with Brownian spatial motion, if $d(\beta-1)>4$, then for any $T>0$, Supp $X(T)$ is totally disconnected $\mathbb{P}$-a.s.; if $d(\beta-1)>4+2(\beta-1)$, then $\mathbb{P}$-a.s. Supp $X(t)$ is totally disconnected for all $t>0$.*

More generally, for the d-dimensional Λ-Fleming-Viot process X with Brownian spatial motion whose coalescence measure Λ has a density that is regularly varying at 0 with index $-\gamma$, if $d\gamma>4$, then for any $T>0$ Supp $X(T)$ is totally disconnected $\mathbb{P}$-a.s.; if $d\gamma>4+2\gamma$, then $\mathbb{P}$-a.s. Supp $X(t)$ is totally disconnected for all $t>0$.

We now remark on the connection between the Perkins disintegration theorem and our results. It is well known that a superBrownian motion, when re-normalized and conditioned on non-extinction, can be transformed into a Fleming-Viot like process. We refer to Konno and Shiga [14], Etheridge and March [13] and Perkins [17] for such relations between the superBrownian motion with binary branching and the classical Fleming-Viot process associated with Kingman coalescent, and Birkner et al [7] for a similar relationship between non-spacial superBrownian motion with stable branching and non-spacial Λ-Fleming-Viot process associated with Beta coalescent. Also see Section II.6 of Ruscher [21] for more detailed discussions. But such a transform typically results in a Fleming-Viot process with either time-changed reproduction mechanism or time inhomogeneous spatial motion, which have not been well studied before. There is still a lot of work to be done in order to make use of this relationship to establish the Fleming-Viot support properties from those known support properties for the corresponding superBrownian motion.

Further, notice that a continuous-state $(1+\alpha)$-stable branching process is associated to a motionless and time-changed Beta$(2-\beta,\beta)$-Fleming-Viot process with $\beta=1+\alpha$ via a non-spatial Perkins' disintegration type result in [7]. In this sense, the condition of Theorem 3.1 for the Beta$(2-\beta,\beta)$-Fleming-Viot process with Brownian spatial motion exactly corresponds to the condition of Theorem 2.4 of [10] for the superBrownian motion with $(1+\alpha)$-stable branching mechanism.

We do not expect the results of Theorems 3.1 and 3.3 in this paper to be sharp. Similar to the superBrownian motions, the disconnectedness of the Λ-Fleming-Viot supports remains an open problem in low dimensions except dimension one. For the d-dimensional Λ-Fleming-Viot process with Λ being the Kingman coalescent, we conjecture that its support is totally disconnected at any fixed positive time for $d\geq 2$.

The next result concerns the uniform disjointness of independent Λ_X-Fleming-Viot process X and Λ_Y-Fleming-Viot process Y both with Brownian spatial motion in $\mathbb{R}^d$. Its proof combines the ideas from those of Theorem 3.2 and Theorem 3.3.

Proposition 3.5. *Suppose that the Fleming-Viot processes X and Y are independent and condition (2.6) holds with constants α_X and α_Y, respectively, for the associated Λ_X-coalescent and Λ_Y-coalescent. If $d>2+\frac{2}{\alpha_X}+\frac{2}{\alpha_Y}$, we have $\mathbb{P}$-a.s.*

$$\text{Supp}\, X(t)\cap \text{Supp}\, Y(t)=\emptyset \text{ for all } t>0.$$

Proof of Proposition 3.5. For any $0<\delta<1$, by first estimating

$$\mathbb{P}\left\{\bigcup_{\delta 2^n-2<k<\delta^{-1}2^n}\ \bigcup_{i=1}^{N_X(\frac{k}{2^n},\frac{k+1}{2^n})}\ \bigcup_{j=1}^{N_Y(\frac{k}{2^n},\frac{k+1}{2^n})}\{|X_i(k/2^n)-Y_j(k/2^n)|<2^{n(-1/2+\varepsilon)}\}\right\}$$

similarly to Proposition 3.2 and then applying the Borel-Cantelli lemma and the modulus of continuity (2.9) for the ancestry process, we can show that uniformly in $t\in(\delta,\delta^{-1})$ and for any i,j,

$$|X_i(t)-Y_j(t)|>2^{n(-1/2+\varepsilon/2)}$$

for n large enough. Then we finish the proof by letting $\delta\to 0$. □

Write

$$\mathcal{R}_X(t_1,t_2):=\overline{\bigcup_{t_1<s<t_2}\text{Supp}\, X(s)},\ \ 0\leq t_1<t_2,$$

for the *range process* of X. We can define the range process $\mathcal{R}_Y(t_1,t_2)$ similarly for the process Y.

Proposition 3.6. *If the Fleming-Viot processes X and Y are independent and condition (2.6) holds with constants α_X and α_Y, respectively, for the associated Λ_X-coalescent and Λ_Y-coalescent. If $d>4+2/\alpha_X+2/\alpha_Y$, we have*

$$\mathcal{R}_X(0,\infty)\cap\mathcal{R}_Y(0,\infty)=\emptyset\ \mathbb{P}\text{-}a.s.$$

Proof of Proposition 3.6. We only need to show that for any $0<\delta<1$,

$$\mathcal{R}_X(\delta,\delta^{-1})\cap\mathcal{R}_Y(\delta,\delta^{-1})=\emptyset\ \ \mathbb{P}\text{-}a.s.$$

To this end, one needs to estimate the probability

$$\mathbb{P}\left\{\bigcup_{\delta 2^n-2<k,k'<\delta^{-1}2^n}\ \bigcup_{i=1}^{N_X(\frac{k}{2^n},\frac{k+1}{2^n})}\ \bigcup_{j=1}^{N_Y(\frac{k'}{2^n},\frac{k'+1}{2^n})}\{|X_i(k/2^n)-Y_j(k'/2^n)|<2^{n(-1/2+\varepsilon)}\}\right\}$$

using estimates on Λ-coalescents and modulus of continuity for the ancestry processes. We leave the details to interested readers. □

We omit the proof for the following result.

Corollary 3.7. *For independent d-dimensional Beta$(2-\beta_X,\beta_X)$-Fleming-Viot process X and Beta$(2-\beta_Y,\beta_Y)$-Fleming-Viot process Y both with Brownian spatial motion, if $d>\frac{2}{\beta_X-1}+\frac{2}{\beta_Y-1}$, then for any $T>0$*

$$\textit{Supp}\,X(T)\cap\textit{Supp}\,Y(T)=\emptyset\ \ \mathbb{P}\text{-}a.s.;$$

if $d>2+\frac{2}{\beta_X-1}+\frac{2}{\beta_Y-1}$, then $\mathbb{P}$-a.s.

$$\textit{Supp}\,X(t)\cap\textit{Supp}\,Y(t)=\emptyset\ \ \textit{for all } t>0;$$

if $d>4+\frac{2}{\beta_X-1}+\frac{2}{\beta_Y-1}$, then

$$\mathcal{R}_X(0,\infty)\cap\mathcal{R}_Y(0,\infty)=\emptyset\ \ \mathbb{P}\text{-}a.s.$$

More generally, for two independent d-dimensional Λ-Fleming-Viot processes X and Y with Brownian spatial motion whose coalescence measures have densities that are regularly varying at 0 with indices $-\gamma_X$ and $-\gamma_Y$, respectively, if $d>\frac{2}{\gamma_X}+\frac{2}{\gamma_Y}$, then for any $T>0$

$$\textit{Supp}\,X(T)\cap\textit{Supp}\,Y(T)=\emptyset\ \ \mathbb{P}\text{-}a.s.;$$

if $d>2+\frac{2}{\gamma_X}+\frac{2}{\gamma_Y}$, then

$$\textit{Supp}\,X(t)\cap\textit{Supp}\,Y(t)=\emptyset\ \ \textit{for all } t>0;$$

if $d>4+\frac{2}{\gamma_X}+\frac{2}{\gamma_Y}$, then

$$\mathcal{R}_X(0,\infty)\cap\mathcal{R}_Y(0,\infty)=\emptyset\ \ \mathbb{P}\text{-}a.s.$$

References

[1] Abraham, R.: On the connected components of the support of super Brownian motion and of its exit measure. *Stochastic Processes and their Applications* **60**, (1995), 227–245. MR-1376802

[2] Berestycki, J., Berestycki, N. and Limic, V.: The Λ-coalescent speed of coming down from infinity. *Annals of Probability* **38**, (2010), 207–233. MR-2599198

[3] Bertoin, J. and Le Gall, J.-F.: Stochastic flows associated to coalescent processes. *Probability Theory and Related Fields* **126**, (2003), 261–288. MR-1990057

[4] Bertoin, J. and Le Gall, J.-F.: Stochastic flows associated to coalescent processes II: Stochastic differential equations. *Annales de I'Institut Henri Poincaré (B), Probabilités et Statistiques* **41**, (2005), 307–333. MR-2139022

[5] Bertoin, J. and Le Gall, J.-F.: Stochastic flows associated to coalescent processes. III. Limit theorems. *Illinois Journal of Mathematics* **50**, (2006), 147–181. MR-2247827

[6] Birkner, M. and Blath, J.: Measure-valued diffusions, general coalescents and population genetic inference, Trends in Stochastic Analysis, LMS 353, *Cambridge University Press*, 2009, 329–363. MR-2562160

[7] Birkner, M., Blath, J., Capaldo, M., Etheridge, A., Mohle, M., Schweinsberg, J. and Wakolbinger, A.: Alpha-stable branching and Beta coalescents. *Electronic Journal of Probability* **10**, (2005), 303–325. MR-2120246

[8] Blath, J.: Measure-valued Process, Self-similarity and Flickering Random Measure. *Progress in Probability* **61**, (2009), 175–196. MR-2762677

[9] Dawson, D. and Vinogradov, V.: Almost-sure path properties of $(2, d, \alpha)$-superprocesses. *Stochastic Processes and their Applications* **51**, (1994), 221–258. MR-1288290

[10] Delmas, J.-F.: Path properties of superprocesses with a general branching mechanism. *The Annals of Probability* **27**, (1999), 1099–1134. MR-1733142

[11] Donnelly, P. and Kurtz, T. G.: A countable representation of the Fleming-Viot measure-valued diffusion. *The Annals of Probability* **24**, (1996), 698–742. MR-1404525

[12] Donnelly, P. and Kurtz, T. G.: Particle representations for measure-valued population models, *The Annals of Probability* **27**, (1999), 166–205. MR-1681126

[13] Etheridge, A. and March, P.: A note on superprocesses. *Probability Theory and related Fields* **89**, (1991), 141–147. MR-1110534

[14] Konno, N. and Shiga, T.: Stochastic differential equations for some measure-valued diffusion. *Probability Theory and related Fields* **79**, (1988), 201–225. MR-0958288

[15] Liu, H. and Zhou, X.: Compact support property of the Λ-Fleming-Viot process with underlying Brownian motion. *Electronic Journal of Probability* **17**, (2012), 1–20. MR-2968680

[16] Liu, H. and Zhou, X.: Some support properties for a class of Λ-Fleming-Viot processes. Accepted by *Annales de I'Institut Henri Poincaré (B), Probabilités et Statistiques*. Available at http://arxiv.org/abs/1307.3990.

[17] Perkins, E.: Conditional Dawson-Watanabe processes and Fleming-Viot processes. Seminar on Stochastic Processes 1991, 142–155, *Birkhäuser*, Boston. MR-1172149

[18] Perkins, E.: Measure-Valued branching diffusions and interactions. Proceedings of the International Congress of Mathematicians, Zurich, 1994, 1036–1046, *Birkhäuser Verlag*, Basel. MR-1404003

[19] Perkins E.: Dawson-Watanabe Superprocesses and Measure-Valued Diffusions. *Lecture Notes in Mathematics* **1781**, (2002), 125–329. MR-1915445

[20] Pitman J.: Coalescents with multiple collisions. *The Annals of Probability* **27**, (1999), 1870–1902. MR-1742892

[21] Ruscher, J. G.: Properties of superprocesses and interacting particle systems, Diploma Thesis, Technische Universität Berlin, 2009.

[22] Sagitov, S.: The general coalescent with asynchronous mergers of ancestral lines. *Journal of Applied Probability* **36**, (1999), 1116–1125. MR-1742154

[23] Schweinsberg J.: A necessary and sufficient condition for the Λ-coalescent to come down from infinity. *Electronic Communications in Probability* **5**, (2000), 1–11. MR-1736720

[24] Tribe R.: The connected components of the closed support of super-Brownian motion. *Probability Theory and related Fields* **84**, (1991), 75–87. MR-1109475

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